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TRANSCENDENTAL LATTICES AND SUPERSINGULAR REDUCTION LATTICES OF A SINGULAR $K3$ SURFACE

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ABSTRACT. A $K3$ surface X defined over a field k of characteristic 0 is called *singular* if the Néron-Severi lattice $\mathrm{NS}(X)$ of $X \otimes \overline{k}$ is of rank 20. Let X be a singular $K3$ surface defined over a number field F. For each embedding $\sigma : F \hookrightarrow \mathbb{C}$, we denote by $T(X^\sigma)$ the transcendental lattice of the complex $K3$ surface X^σ obtained from X by σ. For each prime $\mathfrak{p}$ of F at which X has a supersingular reduction $X_\mathfrak{p}$, we define $L(X, \mathfrak{p})$ to be the orthogonal complement of $\mathrm{NS}(X)$ in $\mathrm{NS}(X_\mathfrak{p})$. We investigate the relation between these lattices $T(X^\sigma)$ and $L(X, \mathfrak{p})$. As an application, we give a lower bound for the degree of a number field over which a singular $K3$ surface with a given transcendental lattice can be defined.

1. INTRODUCTION

For a smooth projective surface X defined over a field k, we denote by $\mathrm{Pic}(X)$ the Picard group of X, and by $\mathrm{NS}(X)$ the Néron-Severi lattice of $X \otimes \bar{k}$, where $\bar{k}$ is the algebraic closure of k. When X is a $K3$ surface, we have a natural isomorphism $\mathrm{Pic}(X \otimes \bar{k}) \cong \mathrm{NS}(X)$. We say that a $K3$ surface X in characteristic 0 is *singular* if $\mathrm{NS}(X)$ is of rank 20, while a $K3$ surface X in characteristic $p > 0$ is *supersingular* if $\mathrm{NS}(X)$ is of rank 22. It is known ([17], [30], [31]) that every complex singular $K3$ surface is defined over a number field.

For a number field F, we denote by $\mathrm{Emb}(F)$ the set of embeddings of F into $\mathbb{C}$, by $\mathbb{Z}_F$ the integer ring of F, and by $\pi_F : \operatorname{Spec} \mathbb{Z}_F \to \operatorname{Spec} \mathbb{Z}$ the natural projection. Let X be a singular $K3$ surface defined over a number field F, and let $\mathcal{X} \to U$ be a smooth proper family of $K3$ surfaces over a non-empty open subset U of $\operatorname{Spec} \mathbb{Z}_F$ such that the generic fiber is isomorphic to X. We put

$$d(X) := \mathrm{disc}(\mathrm{NS}(X)).$$

Remark that we have $d(X) < 0$ by the Hodge index theorem. For $\sigma \in \mathrm{Emb}(F)$, we denote by X^σ the complex analytic $K3$ surface obtained from X by σ. The *transcendental lattice* $T(X^\sigma)$ of X^σ is defined to be the orthogonal complement of $\mathrm{NS}(X) \cong \mathrm{NS}(X^\sigma)$ in the second Betti cohomology group $\mathrm{H}^2(X^\sigma, \mathbb{Z})$, which we regard as a lattice by the cup-product. Then $T(X^\sigma)$ is an even positive-definite lattice of rank 2 with discriminant $-d(X)$. For a closed point $\mathfrak{p}$ of U, we denote by $X_\mathfrak{p}$ the reduction of $\mathcal{X}$ at $\mathfrak{p}$. Then $X_\mathfrak{p}$ is a $K3$ surface defined over the finite field

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$\kappa_{\mathfrak{p}} := \mathbb{Z}_F/\mathfrak{p}$. For a prime integer p, we put

$$\mathcal{S}_p(\mathcal{X}) \ := \ \{ \ \mathfrak{p} \in U \ \mid \ \pi_F(\mathfrak{p}) = p \text{ and } X_{\mathfrak{p}} \text{ is supersingular } \}.$$

For each $\mathfrak{p} \in \mathcal{S}_p(\mathcal{X})$, we have the specialization homomorphism

$$\rho_{\mathfrak{p}} : \mathrm{NS}(X) \ \to \ \mathrm{NS}(X_{\mathfrak{p}}),$$

which preserves the intersection pairing (see [2, Exp. X], [11, §4] or [12, §20.3]), and hence is injective. We denote by $L(\mathcal{X}, \mathfrak{p})$ the orthogonal complement of $\mathrm{NS}(X)$ in $\mathrm{NS}(X_{\mathfrak{p}})$, and call $L(\mathcal{X}, \mathfrak{p})$ the *supersingular reduction lattice* of $\mathcal{X}$ at $\mathfrak{p}$. Then $L(\mathcal{X}, \mathfrak{p})$ is an even negative-definite lattice of rank 2. We will see that, if $p \nmid 2d(X)$, then the discriminant of $L(\mathcal{X}, \mathfrak{p})$ is $-p^2 d(X)$. For an odd prime integer p not dividing $x \in \mathbb{Z}$, we denote by

$$\chi_p(x) \ := \ \left(\frac{x}{p}\right) \in \{1, -1\}$$

the Legendre character. In [26, Proposition 5.5], we have proved the following. (See Theorem-Definition 1.0.4 for the definition of the Artin invariant.)

Proposition 1.0.1. *Suppose that $p \nmid 2d(X)$.*
(1) *If $\chi_p(d(X)) = 1$, then $\mathcal{S}_p(\mathcal{X})$ is empty.*
(2) *If $\mathfrak{p} \in \mathcal{S}_p(\mathcal{X})$, then the Artin invariant of $X_{\mathfrak{p}}$ is 1.*

The first main result of this paper, which will be proved in §6.5, is as follows:

Theorem 1. *There exists a finite set N of prime integers containing the prime divisors of $2d(X)$ such that the following holds:*

$$p \notin N \ \ \Rightarrow \ \ \mathcal{S}_p(\mathcal{X}) = \begin{cases} \emptyset & \text{if } \chi_p(d(X)) = 1, \\ \pi_F^{-1}(p) & \text{if } \chi_p(d(X)) = -1. \end{cases} \tag{1.0.1}$$

We put $\mathbb{Z}_\infty := \mathbb{R}$. Let R be $\mathbb{Z}$ or $\mathbb{Z}_l$, where l is a prime integer or ∞. An *R-lattice* is a free R-module Λ of finite rank with a non-degenerate symmetric bilinear form

$$(\ , \ ) : \Lambda \times \Lambda \ \to \ R.$$

The *discriminant* $\mathrm{disc}(\Lambda) \in R/(R^\times)^2$ of an R-lattice Λ is the determinant modulo $(R^\times)^2$ of a symmetric matrix expressing $(\ , \ )$.

A $\mathbb{Z}$-lattice is simply called a *lattice*. For a lattice Λ and a non-zero integer n, we denote by $\Lambda[n]$ the lattice obtained from Λ by multiplying the symmetric bilinear form $(\ , \ )$ by n. A lattice Λ is said to be *even* if $(v, v) \in 2\mathbb{Z}$ holds for any $v \in \Lambda$. Let Λ and Λ' be lattices. We denote by $\Lambda \perp \Lambda'$ the orthogonal direct sum of Λ and Λ'. A homomorphism $\Lambda \to \Lambda'$ preserving the symmetric bilinear form is called an *isometry*. Note that an isometry is injective because of the non-degeneracy of the symmetric bilinear forms. An isometry $\Lambda \hookrightarrow \Lambda'$ (or a *sublattice* Λ of Λ') is said to be *primitive* if the cokernel Λ'/Λ is torsion-free. The *primitive closure* of a sublattice $\Lambda \hookrightarrow \Lambda'$ is the intersection of $\Lambda \otimes \mathbb{Q}$ and Λ' in $\Lambda' \otimes \mathbb{Q}$. For an isometry $\Lambda \hookrightarrow \Lambda'$, we put

$$(\Lambda \hookrightarrow \Lambda')^\perp \ := \ \{ \ x \in \Lambda' \ \mid \ (x, y) = 0 \quad \text{for all } y \in \Lambda \ \}.$$

Note that $(\Lambda \hookrightarrow \Lambda')^\perp$ is primitive in Λ'. Let r be a positive integer, and d a non-zero integer. We denote by $\mathcal{L}(r, d)$ the set of isomorphism classes of lattices of rank r with discriminant d, and by $[\Lambda] \in \mathcal{L}(r, d)$ the isomorphism class of a lattice Λ. If $[\Lambda] \in \mathcal{L}(r, d)$, then we have $[\Lambda[n]] \in \mathcal{L}(r, n^r d)$, and the map $\mathcal{L}(r, d) \to \mathcal{L}(r, n^r d)$

given by $[\Lambda] \mapsto [\Lambda[n]]$ is injective. We denote by $\mathcal{L}^{\mathrm{even}}(r,d)$ (resp. $\mathcal{L}^{\mathrm{pos}}(r,d)$) the set of isomorphism classes in $\mathcal{L}(r,d)$ of even lattices (resp. of positive-definite lattices). We recall the notion of *genera* of lattices. See [4], for example, for details. Two lattices Λ and Λ' are said to be *in the same genus* if $\Lambda \otimes \mathbb{Z}_l$ and $\Lambda' \otimes \mathbb{Z}_l$ are isomorphic as $\mathbb{Z}_l$-lattices for any l (including ∞). If Λ and Λ' are in the same genus, then we have $\mathrm{rank}(\Lambda) = \mathrm{rank}(\Lambda')$ and $\mathrm{disc}(\Lambda) = \mathrm{disc}(\Lambda')$. Therefore the set $\mathcal{L}(r,d)$ is decomposed into the disjoint union of genera. For each non-zero integer n, Λ and Λ' are in the same genus if and only if $\Lambda[n]$ and $\Lambda'[n]$ are in the same genus. Moreover, if Λ'' is in the same genus as $\Lambda[n]$, then there exists Λ' in the same genus as Λ such that $[\Lambda''] = [\Lambda'[n]]$ holds. Therefore, for each genus $\mathcal{G} \subset \mathcal{L}(r,d)$, we can define the genus $\mathcal{G}[n] \subset \mathcal{L}(r, n^r d)$ by

$$\mathcal{G}[n] \ := \ \{\ [\Lambda[n]] \ \mid \ [\Lambda] \in \mathcal{G}\ \}.$$

The map from the set of genera in $\mathcal{L}(r,d)$ to the set of genera in $\mathcal{L}(r,n^r d)$ given by $\mathcal{G} \mapsto \mathcal{G}[n]$ is injective. Suppose that Λ and Λ' are in the same genus. If Λ is even (resp. positive-definite), then so is Λ'. Hence $\mathcal{L}^{\mathrm{even}}(r,d)$ and $\mathcal{L}^{\mathrm{pos}}(r,d)$ are also disjoint unions of genera. We say that a genus $\mathcal{G} \subset \mathcal{L}(r,d)$ is *even* (resp. *positive-definite*) if $\mathcal{G} \subset \mathcal{L}^{\mathrm{even}}(r,d)$ (resp. $\mathcal{G} \subset \mathcal{L}^{\mathrm{pos}}(r,d)$) holds.

We review the theory of discriminant forms due to Nikulin [20]. Let Λ be an even lattice. We put $\Lambda^\vee := \mathrm{Hom}(\Lambda, \mathbb{Z})$. Then Λ is embedded into $\Lambda^\vee$ naturally as a submodule of finite index, and there exists a unique $\mathbb{Q}$-valued symmetric bilinear form on $\Lambda^\vee$ that extends the $\mathbb{Z}$-valued symmetric bilinear form on Λ. We put

$$D_\Lambda \ := \ \Lambda^\vee/\Lambda,$$

which is a finite abelian group of order $|\mathrm{disc}(\Lambda)|$, and define a quadratic form

$$q_\Lambda : \ D_\Lambda \ \to \ \mathbb{Q}/2\mathbb{Z}$$

by $q_\Lambda(x+\Lambda) := (x,x) + 2\mathbb{Z}$ for $x \in \Lambda^\vee$. The finite quadratic form (D_Λ, q_Λ) is called the *discriminant form* of Λ.

Theorem-Definition 1.0.2 (Corollary 1.9.4 in [20])**.** *Let Λ and Λ' be even lattices. Then Λ and Λ' are in the same genus if and only if the following hold*:

(i) *$\Lambda \otimes \mathbb{Z}_\infty$ and $\Lambda' \otimes \mathbb{Z}_\infty$ are isomorphic as $\mathbb{Z}_\infty$-lattices, and*
(ii) *the finite quadratic forms (D_Λ, q_Λ) and $(D_{\Lambda'}, q_{\Lambda'})$ are isomorphic.*

Therefore, for an even genus $\mathcal{G}$, we can define the discriminant form $(D_\mathcal{G}, q_\mathcal{G})$ of $\mathcal{G}$.

Next, we define Rudakov-Shafarevich lattices.

Theorem-Definition 1.0.3 (Section 1 of [23])**.** *For each odd prime p and a positive integer $\sigma \leq 10$, there exists, uniquely up to isomorphism, an even lattice $\Lambda_{p,\sigma}$ of rank 22 with signature $(1,21)$ such that the discriminant group is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{\oplus 2\sigma}$. We call $\Lambda_{p,\sigma}$ a* Rudakov-Shafarevich lattice.

Theorem-Definition 1.0.4 (Artin [1], Rudakov-Shafarevich [23])**.** *For a supersingular $K3$ surface Y in odd characteristic p, there exists a positive integer $\sigma \leq 10$, which is called the* Artin invariant of Y*, such that* $\mathrm{NS}(Y)$ *is isomorphic to the Rudakov-Shafarevich lattice $\Lambda_{p,\sigma}$.*

We denote by $(D^{\mathrm{RS}}_{p,\sigma}, q^{\mathrm{RS}}_{p,\sigma})$ the discriminant form of the Rudakov-Shafarevich lattice $\Lambda_{p,\sigma}$. The finite quadratic form $(D^{\mathrm{RS}}_{p,\sigma}, q^{\mathrm{RS}}_{p,\sigma})$ has been calculated explicitly in our previous paper [26, Proof of Proposition 4.2].

Our second main result, which will be proved in §2, is as follows:

Theorem 2. *Let X be a singular $K3$ surface defined over a number field F, and let $\mathcal{X} \to U$ be a smooth proper family of $K3$ surfaces over a non-empty open subset U of $\operatorname{Spec} \mathbb{Z}_F$ such that the generic fiber is isomorphic to X. We put $d(X) := \operatorname{disc}(\mathrm{NS}(X))$.*

(T) *There exists a unique genus $\mathcal{G}_{\mathbb{C}}(X) \subset \mathcal{L}(2, -d(X))$ such that $[\, T(X^\sigma)\,]$ is contained in $\mathcal{G}_{\mathbb{C}}(X)$ for any $\sigma \in \operatorname{Emb}(F)$. This genus $\mathcal{G}_{\mathbb{C}}(X)$ is determined by the properties that it is even, positive-definite, and that the discriminant form is isomorphic to $(D_{\mathrm{NS}(X)}, -q_{\mathrm{NS}(X)})$.*

(L) *Let p be a prime integer not dividing $2d(X)$. Suppose that $\mathcal{S}_p(\mathcal{X}) \neq \emptyset$. Then there exists a unique genus $\mathcal{G}_p(\mathcal{X}) \subset \mathcal{L}(2, -d(X))$ such that $[L(\mathcal{X}, \mathfrak{p})]$ is contained in $\mathcal{G}_p(\mathcal{X})[-p]$ for any $\mathfrak{p} \in \mathcal{S}_p(\mathcal{X})$. This genus $\mathcal{G}_p(\mathcal{X})$ is determined by the properties that it is even, positive-definite, and that the discriminant form of $\mathcal{G}_p(\mathcal{X})[-p]$ is isomorphic to $(D^{\mathrm{RS}}_{p,1}, q^{\mathrm{RS}}_{p,1}) \oplus (D_{\mathrm{NS}(X)}, -q_{\mathrm{NS}(X)})$.*

To ease notation, we put

$$M[a,b,c] \;:=\; \begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix}.$$

Let D be a negative integer. We then put

$$\mathcal{Q}_D \;:=\; \{\; M[a,b,c] \;\mid\; a,b,c \in \mathbb{Z},\;\; a>0,\;\; c>0,\;\; b^2-4ac = D \;\}, \tag{1.0.2}$$

$$\mathcal{Q}^*_D \;:=\; \{\; M[a,b,c] \in \mathcal{Q}_D \;\mid\; \gcd(a,b,c) = 1 \;\}. \tag{1.0.3}$$

The group $GL_2(\mathbb{Z})$ acts on $\mathcal{Q}_D$ from the right by $(M, g) \mapsto g^T M g$ for $M \in \mathcal{Q}_D$ and $g \in GL_2(\mathbb{Z})$, and the subset $\mathcal{Q}^*_D$ of $\mathcal{Q}_D$ is stable by this action. We put

$$\begin{aligned} \mathcal{L}_D \;&:=\; \mathcal{Q}_D/\, GL_2(\mathbb{Z}), \quad \mathcal{L}^*_D \;:=\; \mathcal{Q}^*_D/\, GL_2(\mathbb{Z}), \\ \widetilde{\mathcal{L}}_D \;&:=\; \mathcal{Q}_D/\, SL_2(\mathbb{Z}), \quad \widetilde{\mathcal{L}}^*_D \;:=\; \mathcal{Q}^*_D/\, SL_2(\mathbb{Z}). \end{aligned}$$

Then we have a natural identification

$$\mathcal{L}_D \;=\; \mathcal{L}^{\mathrm{pos}}(2, -D) \cap \mathcal{L}^{\mathrm{even}}(2, -D),$$

and $\widetilde{\mathcal{L}}_D$ is regarded as the set of isomorphism classes of even positive-definite *oriented* lattices of rank 2 with discriminant $-D$.

Let S be a complex $K3$ surface or a complex abelian surface. Suppose that the transcendental lattice $T(S) := (\mathrm{NS}(S) \hookrightarrow \mathrm{H}^2(S, \mathbb{Z}))^\perp$ of S is of rank 2. Then $T(S)$ is even, positive-definite and of discriminant $-d(S)$, where $d(S) := \operatorname{disc}(\mathrm{NS}(S))$. By the Hodge structure

$$T(S) \otimes \mathbb{C} = \mathrm{H}^{2,0}(S) \oplus \mathrm{H}^{0,2}(S)$$

of $T(S)$, we can define a *canonical orientation* on $T(S)$ as follows. An ordered basis (e_1, e_2) of $T(S)$ is said to be *positive* if the imaginary part of $(e_1, \omega_S)/(e_2, \omega_S) \in \mathbb{C}$ is positive, where ω_S is a basis of $\mathrm{H}^{2,0}(S)$. We denote by $\widetilde{T}(S)$ the *oriented transcendental lattice* of S, and by $[\, \widetilde{T}(S)\,] \in \widetilde{\mathcal{L}}_{d(S)}$ the isomorphism class of $\widetilde{T}(S)$. We have the following important theorem due to Shioda and Inose [30]:

Theorem 1.0.5 ([30])**.** *The map $S \mapsto [\, \widetilde{T}(S)\,]$ gives rise to a bijection from the set of isomorphism classes of complex singular $K3$ surfaces S to the set of isomorphism classes of even positive-definite oriented lattices of rank 2.*

If a genus $\mathcal{G} \subset \mathcal{L}_D$ satisfies $\mathcal{G} \cap \mathcal{L}_D^* \neq \emptyset$, then $\mathcal{G} \subset \mathcal{L}_D^*$ holds. Therefore $\mathcal{L}_D^*$ is a disjoint union of genera. For a genus $\mathcal{G} \subset \mathcal{L}_D$, we denote by $\widetilde{\mathcal{G}}$ the pull-back of $\mathcal{G}$ by the natural projection $\widetilde{\mathcal{L}}_D \to \mathcal{L}_D$, and call $\widetilde{\mathcal{G}} \subset \widetilde{\mathcal{L}}_D$ a *lifted genus*.

A negative integer D is called a *fundamental discriminant* if it is the discriminant of an imaginary quadratic field.

Our third main result, which will be proved in §§6.6 and 6.7, is as follows:

Theorem 3. *Let S be a complex singular $K3$ surface. Suppose that $D := \mathrm{disc}(\mathrm{NS}(S))$ is a fundamental discriminant, and that $[T(S)]$ is contained in $\mathcal{L}_D^*$.*

(T) *There exists a singular $K3$ surface X defined over a number field F such that $\{[\,\widetilde{T}(X^\sigma)\,] \,|\, \sigma \in \mathrm{Emb}(F)\}$ is equal to the lifted genus in $\widetilde{\mathcal{L}}_D^*$ that contains $[\widetilde{T}(S)]$. In particular, there exists $\sigma_0 \in \mathrm{Emb}(F)$ such that X^{σ_0} is isomorphic to S over $\mathbb{C}$.*

(L) *Suppose further that D is odd. Then there exists a smooth proper family $\mathcal{X} \to U$ of $K3$ surfaces over a non-empty open subset U of $\operatorname{Spec}\mathbb{Z}_F$, where F is a number field, such that the following hold:*

(i) *the generic fiber X of $\mathcal{X} \to U$ satisfies the property in* (T) *above,*
(ii) *if $p \in \pi_F(U)$, then $p \nmid 2D$, and*
(iii) *if $p \in \pi_F(U)$ and $\chi_p(D) = -1$, then $\mathcal{S}_p(\mathcal{X}) = \pi_F^{-1}(p)$ holds, and the set $\{[\,L(\mathcal{X}, \mathfrak{p})\,] \,|\, \mathfrak{p} \in \mathcal{S}_p(\mathcal{X})\}$ coincides with a genus in $\mathcal{L}(2, -p^2D)$.*

Suppose that D is a negative fundamental discriminant. The set $\widetilde{\mathcal{L}}_D^*$ and its decomposition into lifted genera are very well understood by the work of Gauss. We review the theory briefly. We put $K := \mathbb{Q}(\sqrt{D})$, and denote by $\mathcal{I}_D$ the multiplicative group of non-zero fractional ideals of K, by $\mathcal{P}_D \subset \mathcal{I}_D$ the subgroup of non-zero principal fractional ideals, and by $Cl_D := \mathcal{I}_D/\mathcal{P}_D$ the ideal class group of K. Let I be an element of $\mathcal{I}_D$. We denote by $[I] \in Cl_D$ the ideal class of I. We put

$$N(I) \;:=\; [\mathbb{Z}_K : nI]/n^2,$$

where n is an integer $\neq 0$ such that $nI \subset \mathbb{Z}_K$, and define a bilinear form on I by

$$(x, y) \;:=\; (x\bar{y} + y\bar{x})/N(I) \;=\; \mathrm{Tr}_{K/\mathbb{Q}}(x\bar{y})/N(I). \tag{1.0.4}$$

We say that an ordered basis (ω_1, ω_2) of I as a $\mathbb{Z}$-module is positive if

$$(\omega_1\bar{\omega}_2 - \omega_2\bar{\omega}_1)/\sqrt{D} > 0. \tag{1.0.5}$$

By the bilinear form (1.0.4) and the orientation (1.0.5), the $\mathbb{Z}$-module I of rank 2 obtains a structure of an even positive-definite oriented lattice with discriminant $-D$. The isomorphism class of this oriented lattice is denoted by $\psi(I) \in \widetilde{\mathcal{L}}_D$. For the following, see [5, Theorems 5.2.8 and 5.2.9] and [7, Theorem 3.15], for example.

