Transcendence of Generalized Euler Constants

M. Ram Murty and Anastasia Zaytseva

Abstract. We consider a class of analogues of Euler's constant γ and use Baker's theory of linear forms in logarithms to study its arithmetic properties. In particular, we show that with at most one exception, all of these analogues are transcendental.

1. INTRODUCTION. There are two types of numbers: algebraic and transcendental. A complex number is called *algebraic* if it is a root of some algebraic equation with integer coefficients. A complex number that is not algebraic, is called *transcendental*. The theory of transcendental numbers arose in connection with other fundamental questions in mathematics. For example, we can show that the solution of the ancient problem of squaring a circle follows from the fact that the number π is transcendental. Despite the early origin of questions on the arithmetic nature of numbers, the existence of transcendental numbers was not proven until 1844, when Liouville showed that an algebraic number cannot be approximated "too well" by rationals. In other words, if the number is too well approximated by rational numbers, it cannot be algebraic. The first examples of transcendental numbers were constructed in this way. We suggest any book on transcendental number theory for additional details (see [**7**]). Later, in 1874, Cantor showed that the set of all algebraic numbers is countable, whereas the set of all transcendental numbers is uncountable. By this argument, "almost all" numbers are transcendental.

In 1850, Eisenstein (see p. 445 of [**8**]) showed that the set of algebraic numbers forms a field. In particular, the sum and product of two algebraic numbers is again algebraic.

Given some particular constant (for example, a constant obtained as result of some limiting process), it is often very hard to determine whether it is algebraic or transcendental. Let us consider three famous mathematical constants, π, e, and the Euler constant γ, often referred to as the "holy trinity." The transcendence of π and e was shown by Ferdinand Lindemann in 1882 and Charles Hermite in 1873, respectively. On the other hand, despite the efforts of many mathematicians, the arithmetic nature of γ remains a complete mystery even now. We still do not know whether it is algebraic or even irrational.

In this paper we will look at the Euler constant γ and a class of its analogues $\gamma(\Omega)$ defined below, indexed by finite sets of distinct prime numbers such that $\gamma(\varnothing) = \gamma$. These constants $\gamma(\Omega)$ appeared in [**3**] in connection with the Riemann hypothesis. Though we still can't say whether γ is algebraic, we will show that in the infinite list $\{\gamma(\Omega)\}$ all numbers are transcendental with at most one exception. In particular, this implies that if γ is algebraic, then all numbers $\gamma(\Omega)$, $\Omega \neq \varnothing$ are transcendental.

Euler's constant gamma and generalizations. About 280 years ago, Euler introduced an important mathematical constant γ, defined as the following limit:

$$\gamma := \lim_{x \to \infty} \left(\sum_{n=1}^{x} \frac{1}{n} - \log x \right) = 0.577215\ldots.$$

Though less well-known than the numbers e and π, γ is ubiquitous in mathematics. In analysis, the Euler constant appears in the theory of the Γ-function, $\Gamma(z)$. A good example of this is the famous Weierstrass product formula:

$$\frac{1}{\Gamma(z+1)} = e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n} \right) e^{-z/n}.$$

In number theory, γ has connection with the theory of Riemann's ζ-function. Indeed, the constant term in the Laurent expansion of $\zeta(s)$ at $s = 1$ is γ:

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1).$$

As another example, consider the representation

$$\gamma = \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n}$$

where $\zeta(s)$ denotes the Riemann zeta function.

There are many series and integral representations of γ. For more examples we suggest [**4**], which is a new book by Havil devoted to the Euler constant.

There are several generalizations of γ. In 1975, Lehmer [**5**] introduced the class of analogues of the Euler constant described below. Let us fix any natural number q; then for every integer a with $0 < a \leq q$, the limit

$$\gamma(a, q) := \lim_{x \to \infty} \left(\sum_{\substack{n=1 \\ n \equiv a \,(\mathrm{mod}\, q)}}^{x} \frac{1}{n} - \frac{\log x}{q} \right)$$

exists. These numbers, $\gamma(a, q)$, are called Euler–Lehmer constants and have many interesting properties. The constants $\gamma(a, q)$ are closely connected with γ. In particular, $\gamma = 4\gamma(2, 4)$.

At present it is still unknown whether γ or the $\gamma(a, q)$ are transcendental. Murty and Saradha made progress in the study of the arithmetic nature of these numbers in 2010 [**6**]. Using Baker's theory of linear forms of logarithms, they proved the following theorem.

Theorem 1. *In the infinite list of numbers $\{\gamma(a, q)\}$ with $1 \leq a < q$, $q \geq 2$, at most one number is algebraic.*

In particular, though we still do not know whether γ is algebraic or not, Theorem 1 implies that if γ is algebraic, then all numbers $\gamma(a, q)$, $(a, q) \neq (2, 4)$, are transcendental.

