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Transfer Function Approach to Disturbance Decoupling Problem — Source link

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TRANSFER FUNCTION APPROACH TO DISTURBANCE DECOUPLING PROBLEM

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Abstract. We give a necessary and sufficient condition for existence of a controller which decouples the disturbance signal. The condition is based on state space computations. If it is satisfied, we parametrize the set of all disturbance decoupling controllers.

Key words. disturbance decoupling problem, generalized inversion

AMS(MOS) subject classifications. 16G70, 93B20, 93C45

1. Introduction.

Consider the system

$$\begin{cases} x'(t) = Ax(t) + Bu_c(t) + B_d u_d(t) \\ y(t) = Cx(t) \\ y_m(t) = C_m x(t) \end{cases} \tag{1.1}$$

where A, B, B_d, C, and C_m are matrices, u_d is the disturbance, y the output, and y_m the measurement output. The *disturbance decoupling problem* is to find a controller (or a compensator)

$$\begin{cases} x_c'(t) = A_c x_c(t) + B_c y_m(t) \\ u_c(t) = C_c x_c(t) + D_c y_m(t) \end{cases} \tag{1.2}$$

so that the transfer function from u_d to y is identically zero.

Suppose a controller (1.2) is given. Equations (1.1) and (1.2) can be combined into

$$\begin{cases} \begin{bmatrix} x'(t) \\ x_c'(t) \end{bmatrix} = \begin{bmatrix} A + BD_cC_m & BC_c \\ B_cC_m & A_c \end{bmatrix} \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix} + \begin{bmatrix} B_d \\ 0 \end{bmatrix} u_d(t) \\ y(t) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}. \end{cases} \tag{1.3}$$

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Thus, if the disturbance decoupling problem has a solution, $\operatorname{im}[\,B_d^T \quad 0\,]^T$ is contained in a subspace V such that V is invariant under

$$(1.4)\qquad \begin{bmatrix} A+BD_cC_m & BC_c \\ B_cC_m & A_c \end{bmatrix}$$

and $[\,C \quad 0\,]V = (0)$. Partition the state space of the system (1.3) as $X\dot{+}X_c$ where X and X_c are the state spaces of the systems (1.1) and (1.2), and let π be the projection onto X along X_c. Let

$$V_i = \pi(V \cap (X\dot{+}(0))),$$

and let

$$V_p = \pi V.$$

Then [29] $\operatorname{im} B_d \subset V_i$, $V_p \subset \ker C$, and a straightforward computation shows that (V_i, V_p) is a (C_m, A, B)-pair, where in general a (C, A, B)-pair is a pair of subspaces (V_1, V_2) such that

(i) $V_1 \subset V_2$,

(i) $A(V_1 \cap \ker C) \subset V_1$,

(iii) $A(V_2) \subset V_2 + \operatorname{im} B$.

It is well known that condition (ii) is equivalent to

(ii') $\exists G$ such that $(A + GC)V_1 \subset V_1$,

and condition (iii) is equivalent to

(iii') $\exists F$ such that $(A + FB)V_2 \subset V_2$.

A subspace V_1 which satisfies condition (ii) and (ii') is said to be (C, A)-*invariant*, and a subspace V_2 which satisfies conditions (iii) and (iii') is said to be (A, B)-*invariant.*[1]

It has been shown in [29] (see also [30]) that the preceding argument can be reversed. That is, suppose we have a system (1.1), there exists a (C_m, A, B)-pair (V_1, V_2) such that $\operatorname{im} B_d \subset V_1$ and $V_2 \subset \ker C$, and let $W = \mathbf{C}^{\dim V_2 - \dim V_1}$. Then there exist linear maps A_c, B_c, C_c and D_c, and a subspace V of $X\dot{+}W$, such that V is invariant under the operator (1.4) and $[\,C \quad 0\,]V = (0)$, and so the disturbance decoupling problem has a solution. Since the sum of (A, B)-invariant subspaces is an (A, B)-invariant subspace, there is a largest (A, B)-invariant subspace contained in $\ker C$, $V^*(\ker C)$. Similarly, since the intersection of (C, A)-invariant subspaces is a (C, A)-invariant subspace, there is a smallest (C, A)-invariant subspace containing $\operatorname{im} B_d$, $V_*(\operatorname{im} B_d)$. Thus, the disturbance decoupling problem has a solution if and only if

$$(1.5)\qquad V_*(\operatorname{im} B_d) \subset V^*(\ker C).$$

[1] The name "(C, A, B)-pair" is in a sense better than the name "(C, A)-" or "(A, B)-invariant subspace" in that it does not depend on how the matrices A, B and C are labeled.

If the condition (1.5) is satisfied, the minimum dimension of the state space of the controller equals [29]

$$\min\{\dim V_2 - \dim V_1 : \operatorname{im} B_d \subset V_1,\ V_2 \subset \ker C, \text{ and } (V_1, V_2) \text{ is a } (C_m, A, B) - \text{pair}\}.$$

Below, we present a different solution of the disturbance decoupling problem which is based on the transfer functions. More specifically, we reformulate the problem in terms of the causal (that is, proper or analytic at infinity) solutions of equation

$$GXH = R, \tag{1.6}$$

where G, H and R are rational matrix functions (Section 2). In Section 3 we discuss a general solution of equation (1.6), and in Section 4 we parametrize all causal solutions of the equation in terms of a free parameter Y which runs over all proper rational matrix functions of appropriate size. Out approach is based on the formulas for pointwise generalized inverses of rational matrix functions [23, 24].