Proposition 1.0.6. (1) *The map $\psi : \mathcal{I}_D \to \widetilde{\mathcal{L}}_D$ defined above induces a bijection $\Psi : Cl_D \cong \widetilde{\mathcal{L}}_D^*$ with the inverse given by the following. Let $[\Lambda] \in \widetilde{\mathcal{L}}_D^*$ be represented by $M[a, b, c] \in \mathcal{Q}_D^*$, and let $I \in \mathcal{I}_D$ be the fractional ideal generated by $\omega_1 = (-b + \sqrt{D})/2$ and $\omega_2 = a$. Then $\Psi([I]) = [\psi(I)]$ is equal to $[\Lambda]$.*

(2) *Let $[I]$ and $[J]$ be elements of Cl_D. Then $\Psi([I])$ and $\Psi([J])$ are in the same lifted genus if and only if $[I][J]^{-1}$ is contained in $Cl_D^2 := \{x^2 \,|\, x \in Cl_D\}$. In particular, every lifted genus in $\widetilde{\mathcal{L}}_D^*$ consists of the same number of isomorphism classes, and the cardinality is equal to $|Cl_D^2|$.*

Using Theorems 1.0.5 and 3(T), we obtain the following:

Corollary 4. *Let S be a complex singular $K3$ surface such that $D := \mathrm{disc}(\mathrm{NS}(S))$ is a fundamental discriminant and such that $[T(S)]$ is contained in $\mathcal{L}_D^*$. Let Y be a $K3$ surface defined over a number field L such that Y^{τ_0} is isomorphic to S over $\mathbb{C}$ for some $\tau_0 \in \mathrm{Emb}(L)$. Then we have $[L : \mathbb{Q}] \geq |Cl_D^2|$.*

Proof. Let X be the $K3$ surface defined over a number field F given in Theorem 3(T). Then the complex $K3$ surfaces X^{σ_0} and Y^{τ_0} are isomorphic over $\mathbb{C}$, and hence there exists a number field $M \subset \mathbb{C}$ containing both of $\sigma_0(F)$ and $\tau_0(L)$ such that $X \otimes M$ and $Y \otimes M$ are isomorphic over M. Therefore, for each $\sigma \in \mathrm{Emb}(F)$, there exists $\tau \in \mathrm{Emb}(L)$ such that X^σ is isomorphic to Y^τ over $\mathbb{C}$. Since there exist exactly $|Cl_D^2|$ isomorphism classes of complex $K3$ surfaces among X^σ ($\sigma \in \mathrm{Emb}(F)$), we have $|\mathrm{Emb}(L)| \geq |Cl_D^2|$. □

The proof of Theorem 2 is in fact an easy application of Nikulin's theory of discriminant forms, and is given in §2. The main tool of the proof of Theorems 1 and 3 is the Shioda-Inose-Kummer construction [30]. This construction forms a singular $K3$ surface Y from a pair of elliptic curves E' and E. Shioda and Inose [30] proved that, over $\mathbb{C}$, the transcendental lattices of Y and $E' \times E$ are isomorphic. We present their construction in our setting, and show that, over a number field, the supersingular reduction lattices of Y and $E' \times E$ are also isomorphic under certain assumptions. The supersingular reduction lattice of $E' \times E$ is calculated by the specialization homomorphism $\mathrm{Hom}(E', E) \to \mathrm{Hom}(E'_{\mathfrak{p}}, E_{\mathfrak{p}})$. In §3, we investigate the Hom-lattices of elliptic curves. After examining the Kummer construction in §4 and the Shioda-Inose construction in §5, we prove Theorems 1 and 3 in §6. For Theorem 3(T), we use the Shioda-Mitani theory [33]. For Theorem 3(L), we need a description of embeddings of $\mathbb{Z}_K$ into maximal orders of a quaternion algebra over $\mathbb{Q}$. We use Dorman's description [9], which we expound in §7.

In [25], Shafarevich studied, by means of the Shioda-Inose-Kummer construction, number fields over which a singular $K3$ surface with a prescribed Néron-Severi lattice can be defined, and proved a certain finiteness theorem.

The supersingular reduction lattices and their relation to the transcendental lattice were first studied by Shioda [32] for certain $K3$ surfaces. Thanks are due to Professor Tetsuji Shioda for stimulating conversations and many comments.

After the first version of this paper appeared on the e-print archive, Schütt [24] has succeeded in removing the assumptions in Theorem 3(T) and Corollary 4 that $D = \mathrm{disc}(\mathrm{NS}(S))$ be a fundamental discriminant, and that $[T(S)]$ be in $\mathcal{L}_D^*$. Interesting examples of singular $K3$ surfaces defined over number fields are also given in [24, §7].

Applications of Theorem 3(T) to topology and its generalization by Schütt [24] are given in [27] and [28].

The author expresses gratitude to the referee for many comments and suggestions improving the exposition.

Let W be a Dedekind domain. For $P \in \mathrm{Spec}\, W$, we put

$$\kappa_P := \begin{cases} \text{the quotient field of } W & \text{if } P \text{ is the generic point,} \\ W/\mathfrak{p} & \text{if } P \text{ is a closed point } \mathfrak{p}. \end{cases} \tag{1.0.6}$$

2. Proof of Theorem 2

2.1. **The discriminant form of an orthogonal complement.** The following can be derived from [20, Proposition 1.5.1]. We give a simple and direct proof.

Proposition 2.1.1. *Let L be an even lattice, and $M \subset L$ a primitive sublattice. We put $N := (M \hookrightarrow L)^{\perp}$. Suppose that* $\operatorname{disc}(M)$ *and* $\operatorname{disc}(L)$ *are prime to each other. Then there exists an isomorphism*

$$(D_N, q_N) \cong (D_L, q_L) \oplus (D_M, -q_M)$$

of finite quadratic forms. In particular, we have $\operatorname{disc}(N) = \operatorname{disc}(L)\operatorname{disc}(M)$.

Proof. We put $d_L := |\operatorname{disc}(L)| = |D_L|$. The multiplication by d_L induces an automorphism $\delta_L : D_M \xrightarrow{\sim} D_M$ of D_M by the assumption. We regard L, M, N and $L^\vee$, $M^\vee$, $N^\vee$ as submodules of $L \otimes \mathbb{Q} = (M \otimes \mathbb{Q}) \oplus (N \otimes \mathbb{Q})$. First we show that

$$L^\vee \cap M^\vee = M. \tag{2.1.1}$$

The inclusion $\supseteq$ is obvious. Suppose that $x \in L^\vee \cap M^\vee$. Then we have $d_L x \in L$. Since M is primitive in L, we have $L \cap M^\vee = M$, and hence $\delta_L(x + M) = 0$ holds in D_M. Because δ_L is an automorphism of D_M, we have $x \in M$. Next we show that the composite of natural homomorphisms

$$L \hookrightarrow L^\vee \to M^\vee \to D_M \tag{2.1.2}$$

is surjective. Let $\xi \in D_M$ be given. There exists $\eta \in D_M$ such that $\delta_L(\eta) = \xi$. Since $L^\vee \to M^\vee$ is surjective by the primitivity of $M \hookrightarrow L$, there exists a $y \in L^\vee$ that is mapped to η. Then $x := d_L y$ is in L and is mapped to ξ. We define a homomorphism $\tau : D_N \to D_L \oplus D_M$ as follows. Let $x \in N^\vee$ be given. Since $L^\vee \to N^\vee$ is surjective by the primitivity of $N \hookrightarrow L$, there exists a $z \in L^\vee$ that is mapped to x. Let $y \in M^\vee$ be the image of z by $L^\vee \to M^\vee$. We put

$$\tau(x + N) := (z + L, y + M).$$

The well-definedness of τ follows from the formula (2.1.1). Since $z = (y, x)$ in $L^\vee \subset M^\vee \oplus N^\vee$, we have $q_N(x + N) = q_L(z + L) - q_M(y + M)$. The injectivity of τ follows from $L \cap N^\vee = N$. Since the homomorphism (2.1.2) is surjective, the homomorphism τ is also surjective. □

2.2. **The cokernel of the specialization isometry.** Let W be a Dedekind domain with the quotient field F being a number field, and let $\mathcal{X} \to U := \operatorname{Spec} W$ be a smooth proper family of $K3$ surfaces. We put $X := \mathcal{X} \otimes F$. In this subsection, we do *not* assume that $\operatorname{rank}(\mathrm{NS}(X)) = 20$. Let $\mathfrak{p}$ be a closed point of U such that $X_0 := \mathcal{X} \otimes \kappa_{\mathfrak{p}}$ is supersingular. We consider the specialization isometry

$$\rho : \mathrm{NS}(X) = \operatorname{Pic}(X \otimes \overline{F}) \hookrightarrow \mathrm{NS}(X_0) = \operatorname{Pic}(X_0 \otimes \bar{\kappa}_{\mathfrak{p}}),$$

whose definition is given in [2, Exp. X] or [11, §4]. We put $p := \operatorname{char} \kappa_{\mathfrak{p}}$.

Proposition 2.2.1. *Every torsion element of* $\operatorname{Coker}(\rho)$ *has order a power of p.*

Proof. We denote by $\hat{F}$ the completion of F at $\mathfrak{p}$, and by $\hat{A}$ the valuation ring of $\hat{F}$ with the maximal ideal $\hat{\mathfrak{p}}$. Let $\hat{L}$ be a finite extension of $\hat{F}$ with the valuation ring $\hat{B}$, the maximal ideal $\hat{\mathfrak{m}}$, and the residue field $\kappa_{\hat{\mathfrak{m}}}$ such that there exist natural isomorphisms $\operatorname{Pic}(X \otimes \hat{L}) \cong \mathrm{NS}(X)$ and $\operatorname{Pic}(X_0 \otimes \kappa_{\hat{\mathfrak{m}}}) \cong \mathrm{NS}(X_0)$. Then ρ is obtained from the restriction isomorphism

$$\operatorname{Pic}(\mathcal{X} \otimes \hat{B}) \xrightarrow{\sim} \operatorname{Pic}(X \otimes \hat{L})$$

to the generic fiber, whose inverse is given by taking the closure of divisors, and the restriction homomorphism

$$\mathrm{Pic}(\mathcal{X} \otimes \hat{B}) \ \to \ \mathrm{Pic}(X_0 \otimes \kappa_{\hat{\mathfrak{m}}}) \tag{2.2.1}$$

to the central fiber. Therefore it is enough to show that the order of any torsion element of the cokernel of the homomorphism (2.2.1) is a power of p. We put

$$\mathcal{Y} := \mathcal{X} \otimes \hat{B} \quad \text{and} \quad Y_n := \mathcal{Y} \otimes (\hat{B}/\hat{\mathfrak{m}}^{n+1}).$$

Let $\hat{Y}$ be the formal scheme obtained by completing $\mathcal{Y}$ along $Y_0 = X_0 \otimes \kappa_{\hat{\mathfrak{m}}}$. Note that $(\mathcal{Y}, Y_0)$ satisfies the effective Lefschetz condition $\mathrm{Leff}(\mathcal{Y}, Y_0)$ in [15, Exp. X]. (See [16, Theorem 9.7 in Chap. II].) Hence, by [15, Proposition 2.1 in Exp. XI], we have $\mathrm{Pic}(\mathcal{Y}) \cong \mathrm{Pic}(\hat{Y})$. On the other hand, we have $\mathrm{Pic}(\hat{Y}) = \mathrm{proj}\lim_n \mathrm{Pic}(Y_n)$ by [16, Exercise 9.6 in Chap. II]. Let $\mathcal{O}_n$ denote the structure sheaf of Y_n. From the natural exact sequence $0 \to \mathcal{O}_0 \to \mathcal{O}_{n+1}^{\times} \to \mathcal{O}_n^{\times} \to 1$ (see [15, Exp. XI]), we obtain an exact sequence

$$0 \ \to \ \mathrm{Pic}(Y_{n+1}) \ \to \ \mathrm{Pic}(Y_n) \ \to \ \mathrm{H}^2(Y_0, \mathcal{O}_0).$$

In particular, the projective limit of $\mathrm{Pic}(Y_n)$ is equal to $\bigcap_n \mathrm{Pic}(Y_n)$. Since every non-zero element of $\mathrm{H}^2(Y_0, \mathcal{O}_0)$ is of order p, every torsion element of $\mathrm{Pic}(Y_0)/\bigcap_n \mathrm{Pic}(Y_n)$ is of order a power of p. □

Let $\overline{\mathrm{NS}(X)}$ be the primitive closure of $\mathrm{NS}(X)$ in $\mathrm{NS}(X_0)$. Then the index of $\mathrm{NS}(X)$ in $\overline{\mathrm{NS}(X)}$ is a divisor of $\mathrm{disc}(\mathrm{NS}(X))$. Therefore we obtain the following:

Corollary 2.2.2. *If p does not divide* $\mathrm{disc}(\mathrm{NS}(X))$*, then the specialization isometry $\rho : \mathrm{NS}(X) \hookrightarrow \mathrm{NS}(X_0)$ is primitive.*

Remark 2.2.3. Artin [1, §1] showed a similar result over an equal characteristic base. Note that the definition of supersingularity in [1, Definition (0.3)] differs from ours.

2.3. **Proof of Theorem 2.** Let $X \to \mathrm{Spec}\, F$ and $\mathcal{X} \to U$ be as in the statement of Theorem 2. Note that $\mathrm{NS}(X)$ is of signature $(1, 19)$, while the lattice $\mathrm{H}^2(X^\sigma, \mathbb{Z})$ is even, unimodular and of signature $(3, 19)$ for any $\sigma \in \mathrm{Emb}(F)$. Hence $T(X^\sigma)$ is even, positive-definite of rank 2, and its discriminant form is isomorphic to $(D_{\mathrm{NS}(X)}, -q_{\mathrm{NS}(X)})$ by Proposition 2.1.1. Therefore $[T(X^\sigma)]$ is contained in the genus $\mathcal{G} \subset \mathcal{L}_{d(X)}$ characterized by $(D_{\mathcal{G}}, q_{\mathcal{G}}) \cong (D_{\mathrm{NS}(X)}, -q_{\mathrm{NS}(X)})$.

Let $\mathfrak{p}$ be a point of $\mathcal{S}_p(\mathcal{X})$ with $p \nmid 2d(X)$. Since the Artin invariant of $X_{\mathfrak{p}}$ is 1 by Proposition 1.0.1, we have $\mathrm{NS}(X_{\mathfrak{p}}) \cong \Lambda_{p,1}$ by Theorem 1.0.4. Therefore $L(\mathcal{X}, \mathfrak{p})$ is even and negative-definite of rank 2. On the other hand, Corollary 2.2.2 implies that the specialization isometry ρ is primitive, and hence the discriminant form of $L(\mathcal{X}, \mathfrak{p})$ is isomorphic to $(D_{p,1}^{\mathrm{RS}}, q_{p,1}^{\mathrm{RS}}) \oplus (D_{\mathrm{NS}(X)}, -q_{\mathrm{NS}(X)})$ by Proposition 2.1.1. It remains to show that there exists $[M] \in \mathcal{L}_{d(X)}$ such that $L(\mathcal{X}, \mathfrak{p}) \cong M[-p]$, or equivalently, we have $(x, y) \in p\mathbb{Z}$ for any $x, y \in L(\mathcal{X}, \mathfrak{p})$. This follows from the following lemma, whose proof was given in [29].

Lemma 2.3.1. *Let p be an odd prime integer, and L an even lattice of rank* 2*. If the p-part of D_L is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{\oplus 2}$, then $(x, y) \in p\mathbb{Z}$ holds for any $x, y \in L$.*

3. Hom-lattice

3.1. **Preliminaries.** Let E' and E be elliptic curves defined over a field k. We denote by $\mathrm{Hom}_k(E', E)$ the $\mathbb{Z}$-module of homomorphisms from E' to E defined over k, and put

$$\mathrm{Hom}(E', E) := \mathrm{Hom}_{\bar{k}}(E' \otimes \bar{k}, E \otimes \bar{k}),$$
$$\mathrm{End}_k(E) := \mathrm{Hom}_k(E, E) \quad \text{and} \quad \mathrm{End}(E) := \mathrm{Hom}(E, E) = \mathrm{End}_{\bar{k}}(E \otimes \bar{k}).$$

The Zariski tangent space $T_O(E)$ of E at the origin O is a one-dimensional k-vector space, and hence $\mathrm{End}_k(T_O(E))$ is canonically isomorphic to k. By the action of $\mathrm{End}_k(E)$ on $T_O(E)$, we have a representation

$$\mathrm{Lie} : \mathrm{End}_k(E) \to \mathrm{End}_k(T_O(E)) = k.$$

According to [34, §6 in Chap. III], we define a lattice structure on $\mathrm{Hom}(E', E)$ by

$$(f, g) := \deg(f + g) - \deg(f) - \deg(g).$$

We consider the product abelian surface

$$A := E' \times E.$$

Let $O' \in E'$ and $O \in E$ be the origins. We put

$$\xi := [E' \times \{O\}] \in \mathrm{NS}(A), \qquad \eta := [\{O'\} \times E] \in \mathrm{NS}(A), \tag{3.1.1}$$

and denote by $U(A)$ the sublattice of $\mathrm{NS}(A)$ spanned by ξ and η, which is even, unimodular and of signature $(1, 1)$. The following is classical. See [37], for example.

Proposition 3.1.1. *The lattice* $\mathrm{NS}(A)$ *is isomorphic to* $U(A) \perp \mathrm{Hom}(E', E)[-1]$. *In particular, the lattice* $\mathrm{Hom}(E', E)$ *is even and positive-definite.*

One can easily prove the following propositions by means of, for example, the results in [34, §9 in Chap. III] and [36, §3].

Proposition 3.1.2. *Suppose that* $\mathrm{char}\, k = 0$. *Then the following are equivalent:*

(i) $\mathrm{rank}(\mathrm{Hom}(E', E)) = 2$.
(ii) E' *and* E *are isogenous over* $\bar{k}$, *and* $\mathrm{rank}(\mathrm{End}(E')) = 2$.
(iii) *There exists an imaginary quadratic field* K *such that both of* $\mathrm{End}(E') \otimes \mathbb{Q}$ *and* $\mathrm{End}(E) \otimes \mathbb{Q}$ *are isomorphic to* K.

Proposition 3.1.3. *Suppose that* $\mathrm{char}\, k > 0$. *Then the following are equivalent:*

(i) $\mathrm{rank}(\mathrm{Hom}(E', E)) = 4$.
(ii) E' *and* E *are isogenous over* $\bar{k}$, *and* $\mathrm{rank}(\mathrm{End}(E')) = 4$.
(iii) *Both of* E' *and* E *are supersingular.*

3.2. **The elliptic curve E^J.** To the end of §3.4, we work over an algebraically closed field k. For an elliptic curve E, we denote by $k(E)$ the function field of E.

Definition 3.2.1. Two non-zero isogenies $\phi_1 : E \to E_1$ and $\phi_2 : E \to E_2$ are *isomorphic* if there exists an isomorphism $\psi : E_1 \xrightarrow{\sim} E_2$ such that $\psi \circ \phi_1 = \phi_2$ holds, or equivalently, if the subfields $\phi_1^* k(E_1)$ and $\phi_2^* k(E_2)$ of $k(E)$ are equal.

For a non-zero endomorphism $a \in \mathrm{End}(E)$, we denote by E^a the image of a; that is, E^a is an elliptic curve isomorphic to E with an isogeny $a : E \to E^a$. The function field $k(E^a)$ is canonically identified with the subfield $a^* k(E) = \{a^* f \mid f \in k(E)\}$ of $k(E)$, and we have $[k(E) : k(E^a)] = \deg a$.

Definition 3.2.2. Let $J \subset \mathrm{End}(E)$ be a non-zero left-ideal of $\mathrm{End}(E)$. We denote by $k(E^J) \subset k(E)$ the composite of the subfields $k(E^a)$ for all non-zero $a \in J$. Then $k(E^J)$ is a function field of an elliptic curve E^J. We denote by

$$\phi^J : E \to E^J$$

the isogeny corresponding to $k(E^J) \hookrightarrow k(E)$.

Remark 3.2.3. Let $a, b \in \mathrm{End}(E)$ be non-zero. Since $ba(x) = b(a(x))$, we have the canonical inclusions $k(E^{ba}) \subset k(E^a) \subset k(E)$. Hence, if the left ideal J is generated by non-zero elements $a_1, \dots, a_t$, then $k(E^J)$ is the composite of $k(E^{a_1}), \dots, k(E^{a_t})$.

Remark 3.2.4. The isogeny $\phi^J : E \to E^J$ is characterized by the following properties: (i) every $a \in J$ factors through ϕ^J, and (ii) if every $a \in J$ factors through an isogeny $\psi : E \to E'$, then ϕ^J factors through ψ.

3.3. **The** Hom**-lattice in characteristic** 0**.** In this subsection, we assume that $k = \bar{k}$ is of characteristic 0, and that the conditions in Proposition 3.1.2 are satisfied. We denote by D the discriminant of the imaginary quadratic field K in the condition (iii) of Proposition 3.1.2. Note that $\mathrm{End}(E)$ is isomorphic to a $\mathbb{Z}$-subalgebra of $\mathbb{Z}_K$ with $\mathbb{Z}$-rank 2, and that there exist two embeddings of $\mathrm{End}(E)$ into $\mathbb{Z}_K$ as a $\mathbb{Z}$-subalgebra that are conjugate over $\mathbb{Q}$. Each embedding $\mathrm{End}(E) \hookrightarrow \mathbb{Z}_K$ is an isometry of lattices, where $\mathbb{Z}_K$ is considered as a lattice by the formula (1.0.4), because the dual endomorphism corresponds to the conjugate element over $\mathbb{Q}$.

Proposition 3.3.1. *There exist non-zero integers m and n such that*

$$m^2 \operatorname{disc}(\mathrm{Hom}(E', E)) = -n^2 D. \tag{3.3.1}$$

Proof. There exists a non-zero isogeny $\alpha : E \to E'$. Then the map $g \mapsto g \circ \alpha$ induces an isometry Φ_α from $\mathrm{Hom}(E', E)[\deg \alpha]$ to $\mathrm{End}(E)$. Putting $m := \deg \alpha$ and $n := [\mathbb{Z}_K : \mathrm{End}(E)] \cdot |\operatorname{Coker} \Phi_\alpha|$, we obtain the equality (3.3.1). □

Definition 3.3.2. Since $\bar{k} = k$, we have $\mathrm{Lie} : \mathrm{End}(E) \to k$. Suppose that an embedding $i : K \hookrightarrow k$ is fixed. Then an embedding $\iota : \mathrm{End}(E) \hookrightarrow \mathbb{Z}_K$ as a $\mathbb{Z}$-subalgebra is called Lie-*normalized* if $\mathrm{Lie} : \mathrm{End}(E) \to k$ coincides with the composite of $\iota : \mathrm{End}(E) \hookrightarrow \mathbb{Z}_K$, the inclusion $\mathbb{Z}_K \hookrightarrow K$ and $i : K \hookrightarrow k$.

Definition 3.3.3. Suppose that $k = \mathbb{C}$, and that $\mathrm{End}(E) \cong \mathbb{Z}_K$. We fix an embedding $K \hookrightarrow \mathbb{C}$. Let $\Lambda \subset \mathbb{C}$ be a $\mathbb{Z}$-submodule of rank 2 such that $E \cong \mathbb{C}/\Lambda$ as a Riemann surface. For an ideal class $[I]$ of $\mathbb{Z}_K$ represented by a fractional ideal $I \subset K \subset \mathbb{C}$, we denote by $[I] * E$ the complex elliptic curve $\mathbb{C}/I^{-1}\Lambda$, where $I^{-1}\Lambda$ is the $\mathbb{Z}$-submodule of $\mathbb{C}$ generated by $x\lambda$ $(x \in I^{-1}, \lambda \in \Lambda)$. When $I \subseteq \mathbb{Z}_K$, we have $I^{-1}\Lambda \supset \Lambda$, and the identity map $\mathrm{id}_{\mathbb{C}}$ of $\mathbb{C}$ induces an isogeny

$$^{\mathrm{an}}\phi^I : E = \mathbb{C}/\Lambda \to [I] * E = \mathbb{C}/I^{-1}\Lambda.$$

Proposition 3.3.4. *Suppose that $k = \mathbb{C}$, and that $\mathrm{End}(E) \cong \mathbb{Z}_K$. For an ideal $J \subset \mathrm{End}(E)$, the isogeny $\phi^J : E \to E^J$ is isomorphic to $^{\mathrm{an}}\phi^J : E \to [J] * E$, where J is regarded as an ideal of $\mathbb{Z}_K$ by the* Lie*-normalized isomorphism $\mathrm{End}(E) \cong \mathbb{Z}_K$.*

Proof. Suppose that $E = \mathbb{C}/\Lambda$. We choose $\Lambda' \subset \mathbb{C}$ such that $E^J = \mathbb{C}/\Lambda'$ and such that $\phi^J : E = \mathbb{C}/\Lambda \to E^J = \mathbb{C}/\Lambda'$ is given by $\mathrm{id}_{\mathbb{C}}$. For a non-zero $a \in J$, we have $(1/a)\Lambda \supset \Lambda$ and there exists a canonical isomorphism $E^a = \mathbb{C}/(1/a)\Lambda$ such that $a : E \to E^a$ is given by $\mathrm{id}_{\mathbb{C}}$. Therefore Λ' is the largest $\mathbb{Z}$-submodule of $\mathbb{C}$ that is contained in $(1/a)\Lambda$ for any non-zero $a \in J$. Hence we have $\Lambda' = J^{-1}\Lambda$. □

From this analytic description of $\phi^J : E \to E^J$, we obtain the following, which holds in any field of characteristic 0.