Studying the Riemann hypothesis in 2007, Diamond and Ford [**3**] introduced another generalization of Euler's constant arising from integers sieved by finite sets of

primes. More concretely, for a fixed finite set of distinct primes Ω, denote

$$1_\Omega(n) = \begin{cases} 1 & \text{if } \gcd\left(n, \prod_{p \in \Omega} p\right) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\delta_\Omega = \lim_{x \to \infty} \frac{\sum_{n=1}^{x} 1_\Omega(n)}{x}.$$

Then the constant $\gamma(\Omega)$ is defined as the following limit:

$$\gamma(\Omega) := \lim_{x \to \infty} \left(\sum_{n \leq x} \frac{1_\Omega(n)}{n} - \delta_\Omega \log x \right).$$

Note that $\gamma = \gamma(\Omega)$ for $\Omega = \varnothing$. In the case when Ω consists of the first r primes $\{p_1, \ldots, p_r\}$, we replace Ω by Ω_r in the previous notation. We can show that

$$\lim_{r \to \infty} \gamma(\Omega_r) = e^{-\gamma}.$$

We record here two interesting theorems of Diamond and Ford [**3**], connecting the Riemann hypothesis (RH for short) with the generalized Euler constants.

Theorem 2. *Assume RH. Then $\gamma(\Omega_r) > e^{-\gamma}$ for all $r \geq 0$.*

Theorem 3. *Assume RH is false. Then $\gamma(\Omega_r) < e^{-\gamma}$ for infinitely many r.*

In other words, the Riemann hypothesis is equivalent to the statement $\gamma(\Omega_r) > e^{-\gamma}$ for all $r > 0$.

Though we also do not know whether the constants $\{\gamma(\Omega)\}$ are algebraic, the method developed in [**6**] allows us to get a result for the infinite list of numbers $\{\gamma(\Omega)\}$ with the same "all but maybe one" flavor. The goal of this paper is to prove the following theorem.

Theorem 4. *Let S be the set of numbers $\{\gamma(\Omega)\}$, as Ω ranges over all finite sets of distinct primes. Then all numbers of S are transcendental with at most one exception.*

The proof of Theorem 4 involves Baker's theory on linear form of logarithms and some auxillary results on arithmetic functions, which we review in the next section.

2. PRELIMINARIES. We will use the following fundamental theorem [**2**].

Theorem 5 (A. Baker). *Let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ be algebraic numbers with $\alpha_i \neq 0$ for $i = 1, \ldots, n$. Then*

$$\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$$

is either zero or transcendental.

In order to apply Theorem 5, we will get a representation of $\gamma(\Omega)$ in a convenient form. To do this, we will use properties of the Möbius function, which we recall below.

The Möbius function $\mu(n)$, defined as

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes,} \\ 0 & \text{if } p^2 \mid n \text{ for some prime } p, \end{cases}$$

is a multiplicative function, satisfying the following useful relations:

$$\sum_{d\mid n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

and

$$\sum_{d\mid n} \frac{\mu(d)}{d} = \prod_{p\mid n}\left(1 - \frac{1}{p}\right). \tag{2}$$

The following lemma is called the Möbius inversion formula.

Lemma 1 ([1], p. 32). *Let f be any arithmetic multiplicative function. Define a function $h(n)$ by*

$$h(n) = \sum_{d\mid n} f(d).$$

Then the following inversion formula holds:

$$f(n) = \sum_{d\mid n} h(d)\mu\left(\frac{n}{d}\right).$$

If we define von Mangoldt's function $\Lambda(n)$, as

$$\Lambda(n) = \begin{cases} \log p & \text{if for some prime } p \text{ we have } n = p^\alpha, \\ 0 & \text{otherwise,} \end{cases}$$

then $\log n = \sum_{d\mid n} \Lambda(d)$, as is easily verified. Applying the Möbius inversion formula and using (1) gives

$$\Lambda(n) = -\sum_{d\mid n} \mu(d)\log(d).$$

Applying Lemma 1 to the last equation yields:

$$-\mu(n)\log n = \sum_{d\mid n} \Lambda(d)\mu\left(\frac{n}{d}\right). \tag{3}$$

More details on arithmetic functions can be found in [**1**].

Now let us get a representation for $\gamma(\Omega)$ in closed form.