We note that if the function G in (1.6) is identically equal to identity, the problem of finding causal solutions of the equation is called the *causal factorization problem.* Various necessary and sufficient conditions for existence of causal solutions in this case are known [19, 16, 12, 17].

2. Transfer function conditions. After taking Laplace transforms of the signals, and assuming zero initial conditions, equations (1.1) and (1.2) become

$$\begin{cases} sx(x) = Ax(s) + Bu_c(s) + B_d u_d(s) \\ y(s) = Cx(s) \\ y_m(s) = C_m x(s) \end{cases} \tag{2.1}$$

and

$$\begin{cases} sx_c(s) = A_c x_c(s) + B_c y_m(s) \\ u_c(s) = C_c x_c(s) + D_c y_m(s). \end{cases} \tag{2.2}$$

From (2.1), we have the transfer functions of the open loop system

$$\begin{aligned} T_{y\leftarrow u_d}(s) &= C(s-A)^{-1}B_d, & T_{y\leftarrow u_c}(s) &= C(s-A)^{-1}B, \\ T_{y_m\leftarrow u_d}(s) &= C_m(s-A)^{-1}B_d \quad \text{and} & T_{y_m\leftarrow u_c}(s) &= C_m(s-A)^{-1}B \end{aligned} \tag{2.3}$$

The transfer function of the controller (2.2) is

$$T_c(s) = D_c + C_c(s-A)^{-1}B_c, \tag{2.4}$$

and so in the closed loop system

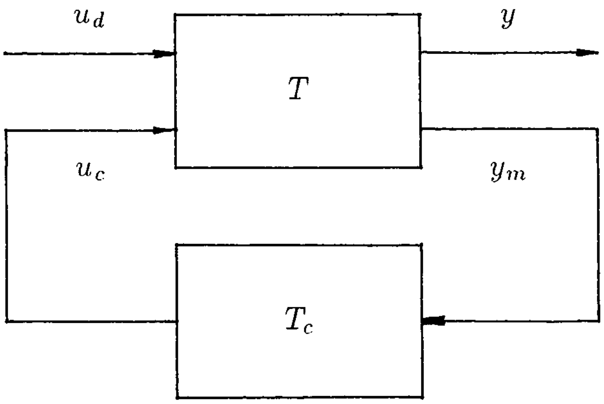

Fig. 2.1

we have

(2.6) $$u_c = T_c(T_{y_m \leftarrow u_d} u_d + T_{y_m \leftarrow u_c} u_c).$$

Hence

(2.8) $$u_c = (I - T_c T_{y_m \leftarrow u_c})^{-1} T_c T_{y_m \leftarrow u_d} u_d,$$

and

(2.8) $$T_{y \leftarrow u_d} + T_{y \leftarrow u_c}(I - T_c T_{y_m \leftarrow u_c})^{-1} T_c T_{y_m \leftarrow u_d}$$

is the closed-loop transfer function from u_d to y. Thus, if there exists a controller (1.2) which solves the disturbance decoupling problem, the function $T_{y \leftarrow u_d}$ can be factored as

(2.9) $$T_{y \leftarrow u_d} = T_{y \leftarrow u_c} X T_{y_m \leftarrow u_d}$$

where

(2.10) $$X = (T_c T_{y_m \leftarrow u_c} - I)^{-1} T_c$$

is proper.

Conversely, suppose we have strictly proper functions (2.3) and equation (2.9) has a proper solution X. Then we can solve equation (2.10) for T_c to obtain the controller

(2.11) $$T_c = X(T_{y_m \leftarrow u_c} X - I)^{-1}.$$

Thus, proper solutions of equation (2.9) provide disturbance decoupling controllers.

In [21], a simple necessary condition for existence of proper solutions of equation (2.9) has been indicated. To simplify notation, write (2.9) as

(2.9′) $$GXH = R$$

and let $G_{i,*}$ and $H_{*,j}$ denote the i^{th} row and j^{th} column of G and H. Let $\mathcal{O}(W)$ be the order of a rational matrix function W at infinity, that is, $\mathcal{O}(0) = -\infty$ and, if $W \neq 0$, $z^{-\mathcal{O}(W)}W(z)$ is analytic and does not vanish at infinity. If equation (2.9') has a proper solution, then

$$R_{ij} = G_{i,*}XH_{*,j}$$

and hence, for all i, j
(i) $G_{i,*} \neq 0$ and $H_{*,j} \neq 0$ whenever $R_{ij} \neq 0$,
(ii) $\mathcal{O}(R_{ij}) \leq \mathcal{O}(G_{i,*}) + \mathcal{O}(H_{*,j})$.
If the matrix G_0 (resp. H_0) formed by the leading coefficients in the Laurent expansions at infinity of the rows (resp. columns) of G (resp. H) has full row (resp. column) rank, conditions (i) and (ii) are also sufficient for existence of a proper solution of equation (2.9'). The general case can be reduced to the case when G_0 (resp. H_0) has full row (resp. column) rank by operations linear over the field of scalar rational functions.