Proposition 3.3.5. *Suppose that* $\operatorname{char} k = 0$ *and that* $\operatorname{End}(E) \cong \mathbb{Z}_K$. *Let* J *be an ideal of* $\operatorname{End}(E)$. *Then* $\operatorname{End}(E^J)$ *is also isomorphic to* $\mathbb{Z}_K$. *Moreover,* $\deg \phi^J$ *is equal to* $|\operatorname{End}(E)/J|$, *and the image of the map*

$$\Phi^J : \operatorname{Hom}(E^J, E) \to \operatorname{End}(E)$$

given by $g \mapsto g \circ \phi^J$ *coincides with* J.

3.4. **The** Hom**-lattice of supersingular elliptic curves.** In this subsection, we assume that $k = \bar{k}$ is of characteristic $p > 0$, and that the conditions in Proposition 3.1.3 are satisfied. In particular, E is a supersingular elliptic curve.

We denote by B the quaternion algebra over $\mathbb{Q}$ that ramifies exactly at p and ∞. It is well known that B is unique up to isomorphism. We denote by $x \mapsto x^*$ the canonical involution of B. Then B is equipped with a positive-definite $\mathbb{Q}$-valued symmetric bilinear form defined by

$$(x, y) := xy^* + yx^*. \tag{3.4.1}$$

A subalgebra of B is called an *order* if its $\mathbb{Z}$-rank is 4. An order is said to be *maximal* if it is maximal among orders with respect to the inclusion. If R is an order of B, then the bilinear form (3.4.1) takes values in $\mathbb{Z}$ on R, and R becomes an even lattice. It is known that R is maximal if and only if the discriminant of R is p^2. The following are the classical results due to Deuring [8]. (See also [18, Chapter 13, Theorem 9].)

Proposition 3.4.1. *There exists a maximal order* R *of* B *such that* $\operatorname{End}(E)$ *is isomorphic to* R *as a* $\mathbb{Z}$*-algebra. The canonical involution of* R *corresponds to the involution* $\phi \mapsto \phi^*$ *of* $\operatorname{End}(E)$, *where* ϕ^* *is the dual endomorphism. Hence the lattice* $\operatorname{End}(E)$ *is isomorphic to the lattice* R, *and we have* $\operatorname{disc}(\operatorname{End}(E)) = p^2$.

Conversely, we have the following:

Proposition 3.4.2. *Let* R *be a maximal order of* B. *Then there exists a supersingular elliptic curve* E_R *such that* $\operatorname{End}(E_R)$ *is isomorphic to* R *as a* $\mathbb{Z}$*-algebra.*

We fix an isomorphism $\operatorname{End}(E) \otimes \mathbb{Q} \cong B$ such that $\operatorname{End}(E)$ is mapped to a maximal order R of B. Let J be a non-zero left-ideal of $\operatorname{End}(E)$. Consider the left- and right-orders

$$O_l(J) := \{x \in B \mid xJ \subset J\}, \quad O_r(J) := \{x \in B \mid Jx \subset J\}$$

of J. Since $O_l(J)$ contains R and R is maximal, $O_l(J)$ is maximal, and hence $O_r(J)$ is also maximal by [22, Theorem (21.2)]. In other words, J is a *normal ideal* of B. We denote by $\operatorname{nr}(J)$ the greatest common divisor of the integers

$$\operatorname{nr}(\phi) := \phi\phi^* = \deg \phi \qquad (\phi \in J).$$

(See [22, Corollary (24.12)].) Then, by [22, Theorem (24.11)], we have

$$\operatorname{nr}(J)^2 = |R/J|. \tag{3.4.2}$$

On the other hand, Deuring [8, (2.3)] proved the following:

$$\deg \phi^J = \operatorname{nr}(J). \tag{3.4.3}$$

Proposition 3.4.3. *The image of the map* $\Phi^J : \mathrm{Hom}(E^J, E) \to \mathrm{End}(E)$ *given by* $g \mapsto g \circ \phi^J$ *is equal to* J.

Proof. By Remark 3.2.4, we have $J \subseteq \mathrm{Im}\,\Phi^J$. Suppose that there exists $a \in \mathrm{Im}\,\Phi^J$ such that $a \notin J$. Let J' be the left-ideal of $\mathrm{End}(E)$ generated by J and a. Then we have $\mathrm{nr}(J') < \mathrm{nr}(J)$ by formula (3.4.2). On the other hand, since a factors through ϕ^J, we have $k(E^a) \subset k(E^J)$ and hence $k(E^{J'}) = k(E^J)$. This contradicts Deuring's formula (3.4.3). □

Proposition 3.4.4. *Let* $\psi : E \to E''$ *be a non-zero isogeny, and let*

$$\Psi : \mathrm{Hom}(E'', E) \to \mathrm{End}(E)$$

be the homomorphism of $\mathbb{Z}$*-modules given by* $g \mapsto g \circ \psi$. *We denote by* J_ψ *the image of* Ψ, *which is a left-ideal of* $\mathrm{End}(E)$. *Then* ψ *is equal to* ϕ^{J_ψ}.

Proof. Since $k(E^{g\circ\psi}) \subset k(E'')$ as subfields of $k(E)$ for any non-zero $g \in \mathrm{Hom}(E'', E)$, we have $k(E^{J_\psi}) \subset k(E'')$, and hence $\phi^{J_\psi} : E \to E^{J_\psi}$ factors through $\psi : E \to E''$. The greatest common divisor of the degrees of $g \in \mathrm{Hom}(E'', E)$ is 1 by Proposition 3.6.1 in the next subsection. Hence we have $\mathrm{nr}(J_\psi) = \deg \psi$ by the definition of nr. Since $\deg \phi^{J_\psi} = \mathrm{nr}(J_\psi)$ by Deuring's formula (3.4.3), we have $\psi = \phi^{J_\psi}$ and $E'' = E^{J_\psi}$. □

Corollary 3.4.5. *The map* $J \mapsto \phi^J$ *establishes a one-to-one correspondence between the set of non-zero left-ideals of* $\mathrm{End}(E)$ *and the set of isomorphism classes of non-zero isogenies from* E.

Proposition 3.4.6. *Let* E' *and* E *be supersingular. Then the discriminant of the lattice* $\mathrm{Hom}(E', E)$ *is equal to* p^2.

Proof. Since E' and E are isogenous, there exists a non-zero left-ideal J of $\mathrm{End}(E)$ such that $E' \cong E^J$. Then we have an isomorphism $\mathrm{Hom}(E', E) \cong J$ of $\mathbb{Z}$-modules given by $g \mapsto g \circ \phi^J$, and hence we have $\mathrm{Hom}(E', E)[\deg \phi^J] \cong J$ as a lattice, from which we obtain

$$\mathrm{disc}(\mathrm{Hom}(E', E)) = \frac{\mathrm{disc}(J)}{(\deg \phi^J)^4} = \frac{\mathrm{disc}(\mathrm{End}(E)) \cdot [\mathrm{End}(E) : J]^2}{(\deg \phi^J)^4} = \mathrm{disc}(\mathrm{End}(E))$$

by the formulae (3.4.2) and (3.4.3). Thus we have $\mathrm{disc}(\mathrm{Hom}(E', E)) = p^2$ by Proposition 3.4.1. □

3.5. **The specialization isometry of** Hom**-lattices.** Let E be an elliptic curve defined over a finite extension $L \subset \overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ such that the j-invariant $j(E) \in L$ is integral over $\mathbb{Z}_p$. This condition is satisfied, for example, if $\mathrm{rank}(\mathrm{End}(E)) = 2$. Then E has potentially good reduction; that is, there exist a finite extension $M \subset \overline{\mathbb{Q}}_p$ of L and a smooth proper morphism $\mathcal{E}_M \to \mathrm{Spec}\,\mathbb{Z}_M$ over the valuation ring $\mathbb{Z}_M$ of M such that $\mathcal{E}_M \otimes M$ is isomorphic to $E \otimes M$. Let E_0 be the central fiber of $\mathcal{E}_M$. Then we have a specialization isometry

$$\rho : \mathrm{End}(E) \hookrightarrow \mathrm{End}(E_0),$$

which is obtained from the specialization isometry $\mathrm{NS}(E \times E) \hookrightarrow \mathrm{NS}(E_0 \times E_0)$ and Proposition 3.1.1. The following follows, for example, from the existence and the uniqueness of the Néron model [35, Chap. IV].

Proposition 3.5.1. *The isomorphism class of* E_0 *over* $\overline{\mathbb{F}}_p$ *and the specialization isometry* ρ *do not depend on the choice of* M *and* $\mathcal{E}_M$.

Replacing L by a finite extension if necessary, we assume that

$$\mathrm{End}(E) \;=\; \mathrm{End}_L(E),$$

so that $\mathrm{Lie} : \mathrm{End}(E) \to L$ is defined.

Let E' be another elliptic curve defined over a finite extension $L' \subset \overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ such that $j(E') \in L'$ is integral over $\mathbb{Z}_p$. Then we have a specialization isometry $\rho' : \mathrm{End}(E') \to \mathrm{End}(E'_0)$, where E'_0 is the central fiber of a Néron model of E'. Replacing L' by a finite extension, we assume that $\mathrm{End}(E') = \mathrm{End}_{L'}(E')$. The following is easy to prove.

Proposition 3.5.2. *Suppose that there exists a $g \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ such that $j(E') = j(E)^g$. Then there exist isomorphisms $\mathrm{End}(E) \cong \mathrm{End}(E')$ and $\mathrm{End}(E_0) \cong \mathrm{End}(E'_0)$ induced from g such that the following diagram is commutative:*

$$\begin{array}{ccccccc}
\overline{\mathbb{Q}}_p & \longleftarrow & L & \xleftarrow{\mathrm{Lie}} & \mathrm{End}(E) & \xrightarrow{\rho} & \mathrm{End}(E_0) \\
{\scriptstyle g}\downarrow\wr & & & & \downarrow\wr & & \downarrow\wr \\
\overline{\mathbb{Q}}_p & \longleftarrow & L' & \xleftarrow{\mathrm{Lie}} & \mathrm{End}(E') & \xrightarrow{\rho'} & \mathrm{End}(E'_0).
\end{array}$$

Suppose that $\mathrm{End}(E) \otimes \mathbb{Q}$ is isomorphic to an imaginary quadratic field K. The following result is again due to Deuring [8]. (See also [18, Chapter 13, Theorem 12].)

Proposition 3.5.3. *The elliptic curve E_0 is supersingular if and only if p is inert or ramifies in K.*

We now work over $\overline{\mathbb{Q}}_p$ and assume that $\mathrm{End}(E)$ is isomorphic to $\mathbb{Z}_K$. Suppose that E_0 is supersingular. We put $R := \mathrm{End}(E_0)$. Let J be an ideal of $\mathrm{End}(E)$ and consider the elliptic curve E^J. Since $\mathrm{End}(E^J)$ is also isomorphic to $\mathbb{Z}_K$ by Proposition 3.3.5, the reduction $(E^J)_0$ of E^J is supersingular by Proposition 3.5.3, and we have a reduction

$$\rho(\phi^J) : E_0 \to (E^J)_0$$

of the isogeny $\phi^J : E \to E^J$. On the other hand, we have the left-ideal $R \cdot \rho(J)$ of R generated by $\rho(J) \subset R$, and the associated isogeny

$$\phi^{RJ} : E_0 \to (E_0)^{R\rho(J)}.$$

Proposition 3.5.4. *The isogenies $\rho(\phi^J)$ and ϕ^{RJ} are isomorphic.*

Proof. We choose $a_1, \dots, a_t \in J$ such that J is generated by $a_1, \dots, a_t$ and such that $[\mathrm{End}(E) : J]$ is equal to the greatest common divisor of $\deg a_1, \dots, \deg a_t$. By Proposition 3.3.5, we have $\deg \rho(\phi^J) = \deg \phi^J = [\mathrm{End}(E) : J]$. By Deuring's formula (3.4.3), we see that $\deg \phi^{RJ}$ is a common divisor of $\deg \rho(a_i) = \deg a_i$ for $i = 1, \dots, t$, and hence $\deg \phi^{RJ}$ divides $\deg \rho(\phi^J)$. On the other hand, the left-ideal $R \cdot \rho(J)$ is generated by $\rho(a_1), \dots, \rho(a_t)$, and hence, by Remarks 3.2.3 and 3.2.4, we see that ϕ^{RJ} factors through $\rho(\phi^J)$. Therefore we obtain $\rho(\phi^J) = \phi^{RJ}$. □

By Proposition 3.5.4, the following diagram is commutative:

$$\begin{array}{ccc}
\mathrm{Hom}(E^J, E) & \hookrightarrow & \mathrm{Hom}((E^J)_0, E_0) \\
{\scriptstyle \Phi^J}\downarrow & & \downarrow{\scriptstyle \Phi^{RJ}} \\
\mathrm{End}(E) & \hookrightarrow & \mathrm{End}(E_0),
\end{array}$$

where the horizontal arrows are the specialization isometries. By Propositions 3.3.5 and 3.4.3, we obtain the following:

Proposition 3.5.5. *We put* $d_J := \deg \phi^J = \deg \rho(\phi^J) = \deg \phi^{RJ}$. *Then we have an isomorphism of lattices*

$$(\mathrm{Hom}(E^J, E) \hookrightarrow \mathrm{Hom}((E^J)_0, E_0))^{\perp}[d_J] \;\cong\; (J \hookrightarrow R \cdot \rho(J))^{\perp},$$

where, on the right-hand side, J *and* $R \cdot \rho(J)$ *are regarded as sublattices of the lattices* $\mathrm{End}(E) \cong \mathbb{Z}_K$ *and* $\mathrm{End}(E_0) = R$, *respectively, and* $J \hookrightarrow R \cdot \rho(J)$ *is given by the specialization isometry* $\rho : \mathrm{End}(E) \hookrightarrow R$.

Finally, we state the lifting theorem of Deuring [8]. See also [18, Chapter 13, Theorem 14] and [13, Proposition 2.7].

Proposition 3.5.6. *Let* E_0 *be a supersingular elliptic curve defined over a field* κ_0 *of characteristic* p, *and* α_0 *an endomorphism of* E_0. *Then there exist a smooth proper family of elliptic curves* $\mathcal{E} \to \mathrm{Spec}\, \mathbb{Z}_L$ *over the valuation ring* $\mathbb{Z}_L$ *of a finite extension* L *of* $\mathbb{Q}_p$ *and an endomorphism* α *of* $\mathcal{E}$ *over* $\mathbb{Z}_L$ *such that* $(\mathcal{E}, \alpha) \otimes \bar{\kappa}_{\mathfrak{p}}$ *is isomorphic to* $(E_0, \alpha_0) \otimes \bar{\kappa}_0$, *where* $\mathfrak{p}$ *is the closed point of* $\mathrm{Spec}\, \mathbb{Z}_L$.

3.6. **Application of Tate's theorem [36].** In this subsection, we prove the following result, which was used in the proof of Proposition 3.4.4.

Proposition 3.6.1. *Let* E' *and* E *be supersingular elliptic curves. Then the greatest common divisor of the degrees of* $g \in \mathrm{Hom}(E', E)$ *is* 1.

Proof. Without loss of generality, we can assume that E' and E are defined over a finite field $\mathbb{F}_q$ of characteristic p. Replacing $\mathbb{F}_q$ by a finite extension, we can assume that $\mathrm{End}(E') = \mathrm{End}_{\mathbb{F}_q}(E')$, $\mathrm{End}(E) = \mathrm{End}_{\mathbb{F}_q}(E)$ and $\mathrm{Hom}(E', E) = \mathrm{Hom}_{\mathbb{F}_q}(E', E)$ hold. Let l be a prime integer $\neq p$, and consider the l-adic Tate module $T_l(E')$ of E'. By the famous theorem of Tate [36], we see that

$$\mathrm{End}_{\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)}(T_l(E')) \;\cong\; \mathrm{End}_{\mathbb{F}_q}(E') \otimes \mathbb{Z}_l \;=\; \mathrm{End}(E') \otimes \mathbb{Z}_l$$

is of rank 4, and hence we can assume that the q-th power Frobenius morphism $\mathrm{Frob}_{E'}$ acts on $T_l(E')$ as a scalar multiplication by $\sqrt{q}$. In the same way, we can assume that Frob_E acts on $T_l(E)$ as a scalar multiplication by $\sqrt{q}$. Then, by the theorem of Tate [36] again, we have a natural isomorphism

$$\mathrm{Hom}(E', E) \otimes \mathbb{Z}_l \;\cong\; \mathrm{Hom}(T_l(E'), T_l(E)) \;\cong\; \mathrm{End}_{\mathbb{Z}_l}(\mathbb{Z}_l^{\oplus 2}).$$

Hence there exists a $g \in \mathrm{Hom}(E', E)$ such that $\deg g$ is not divisible by l. Therefore the greatest common divisor of the degrees of $g \in \mathrm{Hom}(E', E)$ is a power of p. Let $F : E' \to E'^{(p)}$ be the p-th power Frobenius morphism of E'. If the degree of $g : E' \to E$ is divisible by p, then g factors as $g' \circ F$ with $\deg g' = \deg g/p$. Therefore it is enough to show the following:

Claim. For any supersingular elliptic curve E in characteristic p, there exists a $g \in \mathrm{Hom}(E, E^{(p)})$ such that $\deg g$ is prime to p.

Note that $j(E) \in \mathbb{F}_{p^2}$ and $j(E^{(p)}) = j(E)^p$. By Proposition 3.5.6, there exists an elliptic curve $E^\sharp$ defined over a finite extension L of $\mathbb{Q}_p$ such that $\mathrm{End}(E^\sharp)$ is of rank 2 and such that $E^\sharp$ has a reduction isomorphic to E at the closed point $\mathfrak{p}$ of $\mathbb{Z}_L$. We assume that L is Galois over $\mathbb{Q}_p$, and fix an embedding $L \hookrightarrow \mathbb{C}$. Then $\mathrm{End}(E^\sharp)$ is an order $\mathcal{O}$ of an imaginary quadratic field, and $E^\sharp \otimes \mathbb{C}$ is isomorphic to $\mathbb{C}/I$ as a Riemann surface for some invertible $\mathcal{O}$-ideal I ([7, Corollary 10.20]). Note

that $j(E^\sharp)$ is a root of the Hilbert class polynomial of the order $\mathcal{O}$ ([7, Proposition 13.2]). There exists an element $\gamma \in \mathrm{Gal}(L/\mathbb{Q}_p)$ such that

$$j(E^\sharp)^\gamma \equiv j(E^\sharp)^p \mod \mathfrak{p}.$$

We put $E^\flat := (E^\sharp)^\gamma$. Then $E^\flat$ has a reduction isomorphic to $E^{(p)}$ at $\mathfrak{p}$, and we have $E^\flat \otimes \mathbb{C} \cong \mathbb{C}/J$ as a Riemann surface for some invertible $\mathcal{O}$-ideal J. The degree of homomorphisms in $\mathrm{Hom}(E^\sharp, E^\flat) = \mathrm{Hom}(\mathbb{C}/I, \mathbb{C}/J)$ is given by a primitive binary form corresponding to the ideal class of the proper $\mathcal{O}$-ideal $I^{-1}J$ by [7, Theorem 7.7]. By [7, Lemma 2.25], we see that $\mathrm{Hom}(E^\sharp, E^\flat)$ has an element whose degree is prime to p. Since the specialization homomorphism $\mathrm{Hom}(E^\sharp, E^\flat) \to \mathrm{Hom}(E, E^{(p)})$ preserves the degree, we obtain the proof. □

4. Kummer construction

We denote by k an algebraically closed field of characteristic $\neq 2$.

4.1. **Double coverings.** We work over k. Let W and Z be smooth projective surfaces, and $\phi : W \to Z$ a finite double covering. Let $\iota : W \xrightarrow{\sim} W$ be the deck-transformation of W over Z. Then we have homomorphisms

$$\phi_* : \mathrm{NS}(W) \to \mathrm{NS}(Z) \quad \text{and} \quad \phi^* : \mathrm{NS}(Z) \to \mathrm{NS}(W).$$

Let $\mathrm{NS}(W)^+_{\mathbb{Q}} \subset \mathrm{NS}(W) \otimes \mathbb{Q}$ be the eigenspace of ι_* with the eigenvalue 1. We put

$$\mathrm{NS}(W)^+ \ := \ \mathrm{NS}(W) \cap \mathrm{NS}(W)^+_{\mathbb{Q}}.$$

When the base field k is $\mathbb{C}$, we assume that $\mathrm{H}^2(W, \mathbb{Z})$ and $\mathrm{H}^2(Z, \mathbb{Z})$ are torsion-free, so that they can be regarded as lattices. We have homomorphisms

$$\phi_* : \mathrm{H}^2(W, \mathbb{Z}) \to \mathrm{H}^2(Z, \mathbb{Z}) \quad \text{and} \quad \phi^* : \mathrm{H}^2(Z, \mathbb{Z}) \to \mathrm{H}^2(W, \mathbb{Z}).$$

Note that ϕ^* preserves the Hodge structure. We define $\mathrm{H}^2(W, \mathbb{Z})^+ := \mathrm{H}^2(W, \mathbb{Z}) \cap \mathrm{H}^2(W, \mathbb{Q})^+$ in the same way as $\mathrm{NS}(W)^+$.

Lemma 4.1.1. *The homomorphism ϕ_* induces an isometry*

$$\phi_*^+ : \mathrm{NS}(W)^+[2] \ \hookrightarrow \ \mathrm{NS}(Z)$$

with a finite 2-elementary cokernel. When $k = \mathbb{C}$, ϕ_* induces an isometry

$$\phi_*^+ : \mathrm{H}^2(W, \mathbb{Z})^+[2] \ \hookrightarrow \ \mathrm{H}^2(Z, \mathbb{Z})$$

with a finite 2-elementary cokernel that preserves the Hodge structure.

Proof. The proof follows immediately from the following:

$$\phi^* \circ \phi_*(w) = w + \iota_*(w), \quad \phi_* \circ \phi^*(z) = 2z, \quad \iota_* \circ \phi^*(z) = \phi^*(z),$$
$$(\phi^*(z_1), \phi^*(z_2)) = 2(z_1, z_2), \quad (\iota_*(w_1), \iota_*(w_2)) = (w_1, w_2).$$

The inverse of the isomorphism $\phi_*^+ \otimes \mathbb{Q}$ is given by $(1/2)\phi^* \otimes \mathbb{Q}$. □

4.2. **Disjoint (-2)-curves.** We continue to work over k. Let $C_1, \ldots, C_m$ be (-2)-curves on a $K3$ surface X that are disjoint from each other, $\Delta \subset \mathrm{NS}(X)$ the sublattice generated by $[C_1], \ldots, [C_m]$, and $\overline{\Delta} \subset \mathrm{NS}(X)$ the primitive closure of Δ. The discriminant group D_Δ of Δ is isomorphic to $\mathbb{F}_2^{\oplus m}$ with basis

$$\gamma_i := -[C_i]/2 + \Delta \quad (i = 1, \ldots, m).$$

For $x = x_1\gamma_1 + \cdots + x_m\gamma_m \in D_\Delta$, we denote by $\mathrm{wt}(x)$ the *Hamming weight* of x, that is, the number of $x_i \in \mathbb{F}_2$ with $x_i \neq 0$. Then $q_\Delta : D_\Delta \to \mathbb{Q}/2\mathbb{Z}$ is given by

$$q_\Delta(x) = (-\mathrm{wt}(x)/2) + 2\mathbb{Z} \in \mathbb{Q}/2\mathbb{Z}.$$

Lemma 4.2.1. *We put $H_\Delta := \overline{\Delta}/\Delta \subset D_\Delta$. Then, for every $x \in H_\Delta$, we have $\mathrm{wt}(x) \equiv 0 \bmod 4$ and $\mathrm{wt}(x) \neq 4$.*

Proof. Since H_Δ is totally isotropic with respect to q_Δ, we have $\mathrm{wt}(x) \equiv 0 \bmod 4$ for any $x \in H_\Delta$. Let $\gamma : X \to Y$ be the contraction of $C_1, \ldots, C_m$, and $\mathcal{L}_Y$ a very ample line bundle on the normal $K3$ surface Y. Then $\{[C_1], \ldots, [C_m]\}$ is a fundamental system of roots in the the root system

$$\{\, r \in \mathrm{NS}(X) \mid (r, [\gamma^*\mathcal{L}_Y]) = 0, \ \ (r, r) = -2 \,\} \tag{4.2.1}$$

of type $m\mathbb{A}_1$. (See [26, Proposition 2.4].) If there were $x \in H_\Delta$ with $\mathrm{wt}(x) = 4$, then there would exist a vector r in the set (4.2.1) such that $r \neq \pm[C_i]$ for any i, which is a contradiction. □

4.3. **Double Kummer pencil.** Let E' and E be elliptic curves defined over k. We put $A := E' \times E$, and denote by $\mathrm{Km}(A)$ the Kummer surface associated with A; that is, $\mathrm{Km}(A)$ is the minimal resolution of the quotient surface $A/\langle\iota_A\rangle$, where $\iota_A : A \xrightarrow{\sim} A$ is the inversion automorphism $x \mapsto -x$. Let u'_i and u_j $(1 \leq i, j \leq 4)$ be the points of order ≤ 2 in E' and E, respectively, and let $\beta_A : \widetilde{A} \to A$ be the blowing-up of A at the fixed points (u'_i, u_j) of ι_A. Let $\varphi_A : \widetilde{A} \to \mathrm{Km}(A)$ denote the natural finite double covering. The involution ι_A lifts to an involution $\tilde{\iota}_A$ of $\widetilde{A}$, and φ_A is the quotient morphism $\widetilde{A} \to \widetilde{A}/\langle\tilde{\iota}_A\rangle = \mathrm{Km}(A)$.