Lemma 2. *Let Ω be a finite set of primes. Then*

$$\gamma(\Omega) = \prod_{p \in \Omega} \left(1 - \frac{1}{p}\right) \left\{\gamma + \sum_{p \in \Omega} \frac{\log p}{p-1}\right\}. \tag{4}$$

Remark 1. It is easy to see that formula (4) remains valid for $\Omega = \varnothing$ if we understand $\prod_{p \in \varnothing}(1 - \frac{1}{p}) = 1$ and $\sum_{p \in \varnothing}\frac{\log p}{p-1} = 0$.

Proof. Denote $P = \prod_{p \in \Omega} p$. By (1) and (2),

$$\delta_\Omega = \lim_{x \to \infty} \frac{1}{x} \sum_{n=1}^{x} \sum_{d \mid \gcd(n,P)} \mu(d)$$

$$= \lim_{x \to \infty} \frac{1}{x} \sum_{d \mid P} \mu(d) \sum_{n_1=1}^{x/d} 1$$

$$= \lim_{x \to \infty} \frac{1}{x} \sum_{d \mid P} \mu(d) \left[\frac{x}{d}\right]$$

$$= \lim_{x \to \infty} \frac{1}{x} \sum_{d \mid P} \mu(d) \left(\frac{x}{d} - \left\{\frac{x}{d}\right\}\right)$$

$$= \sum_{d \mid P} \frac{\mu(d)}{d} = \prod_{p \mid P} \left(1 - \frac{1}{p}\right).$$

In particular, the above formula implies that δ_Ω is a rational number. Similarly,

$$\sum_{n=1}^{x} \frac{1_\Omega(n)}{n} = \sum_{n=1}^{x} \frac{1}{n} \sum_{d \mid \gcd(n,P)} \mu(d)$$

$$= \sum_{d \mid P} \frac{\mu(d)}{d} \sum_{m=1}^{x/d} \frac{1}{m}$$

$$= \sum_{d \mid P} \frac{\mu(d)}{d} (\log \frac{x}{d} + \gamma + o(1))$$

$$= \delta_\Omega \log x - \sum_{d \mid P} \frac{\mu(d) \log d}{d} + \gamma \delta_\Omega + o(1).$$

Using (3),

$$-\sum_{d \mid P} \frac{\mu(d) \log d}{d} = \sum_{ab \mid P} \frac{\Lambda(a)\mu(b)}{ab} = \sum_{p \in \Omega} \frac{\log p}{p} \sum_{pb \mid P} \frac{\mu(b)}{b} = \delta_\Omega \sum_{p \in \Omega} \frac{\log p}{p-1}.$$

Putting everything together, we get formula (4). ∎

52   © THE MATHEMATICAL ASSOCIATION OF AMERICA   [Monthly 120

3. PROOF OF THE MAIN THEOREM (THEOREM 4). We proceed by contradiction. Suppose, that for two different finite sets of primes $\Omega_1 = \{p_1, \ldots, p_n\}$ and $\Omega_2 = \{q_1, \ldots, q_m\}$ both numbers, $\gamma(\Omega_1)$ and $\gamma(\Omega_2)$, are algebraic. Then the difference

$$\delta_{\Omega_2} \gamma(\Omega_1) - \delta_{\Omega_1} \gamma(\Omega_2) \tag{5}$$

is also algebraic. By Lemma 2, this quantity is equal to

$$\delta_{\Omega_2} \sum_{p \in \Omega_1} \frac{\log p}{p-1} - \delta_{\Omega_1} \sum_{p \in \Omega_2} \frac{\log p}{p-1}, \tag{6}$$

i.e., is a $\overline{\mathbb{Q}}$-linear form in logarithms of algebraic numbers. By Theorem 5, the last expression is either zero or transcendental. So to finish the proof it suffices to show that (6) does not vanish. Suppose, on the contrary, that

$$\delta_{\Omega_2} \sum_{p \in \Omega_1} \frac{\log p}{p-1} - \delta_{\Omega_1} \sum_{p \in \Omega_2} \frac{\log p}{p-1} = 0. \tag{7}$$

Denote $c_i = \frac{\delta_{\Omega_2}}{p_i - 1}$ and $d_j = \frac{\delta_{\Omega_1}}{q_j - 1}$ for $i = 1, \ldots, n$, $j = 1, \ldots, m$. It is easy to see that these numbers are rational; we define D to be the product of their denominators. Rewrite (7) in the form

$$\sum_{i=1}^{n} c_i \log p_i = \sum_{j=1}^{m} d_j \log q_j.$$

Exponentiation of the left-hand side and right-hand side of the last equality yields:

$$p_1^{c_1} \times \cdots \times p_n^{c_n} = q_1^{d_1} \times \cdots \times q_m^{d_m}. \tag{8}$$

After taking both parts to the Dth power we get

$$(p_1^{c_1} \times \cdots \times p_n^{c_n})^D = (q_1^{d_1} \times \cdots \times q_m^{d_m})^D. \tag{9}$$