In the disturbance decoupling problem, we are especially interested in controllers which stabilize the closed loop system in Fig. 2.1, that is, a controller for which the matrix (1.4) has all eigenvalues in the open left half-plane. Such controllers are called *stabilizing compensators*. In terms of the transfer functions, stability of the system in Fig. 2.1 is equivalent [14] to the stability of the transfer functions from u_d, v_1 and v_2 to y, y_m and u_c in the following diagram.

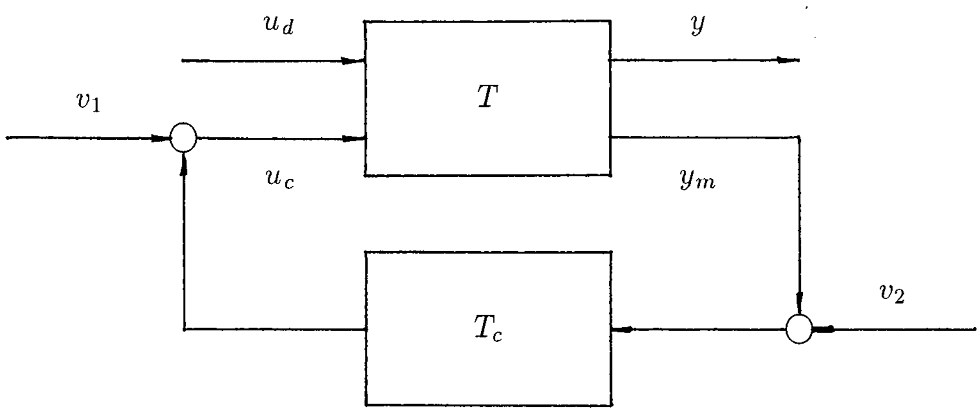

Fig. 2.2

Suppose the plant T is *stabilizable*, that is, there exists a compensator T_c for which the system in Fig. 2.1 is stable, and let M, N, $\tilde{M}$, $\tilde{N}$, U, V, $\tilde{U}$ and $\tilde{V}$ be stable (that is, without poles in the closed right half-plane) rational matrix functions such that

$$T_{y_m \leftarrow u_c} = NM^{-1} = \tilde{M}^{-1}\tilde{N} \tag{2.13}$$

and

$$\begin{bmatrix} \tilde{U} & -\tilde{V} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & V \\ N & U \end{bmatrix} = I. \tag{2.14}$$

Then

$$\{(V - MQ)(U - NQ)^{-1} : Q \text{ is a stable rational matrix function}\} \tag{2.15}$$

is the set of all stabilizing compensators. The parametrization (2.15) of stabilizing compensators is called *Youla parametrization.* An equivalent characterization of stabilizing compensators has been given in [1, 4, 5].

Below, we find all proper solutions of equation (2.9′) without addressing stability of the resulting closed loop plant. That is, we obtain all controllers which decouple the disturbance signal. The set of all stabilizing compensators which solve the disturbance decoupling problem can be obtained by intersecting the set of disturbance decoupling controllers with the set (2.15).

3. Generalized pseudoinverses. In this section we find a general solution of equation $GXH = R$ with G, H and R not necessarily proper rational matrix functions. We begin with a simplified version of the equation where all functions are constant. Consider the equation

$$AXB = C \tag{3.1}$$

where all letters represent complex matrices. If D is a matrix, by a *generalized inverse* of D we will understand any matrix $D^{\ddagger}$ such that

$$DD^{\ddagger}D = D \qquad \text{and} \qquad D^{\ddagger}DD^{\ddagger} = D^{\ddagger}. \tag{3.2}$$

There is no unique terminology concerning generalized inverses. The term generalized inverse is sometimes used to denote a linear operator which satisfies only the first equality in (3.2) (see e.g. [32]), and a matrix which satisfies both equalities (3.2) is called a (1, 2)-inverse of D in [8, 9]. In [22] the term generalized inverse has been used to denote the matrix which in addition to (3.2) satisfies equalities

$$(DD^{\ddagger})^* = DD^{\ddagger} \qquad \text{and} \qquad (D^{\ddagger}D)^* = D^{\ddagger}D. \tag{3.3}$$

This matrix is unique and is called the *Moore-Penrose inverse* of D.

Suppose equation (3.1) has a solution D. Then [22]

$$\begin{aligned} A(A^{\ddagger}CB^{\ddagger})B &= (AA^{\ddagger}A)D(BD^{\ddagger}B) \\ &= A^{\ddagger}DB^{\ddagger} \\ &= C \end{aligned} \tag{3.4}$$

and $X = A^{\ddagger}CB^{\ddagger}$ satisfies the equation. Conversely, if

$$A(A^{\ddagger}CB^{\ddagger})B = C, \tag{3.5}$$

then equation (3.1) has a solution. Thus, equation (3.1) is solvable if and only if (3.5) holds. It has been pointed out in [22] that only the first equality in (3.2) plays role in the preceding computation.