Definition 4.3.1. The diagram

$$\mathrm{Km}(A) \xleftarrow{\varphi_A} \widetilde{A} \xrightarrow{\beta_A} A = E' \times E$$

is called the *Kummer diagram* of E' and E. We denote by $E_{ij} \subset \mathrm{Km}(A)$ the image by φ_A of the exceptional curve of β_A over the point $(u'_i, u_j) \in A$, and by F_j and G_i the image by φ_A of the strict transforms of $E' \times \{u_j\}$ and $\{u'_i\} \times E$, respectively. These (-2)-curves E_{ij}, F_j and G_i on $\mathrm{Km}(A)$ form the configuration depicted in Figure 4.3.1, which is called the *double Kummer pencil* (see [30]).

Let $B_{16} \subset \mathrm{NS}(\widetilde{A})$ be the sublattice generated by the classes of the sixteen (-1)-curves contracted by β_A. Then we have

$$\mathrm{NS}(\widetilde{A}) = \mathrm{NS}(A) \perp B_{16} = U(A) \perp \mathrm{Hom}(E', E)[-1] \perp B_{16}.$$

Since $\tilde{\iota}_A$ acts on $\mathrm{NS}(\widetilde{A})$ trivially, we see that φ_A induces an isometry

$$(\varphi_A)^+_* : U(A)[2] \perp \mathrm{Hom}(E', E)[-2] \perp B_{16}[2] \quad \hookrightarrow \quad \mathrm{NS}(\mathrm{Km}(A))$$

with a finite 2-elementary cokernel. Hence we obtain the following:

Proposition 4.3.2. *We have* $\mathrm{rank}(\mathrm{NS}(\mathrm{Km}(A))) = 18 + \mathrm{rank}(\mathrm{Hom}(E', E))$.

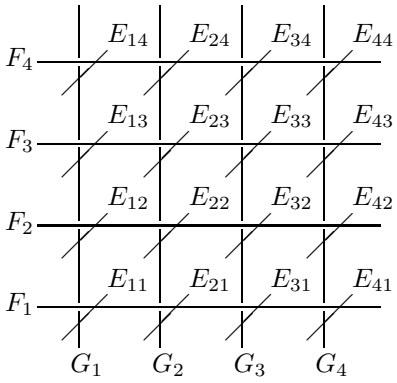

FIGURE 4.3.1. Double Kummer pencil

Recall the basis ξ and η of $U(A)$ defined by (3.1.1). We denote by $\tilde{\xi} \in \mathrm{NS}(\mathrm{Km}(A))$ and $\tilde{\eta} \in \mathrm{NS}(\mathrm{Km}(A))$ the images of ξ and η by $(\varphi_A)_*^+ \circ (\beta_A)^*$, where $(\beta_A)^*$ denotes the total transformation of divisors. Then we have, for any i and j,

$$\tilde{\xi} \,=\, 2[F_j] + \sum_{\mu=1}^{4} [E_{\mu j}] \quad \text{and} \quad \tilde{\eta} \,=\, 2[G_i] + \sum_{\nu=1}^{4} [E_{i\nu}], \tag{4.3.1}$$

and they are orthogonal to $[E_{ij}]$. We have $\tilde{\xi}^2 = \tilde{\eta}^2 = 0$ and $\tilde{\xi}\tilde{\eta} = 2$. We then put

$$N(\mathrm{Km}(A)) \,:=\, \langle \tilde{\xi}, \tilde{\eta} \rangle \perp \langle\, [E_{ij}] \mid 1 \le i, j \le 4 \,\rangle \,\subset\, \mathrm{NS}(\mathrm{Km}(A)),$$

which is the image of $U(A)[2] \perp B_{16}[2]$ by the isometry $(\varphi_A)_*^+$, and we denote by $\overline{N}(\mathrm{Km}(A)) \subset \mathrm{NS}(\mathrm{Km}(A))$ the primitive closure of $N(\mathrm{Km}(A))$.

Proposition 4.3.3. *The lattice $\overline{N}(\mathrm{Km}(A))$ is generated by the classes of (-2)-curves in the double Kummer pencil, and we have $[\overline{N}(\mathrm{Km}(A)) : N(\mathrm{Km}(A))] = 2^7$. In particular, we have $\mathrm{disc}(\overline{N}(\mathrm{Km}(A))) = -2^4$.*

Proof. For simplicity, we put $N := N(\mathrm{Km}(A))$ and $\overline{N} := \overline{N}(\mathrm{Km}(A))$. Let $N' \subset \mathrm{NS}(\mathrm{Km}(A))$ be the sublattice generated by $[E_{ij}]$, $[F_j]$ and $[G_i]$. It is obvious from the equalities (4.3.1) that N' is contained in $\overline{N}$, and it is easy to calculate that $[N' : N] = 2^7$. We will show that $N' = \overline{N}$. Let $\tilde{\xi}^\vee$, $\tilde{\eta}^\vee$ and $[E_{ij}]^\vee (1 \le i, j \le 4)$ be the basis of $N^\vee$ dual to the basis $\tilde{\xi}$, $\tilde{\eta}$ and $[E_{ij}]\,(1 \le i, j \le 4)$ of N. The discriminant group $D_N = N^\vee/N$ is isomorphic to $\mathbb{F}_2^{\oplus 18}$ with basis $\tilde{\xi}^\vee + N$, $\tilde{\eta}^\vee + N$ and $[E_{ij}]^\vee + N$. With respect to this basis, we write an element of D_N by $[\,x, y \,|\, z_{11}, \ldots, z_{44}]$ with $x, y, z_{ij} \in \mathbb{F}_2$. Then $q_N : D_N \to \mathbb{Q}/2\mathbb{Z}$ is given by

$$q_N([\,x, y \,|\, z_{11}, \ldots, z_{44}]) = (\, xy - \mathrm{wt}([z_{11}, \ldots, z_{44}])/2\,) \,+\, 2\mathbb{Z},$$

where $\mathrm{wt}([z_{11}, \ldots, z_{44}])$ is the Hamming weight of $[z_{11}, \ldots, z_{44}] \in \mathbb{F}_2^{\oplus 16}$. We put $H' := N'/N$ and $\overline{H} := \overline{N}/N$. Then we have $H' \subseteq \overline{H}$. By Lemma 4.2.1, if a code word $[\,0, 0 \,|\, z_{11}, \ldots, z_{44}]$ is in $\overline{H}$, then $\mathrm{wt}([z_{11}, \ldots, z_{44}]) \ne 4$ holds. We can confirm by computer that every element v of the finite abelian group D_N of order 2^{18} satisfies the following: if $v \notin H'$, then the linear code $\langle H', v \rangle \subset D_N$ spanned by H' and v is either not totally isotropic with respect to q_N, or containing $[\,0, 0 \,|\, z_{11}, \ldots, z_{44}]$ with $\mathrm{wt}([z_{11}, \ldots, z_{44}]) = 4$. Therefore $\overline{H} = H'$ holds. □

4.4. **The transcendental lattice of** $\mathrm{Km}(A)$**.** In this subsection, we work over $\mathbb{C}$. Then we have $\mathrm{H}^2(\widetilde{A},\mathbb{Z}) = \mathrm{H}^2(A,\mathbb{Z}) \perp B_{16}$. Since $\tilde{\iota}_A$ acts on $\mathrm{H}^2(\widetilde{A},\mathbb{Z})$ trivially, we have an isometry

$$(\varphi_A)_*^+ : \mathrm{H}^2(A,\mathbb{Z})[2] \perp B_{16}[2] \hookrightarrow \mathrm{H}^2(\mathrm{Km}(A),\mathbb{Z})$$

with a finite 2-elementary cokernel. We put

$$\begin{aligned} P(A) &:= (U(A) \perp B_{16} \hookrightarrow \mathrm{H}^2(\widetilde{A},\mathbb{Z}))^\perp = (U(A) \hookrightarrow \mathrm{H}^2(A,\mathbb{Z}))^\perp \quad \text{and} \\ Q(\mathrm{Km}(A)) &:= (\overline{N}(\mathrm{Km}(A)) \hookrightarrow \mathrm{H}^2(\mathrm{Km}(A),\mathbb{Z}))^\perp. \end{aligned}$$

Proposition 4.4.1. *The isometry $(\varphi_A)_*^+$ induces the following commutative diagram, in which the horizontal isomorphisms of lattices preserve the Hodge structure:*

$$\begin{array}{ccc} T(A)[2] & \cong & T(\mathrm{Km}(A)) \\ \big\downarrow & & \big\downarrow \\ P(A)[2] & \cong & Q(\mathrm{Km}(A)). \end{array} \tag{4.4.1}$$

Proof. First we prove that $(\varphi_A)_*^+$ induces $P(A)[2] \cong Q(\mathrm{Km}(A))$. By the definition of $N(\mathrm{Km}(A))$, the isometry $(\varphi_A)_*^+$ maps $(U(A) \perp B_{16})[2]$ to $N(\mathrm{Km}(A))$ isomorphically, and hence $(\varphi_A)_*^+$ induces an isometry from $P(A)[2]$ to $Q(\mathrm{Km}(A))$ with a finite 2-elementary cokernel. Since $U(A) \perp B_{16}$ and $\mathrm{H}^2(\widetilde{A},\mathbb{Z})$ are unimodular, we have $\mathrm{disc}(P(A)[2]) = 2^4$. Since $\mathrm{H}^2(\mathrm{Km}(A),\mathbb{Z})$ is unimodular and $\mathrm{disc}(\overline{N}(\mathrm{Km}(A))) = -2^4$ by Proposition 4.3.3, we have $\mathrm{disc}(Q(\mathrm{Km}(A))) = 2^4$. Therefore the isometry $P(A)[2] \hookrightarrow Q(\mathrm{Km}(A))$ is in fact an isomorphism.

By definition, we have $T(\widetilde{A}) \subset P(A)$ and $T(\mathrm{Km}(A)) \subset Q(\mathrm{Km}(A))$. Since $(\varphi_A)_*^+$ preserves the Hodge structure, the isomorphism $P(A)[2] \cong Q(\mathrm{Km}(A))$ induces $T(A)[2] \cong T(\mathrm{Km}(A))$. □

Remark 4.4.2. The isomorphism $T(A)[2] \cong T(\mathrm{Km}(A))$ was proved in [21, §4] using the sublattice $\langle [E_{ij}] \mid 1 \le i,j \le 4 \rangle$ instead of $N(\mathrm{Km}(A))$. We need the diagram (4.4.1) for the proof of Proposition 4.5.2.

4.5. **The supersingular reduction lattice of** $\mathrm{Km}(A)$**.** Let W be either a number field, or a Dedekind domain with the quotient field F being a number field. We assume that 2 is invertible in W. Let $\mathcal{E}'$ and $\mathcal{E}$ be smooth proper families of elliptic curves over $U := \operatorname{Spec} W$. We put $\mathcal{A} := \mathcal{E}' \times_U \mathcal{E}$.

Definition 4.5.1. A diagram

$$(\mathcal{K}): \quad \mathrm{Km}(\mathcal{A}) \longleftarrow \widetilde{\mathcal{A}} \longrightarrow \mathcal{A} = \mathcal{E}' \times_U \mathcal{E}$$

of schemes and morphisms over U is called the *Kummer diagram* over U of $\mathcal{E}'$ and $\mathcal{E}$ if the following hold:

(i) $\mathrm{Km}(\mathcal{A})$ and $\widetilde{\mathcal{A}}$ are smooth and proper over U,
(ii) $\widetilde{\mathcal{A}} \to \mathcal{A}$ is the blowing-up along the fixed locus (with the reduced structure) of the inversion automorphism $\iota_{\mathcal{A}} : \mathcal{A} \xrightarrow{\sim} \mathcal{A}$ over U, and
(iii) $\mathrm{Km}(\mathcal{A}) \leftarrow \widetilde{\mathcal{A}}$ is the quotient morphism by a lift $\tilde{\iota}_{\mathcal{A}}$ of $\iota_{\mathcal{A}}$.

In this subsection, we consider the case where W is a Dedekind domain.

Suppose that the Kummer diagram $(\mathcal{K})$ over U of $\mathcal{E}'$ and $\mathcal{E}$ is given. Then, at every point P of U (closed or generic, see the definition (1.0.6)), the diagram $(\mathcal{K}) \otimes \bar{\kappa}_P$ is the Kummer diagram of the elliptic curves $\mathcal{E}' \otimes \bar{\kappa}_P$ and $\mathcal{E} \otimes \bar{\kappa}_P$.

Let $\mathfrak{p}$ be a closed point of U with $\kappa := \kappa_{\mathfrak{p}}$ being of characteristic p. Note that $p \neq 2$ by the assumption $1/2 \in W$. We put

$$\begin{array}{llll} E' := \mathcal{E}' \otimes \overline{F}, & E := \mathcal{E} \otimes \overline{F}, & A := E' \times E = \mathcal{A} \otimes \overline{F} & \text{and} \\ E'_0 := \mathcal{E}' \otimes \bar{\kappa}, & E_0 := \mathcal{E} \otimes \bar{\kappa}, & A_0 := E'_0 \times E_0 = \mathcal{A} \otimes \bar{\kappa}. & \end{array} \tag{4.5.1}$$

Then we have $\mathrm{Km}(\mathcal{A}) \otimes \overline{F} = \mathrm{Km}(A)$ and $\mathrm{Km}(\mathcal{A}) \otimes \bar{\kappa} = \mathrm{Km}(A_0)$. We assume that

$$\mathrm{rank}(\mathrm{Hom}(E', E)) = 2 \quad \text{and} \quad \mathrm{rank}(\mathrm{Hom}(E'_0, E_0)) = 4.$$

Then, by Proposition 3.1.1, we have $\mathrm{rank}(\mathrm{NS}(A)) = 4$ and $\mathrm{rank}(\mathrm{NS}(A_0)) = 6$. By Proposition 4.3.2, we see that $\mathrm{Km}(A)$ is singular and $\mathrm{Km}(A_0)$ is supersingular. We consider the supersingular reduction lattices

$$\begin{array}{rcl} L(\mathcal{A}, \mathfrak{p}) & := & (\mathrm{NS}(A) \hookrightarrow \mathrm{NS}(A_0))^{\perp} \quad \text{and} \\ L(\mathrm{Km}(\mathcal{A}), \mathfrak{p}) & := & (\mathrm{NS}(\mathrm{Km}(A)) \hookrightarrow \mathrm{NS}(\mathrm{Km}(A_0)))^{\perp}. \end{array}$$

Note that, by Proposition 3.1.1, we have

$$L(\mathcal{A}, \mathfrak{p}) = (\mathrm{Hom}(E', E) \hookrightarrow \mathrm{Hom}(E'_0, E_0))^{\perp}[-1]. \tag{4.5.2}$$

Proposition 4.5.2. *Suppose that p is prime to* $\mathrm{disc}(\mathrm{NS}(\mathrm{Km}(A)))$. *Then the Kummer diagram $(\mathcal{K})$ induces an isomorphism $L(\mathcal{A}, \mathfrak{p})[2] \cong L(\mathrm{Km}(\mathcal{A}), \mathfrak{p})$.*

We use the following lemma, whose proof is quite elementary and is omitted:

Lemma 4.5.3. *Let $\rho : \Lambda_1 \hookrightarrow \Lambda_2$ be an isometry of lattices. Suppose that ρ maps a sublattice N_1 of Λ_1 to a sublattice N_2 of Λ_2. For $i = 1, 2$, we put $M_i := (N_i \hookrightarrow \Lambda_i)^{\perp}$. If* $\mathrm{rank}(N_1) = \mathrm{rank}(N_2)$, *then ρ maps M_1 into M_2 and we have*

$$(\Lambda_1 \hookrightarrow \Lambda_2)^{\perp} \ = \ (M_1 \hookrightarrow M_2)^{\perp}.$$

Proof of Proposition 4.5.2. Note that the double Kummer pencil is defined on $\mathrm{Km}(\mathcal{A})$ and is flat over U. Hence, by Proposition 4.3.3, the specialization isometry

$$\rho_{\mathrm{Km}(A)} : \mathrm{NS}(\mathrm{Km}(A)) \ \hookrightarrow \ \mathrm{NS}(\mathrm{Km}(A_0))$$

maps $\overline{N}(\mathrm{Km}(A))$ to $\overline{N}(\mathrm{Km}(A_0))$ isomorphically. We put

$$\begin{array}{rcl} \Sigma(\mathrm{Km}(A)) & := & (\overline{N}(\mathrm{Km}(A)) \hookrightarrow \mathrm{NS}(\mathrm{Km}(A)))^{\perp} \quad \text{and} \\ \Sigma(\mathrm{Km}(A_0)) & := & (\overline{N}(\mathrm{Km}(A_0)) \hookrightarrow \mathrm{NS}(\mathrm{Km}(A_0)))^{\perp}. \end{array}$$

Then $\rho_{\mathrm{Km}(A)}$ maps $\Sigma(\mathrm{Km}(A))$ to $\Sigma(\mathrm{Km}(A_0))$, and we have

$$L(\mathrm{Km}(\mathcal{A}), \mathfrak{p}) \ = \ (\Sigma(\mathrm{Km}(A)) \hookrightarrow \Sigma(\mathrm{Km}(A_0)))^{\perp} \tag{4.5.3}$$

by Lemma 4.5.3. The isometry $(\varphi_A)^+_*$ maps $(U(A) \perp B_{16})[2] \subset \mathrm{NS}(A)[2]$ to $N(\mathrm{Km}(A))$ isomorphically. Hence $(\varphi_A)^+_*$ induces an isometry

$$\mathrm{Hom}(E', E)[-2] \ \hookrightarrow \ \Sigma(\mathrm{Km}(A)) \tag{4.5.4}$$

with a finite 2-elementary cokernel. In the same way, we see that $(\varphi_{A_0})^+_*$ induces an isometry

$$\mathrm{Hom}(E'_0, E_0)[-2] \ \hookrightarrow \ \Sigma(\mathrm{Km}(A_0)) \tag{4.5.5}$$

with a finite 2-elementary cokernel. By the equalities (4.5.2) and (4.5.3), it is enough to show that the isometries (4.5.4) and (4.5.5) are isomorphisms.

To show that (4.5.4) is an isomorphism, we choose an embedding σ of $\overline{F}$ into $\mathbb{C}$, and consider the analytic manifolds E'^{σ}, E^{σ}, $\widetilde{A}^{\sigma}$, A^{σ} and $\mathrm{Km}(A^{\sigma}) = \mathrm{Km}(A)^{\sigma}$. By Proposition 3.1.1, we have

$$\mathrm{Hom}(E'^{\sigma}, E^{\sigma})[-1] \;=\; P(A^{\sigma}) \cap \mathrm{NS}(A^{\sigma}) \;=\; (T(A^{\sigma}) \hookrightarrow P(A^{\sigma}))^{\perp},$$

where $P(A^{\sigma})$ is the lattice defined in the previous subsection. By the definition of $\Sigma(\mathrm{Km}(A)^{\sigma})$, we have

$$\Sigma(\mathrm{Km}(A)^{\sigma}) \;=\; Q(\mathrm{Km}(A)^{\sigma}) \cap \mathrm{NS}(\mathrm{Km}(A)^{\sigma}) \;=\; (T(\mathrm{Km}(A)^{\sigma}) \hookrightarrow Q(\mathrm{Km}(A)^{\sigma}))^{\perp},$$

where $Q(\mathrm{Km}(A)^{\sigma})$ is the lattice defined in the previous subsection. Therefore, from (4.4.1), we see that the analytic Kummer diagram $(\mathcal{K})^{\sigma}$ induces

$$\mathrm{Hom}(E'^{\sigma}, E^{\sigma})[-2] \;\cong\; \Sigma(\mathrm{Km}(A)^{\sigma}).$$

Hence the isometry (4.5.4) is an isomorphism.

Since $p \nmid 2\,\mathrm{disc}(\mathrm{NS}(\mathrm{Km}(A)))$, the Artin invariant of $\mathrm{Km}(A_0)$ is 1 by Proposition 1.0.1, and hence $\mathrm{disc}(\mathrm{NS}(\mathrm{Km}(A_0)))$ is equal to $-p^2$. By Proposition 4.3.3, we have $\mathrm{disc}(\overline{N}(\mathrm{Km}(A_0))) = -2^4$, and hence we obtain $\mathrm{disc}(\Sigma(\mathrm{Km}(A_0))) = 2^4p^2$ by Proposition 2.1.1. On the other hand, we have $\mathrm{disc}(\mathrm{Hom}(E'_0, E_0)[-2]) = 2^4p^2$ by Proposition 3.4.6. Comparing the discriminants, we conclude that the isometry (4.5.5) is an isomorphism. □

5. Shioda-Inose construction

We continue to denote by k an algebraically closed field of characteristic $\neq 2$.

5.1. **Shioda-Inose configuration.** Let Z be a $K3$ surface defined over k.

Definition 5.1.1. We say that a pair (C, Θ) of reduced effective divisors on Z is a *Shioda-Inose configuration* if the following hold:

(i) C and Θ are disjoint,
(ii) $C = C_1 + \cdots + C_8$ is an ADE-configuration of (-2)-curves of type $\mathbb{E}_8$,
(iii) $\Theta = \Theta_1 + \cdots + \Theta_8$ is an ADE-configuration of (-2)-curves of type $8\mathbb{A}_1$,
(iv) there exists a class $[\mathcal{L}] \in \mathrm{NS}(Z)$ such that $2[\mathcal{L}] = [\Theta]$.

Let (C, Θ) be a Shioda-Inose configuration on Z. Then there exists a finite double covering $\varphi_Y : \widetilde{Y} \to Z$ that branches exactly along Θ by the condition (iv). Let $T_i \subset \widetilde{Y}$ be the reduced part of the pull-back of Θ_i by φ_Y, which is a (-1)-curve on $\widetilde{Y}$, and let $\beta_Y : \widetilde{Y} \to Y$ be the contraction of $T_1, \ldots, T_8$. Then Y is a $K3$ surface. Let $\tilde{\iota}_Y$ be the deck-transformation of $\widetilde{Y}$ over Z. Then $\tilde{\iota}_Y$ is the lift of an involution $\iota_Y : Y \xrightarrow{\sim} Y$ of Y, which has eight fixed points.

Definition 5.1.2. The diagram

$$(\mathcal{SI}): \quad Y \xleftarrow{\beta_Y} \widetilde{Y} \xrightarrow{\varphi_Y} Z$$

is called the *Shioda-Inose diagram* associated with the Shioda-Inose configuration (C, Θ) on Z. We call Y a *Shioda-Inose surface* of Z.