The right-hand side and the left-hand side in expression (9) are prime factorizations of natural numbers. Since, by assumption, the sets $\Omega_1 = \{p_1, \ldots, p_n\}$ and $\Omega_2 = \{q_1, \ldots, q_m\}$ are distinct, the last equality contradicts unique factorization. We have shown that (5) does not vanish, so by Theorem 5 it is transcendental. At most one number from the infinite list $\{\gamma(\Omega)\}$ is algebraic. This completes the proof of Theorem 4. ∎

4. REMARKS. We should mention that it is widely believed that the constant γ is transcendental. Theorem 4 guarantees that at most one number from the infinite list $\{\gamma(\Omega)\}$ is algebraic. We expect all of the $\gamma(\Omega)'s$ to be transcendental.

REFERENCES

1. T. M. Apostol, *Introduction to Analytic Number Theory*, Undergraduate Texts in Mathematics, Springer, New York, 1976.
2. A. Baker, *Transcendental Number Theory*, Cambridge University Press, Cambridge, 1990.
3. H. G. Diamond, K. Ford, Generalized Euler constants, *Math. Proc. Cambridge Philos. Soc.* **145** (2008) 27–41.
4. J. Havil, *Gamma: Exploring Euler's Constant*, Princeton University Press, Princeton, NJ, 2009.
5. D. H. Lehmer, Euler constants for arithmetical progressions, *Acta Arith.* **27** (1975) 125–142.
6. M. R. Murty, N. Saradha, Euler-Lehmer constants and a conjecture of Erdős, *J. Number Theory* **130** (2010) 2671–2682, available at http://dx.doi.org/10.1016/j.jnt.2010.07.004.
7. A. B. Shidlovskii, *Transcendental Numbers*, De Gruyter studies in Mathematics, **012**, Walter de Gruyter, Berlin–New York, 1989.
8. J. Stillwell, *Mathematics and Its History*, third edition, Undergraduate Texts in Mathematics, Springer, 2010.

M. RAM MURTY obtained his Ph.D. from MIT in 1980, under the supervision of Harold Stark. After post-doctoral fellowships at the Institute for Advanced Study in Princeton and the Tata Institute for Fundamental Research in Mumbai, he joined McGill University in 1982. In 1996, he moved to Queen's University, where he holds the Queen's Research Chair in Mathematics and Philosophy. He is Fellow of the Royal Society of Canada and the Indian Science Academy.
Department of Mathematics, Queen's University, Kingston, Ontario, K7L 3N6, Canada
murty@mast.queensu.ca

ANASTASIA ZAYTSEVA graduated from Moscow State University in 2009. Currently, she is doing her doctoral studies at Queen's University.
Department of Mathematics, Queen's University, Kingston, Ontario, K7L 3N6, Canada
zaytsevaan@gmail.com

Yet Another Proof That $\left(1 + \frac{1}{n}\right)^{n+(1/2)}$ Is Decreasing

Let $u_n = \left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}}$. Then for $n \geq 2$,

$$
\begin{aligned}
\frac{u_{n-1}}{u_n} &= \exp\left\{\left(n - \frac{1}{2}\right)\log\left(1 + \frac{1}{n-1}\right) - \left(n + \frac{1}{2}\right)\log\left(1 + \frac{1}{n}\right)\right\} \\
&= \exp\left\{-\left(n - \frac{1}{2}\right)\log\left(1 - \frac{1}{n}\right) - \left(n + \frac{1}{2}\right)\log\left(1 + \frac{1}{n}\right)\right\} \\
&= \exp\left\{-n\left(\log\left(1 + \frac{1}{n}\right) + \log\left(1 - \frac{1}{n}\right)\right) - \frac{1}{2}\left(\log\left(1 + \frac{1}{n}\right) - \log\left(1 - \frac{1}{n}\right)\right)\right\} \\
&= \exp\left\{2n\left(\frac{1}{2n^2} + \frac{1}{4n^4} + \frac{1}{6n^6} + \cdots\right) - \left(\frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} + \cdots\right)\right\} \\
&= \exp\left\{\frac{1}{2 \times 3n^3} + \frac{2}{3 \times 5n^5} + \frac{3}{4 \times 7n^7} + \frac{4}{5 \times 9n^9} + \cdots\right\} \\
&> 1,
\end{aligned}
$$

so u_n is decreasing.

Earlier proofs were given by the author in this MONTHLY, Vol. 118, p. 811 (first line of math), and Vol. 117, p. 880 (this proof would look nicer if n were replaced by $n - 1$ throughout), in response to the article by S. K. Khattri, Vol. 117, pp. 273–277.

—Submitted by Michael D. Hirschhorn