Suppose equation (3.1) is solvable and let $A^\ddagger$ and $B^\ddagger$ be generalized inverses of A and B. The particular solution $A^\ddagger CB^\ddagger$ can be used to parametrize all solutions of the equation. Indeed, a matrix Z solves (3.1) if and only if Z differs from $A^\ddagger CB^\ddagger$ by Y where Y satisfies the homogeneous equation $AXB = 0$. Suppose a matrix Y satisfies the homogeneous equation. Then $A^\ddagger(AYB)B^\ddagger = 0$, and Y can be written as

$$Y - A^\ddagger AYBB^\ddagger. \tag{3.7}$$

Conversely, for an arbitrary matrix Y of appropriate size

$$A(Y - A^\ddagger AYBB^\ddagger)B = AYB - AYB = 0.$$

It follows that the general solution of equation (3.1) has the form

$$A^\ddagger CB^\ddagger + Y - A^\ddagger AYBB^\ddagger. \tag{3.9}$$

All these observations can be extended to equations involving rational matrix functions either algebraically, by considering matrices over the field $\mathcal{R}$ of scalar rational functions, or analytically, by considering pointwise generalized inverses of matrix valued functions. Adopting the notation in [6], we will denote a generalized inverse of a linear transformation $W: \mathcal{R}^n \to \mathcal{R}^m$ by $W^\times$. Thus,

$$WW^\times W = W \qquad \text{and} \qquad W^\times WW^\times = W^\times. \tag{3.10}$$

Let $G^\times$ and $H^\times$ be generalized inverses of G and H. Then equation (2.9′) is solvable if and only if $GG^\times RH^\times H = R$, and the general solution of the solvable equation is

$$X = G^\times RH^\times + Y - G^\times GYHH^\times. \tag{3.11}$$

We will regard an element G of $\mathcal{R}^{m\times n}$ as a (meromorphic) function from the extended complex plane $\mathbf{C}_\infty$ into $\mathbf{C}^{m\times n}$. The function $\overline{G}$ such that

$$\overline{G}(z)G(z) = I \qquad \text{and} \qquad G(z)\overline{G}(z) = I$$

is a pointwise inverse of G, and a function $G^\times$ which satisfies (3.10) is a pointwise generalized inverse of G. The functions $\overline{G}$ and $G^\times$ have been called a *pseudoinverse* ("fake inverse") and a *generalized pseudoinverse* in [23].

Suppose a function $G \in \mathcal{R}^{n\times n}$ is analytic, and takes nonsingular value, at infinity, and let (A, B, C, D) be a realization of G. Then the pseudoinverse of G is given by the formula [31, 6]

$$[G(z)]^{-1} = D^{-1} - D^{-1}C(z - A^\times)^{-1}BD^{-1} \tag{3.13}$$

with $A^\times = A - BD^{-1}C$. This formula extends to pointwise generalized inverses as follows.

Theorem 3.1 *Suppose a function $G \in \mathcal{R}^{m\times n}$ has neither a pole nor zero at infinity, and let (A, B, C, D) be a realization of G. Pick a generalized inverse $D^\ddagger$ of D, and let*

$$G^\times(z) = D^\ddagger - D^\ddagger C(z - A^\times)^{-1} BD^\ddagger \tag{3.14}$$

with $A^\times = A - BD^\ddagger C$. Then $G^\times$ is a generalized pseudoinverse of G.

Proof We have

$$\begin{aligned} G^\times(z)G(z)D^\ddagger &= (D^\ddagger - D^\ddagger C(z - A^\times)^{-1}BD^\ddagger)(DD^\ddagger + C(z-A)^{-1}BD^\ddagger) \\ &= D^\ddagger + D^\ddagger C(z-A)^{-1}BD^\ddagger - D^\ddagger C)z - A^\times)^{-1}BD^\ddagger \\ &\qquad - D^\ddagger C(z-A^\times)^{-1}BD^\ddagger C(z-A)^{-1}BD^\ddagger \\ &= D^\ddagger + D^\ddagger C(z-A^\times)^{-1}[z - A^\times - z + A - BD^\ddagger C](z-A)^{-1}BD^\ddagger \\ &= D^\ddagger. \end{aligned}$$

Hence $G^\times G G^\times = G^\times$ and the second equality in (3.10) holds. Also, $GG^\times GG^\times = GG^\times$ and multiplication by $GG^\times$ is a projection onto $\mathrm{im}\,(GG^\times)$. Clearly,

$$\mathrm{rank}_{\mathcal{R}}(GG^\times) \geq \mathrm{rank}_{\mathcal{R}}(G).$$

Since G has neither a zero nor pole at infinity,

$$\mathrm{rank}_{\mathcal{R}}(G) = \mathrm{rank}_{\mathbf{C}}(D) = \mathrm{rank}_{\mathbf{C}}(D^\ddagger) = \mathrm{rank}_{\mathcal{R}}(G^\times)$$

and $\mathrm{rank}_{\mathcal{R}}(GG^\times) = \mathrm{rank}_{\mathcal{R}}(G)$. Thus, $\mathrm{im}\,(GG^\times) = \mathrm{im}\,(G)$ and multiplication by $GG^\times$ is a projection onto $\mathrm{im}\, G$. Hence $GG^\times G = G$. □

Theorem 3.1 has been obtained originally [23] as a corollary to a generalization of the theorem of Bart-Gohberg-Kaashoek-Van Dooren on minimal factorization of regular rational matrix functions [7]. The connection between the spectrum of $A^\times$ and the zeros of G has been indicated in [27]. Note that both properties (3.2) of a generalized inverse of a matrix have been used in the proof.