We denote by $\Gamma \subset \mathrm{NS}(Z)$ the sublattice generated by $[C_1], \ldots, [C_8]$. Then Γ is a negative-definite root lattice of type $\mathbb{E}_8$. In particular, Γ is unimodular. We then denote by $M(Z) \subset \mathrm{NS}(Z)$ the sublattice generated by $[\Theta_1], \ldots, [\Theta_8]$, and by $\overline{M}(Z)$ the primitive closure of $M(Z)$ in $\mathrm{NS}(Z)$. Note that $\overline{M}(Z)$ and Γ are orthogonal in $\mathrm{NS}(Z)$. By condition (iv) of the Shioda-Inose configuration, $\overline{M}(Z)$ contains the

class $[\mathcal{L}] = [\Theta]/2$. In fact, we can see that $\overline{M}(Z)$ is generated by $M(Z)$ and $[\mathcal{L}]$ from the following equality [30, Lemma 3.4]:

$$\operatorname{disc}(\overline{M}(Z)) = 2^6. \tag{5.1.1}$$

Note that one can also prove the equality (5.1.1) easily using Lemma 4.2.1 in the same way as the proof Proposition 4.3.3. We then put

$$\Xi(Z) \ := \ (\Gamma \perp \overline{M}(Z) \hookrightarrow \mathrm{NS}(Z))^{\perp}. \tag{5.1.2}$$

Next we define several sublattices of $\mathrm{NS}(\widetilde{Y})$ and $\mathrm{NS}(Y)$. Since φ_Y is étale in a neighborhood of $C \subset Z$ and C is simply connected, the pull-back of C by φ_Y consists of two connected components $C^{[1]}$ and $C^{[2]}$. Let $\Gamma^{[1]}$ and $\Gamma^{[2]}$ be the sublattices of $\mathrm{NS}(\widetilde{Y})$ generated by the classes of the irreducible components of $C^{[1]}$ and $C^{[2]}$, respectively. We put

$$\widetilde{\Gamma} := \Gamma^{[1]} \perp \Gamma^{[2]}.$$

The sublattice $\widetilde{\Gamma}$ is mapped by β_Y isomorphically to a sublattice of $\mathrm{NS}(Y)$, which we will denote by the same letter $\widetilde{\Gamma}$. We denote by $B_8 \subset \mathrm{NS}(\widetilde{Y})$ the sublattice generated by the classes of the (-1)-curves $[T_1], \ldots, [T_8]$. Then B_8 is orthogonal to $\widetilde{\Gamma}$, and we have a canonical isomorphism $\mathrm{NS}(\widetilde{Y}) \cong \mathrm{NS}(Y) \perp B_8$. We put

$$\Pi(Y) \ := \ (\widetilde{\Gamma} \perp B_8 \hookrightarrow \mathrm{NS}(\widetilde{Y}))^{\perp} = (\widetilde{\Gamma} \hookrightarrow \mathrm{NS}(Y))^{\perp}. \tag{5.1.3}$$

Since $\widetilde{\Gamma}$ and B_8 are unimodular, we have

$$\mathrm{NS}(\widetilde{Y}) = \widetilde{\Gamma} \perp B_8 \perp \Pi(Y) \quad \text{and} \quad \mathrm{NS}(Y) = \widetilde{\Gamma} \perp \Pi(Y). \tag{5.1.4}$$

The action of $\tilde{\iota}_Y$ on $\mathrm{NS}(\widetilde{Y})$ and the action of ι_Y on $\mathrm{NS}(Y)$ preserve the orthogonal direct-sum decompositions (5.1.4), and the action of $\tilde{\iota}_Y$ is trivial on B_8. We put

$$\widetilde{\Gamma}^+ \ := \ \widetilde{\Gamma} \cap \widetilde{\Gamma}^+_{\mathbb{Q}} \quad \text{and} \quad \Pi(Y)^+ \ := \ \Pi(Y) \cap \Pi(Y)^+_{\mathbb{Q}},$$

where $\widetilde{\Gamma}^+_{\mathbb{Q}}$ (resp. $\Pi(Y)^+_{\mathbb{Q}}$) is the eigenspace of $(\tilde{\iota}_Y)_*$ on $\widetilde{\Gamma} \otimes \mathbb{Q}$ (resp. $\Pi(Y) \otimes \mathbb{Q}$) with the eigenvalue 1. Since $\tilde{\iota}_Y$ acts on $\widetilde{\Gamma}$ by interchanging $\Gamma^{[1]}$ and $\Gamma^{[2]}$, we have $\operatorname{rank}(\widetilde{\Gamma}^+) = 8$. By Lemma 4.1.1, we see that φ_Y induces an isometry

$$(\varphi_Y)^+_* : \widetilde{\Gamma}^+[2] \perp B_8[2] \perp \Pi(Y)^+[2] \ \hookrightarrow \ \mathrm{NS}(Z)$$

with a finite 2-elementary cokernel. Since $(\varphi_Y)^+_*$ induces an isometry $\widetilde{\Gamma}^+[2] \hookrightarrow \Gamma$ with a finite 2-elementary cokernel and an isomorphism $B_8[2] \cong M(Z)$, we obtain the following:

Proposition 5.1.3. (1) *We have*

$$\operatorname{rank}(\mathrm{NS}(Y)) = 16 + \operatorname{rank}(\Pi(Y)) \ \geq \ \operatorname{rank}(\mathrm{NS}(Z)) = 16 + \operatorname{rank}(\Pi(Y)^+).$$

(2) *If* $\operatorname{rank}(\mathrm{NS}(Z))$ *is equal to* $\operatorname{rank}(\mathrm{NS}(Y))$, *then* $(\varphi_Y)^+_*$ *induces an isometry* $\Pi(Y)[2] \hookrightarrow \Xi(Z)$ *with a finite* 2*-elementary cokernel.*

5.2. **The transcendental lattice of the Shioda-Inose surface.** In this subsection, we work over $\mathbb{C}$. Note that we have $\mathrm{H}^2(\widetilde{Y}, \mathbb{Z}) = \mathrm{H}^2(Y, \mathbb{Z}) \perp B_8$. We consider the isometry

$$(\varphi_Y)^+_* : \mathrm{H}^2(\widetilde{Y}, \mathbb{Z})^+[2] \ \hookrightarrow \ \mathrm{H}^2(Z, \mathbb{Z})$$

with a finite 2-elementary cokernel. We put

$$\begin{aligned} R(Y) \ &:= \ (\widetilde{\Gamma} \perp B_8 \hookrightarrow \mathrm{H}^2(\widetilde{Y}, \mathbb{Z}))^{\perp} \ = \ (\widetilde{\Gamma} \hookrightarrow \mathrm{H}^2(Y, \mathbb{Z}))^{\perp} \quad \text{and} \\ S(Z) \ &:= \ (\Gamma \perp \overline{M}(Z) \hookrightarrow \mathrm{H}^2(Z, \mathbb{Z}))^{\perp}. \end{aligned}$$

Proposition 5.2.1. *The isometry $(\varphi_Y)^+_*$ induces the following commutative diagram, in which the horizontal isomorphisms of lattices preserve the Hodge structure:*

$$\begin{array}{ccc} T(Y)[2] & \cong & T(Z) \\ \big\downarrow & & \big\downarrow \\ R(Y)[2] & \cong & S(Z). \end{array} \tag{5.2.1}$$

Proof. First we prove $R(Y)[2] \cong S(Z)$. Since $\widetilde{\Gamma}$ and B_8 are unimodular, we have

$$\mathrm{H}^2(\widetilde{Y}, \mathbb{Z}) = \widetilde{\Gamma} \perp B_8 \perp R(Y). \tag{5.2.2}$$

The action of $\tilde{\iota}_Y$ on $\mathrm{H}^2(\widetilde{Y}, \mathbb{Z})$ preserves the decomposition (5.2.2) and is trivial on B_8. Since $\mathrm{rank}(\widetilde{\Gamma}^+) = 8$ and $\mathrm{rank}(\mathrm{H}^2(\widetilde{Y}, \mathbb{Z})^+) = 22$, we see that $\tilde{\iota}_Y$ acts on $R(Y)$ trivially. (This fact was also proved in [30, Lemma 3.2].) We thus obtain an isometry

$$(\varphi_Y)^+_* : \widetilde{\Gamma}^+[2] \perp B_8[2] \perp R(Y)[2] \hookrightarrow \mathrm{H}^2(Z, \mathbb{Z})$$

with a finite 2-elementary cokernel. Since $(\varphi_Y)^+_*$ maps $\widetilde{\Gamma}^+[2]$ to Γ and $B_8[2]$ to $\overline{M}(Z)$ with finite 2-elementary cokernels, it induces an isometry from $R(Y)[2]$ to $S(Z)$ with a finite 2-elementary cokernel. From the decomposition (5.2.2), we have $\mathrm{disc}(R(Y)[2]) = -2^6$. Since $\mathrm{disc}(\overline{M}(Z)) = 2^6$ by the equality (5.1.1), we have $\mathrm{disc}(S(Z)) = -2^6$. Therefore the isometry $R(Y)[2] \hookrightarrow S(Z)$ is in fact an isomorphism.

The proof of the isomorphism $T(Y)[2] \cong T(Z)$ is completely parallel to the second paragraph of the proof of Proposition 4.4.1. □

Remark 5.2.2. The isomorphism $T(Y)[2] \cong T(Z)$ is due to Shioda and Inose [30]. We need the diagram (5.2.1) for the proof of Proposition 5.3.2.

Corollary 5.2.3. *In characteristic* 0*, we have* $\mathrm{rank}(\mathrm{NS}(Y)) = \mathrm{rank}(\mathrm{NS}(Z))$ *and* $2^{22-r}\,\mathrm{disc}(\mathrm{NS}(Y)) = \mathrm{disc}(\mathrm{NS}(Z))$*, where* $r := \mathrm{rank}(\mathrm{NS}(Y)) = \mathrm{rank}(\mathrm{NS}(Z))$.

5.3. **The supersingular reduction lattice of the Shioda-Inose surface.** Let W be either a number field, or a Dedekind domain with the quotient field F being a number field such that $1/2 \in W$. Let $\mathcal{Z}$ be a smooth proper family of $K3$ surfaces over $U := \mathrm{Spec}\, W$.

Definition 5.3.1. A diagram

$$(\mathcal{SI}) : \quad \mathcal{Y} \longleftarrow \widetilde{\mathcal{Y}} \longrightarrow \mathcal{Z}$$

of schemes and morphisms over U is called a *Shioda-Inose diagram* over U if there exists a pair of reduced effective divisors $(\mathcal{C}, \Theta)$ of $\mathcal{Z}$ such that the following hold:

(i) $\mathcal{C}$ and Θ are flat over U,
(ii) $\mathcal{Y}$ and $\widetilde{\mathcal{Y}}$ are smooth and proper over U,
(iii) at each point P of U (closed or generic, see the definition (1.0.6)), the pair of divisors $(\mathcal{C} \otimes \bar{\kappa}_P, \Theta \otimes \bar{\kappa}_P)$ is a Shioda-Inose configuration on $\mathcal{Z} \otimes \bar{\kappa}_P$,
(iv) $\widetilde{\mathcal{Y}} \to \mathcal{Z}$ is a finite double covering that branches exactly along Θ, and
(v) $\mathcal{Y} \leftarrow \widetilde{\mathcal{Y}}$ is a contraction of the inverse image of Θ in $\widetilde{\mathcal{Y}}$.

In this subsection, we consider the case where W is a Dedekind domain.

Suppose that a Shioda-Inose diagram $(\mathcal{SI})$ over U is given. Then, at every point P of U, the diagram $(\mathcal{SI}) \otimes \bar{\kappa}_P$ is a Shioda-Inose diagram of $\mathcal{Z} \otimes \bar{\kappa}_P$.

Let $\mathfrak{p}$ be a closed point of U with $\kappa := \kappa_{\mathfrak{p}}$ being of characteristic p. Note that $p \neq 2$ by the assumption $1/2 \in W$. We put

$$\begin{aligned} Y &:= \mathcal{Y} \otimes \overline{F}, \quad \widetilde{Y} := \widetilde{\mathcal{Y}} \otimes \overline{F}, \quad Z := \mathcal{Z} \otimes \overline{F}, \\ Y_0 &:= \mathcal{Y} \otimes \bar{\kappa}, \quad \widetilde{Y}_0 := \widetilde{\mathcal{Y}} \otimes \bar{\kappa}, \quad Z_0 := \mathcal{Z} \otimes \bar{\kappa}. \end{aligned} \tag{5.3.1}$$

We assume that Z is singular and Z_0 is supersingular. Then Y is singular and Y_0 is supersingular by Proposition 5.1.3. We consider the supersingular reduction lattices

$$L(\mathcal{Z}, \mathfrak{p}) := (\mathrm{NS}(Z) \hookrightarrow \mathrm{NS}(Z_0))^{\perp} \quad \text{and} \quad L(\mathcal{Y}, \mathfrak{p}) := (\mathrm{NS}(Y) \hookrightarrow \mathrm{NS}(Y_0))^{\perp}.$$

By Corollary 5.2.3, we have $4\,\mathrm{disc}(\mathrm{NS}(Y)) = \mathrm{disc}(\mathrm{NS}(Z))$. Since p is odd, the following are equivalent: (i) $p \nmid \mathrm{disc}(\mathrm{NS}(Z))$, and (ii) $p \nmid \mathrm{disc}(\mathrm{NS}(Y))$.

Proposition 5.3.2. *Suppose that p satisfies the conditions* (i) *and* (ii) *above. Then the Shioda-Inose diagram $(\mathcal{SI})$ induces an isomorphism $L(\mathcal{Y}, \mathfrak{p})[2] \cong L(\mathcal{Z}, \mathfrak{p})$.*

Proof. Recall the definition (5.1.3) of Π. Since the specialization isometry $\mathrm{NS}(Y) \hookrightarrow \mathrm{NS}(Y_0)$ maps $\widetilde{\Gamma} \subset \mathrm{NS}(Y)$ to $\widetilde{\Gamma} \subset \mathrm{NS}(Y_0)$ isomorphically, it maps $\Pi(Y)$ to $\Pi(Y_0)$, and we have

$$L(\mathcal{Y}, \mathfrak{p}) \;=\; (\Pi(Y) \hookrightarrow \Pi(Y_0))^{\perp}$$

by Lemma 4.5.3. Recall the definition (5.1.2) of Ξ. Since the specialization isometry $\mathrm{NS}(Z) \hookrightarrow \mathrm{NS}(Z_0)$ maps $\Gamma \subset \mathrm{NS}(Z)$ to $\Gamma \subset \mathrm{NS}(Z_0)$ and $M(Z) \subset \mathrm{NS}(Z)$ to $M(Z_0) \subset \mathrm{NS}(Z_0)$ isomorphically, it maps $\Xi(Z)$ to $\Xi(Z_0)$, and we have

$$L(\mathcal{Z}, \mathfrak{p}) \;=\; (\Xi(Z) \hookrightarrow \Xi(Z_0))^{\perp}$$

by Lemma 4.5.3. By Proposition 5.1.3, we have the isometries

$$\Pi(Y)[2] \;\hookrightarrow\; \Xi(Z) \quad \text{and} \quad \Pi(Y_0)[2] \;\hookrightarrow\; \Xi(Z_0) \tag{5.3.2}$$

with finite 2-elementary cokernels induced by $(\mathcal{SI}) \otimes \overline{F}$ and $(\mathcal{SI}) \otimes \bar{\kappa}$, respectively. It is therefore enough to show that both of the isometries in (5.3.2) are isomorphisms.

We choose an embedding σ of $\overline{F}$ into $\mathbb{C}$, and consider the transcendental lattices $T(Y^{\sigma})$ and $T(Z^{\sigma})$. We have

$$\begin{aligned} \Pi(Y) \;&\cong\; \Pi(Y^{\sigma}) \;=\; R(Y^{\sigma}) \cap \mathrm{NS}(Y^{\sigma}) \;=\; (T(Y^{\sigma}) \hookrightarrow R(Y^{\sigma}))^{\perp} \quad \text{and} \\ \Xi(Z) \;&\cong\; \Xi(Z^{\sigma}) \;=\; S(Z^{\sigma}) \cap \mathrm{NS}(Z^{\sigma}) \;=\; (T(Z^{\sigma}) \hookrightarrow S(Z^{\sigma}))^{\perp}, \end{aligned}$$

where $R(Y^{\sigma})$ and $S(Z^{\sigma})$ are the lattices defined in the previous subsection. Since the analytic Shioda-Inose diagram $(\mathcal{SI})^{\sigma}$ induces the commutative diagram (5.2.1) for Y^{σ} and Z^{σ}, we see that the first isometry of (5.3.2) is an isomorphism.

By Proposition 1.0.1, we have $\mathrm{disc}(\mathrm{NS}(Y_0)) = \mathrm{disc}(\mathrm{NS}(Z_0)) = -p^2$. Since $\widetilde{\Gamma}$ is unimodular, we have $\mathrm{disc}(\Pi(Y_0)) = -p^2$ by Proposition 2.1.1, and hence $\mathrm{disc}(\Pi(Y_0)[2])$ is equal to -2^6p^2. Since Γ is unimodular and $\mathrm{disc}(\overline{M}(Z_0)) = 2^6$ by the equality (5.1.1), we see that $\mathrm{disc}(\Xi(Z_0))$ is equal to -2^6p^2 by Proposition 2.1.1 again. Therefore the second isometry of (5.3.2) is also an isomorphism. □

6. Proof of Theorems 1 and 3

6.1. **Preliminaries.** In this subsection, we quote fundamental facts in algebraic geometry from Grothendieck's FGA [14, no. 221]. See also [10, Chap. 5].

Let S be a noetherian scheme, and let $\mathcal{W}$ and $\mathcal{Z}$ be schemes flat and projective over S. We denote by $\mathfrak{Mor}_S(\mathcal{W},\mathcal{Z})$ the functor from the category $\mathcal{S}ch_S$ of locally noetherian schemes over S to the category of sets such that, for an object T of $\mathcal{S}ch_S$, we have

$$\mathfrak{Mor}_S(\mathcal{W},\mathcal{Z})(T) = \text{the set of } T\text{-morphisms from } \mathcal{W}\times_S T \text{ to } \mathcal{Z}\times_S T.$$

Then we have the following ([14, no. 221, Section 4], [10, Theorem 5.23]):

Theorem 6.1.1. *The functor $\mathfrak{Mor}_S(\mathcal{W},\mathcal{Z})$ is representable by an open subscheme $\mathrm{Mor}_S(\mathcal{W},\mathcal{Z})$ of the Hilbert scheme $\mathrm{Hilb}_{\mathcal{W}\times_S\mathcal{Z}/S}$ parameterizing closed subschemes of $\mathcal{W}\times_S\mathcal{Z}$ flat over S.*

Let F be a number field, and let X and Y be smooth projective varieties defined over F. By the flattening stratification ([10, Theorem 5.12], [19, Lecture 8]), we have a non-empty open subset U of $\operatorname{Spec}\mathbb{Z}_F$ and smooth projective U-schemes $\mathcal{X}$ and $\mathcal{Y}$ such that the generic fibers $\mathcal{X}\times_U F$ and $\mathcal{Y}\times_U F$ are isomorphic to X and Y, respectively. We will consider the schemes

$$\mathrm{Mor}_V(\mathcal{X}_V,\mathcal{Y}_V) = \mathrm{Mor}_U(\mathcal{X},\mathcal{Y})\times_U V$$

for non-empty open subsets V of U, where $\mathcal{X}_V := \mathcal{X}\times_U V$ and $\mathcal{Y}_V := \mathcal{Y}\times_U V$.

Proposition 6.1.2. *Let $\varphi : X \to Y$ be an F-morphism. Then there exist a non-empty open subset $V \subset U$ and a V-morphism $\widetilde{\varphi}_V : \mathcal{X}_V \to \mathcal{Y}_V$ that extends φ. If $\widetilde{\varphi}_{V'} : \mathcal{X}_{V'} \to \mathcal{Y}_{V'}$ is a morphism over a non-empty open subset $V' \subset U$ that extends φ, then $\widetilde{\varphi}_V|_{V\cap V'} = \widetilde{\varphi}_{V'}|_{V\cap V'}$ holds, where $\widetilde{\varphi}_V|_{V\cap V'}$ and $\widetilde{\varphi}_{V'}|_{V\cap V'}$ denote the restrictions of $\widetilde{\varphi}_V$ and $\widetilde{\varphi}_{V'}$ to $\mathcal{X}_{V\cap V'}$.*

Proof. We denote by $[\varphi] : \operatorname{Spec} F \to \mathrm{Mor}_U(\mathcal{X},\mathcal{Y})$ the U-morphism corresponding to $\varphi : X \to Y$. Let Φ be the Hilbert polynomial of the graph $\Gamma(\varphi) \subset X\times_F Y$ of φ with respect to a relatively ample invertible sheaf $\mathcal{O}(1)$ of $\mathcal{X}\times_U\mathcal{Y} \to U$, so that $[\varphi]$ is an F-rational point of $\mathrm{Mor}_U(\mathcal{X},\mathcal{Y}) \cap H^\Phi$, where $H^\Phi := \mathrm{Hilb}^\Phi_{\mathcal{X}\times_U\mathcal{Y}/U}$ is the Hilbert scheme parameterizing closed subschemes of $\mathcal{X}\times_U\mathcal{Y}$ flat over U with the Hilbert polynomial of fibers with respect to $\mathcal{O}(1)$ being equal to Φ. Since H^Φ is projective over U, the morphism $[\varphi]$ extends to a morphism $[\varphi]^\sim_U : U \to H^\Phi$. Since $\mathrm{Mor}_U(\mathcal{X},\mathcal{Y})\cap H^\Phi$ is open in H^Φ, there exists a non-empty open subset V of U such that $[\varphi]$ extends to a U-morphism

$$[\varphi]^\sim_V : V \to \mathrm{Mor}_U(\mathcal{X},\mathcal{Y}).$$

Hence the existence of a morphism $\widetilde{\varphi}_V : \mathcal{X}_V \to \mathcal{Y}_V$ extending φ over some non-empty open subset $V \subset U$ is proved. The equality $\widetilde{\varphi}_V|_{V\cap V'} = \widetilde{\varphi}_{V'}|_{V\cap V'}$ follows from the fact that $H^\Phi \to U$ is separated. □

We call $\widetilde{\varphi}_V$ the *extension of φ over V*. By the uniqueness of the extension, we obtain the following:

Corollary 6.1.3. *Let $\mathcal{Z}$ be a smooth projective U-scheme with the generic fiber Z. Let $\psi : Y \to Z$ be an F-morphism, and let $\widetilde{\psi}_{V'} : \mathcal{Y}_{V'} \to \mathcal{Z}_{V'}$ be the extension of ψ over a non-empty open subset $V' \subset U$. Then $(\widetilde{\psi}_{V'}|_{V\cap V'}) \circ (\widetilde{\varphi}_V|_{V\cap V'})$ is the extension of $\psi\circ\varphi : X \to Z$ over $V \cap V'$.*

Applying Corollary 6.1.3 to an F-isomorphism and its inverse, we obtain the following, which plays a key role in the proof of Theorem 1:

Corollary 6.1.4. *If X and Y are isomorphic over F, then there exists a non-empty open subset $V \subset U$ such that $\mathcal{X}_V$ and $\mathcal{Y}_V$ are isomorphic over V.*

We give three applications that will be used in the proof of Proposition 6.3.2.

Example 6.1.5. Let Q be an F-rational point of Y, and $\varphi : X \to Y$ the blowing-up of Y at Q, which is defined over F. By shrinking U if necessary, we can assume that Q is the generic fiber of a closed subscheme $\mathcal{Q} \subset \mathcal{Y}$ that is smooth over U. Let $\beta_U : \mathcal{X}' \to \mathcal{Y}$ be the blowing-up of $\mathcal{Y}$ along $\mathcal{Q}$, which is defined over U. Then the restriction $\beta_\eta : X' \to Y$ of β_U to the generic fiber X' of $\mathcal{X}' \to U$ is isomorphic to φ; that is, there exists an F-isomorphism $\tau : X' \xrightarrow{\sim} X$ such that $\beta_\eta = \phi \circ \tau$. Hence, by Corollaries 6.1.3 and 6.1.4, there exists a non-empty open subset $V \subset U$ such that the restriction $\beta_V : \mathcal{X}'_V \to \mathcal{Y}_V$ of β_U to $\mathcal{X}'_V$ coincides with the composite of the V-isomorphism $\widetilde{\tau}_V : \mathcal{X}'_V \xrightarrow{\sim} \mathcal{X}_V$ and the extension $\widetilde{\varphi}_V : \mathcal{X}_V \to \mathcal{Y}_V$ of φ over V.

Example 6.1.6. Let D be a reduced smooth divisor of Y such that every irreducible component D_i of D is defined over F. By shrinking U if necessary, we can assume that each D_i is the generic fiber of a closed subscheme $\mathcal{D}_i \subset \mathcal{Y}$ that is smooth over U. We can also assume that these $\mathcal{D}_i$ are mutually disjoint. Then $\mathcal{D} := \sum \mathcal{D}_i$ is smooth over U.