The pointwise generalized inverses of G which arise through formula (3.14) are precisely those which have zero defect [23, 24], where the *defect* of a rational matrix function is the difference between the McMillan degree of the function and the sum of multiplicities of all its zeros [17]. The defect of a function $W \in \mathcal{R}^{m\times n}$ can be also characterized as follows. Let $\Omega\mathcal{R}$ and $\Omega_\infty\mathcal{R}$ be the subrings of $\mathcal{R}$ formed by functions analytic in the complex plane and at infinity, respectively, and denote by $\Omega\mathcal{R}^i$ and $\Omega_\infty\mathcal{R}^i$ appropriate direct sums. Let V be a subspace of $\mathcal{R}^n$. A Wedderburn-Forney space $\mathcal{W}(V)$ associated with V is the finite dimensional space over $\mathbf{C}$ [35, 36]

$$\mathcal{W}(V) = \frac{\pi_- V}{V \cap \pi_- \mathcal{R}^n}$$

where π_- is the projection onto $z^{-1}\Omega_\infty\mathcal{R}^n$ along $\Omega\mathcal{R}^n$. A function $G \in \mathcal{R}^{m\times n}$ has zero defect if and only if $\mathcal{W}(\ker G) = (0)$ and $\mathcal{W}(\mathrm{im}\, G) = (0)$.

Let

$$Z(G) = \frac{G^{-1}(\Omega\mathcal{R}^m) + \Omega\mathcal{R}^n}{\ker G + \Omega\mathcal{R}^n}$$

and

$$Z_\infty(G) = \frac{G^{-1}(\Omega_\infty\mathcal{R}^m) + \Omega_\infty\mathcal{R}^n}{\ker G + \Omega_\infty\mathcal{R}^n}$$

be the zero modules of G in the finite plane and at infinity, and let

$$\mathcal{Z}(G) = Z(G) \oplus Z_\infty(G)$$

be the global zero space of G. Let

$$P(G) = \frac{\Omega\mathcal{R}^n}{G^{-1}(\Omega\mathcal{R}^m) \cap \Omega\mathcal{R}^n}$$

and

$$P_\infty(G) = \frac{\Omega_\infty\mathcal{R}^n}{G^{-1}(\Omega_\infty\mathcal{R}^m) \cap \Omega_\infty\mathcal{R}^n}$$

be the pole modules of G in the finite plane and at infinity, and let $\mathcal{P}(G) = P(G) \oplus P_\infty(G)$ be the global pole space of G. Then [35, 36] there exists an exact sequence of vector spaces and vector space homomorphisms

$$0 \to \mathcal{Z}(G) \to \frac{\mathcal{P}(G)}{\mathcal{W}(\ker G)} \to \mathcal{W}(\operatorname{im} G) \to 0.$$

Thus, if G has no defect, $\mathcal{Z}(G) \cong \mathcal{P}(G)$.

The assumption essential in Theorem 3.1 is that G have neither a pole nor zero at infinity. The functions G and H in (2.9′) are strictly proper. Also, equation (2.9′) can be of interest in the case when G or H have a pole at infinity. To write down a formula for a generalized pseudoinverse of an arbitrary rational matrix function, we need a more general representation tool. An arbitrary function $W \in \mathcal{R}^{m\times n}$ can be represented in the generalized state space form [34]

$$W(z) = D + C(zE - A)^{-1}B. \tag{3.16}$$

In fact [34], matrices A, B, C and E can be chosen so that $D = 0$.

The function W can be also represented in the form [28, 11, 1]

$$W(z) = C_{\mathrm{C}}(z - A_{\mathrm{C}})^{-1}B_{\mathrm{C}} + D_\infty + C_\infty(z^{-1} - A_\infty)^{-1}B_\infty \tag{3.17}$$

with A_∞ nilpotent, where the last two terms realize the polynomial part of W. In [15], nonproper rational matrix functions have been represented in the form

$$W(z) = D + (z - \alpha)C(zG - A)^{-1}B, \tag{3.18}$$

where α is a point in the complex plane. The representation (3.18) has been modified in [25] to the form

$$W(z) = D + C\left(\frac{1}{z - \alpha} - \tilde{A}\right)^{-1}\tilde{B}, \tag{3.19}$$

called a *centered realization*. It is written as $(\tilde{A}, \tilde{B}, C, D, \alpha)$.

A realization (A, B, C, D, α) of a rational matrix function W is *minimal* if the dimension of the state space (i.e. the domain of A) equals the McMillan degree of W. For a fixed α, a minimal realization is unique up to similarity. We note that centered realizations obey the usual rules of realization arithmetic. That is, if $(A_i, B_i, C_i, D_i, \alpha)$ is a realization of a function G_i $(i = 1, 2)$ and the number of columns of G_1 equals the number of rows of G_2, then

$$\left(\begin{bmatrix} A_1 & B_1C_2 \\ & A_2 \end{bmatrix}, \begin{bmatrix} B_1D_2 \\ B_2 \end{bmatrix}, [\,C_1 \quad D_1C_2\,],\, D_1D_2,\, \alpha\right)$$

is a realization of the function G_1G_2. Also, if G_1 and G_2 have equal sizes,

$$\left(\begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, [\,C_1 \quad C_2\,],\, D_1 + D_2,\, \alpha\right)$$

is a realization of the function $G_1 + G_2$.

The advantage of representing nonproper rational matrix functions by means of centered realizations is that centered realizations involve a single state space operator rather than the operator pencil. We recall from [25] the connection between the matrices in (3.17) and those in (3.19).

Proposition 3.2 *Suppose* (3.17) *holds with* A_∞ *nilpotent. Pick a point* $\alpha \in \mathbf{C}$ *such that the matrices* $(\alpha - A_{\mathrm{C}})$ *and* $(I - \alpha A_\infty)$ *are invertible, and let*

$$A = \begin{bmatrix} -(\alpha - A_{\mathrm{C}})^{-1} & \\ & (I - \alpha A_\infty)^{-1}A_\infty \end{bmatrix}, \quad B = \begin{bmatrix} -(\alpha - A_{\mathrm{C}})^{-2}B_{\mathrm{C}} \\ (I - \alpha A_\infty)^{-2}B_\infty \end{bmatrix},$$
$$C = [\,C_{\mathrm{C}} \quad C_\infty\,], \quad D = D_\infty + C_{\mathrm{C}}(\alpha - A_{\mathrm{C}})^{-1}B_{\mathrm{C}} + \alpha \mathbf{C}_\infty(I - \alpha A_\infty)^{-1}B_\infty.$$

Then

$$W(z) = D + C\left(\frac{1}{z - \alpha} - A\right)^{-1} B. \tag{3.20}$$

For the sake of completeness, we include the computation which underlies Proposition 3.2. Suppose (3.17) holds, and let $\alpha \in \mathbf{C}$ be such that the matrices $(\alpha - A_{\mathrm{C}})$ and $(I - \alpha A_\infty)$ are invertible. Then [18]

$$\begin{aligned} W(z) &= D_\infty + C_{\mathbf{C}}(\alpha - A_{\mathbf{C}})^{-1}B_{\mathbf{C}} + \alpha C_\infty(I - \alpha A_\infty)^{-1}B_\infty \\ &\quad + (z - \alpha)[\,C_{\mathbf{C}} \quad C_\infty\,]\left(z\begin{bmatrix} I & \\ & A_\infty \end{bmatrix} - \begin{bmatrix} A_{\mathbf{C}} & \\ & I \end{bmatrix}\right)^{-1} \begin{bmatrix} -(\alpha - A_{\mathbf{C}})^{-1}B_{\mathbf{C}} \\ -(I - \alpha A_\infty)^{-1}B_\infty \end{bmatrix} \\ &:= D + (z - \alpha)C(zG - A_\alpha)^{-1}B_\alpha \end{aligned}$$

where the matrix $\alpha G - A_\alpha$ is invertible. Hence

$$\begin{aligned}
W(z) &= D + C\left(\frac{z}{z-\alpha}G - \frac{1}{z-\alpha}A_\alpha\right)^{-1} B_\alpha \\
&= D + C\left(G + \frac{\alpha}{z-\alpha}G - \frac{1}{z-\alpha}A_\alpha\right)^{-1} B_\alpha \\
&= D + C\left(\frac{1}{z-\alpha}(\alpha G - A_\alpha) + G\right)^{-1} B_\alpha \\
&= D + C\left(\frac{1}{z-\alpha} + (\alpha G - A_\alpha)^{-1}G\right)^{-1} (\alpha G - A_\alpha)^{-1}B_\alpha \\
&:= D + C\left(\frac{1}{z-\alpha} - A\right)^{-1} B.
\end{aligned}$$

Proposition 3.2 has the following converse.

Proposition 3.3 *Let (A, B, C, D, α) be a realization of a function $W \in \mathcal{R}^{m\times n}$. Let S be a nonsingular matrix such that*

$$SAS^{-1} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$$

with the matrix A_{11} invertible and A_{22} nilpotent, and partition

$$SB = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}, \quad CS^{-1} = [\, C_1 \quad C_2 \,]$$

conformably. Then

$$W(z) = D_1 + C_1(z - A_1)^{-1}B_1 + C_2\left(\frac{1}{z} - A_2\right)^{-1} B_2,$$

where

$$\begin{aligned}
A_1 &= \alpha + A_{11}^{-1}, & A_2 &= A_{22}(I + \alpha A_{22})^{-1}, \\
B_1 &= -A_{11}^{-2}\tilde{B}_1, & B_2 &= (I + \alpha A_{22})^{-2}\tilde{B}_2,
\end{aligned}$$

and $D_1 = D - C_1 A_{11}^{-1}\tilde{B}_1 - \alpha C_2(I + \alpha A_{22})^{-1}\tilde{B}_2$.

In [23], the formulas for a generalized pseudoinverse of an arbitrary rational matrix function has been given in terms the realization (3.18). The formulas take the following form [24, 10] for a realization (3.20).