Proposition 6.1.7. *Let $\varphi : X \to Y$ be an F-morphism that is a double covering branching exactly along D. Then there exists an open subset $V \neq \emptyset$ of U such that the extension of φ over V is a double covering of $\mathcal{Y}_V$ branching exactly along $\mathcal{D}_V$.*

Proof. Let L be an invertible sheaf on Y defined by the exact sequence

$$0 \ \to \ \mathcal{O}_Y \ \to \ \varphi_* \mathcal{O}_X \ \to \ L^{-1} \ \to 0.$$

Then L is defined over F, and we have an isomorphism

$$\rho \ : \ L^{\otimes 2} \ \xrightarrow{\sim} \ \mathcal{O}_Y(D)$$

on Y that corresponds to the double covering φ in the way described in [6, Chap. 0]. There exist a non-empty open subset V of U and an invertible sheaf $\mathcal{L}$ on $\mathcal{Y}_V$ such that $\mathcal{L} \otimes_{\mathcal{O}_{\mathcal{Y}_V}} \mathcal{O}_Y = L$. We consider the invertible sheaves

$$\mathcal{M} := \mathcal{H}om_{\mathcal{O}_{\mathcal{Y}_V}}(\mathcal{L}^{\otimes 2}, \mathcal{O}_{\mathcal{Y}_V}(\mathcal{D}_V)) \quad \text{and} \quad M := \mathcal{H}om_{\mathcal{O}_Y}(L^{\otimes 2}, \mathcal{O}_Y(D))$$

on $\mathcal{Y}_V$ and Y, respectively. Then we have $M = \mathcal{M} \otimes_{\mathcal{O}_{\mathcal{Y}_V}} \mathcal{O}_Y \cong \mathcal{O}_Y$. By [16, Proposition 9.3 in Chap. III], the restriction homomorphisms

$$H^0(\mathcal{Y}_V, \mathcal{M}) \to H^0(Y, M) \quad \text{and} \quad H^0(\mathcal{Y}_V, \mathcal{M}^{-1}) \to H^0(Y, M^{-1})$$

to the generic fiber Y induce isomorphisms

$$H^0(\mathcal{Y}_V, \mathcal{M}) \otimes_R F \cong H^0(Y, M) \quad \text{and} \quad H^0(\mathcal{Y}_V, \mathcal{M}^{-1}) \otimes_R F \cong H^0(Y, M^{-1}),$$

where $R := \Gamma(V, \mathcal{O}_V)$. Hence, by shrinking $V = \operatorname{Spec} R$, we have elements $f \in H^0(\mathcal{Y}_V, \mathcal{M})$ and $g \in H^0(\mathcal{Y}_V, \mathcal{M}^{-1})$ that restrict to ρ and ρ^{-1}, respectively. Then the composites $f \circ g$ and $g \circ f$, considered as elements of $H^0(\mathcal{Y}_V, \mathcal{M} \otimes \mathcal{M}^{-1}) = R$, are mapped to the $1 \in H^0(Y, M \otimes M^{-1}) = F$. Since $R \hookrightarrow F$, we see that f and g are isomorphisms. Thus ρ extends to an isomorphism

$$\widetilde{\rho} \ : \ \mathcal{L}^{\otimes 2} \ \xrightarrow{\sim} \ \mathcal{O}_{\mathcal{Y}_V}(\mathcal{D}_V).$$

By means of $\widetilde{\rho}$, a double covering $\delta_V : \mathcal{X}'_V \to \mathcal{Y}_V$ that branches exactly along $\mathcal{D}_V$ is constructed as a closed subscheme of the line bundle on $\mathcal{Y}_V$ corresponding to the invertible sheaf $\mathcal{L}$. By construction, the restriction $\delta_\eta : X' \to Y$ of δ_V to the generic fiber is isomorphic to $\varphi : X \to Y$. By Corollaries 6.1.3 and 6.1.4, it follows that, making V smaller if necessary, we have a V-isomorphism $\mathcal{X}'_V \cong \mathcal{X}_V$ under which δ_V coincides with the extension $\widetilde{\varphi}_V$ of φ over V. □

Example 6.1.8. In this example, we assume $1/2 \in R := \Gamma(U, \mathcal{O}_U)$. Let $\iota : X \xrightarrow{\sim} X$ be an involution defined over F, and $\varphi : X \to Y$ the quotient morphism by the group $\langle \iota \rangle$. Suppose that the extension $\widetilde{\iota}_U : \mathcal{X} \xrightarrow{\sim} \mathcal{X}$ of ι over U exists. Then $\widetilde{\iota}_U$ is an involution over U by Corollary 6.1.3. Let $q_U : \mathcal{X} \to \mathcal{Y}'$ be the quotient morphism by the group $\langle \widetilde{\iota}_U \rangle$, which is defined over U by $1/2 \in R$. Then, by Corollaries 6.1.3, 6.1.4 and Lemma 6.1.9 below, we have a non-empty open subset $V \subset U$ and a V-isomorphism $\mathcal{Y}_V \cong \mathcal{Y}'_V$ under which the extension $\widetilde{\varphi}_V$ of φ over V coincides with the restriction $q_V : \mathcal{X}_V \to \mathcal{Y}'_V$ of q_U to $\mathcal{X}_V$.

Lemma 6.1.9. *Let A be an R-algebra on which an involution i acts. Then we have $A^{\langle i \rangle} \otimes_R F = (A \otimes_R F)^{\langle i \rangle}$, where $A^{\langle i \rangle} := \{a \in A \mid i(a) = a\}$.*

Proof. Since $1/2 \in R$, we see that the R-module A is the direct-sum of $A^{\langle i \rangle} = \{(a + i(a))/2 \mid a \in A\}$ and $\{(a - i(a))/2 \mid a \in A\}$. □

6.2. **Shioda-Inose configuration on** $\mathrm{Km}(E' \times E)$**.** The following result is due to Shioda and Inose [30]. We briefly recall the proof.

Proposition 6.2.1. *Let E' and E be elliptic curves defined over an algebraically closed field k of characteristic 0. Then there exists a Shioda-Inose configuration (C, Θ) on the Kummer surface $\mathrm{Km}(E' \times E)$.*

Proof. Let E_{ij}, F_j and G_i $(1 \le i, j \le 4)$ be the (-2)-curves in the double Kummer pencil (Figure 4.3.1) on $\mathrm{Km}(E' \times E)$. We consider the divisor

$$H := E_{12} + 2F_2 + 3E_{32} + 4G_3 + 5E_{31} + 6F_1 + 3E_{21} + 4E_{41} + 2G_4, \tag{6.2.1}$$

and let C be the reduced part of $H - E_{12}$:

$$C := F_2 + E_{32} + G_3 + E_{31} + F_1 + E_{21} + E_{41} + G_4, \tag{6.2.2}$$

which is an ADE-configuration of (-2)-curves of type $\mathbb{E}_8$. The complete linear system $|H|$ defines an elliptic pencil

$$\Phi : \mathrm{Km}(E' \times E) \to \mathbb{P}^1$$

with a section G_1. Since $HE_{13} = 0$ and $HE_{14} = 0$, each of E_{13} and E_{14} is contained in a fiber of Φ. We put $t_0 := \Phi(H)$, $t_1 := \Phi(E_{13})$ and $t_2 := \Phi(E_{14})$. Note that $t_0 \ne t_1 \ne t_2 \ne t_0$, because H, E_{13} and E_{14} intersect G_1 at distinct points. By [30, Theorem 1], the fibers of Φ over t_1 and t_2 are either (a) of type $\mathrm{I}^*_{b_1}$ and $\mathrm{I}^*_{b_2}$ with $b_1 + b_2 \le 2$, or (b) of type I^*_0 and IV^*. Hence there exist exactly eight (-2)-curves $\Theta_1, \ldots, \Theta_8$ in $\Phi^{-1}(t_1)$ and $\Phi^{-1}(t_2)$ that appear in the fiber with odd multiplicity. We denote by Θ the sum of $\Theta_1, \ldots, \Theta_8$. Let Δ be a projective line, and $f : \Delta \to \mathbb{P}^1$ the double covering that branches exactly at t_1 and t_2. Let $\widetilde{Y}$ be the normalization of $\mathrm{Km}(E' \times E) \times_{\mathbb{P}^1} \Delta$. Then $\widetilde{Y} \to \mathrm{Km}(E' \times E)$ is a finite double covering that branches exactly along Θ. Hence (C, Θ) is a Shioda-Inose configuration. □

6.3. **The SIK diagram.** Let W be either a number field, or a Dedekind domain with the quotient field F being a number field such that $1/2 \in W$.

Definition 6.3.1. Let $\mathcal{E}'$ and $\mathcal{E}$ be smooth proper families of elliptic curves over $U := \operatorname{Spec} W$. We put $\mathcal{A} := \mathcal{E}' \times_U \mathcal{E}$. A diagram

$$(\mathcal{SIK}): \ \mathcal{Y} \ \longleftarrow \ \widetilde{\mathcal{Y}} \ \longrightarrow \ \mathrm{Km}(\mathcal{A}) \ \longleftarrow \ \widetilde{\mathcal{A}} \ \longrightarrow \ \mathcal{A}$$

of schemes and morphisms over U is called an *SIK diagram* of $\mathcal{E}'$ and $\mathcal{E}$ if the left half $\mathcal{Y} \leftarrow \widetilde{\mathcal{Y}} \rightarrow \mathrm{Km}(\mathcal{A})$ is a Shioda-Inose diagram over U, and the right half $\mathrm{Km}(\mathcal{A}) \leftarrow \widetilde{\mathcal{A}} \rightarrow \mathcal{A}$ is the Kummer diagram of $\mathcal{E}'$ and $\mathcal{E}$ over U.

Proposition 6.3.2. *Let E' and E be elliptic curves defined over a number field L.*

(1) *There exist a finite extension F of L and an SIK diagram*

$$(\mathcal{SIK})_F: \ Y \ \longleftarrow \ \widetilde{Y} \ \longrightarrow \ \mathrm{Km}(A) \ \longleftarrow \ \widetilde{A} \ \longrightarrow \ A := (E' \times E) \otimes F$$

of $E' \otimes F$ and $E \otimes F$ over F.

(2) *Moreover, there exist a non-empty open subset U of $\operatorname{Spec} \mathbb{Z}_F[1/2]$, smooth proper families $\mathcal{E}'$ and $\mathcal{E}$ of elliptic curves over U with the generic fibers being isomorphic to $E' \otimes F$ and $E \otimes F$, respectively, and an SIK diagram*

$$(\mathcal{SIK})_U: \ \mathcal{Y} \ \longleftarrow \ \widetilde{\mathcal{Y}} \ \longrightarrow \ \mathrm{Km}(\mathcal{A}) \ \longleftarrow \ \widetilde{\mathcal{A}} \ \longrightarrow \ \mathcal{A} := \mathcal{E}' \times_U \mathcal{E}$$

of $\mathcal{E}'$ and $\mathcal{E}$ over U such that $(\mathcal{SIK})_U \otimes F$ is equal to the SIK diagram $(\mathcal{SIK})_F$ over F in (1) *above.*

Proof. Our argument for the proof of the assertion (1) is similar to [30, §6]. We use the notation in the proof of Proposition 6.2.1. Let F be a finite extension of L such that every 2-torsion point $Q_{ij} := (u'_i, u_j)$ of $A := (E' \times E) \otimes F$ is rational over F. Then the blowing-up $\widetilde{A} \rightarrow A$ and the involution $\tilde{\iota}_A$ of $\widetilde{A}$ are defined over F. Therefore the quotient morphism $\widetilde{A} \rightarrow \mathrm{Km}(A)$ is defined over F, and every irreducible component of the double Kummer pencil on $\mathrm{Km}(A)$ is rational over F. Since the divisor H is defined over F, the elliptic pencil Φ on $\mathrm{Km}(A)$ is defined over F. Moreover, the points $t_1, t_2 \in \mathbb{P}^1$ are F-rational. Replacing F by a finite extension, we can assume that $\widetilde{Y}$ is defined over F, and that $\Theta_1, \ldots, \Theta_8$ are rational over F. Then the (-1)-curves $T_1, \ldots, T_8$ on $\widetilde{Y}$ are rational over F, and the contraction $\widetilde{Y} \rightarrow Y$ is defined over F. Moreover, the image $R_i \in Y$ of $T_i \subset \widetilde{Y}$ is an F-rational point of Y. Thus we have obtained an SIK diagram $(\mathcal{SIK})_F$ of E' and E over F, and the assertion (1) is proved. Moreover, $(\mathcal{SIK})_F$ has the following properties:

(i) Each of the centers Q_{ij} of the blowing-up $\widetilde{A} \rightarrow A$ is rational over F, and each of the centers R_i of the blowing-up $Y \leftarrow \widetilde{Y}$ is rational over F.

(ii) Each irreducible component of the double Kummer pencil on $\mathrm{Km}(A)$ is rational over F. In particular, each irreducible component C_i of the $\mathbb{E}_8$-configuration C is rational over F. (See (6.2.2).)

(iii) Each irreducible component Θ_i of the branch curve of the double covering $\widetilde{Y} \rightarrow \mathrm{Km}(A)$ is rational over F.

We choose a non-empty open subset U of $\operatorname{Spec} \mathbb{Z}_F[1/2]$, construct smooth proper families $\mathcal{E}'$ and $\mathcal{E}$ of elliptic curves over U with the generic fibers being isomorphic to $E' \otimes F$ and $E \otimes F$, respectively, and make a diagram $(\mathcal{SIK})_U$ of schemes and morphisms over U such that each scheme is smooth and projective over U and such

that $(\mathcal{SIK})_U \otimes F$ is equal to the SIK diagram $(\mathcal{SIK})_F$ over F. We will show that, after deleting finitely many closed points from U, the diagram $(\mathcal{SIK})_U$ becomes an SIK diagram over U. Note that, since $\mathcal{E}'$ and $\mathcal{E}$ are families of elliptic curves (that is, with a section over U), the inversion automorphism $\iota_{\mathcal{A}}$ of $\mathcal{A}$ is defined over U. We can make U so small that the following hold:

- Each $Q_{ij} \in A$ is the generic fiber of a closed subscheme $\mathcal{Q}_{ij}$ of $\mathcal{A}$ that is smooth over U, and these $\mathcal{Q}_{ij}$ are mutually disjoint. Then $\bigcup \mathcal{Q}_{ij}$ is the fixed locus of $\iota_{\mathcal{A}}$, and $\widetilde{\mathcal{A}} \to \mathcal{A}$ is the blowing-up along $\bigcup \mathcal{Q}_{ij}$ by Example 6.1.5.
- The involution $\tilde{\iota}_A$ of $\widetilde{A}$ extends to an involution $(\tilde{\iota}_A)^{\sim}_U$ of $\widetilde{\mathcal{A}}$ over U, which is a lift of $\iota_{\mathcal{A}}$ by Corollary 6.1.3. By Example 6.1.8, the morphism $\mathrm{Km}(\mathcal{A}) \leftarrow \widetilde{\mathcal{A}}$ is the quotient morphism by $\langle (\tilde{\iota}_A)^{\sim}_U \rangle$.
- Each $\Theta_i \subset \mathrm{Km}(A)$ is the generic fiber of a closed subscheme $\mathit{\Theta}_i$ of $\mathrm{Km}(\mathcal{A})$ that is smooth over U. By the specialization isometry from $\mathrm{NS}(\mathrm{Km}(A))$ to $\mathrm{NS}(\mathrm{Km}(\mathcal{A}) \otimes \kappa_{\mathfrak{p}})$ for closed points $\mathfrak{p}$ of U, we see that these $\mathit{\Theta}_i$ are mutually disjoint. By Example 6.1.6, the morphism $\mathcal{Y} \to \mathrm{Km}(\mathcal{A})$ is a double covering branching exactly along $\mathit{\Theta} := \sum \mathit{\Theta}_i$.
- Each irreducible component C_i of C is the generic fiber of a closed subscheme $\mathcal{C}_i$ of $\mathrm{Km}(\mathcal{A})$ that is smooth over U. We put $\mathcal{C} := \sum \mathcal{C}_i$. Considering the specialization isometry $\mathrm{NS}(\mathrm{Km}(A)) \hookrightarrow \mathrm{NS}(\mathrm{Km}(\mathcal{A}) \otimes \kappa_{\mathfrak{p}})$ for closed points $\mathfrak{p}$ of U, we see that $\mathcal{C}$ is a flat family of $\mathbb{E}_8$-configurations of (-2)-curves over U, and that $\mathit{\Theta}$ and $\mathcal{C}$ are disjoint. Hence $(\mathcal{C}, \mathit{\Theta}) \otimes \kappa_P$ is a Shioda-Inose configuration on $\mathrm{Km}(\mathcal{A}) \otimes \kappa_P$ for every point P of U.
- Each $R_i \in Y$ is the generic fiber of a closed subscheme $\mathcal{R}_i$ of $\mathcal{Y}$ that is smooth over U, and these $\mathcal{R}_i$ are mutually disjoint. The morphism $\widetilde{\mathcal{Y}} \leftarrow \mathcal{Y}$ is the blowing-up along $\bigcup \mathcal{R}_i$ by Example 6.1.5.

Hence $(\mathcal{SIK})_U$ is an SIK diagram over U. □

We consider the SIK diagram $(\mathcal{SIK})_U$ over a non-empty open subset $U \subset \operatorname{Spec} \mathbb{Z}_F[1/2]$, and the SIK diagram $(\mathcal{SIK})_F = (\mathcal{SIK})_U \otimes F$ over F, as in Proposition 6.3.2. (Remark that we have changed the notation from (4.5.1) and (5.3.1) to $Y := \mathcal{Y} \otimes F$, $E' := \mathcal{E}' \otimes F$ and $E := \mathcal{E} \otimes F$.) By the isomorphisms of Propositions 4.4.1 and 5.2.1, we obtain the following:

Proposition 6.3.3. *For each $\sigma \in \mathrm{Emb}(F)$, the diagram $(\mathcal{SIK}) \otimes \mathbb{C}$ obtained from $(\mathcal{SIK}) \otimes F$ by $\sigma : F \hookrightarrow \mathbb{C}$ induces an isomorphism of lattices $T(Y^\sigma) \cong T(A^\sigma)$ that preserves the Hodge structure.*

We assume the following, which are equivalent by Proposition 4.3.2 and Corollary 5.2.3: (i) $\mathrm{rank}(\mathrm{Hom}(E', E)) = 2$. (ii) $\mathrm{Km}(A)$ is singular. (iii) Y is singular.

Proposition 6.3.4. *We put $d(Y) := \mathrm{disc}(\mathrm{NS}(Y))$. There exists a finite set N of prime integers containing the prime divisors of $2d(Y)$ such that the following holds:*

$$p \notin N \;\Rightarrow\; \mathcal{S}_p(\mathcal{Y}) = \begin{cases} \emptyset & \text{if } \chi_p(d(Y)) = 1, \\ \pi_F^{-1}(p) & \text{if } \chi_p(d(Y)) = -1. \end{cases} \tag{6.3.1}$$

Proof. By Proposition 3.1.2, there exists an imaginary quadratic field K such that $K \cong \mathrm{End}(E') \otimes \mathbb{Q} \cong \mathrm{End}(E) \otimes \mathbb{Q}$. We denote by D the discriminant of K. We choose N in such a way that N contains all the prime divisors of $2d(Y)D$, and that

if $p \notin N$, then $\pi_F^{-1}(p) \subset U$ holds. By Propositions 6.3.3 and 3.1.1, we have

$$\begin{aligned} d(Y) := \operatorname{disc}(\mathrm{NS}(Y)) &= -\operatorname{disc}(T(Y^\sigma)) = -\operatorname{disc}(T(A^\sigma)) \\ &= \operatorname{disc}(\mathrm{NS}(A)) = -\operatorname{disc}(\mathrm{Hom}(E', E)). \end{aligned}$$

By Proposition 3.3.1, we have $m^2 d(Y) = n^2 D$ for some non-zero integers m and n. We can assume that $\gcd(m, n) = 1$. Then any $p \notin N$ is prime to mn, and hence

$$p \notin N \;\Rightarrow\; \chi_p(d(Y)) = \chi_p(D)$$

holds. Let p be a prime integer not in N. If $\chi_p(d(Y)) = 1$, then $\mathcal{S}_p(\mathcal{Y}) = \emptyset$ by Proposition 1.0.1. Suppose that $\chi_p(d(Y)) = -1$, and let $\mathfrak{p}$ be a point of $\pi_F^{-1}(p) \subset U$. Since $\chi_p(D) = -1$, both of $E'_\mathfrak{p} := \mathcal{E}' \otimes \kappa_\mathfrak{p}$ and $E_\mathfrak{p} := \mathcal{E} \otimes \kappa_\mathfrak{p}$ are supersingular by Proposition 3.5.3, and hence $\mathrm{Km}(\mathcal{A}) \otimes \kappa_\mathfrak{p} = \mathrm{Km}(E'_\mathfrak{p} \times E_\mathfrak{p})$ is supersingular by Propositions 3.1.3 and 4.3.2. By Proposition 5.1.3, we see that $\mathcal{Y} \otimes \kappa_\mathfrak{p}$ is also supersingular. Hence $\pi_F^{-1}(p) = \mathcal{S}_p(\mathcal{Y})$ holds. □

From the equality (4.5.2) and Propositions 4.5.2, 5.3.2, we obtain the following:

Proposition 6.3.5. *Let N be a finite set of prime integers with the properties given in Proposition* 6.3.4. *Suppose that $p \notin N$ satisfies $\chi_p(d(Y)) = -1$, and let $\mathfrak{p}$ be a point of $\pi_F^{-1}(p) = \mathcal{S}_p(\mathcal{Y})$. Then the diagram $(\mathcal{SIK})_U$ induces an isomorphism of lattices*

$$L(\mathcal{Y}, \mathfrak{p}) \;\cong\; (\mathrm{Hom}(E', E) \hookrightarrow \mathrm{Hom}(E'_\mathfrak{p}, E_\mathfrak{p}))^\perp[-1],$$

where $E'_\mathfrak{p} := \mathcal{E}' \otimes \kappa_\mathfrak{p}$ and $E_\mathfrak{p} := \mathcal{E} \otimes \kappa_\mathfrak{p}$.

6.4. **Shioda-Mitani theory.** In this subsection, we work over $\mathbb{C}$, and review the Shioda-Mitani theory [33] on product abelian surfaces. Let $M[a, b, c]$ be a matrix in the set $\mathcal{Q}_D$ defined in (1.0.2), where $D = b^2 - 4ac$ is a negative integer. Let $\sqrt{D} \in \mathbb{C}$ be in the upper half-plane, and put

$$\tau' := (-b + \sqrt{D})/(2a), \quad \tau := (b + \sqrt{D})/2. \tag{6.4.1}$$

We consider the complex elliptic curves

$${}^{\mathrm{an}}E' := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau'), \quad {}^{\mathrm{an}}E := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau). \tag{6.4.2}$$

Proposition 6.4.1 (§3 in [33])**.** *The oriented transcendental lattice $\widetilde{T}({}^{\mathrm{an}}E' \times {}^{\mathrm{an}}E)$ of the product abelian surface ${}^{\mathrm{an}}E' \times {}^{\mathrm{an}}E$ is represented by $M[a, b, c] \in \mathcal{Q}_D$.*

Suppose that D is a negative fundamental discriminant and that $M[a, b, c]$ is in the set $\mathcal{Q}_D^*$ defined in (1.0.3). We put $K := \mathbb{Q}(\sqrt{D}) \subset \mathbb{C}$. Then

$$I_0 := \mathbb{Z} + \mathbb{Z}\tau'$$

is a fractional ideal of K, and $\mathbb{Z} + \mathbb{Z}\tau$ is equal to $\mathbb{Z}_K$.