Theorem 3.4 *Let $W \in \mathcal{R}^{m\times n}$, pick a point α which is neither a pole nor zero of W, and compute a centered realization (A, B, C, D, α) of W. Then*

$$W^\times(z) = D^\ddagger - D^\ddagger C\left(\frac{1}{z-\alpha} - A^\times\right)^{-1} BD^\ddagger,$$

with $A^\times = A - BD^\ddagger C$, is a generalized pseudoinverse of W whenever $D^\ddagger$ is a generalized inverse of D.

4. Parametrization of all disturbance decoupling controllers.

4. Parametrization of all disturbance decoupling controllers. Theorem 3.4 and formula (3.11) provide the general solution of equation (2.9′). To characterize the proper solutions, we will need certain elements of valuation theory. The valuation-theoretic tools have been frequently used in systems theory [13, 33, 17]. We recall now the definition of orthogonality in $\mathcal{R}^n$ adopted in [2, 3].

Pick a point $\lambda \in \mathbf{C}_\infty$. Let $|0|_\lambda = 0$, and if r is a nonzero element of $\mathcal{R}$, let

$$|r|_\lambda = e^\eta \tag{4.1}$$

where η is the integer such that $(z-\alpha)^\eta r(z)$ (or $z^{-\eta} r(z)$ if $\lambda = \infty$) is analytic, and does not vanish, at λ. The function $|\cdot|_\lambda$ determines the non-Archimedean valuation of $\mathcal{R}$. If $x = (x_1, x_2, \ldots, x_n) \in \mathcal{R}^n$, let

$$\|x\|_\lambda = \max\{|x_1|_\lambda, |x_2|_\lambda, \ldots, |x_n|_\lambda\}. \tag{4.2}$$

Then $(\mathcal{R}^n, \|\cdot\|_\lambda)$ is a non-Archimedean normed space over the real valued field $(\mathcal{R}, |\cdot|_\lambda)$. The space $(\mathcal{R}^n, \|\cdot\|_\lambda)$ is not complete, but the concept of orthogonality [20] still applies. If U and V are subspaces of $\mathcal{R}^n$,

$$\|u+v\|_\lambda \leq \|u\|_\lambda + \|v\|_\lambda \tag{4.3}$$

for all $u \in U$ and $v \in V$. Subspaces U and V are orthogonal with respect to $\|\cdot\|_\lambda$ if (4.3) is always an equality. If this happens, we will say then that U and V are *orthogonal at* λ. Subspaces U and V of $\mathcal{R}^n$ are orthogonal on a set $\sigma \in \mathbf{C}_\infty$ if they are orthogonal at every point of σ.

Orthogonality of subspaces U and V of $\mathcal{R}^n$ at λ can be characterized in terms of linear algebra as follows. If Ω is a subspace of $\mathcal{R}^n$, let

$$\Omega(\lambda) = \{\phi(\lambda) : \ \phi \in \Omega \text{ is analytic at } \lambda\}. \tag{4.5}$$

Then $\Omega(\lambda)$ is a subspace of $\mathbf{C}^n$, and

$$\dim_{\mathbf{C}} \Omega(\lambda) = \dim_{\mathcal{R}} \Omega. \tag{4.5}$$

The space $\Omega(\lambda)$ has been called the *value* of Ω at λ. It can be alternatively characterized as the linear span of the leading coefficients in the Laurent expansions at λ of the functions in Ω. The subspaces U and V of $\mathcal{R}^n$ are orthogonal at λ [2, 3] if

$$U(\lambda) \cap V(\lambda) = (0). \tag{4.6}$$

Let $G^\times$ be a pointwise generalized inverse of a function $G \in \mathcal{R}^{m\times n}$. Then

$$WW^\times WW^\times = WW^\times$$

and the map $\pi : \ \mathcal{R}^m \to \mathcal{R}^n$ defined by

$$\pi\phi = WW^\times \phi \tag{4.7}$$

is a projection onto the subspace U spanned by the columns of $WW^{\times}$ along the subspace spanned by the columns of $(I-WW^{\times})$. Since $\operatorname{rank}_{\mathcal{R}} W = \operatorname{rank}_{\mathcal{R}}(WW^{\times})$, V is spanned by the columns of W. Similarly, postmultiplication by $WW^{\times}$ is a projection of $\mathcal{R}^{1\times n}$ onto the row span of $W^{\times}$ along the subspace spanned by the rows of $(I - WW^{\times})$.

Pick complementary subspaces U and V of $\mathcal{R}^n$, and let π be the projection onto U along V. The projection π is said to be *orthogonal* at λ [2] if U and V are orthogonal at λ. By Proposition 2.1 in [26] (see also Proposition 4.3 in [2]), the following statements are equivalent:

(i) the projection π is orthogonal at λ,

(ii) $\pi(\phi)$ is analytic at λ whenever ϕ is analytic at λ.