Proposition 6.4.2 ((4.14) in [33])**.** *Let J_1 and J_2 be fractional ideals of K. Then the product abelian surface $\mathbb{C}/J_1 \times \mathbb{C}/J_2$ is isomorphic to ${}^{\mathrm{an}}E' \times {}^{\mathrm{an}}E = \mathbb{C}/I_0 \times \mathbb{C}/\mathbb{Z}_K$ if and only if $[J_1][J_2] = [I_0]$ holds in the ideal class group Cl_D.*

Recall the definition of $\Psi : Cl_D \xrightarrow{\sim} \widetilde{\mathcal{L}}_D^*$ in Proposition 1.0.6. The image of $[I_0] \in Cl_D$ by Ψ is represented by $M[a, b, c]$. Hence we obtain the following:

Corollary 6.4.3. *For fractional ideals J_1 and J_2 of K, we have*

$$[\,\widetilde{T}(\mathbb{C}/J_1 \times \mathbb{C}/J_2)\,] \;=\; \Psi([J_1][J_2]).$$

6.5. **Proof of Theorem 1.** Let $X \to \operatorname{Spec} F$ and $\mathcal{X} \to U$ be as in §1. We choose $\sigma \in \operatorname{Emb}(F)$, and let $M[a,b,c] \in \mathcal{Q}_{d(X)}$ be a matrix representing $[\widetilde{T}(X^\sigma)] \in \widetilde{\mathcal{L}}_{d(X)}$, where $d(X) := \operatorname{disc}(\operatorname{NS}(X))$. We define complex elliptic curves ${}^{\mathrm{an}}E'$ and ${}^{\mathrm{an}}E$ by (6.4.1) and (6.4.2). Then there exist elliptic curves E' and E defined over a number field $L \subset \mathbb{C}$ such that $E' \otimes \mathbb{C}$ and $E \otimes \mathbb{C}$ are isomorphic to ${}^{\mathrm{an}}E'$ and ${}^{\mathrm{an}}E$, respectively. By replacing L with a finite extension, if necessary, we have an SIK diagram

$$\mathcal{Y} \longleftarrow \widetilde{\mathcal{Y}} \longrightarrow \operatorname{Km}(\mathcal{A}) \longleftarrow \widetilde{\mathcal{A}} \longrightarrow \mathcal{A} := \mathcal{E}' \times_{U_L} \mathcal{E}$$

over a non-empty open subset U_L of $\operatorname{Spec} \mathbb{Z}_L[1/2]$ such that the generic fibers of $\mathcal{E}'$ and $\mathcal{E}$ are isomorphic to E' and E, respectively. We put

$$A := \mathcal{A} \otimes L = E' \times E \quad \text{and} \quad Y := \mathcal{Y} \otimes L.$$

Then we see from Proposition 6.4.1 that $[\widetilde{T}(A \otimes \mathbb{C})]$ is represented by the matrix $M[a,b,c]$. Therefore we have $[\widetilde{T}(A \otimes \mathbb{C})] = [\widetilde{T}(X^\sigma)]$. On the other hand, we have $[\widetilde{T}(Y \otimes \mathbb{C})] = [\widetilde{T}(A \otimes \mathbb{C})]$ by Proposition 6.3.3. Hence we obtain

$$[\widetilde{T}(Y \otimes \mathbb{C})] = [\widetilde{T}(X^\sigma)].$$

By the Torelli theorem for $K3$ surfaces [21] or the Shioda-Inose theorem (Theorem 1.0.5), the complex $K3$ surfaces $Y \otimes \mathbb{C}$ and X^σ are isomorphic. Hence $d(Y) := \operatorname{disc}(\operatorname{NS}(Y))$ is equal to $d(X)$. Moreover, there exists a number field $M \subset \mathbb{C}$ containing both of $\sigma(F) \subset \mathbb{C}$ and $L \subset \mathbb{C}$ such that $X \otimes M$ and $Y \otimes M$ are isomorphic over M. Then $\mathcal{X} \times \operatorname{Spec} \mathbb{Z}_M$ and $\mathcal{Y} \times \operatorname{Spec} \mathbb{Z}_M$ are isomorphic over the generic point of $\operatorname{Spec} \mathbb{Z}_M$, and hence there exists a non-empty open subset V of $\operatorname{Spec} \mathbb{Z}_M$ such that

$$\mathcal{X}_V := \mathcal{X} \otimes V \quad \text{and} \quad \mathcal{Y}_V := \mathcal{Y} \otimes V$$

are isomorphic over V by Corollary 6.1.4. Let

$$\pi_{M,F} : \operatorname{Spec} \mathbb{Z}_M \to \operatorname{Spec} \mathbb{Z}_F \quad \text{and} \quad \pi_{M,L} : \operatorname{Spec} \mathbb{Z}_M \to \operatorname{Spec} \mathbb{Z}_L$$

be the natural projections. By deleting finitely many closed points from V, we can assume that $\pi_{M,F}(V) \subset U$ and $\pi_{M,L}(V) \subset U_L$. Then we have

$$\pi_{M,F}^{-1}(\mathcal{S}_p(\mathcal{X})) \cap V = \mathcal{S}_p(\mathcal{X}_V) = \mathcal{S}_p(\mathcal{Y}_V) = \pi_{M,L}^{-1}(\mathcal{S}_p(\mathcal{Y})) \cap V$$

for any $p \in \pi_M(V)$. We choose a finite set N of prime integers in such a way that the following hold:

(i) N contains all the prime divisors of $2d(X) = 2d(Y)$,
(ii) if $p \notin N$, then $\pi_M^{-1}(p) \subset V$, and hence $\pi_F^{-1}(p) \subset U$ and $\pi_L^{-1}(p) \subset U_L$ hold,
(iii) N satisfies the condition (6.3.1) for $\mathcal{Y}$.

Then N satisfies the condition (1.0.1) for $\mathcal{X}$. Hence Theorem 1 is proved.

6.6. **Proof of Theorem 3(T).** Let S be as in the statement of Theorem 3. Since $D = \operatorname{disc}(\operatorname{NS}(S))$ is assumed to be a fundamental discriminant, there exists an imaginary quadratic field K with discriminant D. We fix an embedding $K \hookrightarrow \mathbb{C}$ once and for all. For a finite extension L of K, we denote by $\operatorname{Emb}(L/K)$ the set of embeddings of L into $\mathbb{C}$ whose restrictions to K are the fixed one.

We recall the theory of complex multiplications. See [35, Chap. II], for example, for the details. Let $\overline{\mathbb{Q}} \subset \mathbb{C}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, and let $\mathcal{ELL}(\mathbb{Z}_K)$ be the set of $\overline{\mathbb{Q}}$-isomorphism classes $[E]$ of elliptic curves E defined over $\overline{\mathbb{Q}}$ such that

$\mathrm{End}(E) \cong \mathbb{Z}_K$. Then $\mathcal{ELL}(\mathbb{Z}_K)$ consists of h elements, where h is the class number $|Cl_D|$ of $\mathbb{Z}_K$. We denote by

$$\alpha_1, \ldots, \alpha_h \ \in \ \overline{\mathbb{Q}} \ \subset \ \mathbb{C}$$

the j-invariants $j(E)$ of the isomorphism classes $[E] \in \mathcal{ELL}(\mathbb{Z}_K)$, and put

$$\Phi_D(t) \ := \ (t-\alpha_1)\cdots(t-\alpha_h).$$

Then $\Phi_D(t)$ is a polynomial in $\mathbb{Z}[t]$, which is called the *Hilbert class polynomial* of $\mathbb{Z}_K$. It is known that $\Phi_D(t)$ is irreducible in $K[t]$. The field $\mathcal{H} := K(\alpha_1) \subset \mathbb{C}$ is the maximal unramified abelian extension of K, which is called the *Hilbert class field* of K. We define an action of Cl_D on $\mathcal{ELL}(\mathbb{Z}_K)$ by

$$[I] * [E] \ := \ [\mathbb{C}/I^{-1}I_E] \qquad \text{for } \ [I] \in Cl_D \ \text{ and } \ [E] \in \mathcal{ELL}(\mathbb{Z}_K),$$

where $I_E \subset K$ is a fractional ideal such that $E \cong \mathbb{C}/I_E$. On the other hand, for an elliptic curve E defined over $\overline{\mathbb{Q}}$ and $\gamma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)$, we denote by E^γ the elliptic curve obtained from E by letting γ act on the defining equation for E. Then $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ acts on $\mathcal{ELL}(\mathbb{Z}_K)$ by $[E]^\gamma := [E^\gamma]$. The following is the central result in the theory of complex multiplications.

Theorem 6.6.1. *There exists a homomorphism $F : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to Cl_D$ such that $[E]^\gamma = F(\gamma) * [E]$ holds for any $[E] \in \mathcal{ELL}(\mathbb{Z}_K)$ and $\gamma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)$. Moreover, this homomorphism F induces an isomorphism $\mathrm{Gal}(\mathcal{H}/K) \cong Cl_D$.*

We put $H := K[t]/(\Phi_D)$, and denote by $\alpha \in H$ the class of $t \in K[t]$ modulo the ideal (Φ_D). Then we have $\mathrm{Emb}(H/K) = \{\sigma_1, \ldots, \sigma_h\}$, where σ_i is given by $\sigma_i(\alpha) = \alpha_i$. Moreover, we have $\mathcal{H} = \sigma_1(H) = \cdots = \sigma_h(H)$ in $\mathbb{C}$. Let E_α be an elliptic curve defined over H such that

$$j(E_\alpha) = \alpha \ \in \ H.$$

A construction of such an elliptic curve is given, for example, in [34, §1 in Chap. III]. For each $\sigma_i \in \mathrm{Emb}(H/K)$, we denote by $E_\alpha^{\sigma_i}$ the elliptic curve defined over $\mathcal{H} = \sigma_i(H) \subset \overline{\mathbb{Q}}$ obtained from E_α by applying σ_i to the coefficients of the defining equation. Then we have $j(E_\alpha^{\sigma_i}) = \alpha_i \in \mathcal{H}$, and there exists a unique ideal class $[I_i] \in Cl_D$ of K such that $E_\alpha^{\sigma_i} \cong \mathbb{C}/I_i$. Moreover, we have

$$\mathcal{ELL}(\mathbb{Z}_K) = \{[E_\alpha^{\sigma_1}], \ldots, [E_\alpha^{\sigma_h}]\} \quad \text{and} \quad Cl_D = \{[I_1], \ldots, [I_h]\}. \tag{6.6.1}$$

Since $\mathrm{Gal}(\mathcal{H}/K)$ is abelian, there exists a canonical isomorphism $\mathrm{Gal}(H/K) \xrightarrow{\sim} \mathrm{Gal}(\mathcal{H}/K)$, which we will denote by $\gamma \mapsto \widetilde{\gamma}$. By Theorem 6.6.1, we have an isomorphism $\mathrm{Gal}(H/K) \cong Cl_D$ denoted by $\gamma \mapsto [I_\gamma]$ such that

$$E_\alpha^{\sigma_i \circ \gamma} = (E_\alpha^\gamma)^{\sigma_i} = (E_\alpha^{\sigma_i})^{\widetilde{\gamma}} \ \cong \ \mathbb{C}/{I_\gamma}^{-1} I_i \tag{6.6.2}$$

holds for any $i = 1, \ldots, h$ and any $\gamma \in \mathrm{Gal}(H/K)$.

There exist a finite extension F of H and a non-empty open subset U of $\mathrm{Spec}\, \mathbb{Z}_F[1/2]$ such that, for each $\gamma \in \mathrm{Gal}(H/K)$, there exist smooth proper families of elliptic curves $\mathcal{E}_\alpha^\gamma$ and $\mathcal{E}_\alpha$ over U whose generic fibers are isomorphic to $E_\alpha^\gamma \otimes F$ and $E_\alpha \otimes F$, respectively, and an SIK diagram

$$(\mathcal{SIK})^\gamma : \ \mathcal{Y}^\gamma \ \longleftarrow \ \widetilde{\mathcal{Y}}^\gamma \ \longrightarrow \ \mathrm{Km}(\mathcal{A}^\gamma) \ \longleftarrow \ \widetilde{\mathcal{A}}^\gamma \ \longrightarrow \ \mathcal{A}^\gamma := \mathcal{E}_\alpha^\gamma \times_U \mathcal{E}_\alpha$$

of $\mathcal{E}_\alpha^\gamma$ and $\mathcal{E}_\alpha$ over U. We then put $Y^\gamma := \mathcal{Y}^\gamma \otimes F$ and $A^\gamma := \mathcal{A}^\gamma \otimes F$. Let σ be an element of $\mathrm{Emb}(F/K)$. If the restriction of σ to H is equal to σ_i, then we have the following equalities in $\widetilde{\mathcal{L}}_D^*$:

$$\begin{aligned}
[\,\widetilde{T}((Y^\gamma)^\sigma)\,] &= [\,\widetilde{T}((A^\gamma)^\sigma)\,] && \text{by Proposition 6.3.3} \\
&= [\,\widetilde{T}(E_\alpha^{\sigma_i\circ\gamma} \times E_\alpha^{\sigma_i})\,] \\
&= [\,\widetilde{T}(\mathbb{C}/(I_\gamma^{-1}I_i) \times \mathbb{C}/I_i)\,] && \text{by (6.6.2)} \\
&= \Psi([I_\gamma]^{-1}[I_i]^2) && \text{by Corollary 6.4.3.}
\end{aligned}$$

Note that the restriction map $\mathrm{Emb}(F/K) \to \mathrm{Emb}(H/K)$ is surjective. Therefore, by Proposition 1.0.6 and the equalities (6.6.1), we see that the subset

$$\{\,[\,\widetilde{T}((Y^\gamma)^\sigma)\,] \mid \sigma \in \mathrm{Emb}(F/K)\,\} \quad = \quad \{\,\Psi([I_\gamma]^{-1}[I_i]^2) \mid i = 1, \dots, h\,\}$$

of $\widetilde{\mathcal{L}}_D^*$ coincides with the lifted genus that contains $\Psi([I_\gamma]^{-1})$. Since the homomorphism $\gamma \mapsto [I_\gamma]$ from $\mathrm{Gal}(H/K)$ to Cl_D is an isomorphism, we have a unique element $\gamma(S) \in \mathrm{Gal}(H/K)$ such that $\Psi([I_{\gamma(S)}]^{-1})$ is equal to $[\,\widetilde{T}(S)\,]$. We put

$$\mathcal{X} := \mathcal{Y}^{\gamma(S)} \quad \text{and} \quad X := Y^{\gamma(S)}.$$

Then X has the property required in Theorem 3(T).

6.7. **Proof of Theorem 3(L).** We continue to use the notation fixed in the previous subsection. We consider E_α as being defined over F. Replacing F by a finite extension, if necessary, we can assume that F is Galois over $\mathbb{Q}$ and that

$$\mathrm{End}_F(E_\alpha) = \mathrm{End}(E_\alpha) \tag{6.7.1}$$

holds so that $\mathrm{Lie} : \mathrm{End}(E_\alpha) \to F$ is defined. Since $j(E_\alpha) = \alpha$ is a root of Φ_D and F contains K, we have the Lie-normalized isomorphism

$$\mathrm{End}(E_\alpha) \;\cong\; \mathbb{Z}_K. \tag{6.7.2}$$

Making the base space U of the SIK diagram $(\mathcal{SIK})^{\gamma(S)}$ smaller if necessary, we can assume the following:

(i) $U = \pi_F^{-1}(\pi_F(U))$, and $p \nmid 2D$ for any $p \in \pi_F(U)$,
(ii) if $p \in \pi_F(U)$, then $\Phi_D(t) \bmod p$ has no multiple roots in $\overline{\mathbb{F}}_p$, and
(iii) for $p \in \pi_F(U)$, we have the following equivalence:

$$\chi_p(D) = -1 \quad \Leftrightarrow \quad \mathcal{S}_p(\mathcal{X}) \neq \emptyset \quad \Leftrightarrow \quad \mathcal{S}_p(\mathcal{X}) = \pi_F^{-1}(p).$$

Let p be a prime integer in $\pi_F(U)$ such that $\chi_p(D) = -1$, so that $\mathcal{S}_p(\mathcal{X}) = \pi_F^{-1}(p)$. We show that, under the assumption that D is odd, the set of isomorphism classes of supersingular reduction lattices $\{[L(\mathcal{X}, \mathfrak{p})] \,|\, \mathfrak{p} \in \pi_F^{-1}(p)\}$ coincides with a genus.

Let B denote the quaternion algebra over $\mathbb{Q}$ that ramifies exactly at p and ∞. We consider pairs (R, Z) of a $\mathbb{Z}$-algebra R and a subalgebra $Z \subset R$ such that R is isomorphic to a maximal order of B and such that Z is isomorphic to $\mathbb{Z}_K$. We say that two such pairs (R, Z) and (R', Z') are isomorphic if there exists an isomorphism $\varphi : R \xrightarrow{\sim} R'$ satisfying $\varphi(Z) = Z'$. We denote by $\mathcal{R}$ the set of isomorphism classes $[R, Z]$ of these pairs. Next we consider pairs (R, ρ) of a $\mathbb{Z}$-algebra R isomorphic to a maximal order of B and an embedding $\rho : \mathbb{Z}_K \hookrightarrow R$ as a $\mathbb{Z}$-subalgebra. We say

that two such pairs (R, ρ) and (R', ρ') are isomorphic if there exists an isomorphism $\varphi : R \xrightarrow{\sim} R'$ satisfying $\varphi \circ \rho = \rho'$. We denote by $\widetilde{\mathcal{R}}$ the set of isomorphism classes $[R, \rho]$ of these pairs. For an embedding $\rho : \mathbb{Z}_K \hookrightarrow R$, we denote by $\bar{\rho}$ the composite of the non-trivial automorphism of $\mathbb{Z}_K$ and ρ. The natural map

$$\Pi_{\mathcal{R}} : \widetilde{\mathcal{R}} \to \mathcal{R}$$

given by $[R, \rho] \mapsto [R, \rho(\mathbb{Z}_K)]$ is surjective, and its fiber consists either of two elements $[R, \rho]$ and $[R, \bar{\rho}]$, or of a single element $[R, \rho] = [R, \bar{\rho}]$.

Let $\mathfrak{p}$ be a point of $\pi_F^{-1}(p)$. We denote by $F_{[\mathfrak{p}]}$ the completion of F at $\mathfrak{p}$, and put

$$E_{[\mathfrak{p}]} := \mathcal{E}_\alpha \otimes F_{[\mathfrak{p}]} \quad \text{and} \quad E_{\mathfrak{p}} := \mathcal{E}_\alpha \otimes \kappa_{\mathfrak{p}}.$$

Then we have canonical isomorphisms

$$\mathrm{End}_{F_{[\mathfrak{p}]}}(E_{[\mathfrak{p}]}) \cong \mathrm{End}(E_{[\mathfrak{p}]}) \cong \mathrm{End}(E_\alpha) \tag{6.7.3}$$

by the assumption (6.7.1), and hence $\mathrm{Lie} : \mathrm{End}(E_{[\mathfrak{p}]}) \to F_{[\mathfrak{p}]}$ is defined. We put

$$R_{\mathfrak{p}} := \mathrm{End}(E_{\mathfrak{p}}),$$

which is isomorphic to a maximal order of B by Proposition 3.4.1, and denote by

$$\rho_{\mathfrak{p}} : \mathrm{End}(E_{[\mathfrak{p}]}) \hookrightarrow R_{\mathfrak{p}}$$

the specialization isometry. Using the isomorphisms (6.7.3) and the Lie-normalized isomorphism (6.7.2), we obtain an element $[R_{\mathfrak{p}}, \rho_{\mathfrak{p}}]$ of $\widetilde{\mathcal{R}}$. We denote by

$$\widetilde{r} : \mathcal{S}_p(\mathcal{X}) \to \widetilde{\mathcal{R}}$$

the map given by $\mathfrak{p} \mapsto [R_{\mathfrak{p}}, \rho_{\mathfrak{p}}]$.

Lemma 6.7.1. *The map $\widetilde{r}$ is surjective.*

Proof. First we show that the map $r := \Pi_{\mathcal{R}} \circ \widetilde{r}$ from $\mathcal{S}_p(\mathcal{X})$ to $\mathcal{R}$ is surjective. Let $[R, Z]$ be an element of $\mathcal{R}$. By Proposition 3.4.2, there exists a supersingular elliptic curve C_0 in characteristic p with an isomorphism $\psi : \mathrm{End}(C_0) \xrightarrow{\sim} R$. Let $\alpha_0 \in \mathrm{End}(C_0)$ be an element such that the subalgebra $\mathbb{Z} + \mathbb{Z}\,\alpha_0$ corresponds to $Z \subset R$ by ψ. By Proposition 3.5.6, there exists a lift (C, α) of (C_0, α_0), where C is an elliptic curve defined over a finite extension of $\mathbb{Q}_p$. Since $\mathbb{Z} + \mathbb{Z}\,\alpha \subseteq \mathrm{End}(C)$ is isomorphic to $\mathbb{Z}_K$, we have $\mathrm{End}(C) \cong \mathbb{Z}_K$, and hence the j-invariant of C is a root of the Hilbert class polynomial Φ_D in $\overline{\mathbb{Q}}_p$. Since the set of roots of Φ_D in $\overline{\mathbb{Q}}_p$ is in one-to-one correspondence with $\pi_H^{-1}(p)$ by the assumption (ii) on U, and U contains $\pi_F^{-1}(p)$ by the assumption (i) on U, there exists $\mathfrak{p} \in \pi_F^{-1}(p) \subset U$ such that

$$j(E_{[\mathfrak{p}]}) = j(C).$$

By applying Proposition 3.5.2 with $g = \mathrm{id}$, we have $r(\mathfrak{p}) = [R, Z]$. To prove that $\widetilde{r}$ is surjective, therefore, it is enough to show that, for each $\mathfrak{p} \in \pi_F^{-1}(p)$, there exists $\mathfrak{p}' \in \pi_F^{-1}(p)$ such that $[R_{\mathfrak{p}'}, \rho_{\mathfrak{p}'}] = [R_{\mathfrak{p}}, \overline{\rho_{\mathfrak{p}}}]$ holds in $\widetilde{\mathcal{R}}$. We choose an element $g \in \mathrm{Gal}(F/\mathbb{Q})$ such that the restriction of g to K is the non-trivial element of

$\mathrm{Gal}(K/\mathbb{Q})$, and let $\mathfrak{p}'$ be the image of $\mathfrak{p}$ by the action of g on $\pi_F^{-1}(p)$. Consider the diagram

$$\begin{array}{ccccccc}
F_{[\mathfrak{p}]} & & \xleftarrow{\mathrm{Lie}} & & \mathrm{End}(E_{[\mathfrak{p}]}) & \xrightarrow{\rho_{\mathfrak{p}}} & \mathrm{End}(E_{\mathfrak{p}}) \\
& \nwarrow & & \nearrow \lambda & & & \\
f_g \downarrow \wr & F & \xleftarrow{\mathrm{Lie}} & \mathrm{End}(E_\alpha) & e_g \downarrow \wr & & E_g \downarrow \wr \\
& \swarrow & & \searrow \lambda' & & & \\
F_{[\mathfrak{p}']} & & \xleftarrow{\mathrm{Lie}} & & \mathrm{End}(E_{[\mathfrak{p}']}) & \xrightarrow{\rho_{\mathfrak{p}'}} & \mathrm{End}(E_{\mathfrak{p}'}),
\end{array}$$

where λ and λ' are the canonical isomorphisms (6.7.3), and the vertical isomorphisms f_g, e_g and E_g are given by the action of g. Then we have $e_g \circ \lambda = \overline{\lambda'}$, where $\overline{\lambda'}$ is the composite of the non-trivial automorphism of $\mathrm{End}(E_\alpha) \cong \mathbb{Z}_K$ and λ'. By Proposition 3.5.2, we have $E_g \circ \rho_{\mathfrak{p}} = \rho_{\mathfrak{p}'} \circ e_g$, and hence $[R_{\mathfrak{p}'}, \rho_{\mathfrak{p}'}] = [R_{\mathfrak{p}}, \overline{\rho_{\mathfrak{p}}}]$. □

Suppose that the ideal class $[I_{\gamma(S)}] \in Cl_D$ is represented by an ideal $J \subset \mathbb{Z}_K$. We can regard J as an ideal of $\mathrm{End}(E_\alpha)$ by the Lie-normalized isomorphism (6.7.2). By [7, Corollary 7.17], we can choose $J \subset \mathbb{Z}_K$ in such a way that

$$d_J := \deg \phi^J = [\mathrm{End}(E_\alpha) : J]$$

is prime to D. (See Remark 6.7.3 (2).) For any $\sigma_i \in \mathrm{Emb}(H/K)$, we have the following isomorphisms of complex elliptic curves:

$$\begin{array}{rcll}
(E_\alpha^{\gamma(S)})^{\sigma_i} & \cong & \mathbb{C}/{I_{\gamma(S)}}^{-1} I_i & \text{by (6.6.2)} \\
& \cong & \mathbb{C}/J^{-1} I_i & \text{by } [J] = [I_{\gamma(S)}] \\
& \cong & (E_\alpha^{\sigma_i})^J & \text{by } E_\alpha^{\sigma_i} \cong \mathbb{C}/I_i \text{ and Proposition 3.3.4} \\
& \cong & (E_\alpha^J)^{\sigma_i} & \text{since the construction of } E \to E^J \text{ is algebraic.}
\end{array}$$

Hence we have

$$E_\alpha^{\gamma(S)} \otimes \overline{F} \;\cong\; E_\alpha^J \otimes \overline{F}. \tag{6.7.4}$$

We then consider J as an ideal of $\mathrm{End}(E_{[\mathfrak{p}]})$ by the canonical isomorphisms (6.7.3), and we consider the left ideal $R_{\mathfrak{p}}\rho_{\mathfrak{p}}(J)$ of $R_{\mathfrak{p}}$ generated by $\rho_{\mathfrak{p}}(J)$. From the isomorphism (6.7.4), we obtain an isomorphism

$$\mathcal{E}_\alpha^{\gamma(S)} \otimes \overline{F}_{[\mathfrak{p}]} \;\cong\; {E_{[\mathfrak{p}]}}^J \otimes \overline{F}_{[\mathfrak{p}]}.$$

Then, by Proposition 3.5.4, we have

$$\mathcal{E}_\alpha^{\gamma(S)} \otimes \bar{\kappa}_{\mathfrak{p}} \;\cong\; {E_{\mathfrak{p}}}^{R_{\mathfrak{p}}\rho_{\mathfrak{p}}(J)} \otimes \bar{\kappa}_{\mathfrak{p}}.$$

Therefore we have the following equalities in the set $\mathcal{L}_{p^2 d_J^2 D}$:

$$\begin{array}{rl}
& [\, L(\mathcal{X}, \mathfrak{p})[-d_J] \,] \\
= & [\, (\mathrm{Hom}({E_{[\mathfrak{p}]}}^J, E_{[\mathfrak{p}]}) \hookrightarrow \mathrm{Hom}({E_{\mathfrak{p}}}^{R_{\mathfrak{p}}\rho_{\mathfrak{p}}(J)}, E_{\mathfrak{p}}))^\perp\, [d_J] \,] \quad \text{by Proposition 6.3.5} \\
= & [\, (J \hookrightarrow R_{\mathfrak{p}}\rho_{\mathfrak{p}}(J))^\perp \,] \quad \text{by Proposition 3.5.5.}
\end{array}$$

By the surjectivity of the map $\widetilde{r}$, we complete the proof of Theorem 3(L) by the following proposition, which will be proved in the next section.

Proposition 6.7.2. *Let J be an ideal of $\mathbb{Z}_K$. Suppose that D is odd and that $d_J = N(J) = [\mathbb{Z}_K : J]$ is prime to D. Then the set*

$$\{\ [(J \hookrightarrow R\rho(J))^{\perp}]\ \mid\ [R,\rho] \in \widetilde{\mathcal{R}}\ \}$$

coincides with a genus in $\mathcal{L}_{p^2 d_J^2 D}$.

Remark 6.7.3. (1) We make use of the assumption that D is odd in Theorem 3(L) only in the proof of Proposition 6.7.2. (2) The condition $\gcd(N(J), D) = 1$ is assumed only in order to simplify the proof of Proposition 6.7.2.

7. The maximal orders of a quaternion algebra

Let K, D, p, B and $\widetilde{\mathcal{R}}$ be as in the previous section. We assume that D is odd. We describe the set $\widetilde{\mathcal{R}}$ following Dorman [9], and prove Proposition 6.7.2.

7.1. **Dorman's description of $\widetilde{\mathcal{R}}$.** Note that D is a square-free negative integer satisfying $D \equiv 1 \mod 4$. We choose a prime integer q that satisfies

$$\chi_l(-pq) = 1 \quad \text{for all prime divisors } l \text{ of } D. \tag{7.1.1}$$

Then the $\mathbb{Q}$-algebra

$$B\ :=\ \{\ [\alpha,\beta]\ \mid\ \alpha,\beta \in K\ \}, \quad \text{where} \quad [\alpha,\beta] := \begin{pmatrix} \alpha & \beta \\ -pq\overline{\beta} & \overline{\alpha} \end{pmatrix},$$

is a quaternion algebra that ramifies exactly at p and ∞. The canonical involution of B is given by $[\alpha,\beta]^* = [\overline{\alpha}, -\beta]$. Hence the bilinear form (3.4.1) on B is given by

$$([\alpha,\beta],[\alpha',\beta'])\ =\ \mathrm{Tr}_{K/\mathbb{Q}}(\alpha\overline{\alpha'}) + pq\,\mathrm{Tr}_{K/\mathbb{Q}}(\beta\overline{\beta'}). \tag{7.1.2}$$

Note that we have

$$[\gamma,0][\alpha,\beta] = [\gamma\alpha,\gamma\beta] \quad \text{and} \quad [\alpha,\beta][\gamma,0] = [\gamma\alpha,\overline{\gamma}\beta]. \tag{7.1.3}$$

For simplicity, we use the following notation:

$$[S,T]\ :=\ \{\ [\alpha,\beta] \in B\ \mid\ \alpha \in S,\ \beta \in T\ \} \qquad \text{for subsets } S \text{ and } T \text{ of } K.$$

For $u \in B^{\times}$, we denote by $\theta_u : B \xrightarrow{\sim} B$ the inner automorphism $\theta_u(x) := uxu^{-1}$. We have a natural embedding $\iota : K \hookrightarrow B$ given by $\iota(\alpha) := [\alpha, 0]$. By the Skolem-Noether theorem ([3, §10 in Chap. 8]), we see that, if $\iota' : K \hookrightarrow B$ is another embedding as a $\mathbb{Q}$-algebra, then there exists a $u \in B^{\times}$ such that $\theta_u \circ \iota = \iota'$ holds. On the other hand, we have $\theta_u \circ \iota = \iota$ if and only if $u \in [K^{\times}, 0]$. Hence we have a canonical identification

$$\widetilde{\mathcal{R}}\ \cong\ [K^{\times},0]\backslash\mathbf{R},$$

where $\mathbf{R}$ is the set of maximal orders R of B such that $R \cap [K,0] = [\mathbb{Z}_K, 0]$ holds. We will examine the set $\mathbf{R}$.

For a $\mathbb{Z}$-submodule $\Lambda \subset B$ of rank 4, we put

$$N_B(\Lambda) := [[\mathbb{Z}_K, \mathbb{Z}_K] : n\Lambda]/n^4,$$

where n is a non-zero integer such that $n\Lambda \subset [\mathbb{Z}_K, \mathbb{Z}_K]$. An order R of B is maximal if and only if R is of discriminant p^2 as a lattice. Since the discriminant of $[\mathbb{Z}_K, \mathbb{Z}_K]$ is $p^2q^2|D|^2$, we obtain the following.

Lemma 7.1.1. *An order R of B is maximal if and only if $N_B(R) = 1/(q|D|)$.*

We denote by $\mathrm{pr}_2 : B \to K$ the projection given by $\mathrm{pr}_2([\alpha, \beta]) := \beta$.

Lemma 7.1.2. *Let R be an element of $\mathbf{R}$. Then $M_R := \mathrm{pr}_2(R)$ is a fractional ideal of K with $N(M_R) = 1/(q|D|)$.*

Proof. It is obvious that $M_R \subset K$ is a finitely generated $\mathbb{Z}_K$-module by the formula (7.1.3). Since $[K, 0] \cap R = [\mathbb{Z}_K, 0]$, we have $N(M_R) = N_B(R) = 1/(q|D|)$ by Lemma 7.1.1. □

From the condition (7.1.1) on q, $\chi_p(D) = -1$ and $D \equiv 1 \bmod 4$, we deduce that q splits completely in K. We choose an ideal $Q \subset \mathbb{Z}_K$ such that $(q) = Q\overline{Q}$. We also denote by $\mathcal{D}$ the principal ideal $(\sqrt{D}) \subset \mathbb{Z}_K$. Let R be an element of $\mathbf{R}$. By Lemma 7.1.2, the fractional ideal

$$I_R := \mathcal{D}QM_R \qquad (M_R := \mathrm{pr}_2(R))$$

satisfies $N(I_R) = 1$. Since $[K, 0] \cap R = [\mathbb{Z}_K, 0]$, we can define a map

$$f_R : M_R \to K/\mathbb{Z}_K$$

by $f_R(\beta) := \alpha + \mathbb{Z}_K$ for $[\alpha, \beta] \in R$. By the formula (7.1.3), we see that f_R is a homomorphism of $\mathbb{Z}_K$-modules, and we have $f_R(\gamma\beta) = f_R(\overline{\gamma}\beta)$ for any $\gamma \in \mathbb{Z}_K$ and $\beta \in M_R$. Therefore $f_R(\sqrt{D}\beta) = \sqrt{D} f_R(\beta) = 0$ holds for any $\beta \in M_R$. Thus f_R induces a homomorphism

$$\widetilde{f}_R : M_R/\mathcal{D}M_R \to \mathcal{D}^{-1}/\mathbb{Z}_K$$

of torsion $\mathbb{Z}_K$-modules.

Lemma 7.1.3. *The homomorphism $\widetilde{f}_R$ is an isomorphism.*

Proof. Since $|M_R/\mathcal{D}M_R| = |\mathcal{D}^{-1}/\mathbb{Z}_K| = |D|$, it is enough to show that $\widetilde{f}_R$ is injective. Let $\mathcal{F}$ be the fractional ideal such that $\mathrm{Ker}(f_R) = \mathcal{F}\mathcal{D}M_R = \mathcal{F}Q^{-1}I_R$. Suppose that $\beta, \beta' \in \mathrm{Ker}(f_R)$. Then $[0, \beta] \in R$ and $[0, \beta'] \in R$ hold, and hence $[0, \beta] \cdot [0, \beta'] = [-pq\beta\overline{\beta'}, 0]$ is also in R. From $[K, 0] \cap R = [\mathbb{Z}_K, 0]$, we have $-pq\beta\overline{\beta'} \in \mathbb{Z}_K$. Since $N(I_R) = 1$, we have

$$pq(\mathcal{F}Q^{-1}I_R)(\overline{\mathcal{F}Q^{-1}I_R}) = p\mathcal{F}\overline{\mathcal{F}} \subseteq \mathbb{Z}_K.$$

Since $\gcd(p, D) = 1$ and $\mathbb{Z}_K \subseteq \mathcal{F} \subseteq \mathcal{D}^{-1}$, we have $\mathcal{F} = \mathbb{Z}_K$. □

Since $\widetilde{f}_R$ is an isomorphism, there exists a unique element

$$\mu_R + \mathcal{D}M_R = \mu_R + Q^{-1}I_R \in M_R/\mathcal{D}M_R$$

such that $\widetilde{f}_R(\mu_R + Q^{-1}I_R) = (1/\sqrt{D}) + \mathbb{Z}_K$.

Lemma 7.1.4. *For any $\beta \in M_R$, we have*

$$f_R(\beta) = pq\sqrt{D}\overline{\mu_R}\beta + \mathbb{Z}_K = pq\sqrt{D}\mu_R\overline{\beta} + \mathbb{Z}_K.$$

Proof. For $\beta \in M_R$, we have $[0, \sqrt{D}\beta] \in R$. Since $[1/\sqrt{D}, \mu_R] \in R$, we have

$$\begin{aligned} [0, \sqrt{D}\beta] \cdot [1/\sqrt{D}, \mu_R] &= [\, -pq\sqrt{D}\overline{\mu_R}\beta,\ -\beta\,] \in R \quad \text{and} \\ [1/\sqrt{D}, \mu_R] \cdot [0, \sqrt{D}\beta] &= [\, pq\sqrt{D}\mu_R\overline{\beta},\ \beta\,] \in R, \end{aligned}$$

from which the desired description of f_R follows. □

By the definition of μ_R, we have $f_R(\mu_R) = pq\sqrt{D}\overline{\mu_R}\mu_R + \mathbb{Z}_K = (1/\sqrt{D}) + \mathbb{Z}_K$. Therefore we have

$$pqD|\mu_R|^2 - 1 \ \in \ \mathcal{D} \cap \mathbb{Q} = D\mathbb{Z},$$

where the second equality follows from the assumption that D is odd.

Lemma 7.1.5. *Let I be a fractional ideal with $N(I) = 1$, and let $x, x' \in \mathcal{D}^{-1}Q^{-1}I$ satisfy $x' - x \in Q^{-1}I$. Then we have $qD|x|^2 \in \mathbb{Z}$ and $qD|x'|^2 \equiv qD|x|^2 \mod D$.*

Proof. Since $I\overline{I} = \mathbb{Z}_K$, we have

$$qD|x|^2 \ \in \ qD(\mathcal{D}^{-2}Q^{-1}I\overline{Q^{-1}I}) \cap \mathbb{Q} = \mathbb{Z}_K \cap \mathbb{Q} = \mathbb{Z}.$$

We put $x' = x + y$ with $y \in Q^{-1}I$. Then we have $q|y|^2 \in \mathbb{Z}_K \cap \mathbb{Q} = \mathbb{Z}$. Since D is odd, we have $\mathcal{D}^{-1} \cap \mathbb{Q} = \mathbb{Z}$, and hence $q(x\overline{y} + y\overline{x}) \in \mathcal{D}^{-1} \cap \mathbb{Q} = \mathbb{Z}$ holds. □

We define $\mathcal{T}$ to be the set of all pairs $(I, \mu + Q^{-1}I)$, where I is a fractional ideal of K such that $N(I) = 1$, and $\mu + Q^{-1}I$ is an element of $\mathcal{D}^{-1}Q^{-1}I/Q^{-1}I$ such that $pqD|\mu|^2 \equiv 1 \mod D$. Then we have a map $\tau : \mathbf{R} \to \mathcal{T}$ given by

$$\tau(R) \ := \ (I_R, \mu_R + Q^{-1}I_R) \ \in \ \mathcal{T}.$$

Proposition 7.1.6. *The map τ is a bijection.*

Proof. The maximal order R is uniquely recovered from $(I_R, \mu_R + Q^{-1}I_R)$ by

$$R \ = \ \{\ [\alpha, \beta] \ \mid \ \beta \in \mathcal{D}^{-1}Q^{-1}I_R,\ \alpha \equiv pq\sqrt{D}\overline{\mu_R}\beta \mod \mathbb{Z}_K\ \}.$$

Hence τ is injective. Let an element $t := (I, \mu + Q^{-1}I)$ of $\mathcal{T}$ be given. We put $M_t := \mathcal{D}^{-1}Q^{-1}I$, and define $f_t : M_t \to \mathcal{D}^{-1}/\mathbb{Z}_K$ by

$$f_t(\beta) \ := \ pq\sqrt{D}\overline{\mu}\beta + \mathbb{Z}_K.$$

Note that the definition of f_t does not depend on the choice of the representative μ of $\mu + Q^{-1}I$. Since $M_t\overline{M_t} = (1/(Dq))$, we see that $\mu\overline{\beta} - \overline{\mu}\beta$ is contained in $\mathcal{D}(1/(Dq)) = \mathcal{D}^{-1}(1/q)$ for any $\beta \in M_t$. (Note that $\gamma - \overline{\gamma} \in \mathcal{D}$ for any $\gamma \in \mathbb{Z}_K$.) Therefore we have

$$f_t(\beta) \ = \ pq\sqrt{D}\mu\overline{\beta} + \mathbb{Z}_K \tag{7.1.4}$$

for any $\beta \in M_t$. We put

$$R_t \ := \ \{\ [\alpha, \beta] \ \mid \ \beta \in M_t,\ \ \alpha \equiv f_t(\beta) \mod \mathbb{Z}_K\ \}.$$

We prove that τ is surjective by showing that $R_t \in \mathbf{R}$. It is obvious that R_t is a $\mathbb{Z}$-module of rank 4 satisfying $R_t \cap [K, 0] = [\mathbb{Z}_K, 0]$. We show that R_t is closed under the product. Since f_t is a homomorphism of $\mathbb{Z}_K$-modules, we have $[\mathbb{Z}_K, 0]R_t = R_t$. By the formula (7.1.4), we have $R_t[\mathbb{Z}_K, 0] = R_t$. Hence it is enough to prove that

$$[\,pq\sqrt{D}\mu\overline{\beta},\ \beta\,] \cdot [\,pq\sqrt{D}\overline{\mu}\beta',\ \beta'\,] = [\,p^2q^2D|\mu|^2\overline{\beta}\beta' - pq\beta\overline{\beta'},\ \ pq\sqrt{D}\mu(\overline{\beta}\beta' - \beta\overline{\beta'})\,]$$

is in R_t for any $\beta, \beta' \in M_t$. Because

$$pq\beta\overline{\beta'} \in pqM_t\overline{M}_t = (p/D) \quad \text{and} \quad 1 - pqD|\mu|^2 \equiv 0 \mod D,$$

we have a congruence $pq\beta\overline{\beta'} \equiv p^2q^2D|\mu|^2\beta\overline{\beta'} \mod \mathbb{Z}_K$. Hence

$$f_t(pq\sqrt{D}\mu(\overline{\beta}\beta' - \beta\overline{\beta'})) \equiv p^2q^2D|\mu|^2\overline{\beta}\beta' - pq\beta\overline{\beta'} \mod \mathbb{Z}_K.$$

Therefore $R_tR_t = R_t$ is proved, and hence R_t is an order. Because $N(M_t) = 1/(q|D|)$, we see that R_t is maximal by Lemma 7.1.1. Hence $R_t \in \mathbf{R}$. □

We make $[K^\times, 0]$ act on the set $\mathcal{T}$ by

$$u \cdot (I, \mu + Q^{-1}I) := (Iu\overline{u}^{-1}, \mu u\overline{u}^{-1} + Q^{-1}Iu\overline{u}^{-1}).$$

Then $\widetilde{\mathcal{R}} \cong [K^\times, 0]\backslash\mathbf{R}$ is canonically identified with $[K^\times, 0]\backslash\mathcal{T}$.

Let $\mathcal{I}_1 \subset \mathcal{I}_D$ denote the group of fractional ideals I with $N(I) = 1$. We put

$$\mathcal{P}_1 := \mathcal{I}_1 \cap \mathcal{P}_D \quad \text{and} \quad \mathcal{C}_1 := \mathcal{I}_1/\mathcal{P}_1.$$

Then $\mathcal{C}_1$ is a subgroup of the ideal class group $Cl_D = \mathcal{I}_D/\mathcal{P}_D$. Since the homomorphism $\mathcal{I}_D \to \mathcal{I}_1$ given by $I \mapsto I\overline{I}^{-1}$ is surjective and $[I][\overline{I}]^{-1}$ is equal to $[I]^2$ in Cl_D, we see that the subgroup $\mathcal{C}_1$ of Cl_D is equal to Cl_D^2.

Lemma 7.1.7. *The map $R \mapsto [I_R]$ from $\mathbf{R}$ to $\mathcal{C}_1 = Cl_D^2$ is surjective.*

Proof. We will show that, for each $I \in \mathcal{I}_1$, there exists $\mu \in \mathcal{D}^{-1}Q^{-1}I$ such that $pqD|\mu|^2 \equiv 1 \bmod D$ holds. Since $I\overline{I} = \mathbb{Z}_K$, there exists an ideal $A \subset \mathbb{Z}_K$ such that $A + \overline{A} = \mathbb{Z}_K$ and $I = A\overline{A}^{-1}$. Since A and $\overline{A}$ have no common prime divisors, the norm $n := N(A)$ is prime to D. Hence there exists an integer m such that $nm \equiv 1 \bmod D$ holds. By the condition (7.1.1) on q, there exists an integer z such that $-pqz^2 \equiv 1 \bmod D$ holds. Therefore we have an element

$$znm \ \in \ A\overline{A} \ \subset \ A \ \subset \ I \ \subset \ Q^{-1}I$$

such that $-pq(znm)^2 \equiv 1 \bmod D$. Then the element $\mu := znm/\sqrt{D}$ of $\mathcal{D}^{-1}Q^{-1}I$ satisfies $pqD|\mu|^2 = -pq(znm)^2 \equiv 1 \bmod D$. □

7.2. **Proof of Proposition 6.7.2.** Let J be a non-zero ideal in $\mathbb{Z}_K$ such that $d_J := N(J)$ is prime to D. Then we have

$$\overline{J} \cap \mathcal{D} = \overline{J}\mathcal{D}. \tag{7.2.1}$$

Lemma 7.2.1. *Let R be an element of $\mathbf{R}$, and let RJ denote the left-ideal of R generated by $[J, 0] \subset R$. Then we have*

$$(J \hookrightarrow RJ)^\perp \ = \ [0, Q^{-1}I_R\overline{J}],$$

where $[0, Q^{-1}I_R\overline{J}]$ is the lattice such that the underlying $\mathbb{Z}$-module is $Q^{-1}I_R\overline{J} \subset K$ and such that the bilinear form is given by $(x, y) := pq\,\mathrm{Tr}_{K/\mathbb{Q}}(x\overline{y})$.

Proof. For simplicity, we put $M := M_R$, $I := I_R$, $f := f_R$ and $\mu := \mu_R$. Since $J \otimes \mathbb{Q} = K$ and $B = [K, 0] \perp [0, K]$ by (7.1.2), we see that $(J \hookrightarrow RJ)^\perp$ is equal to $[0, K] \cap RJ$. Let γ, γ' be a basis of J as a $\mathbb{Z}$-module. For an element $x \in K$, we have the following equivalence:

$[0, x] \in RJ$

$\Leftrightarrow$ there exist $[\alpha, \beta], [\alpha', \beta'] \in R$ such that $\alpha\gamma + \alpha'\gamma' = 0$ and $\beta\overline{\gamma} + \beta'\overline{\gamma'} = x$

$\Leftrightarrow$ there exist $\beta, \beta' \in M$ and $a, a' \in \mathbb{Z}_K$ such that $\beta\overline{\gamma} + \beta'\overline{\gamma'} = x$ and $pq\sqrt{D}\mu(\overline{\beta}\gamma + \overline{\beta'}\gamma') + a\gamma + a'\gamma' = 0$

$\Leftrightarrow$ $x \in \overline{J}M$ and $pq\sqrt{D}\mu\overline{x} \in J$.

Suppose that $x \in \overline{J}M$ and $pq\sqrt{D}\mu\overline{x} \in J$. Then we have $f(x) = 0$ by Lemma 7.1.4, and hence

$$x \ \in \ \overline{J}M \cap \mathrm{Ker}(f) = \overline{J}\mathcal{D}M = Q^{-1}I\overline{J}$$

by Lemma 7.1.3 and the equality (7.2.1). Conversely, suppose that $x \in Q^{-1}I\overline{J}$. Then we have $x \in \overline{J}M$. On the other hand, there exist $\xi, \xi' \in Q^{-1}I$ such that

$x = \xi\overline{\gamma} + \xi'\overline{\gamma'}$. Since $\xi, \xi' \in \mathcal{D}M$, we have $f(\xi) = f(\xi') = 0$, and hence both of $pq\sqrt{D}\mu\overline{\xi}$ and $pq\sqrt{D}\mu\overline{\xi'}$ are in $\mathbb{Z}_K$. Therefore we have

$$pq\sqrt{D}\mu\overline{x} = (pq\sqrt{D}\mu\overline{\xi})\gamma + (pq\sqrt{D}\mu\overline{\xi'})\gamma' \in J,$$

and thus $[0, x] \in RJ$ holds. □

We define an orientation of the $\mathbb{Z}$-module $Q^{-1}I_R\overline{J} \subset K$ by (1.0.5). Then, for each $R \in \mathbf{R}$, we obtain an oriented lattice $[0, Q^{-1}I_R\overline{J}]$ of discriminant

$$(pq)^2 \cdot N(Q^{-1}I_R\overline{J})^2 \cdot \mathrm{disc}(\mathbb{Z}_K) = -p^2 d_J^2 D.$$

On the other hand, recall that $\Psi([Q^{-1}I_R\overline{J}]) \in \widetilde{\mathcal{L}}_D^*$ is represented by an oriented lattice such that the underlying $\mathbb{Z}$-module is $Q^{-1}I_R\overline{J} \subset K$ and such that the bilinear form is given by

$$(x, y) = \frac{1}{N(Q^{-1}I_R\overline{J})} \mathrm{Tr}_{K/\mathbb{Q}}(x\overline{y}) = \frac{q}{d_J} \mathrm{Tr}_{K/\mathbb{Q}}(x\overline{y}).$$

Therefore the isomorphism class of the oriented lattice $(J \hookrightarrow RJ)^\perp = [0, Q^{-1}I_R\overline{J}]$ is equal to

$$\Psi([Q^{-1}I_R\overline{J}])[pd_J] \ \in \ \widetilde{\mathcal{L}}_{p^2 d_J^2 D}.$$

By Lemma 7.1.7, we have $\{[I_R] \,|\, R \in \mathbf{R}\} = Cl_D^2$. Hence Proposition 1.0.6 implies that the subset $\{\Psi([Q^{-1}I_R\overline{J}]) \,|\, R \in \mathbf{R}\}$ of $\widetilde{\mathcal{L}}_D^*$ is a lifted genus $\widetilde{\mathcal{G}}$. Consequently, the set

$$\{\ [(J \hookrightarrow RJ)^\perp] \ \mid \ R \in \mathbf{R}\ \} \ = \ \widetilde{\mathcal{G}}\,[pd_J]$$

is also a lifted genus in $\widetilde{\mathcal{L}}_{p^2 d_J^2 D}$. Thus Proposition 6.7.2 is proved.

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