Orthogonal projections corresponding to multiplication by $WW^{\times}$ can be also characterized as follows [26]. If $W \in \mathcal{R}^{m\times n}$, let

$$W^{o\ell} = \{\phi \in \mathcal{R}^{1\times m} : \phi W = 0\} \tag{4.8}$$

and let

$$W^{or} = \{\psi \in \mathcal{R}^{n\times 1} : W\psi = 0\}. \tag{4.9}$$

Let EGF and $\overline{E}\,\overline{G}\,\overline{F}$ be the Smith's normal forms at λ [1] of W and $W^{\times}$, respectively. Suppose the normal rank of W (and $W^{\times}$) equals k. Then

$$EGF = \begin{bmatrix} E_1 & E_2 \end{bmatrix} \begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F_1 \\ * \end{bmatrix} = E_1 G_1 F_1$$

where G_1 has the size $k \times k$. Similarly, $\overline{E}\,\overline{G}\,\overline{F} = \overline{E}_1\,\overline{G}_1\,\overline{F}_1$. From the first equality in (3.10) it follows that

$$G_1(z)F_1(z)\overline{E}_1(z)\overline{G}_1(z)\overline{F}_1(z)E_1(z) = I,$$

and so $G_1 F_1 \overline{E}_1\,\overline{G}_1$ is the pseudoinverse of $\overline{F}_1 E_1$. Now E_1 and $\overline{F}_1$ are analytic at λ, the column span of E_1 is equal to the column span of W, and the row span of $\overline{F}_1$ is equal to the row span of $W^{\times}$. Consequently, the matrix $\overline{F}_1(\lambda)E_1(\lambda)$ is nonsingular if and only if $W^{\times or}$ and the column span of W are orthogonal at λ. Therefore, the function $WW^{\times} = E_1 G_1 F_1 \overline{E}_1 \overline{G}_1 \overline{F}_1$ is analytic at λ if and only if $W^{\times or}$ and the column span of W are orthogonal at λ. Similarly, the function $W^{\times}W$ is analytic at λ if and only if the row span of W is orthogonal to $W^{\times o\ell}$ at λ. We summarize the results of the preceding discussion.

Proposition 4.1 *Let $W^{\times} \in \mathcal{R}^{n\times m}$ be a generalized pseudoinverse of a function $W \in \mathcal{R}^{m\times n}$, let π_L and π_R be the projections*

$$\pi_L(\phi) = W^{\times}W\phi \qquad \text{and} \qquad \pi_R(\psi) = \psi WW^{\times},$$

and let $\lambda \in \mathbf{C}_\infty$. Then

(i) π_L is orthogonal at λ if and only if the row span of W is orthogonal to $W^{\times o\ell}$ at λ,

(ii) π_R *is orthogonal at* λ *if and only if the column span of* W *is orthogonal to* $W^{\times or}$ *at* λ.

The generalized pseudoinverse (3.14) of the function $G \in \mathcal{R}^{m\times n}$ has zero defect; $G^{\times or}(\lambda)$ and $G^{\times o\ell}(\lambda)$ coincide with the right and left kernels of $D^{\ddagger}$ for all $\lambda \in \mathbf{C}_\infty$. Consequently, we can choose a generalized inverse $D^{\ddagger}$ of D so that the projection determined by left multiplication by $G^{\times}G$ and/or right multiplication by $GG^{\times}$ is orthogonal at a given point $\lambda \in \mathbf{C}_\infty$.

We can give now a necessary and sufficient condition for existence of proper solutions of equation (2.9′). Pick a generalized pseudoinverse $G^{\times}$ of G such that $G^{\times o\ell}$ is orthogonal to the row span of G at infinity, and a generalized pseudoinverse $H^{\times}$ of H such that $H^{\times or}$ is orthogonal to the column span of H at infinity. Equation (2.9′) is solvable if and only if $GG^{\times}RH^{\times}H = R$. If $G^{\times}RH^{\times}$ is proper, then obviously equation (2.9′) has a proper solution. Suppose equation (2.9′) has a proper solution Z. Then $G^{\times}RH^{\times}$ and Z are two solutions of (2.9′), and so there exists a rational matrix function Y such that

$$Z = G^{\times}RH^{\times} + Y - G^{\times}GYHH^{\times}. \tag{4.10}$$

Hence

$$G^{\times}GZHH^{\times} = G^{\times}RH^{\times}, \tag{4.11}$$

and, by Proposition 4.1, the function $G^{\times}RH^{\times}$ is proper. Thus, a solvable equation (2.9′) has a proper solution if and only if $G^{\times}RH^{\times}$ is proper.

Al proper solutions of equation (2.9′) can be parametrized as follows. Let $G^{\times}$ and $H^{\times}$ be as above, and suppose $G^{\times}RH^{\times}$ is a proper solution of equation (2.9′). If Y is a proper rational matrix function of appropriate size, then

$$X = G^{\times}RH^{\times} + Y - G^{\times}GYHH^{\times} \tag{4.12}$$

is a proper solution of equation (2.9′).

Suppose Z is a proper solution of equation (2.9′). Then the function $G^{\times}RH^{\times}$ is proper and

$$Z = G^{\times}RH^{\times} + \tilde{Y} - G^{\times}G\tilde{Y}HH^{\times} \tag{4.13}$$

for some rational matrix function $\tilde{Y}$. Let

$$Y = \tilde{Y} - G^{\times}G\tilde{Y}HH^{\times}. \tag{4.14}$$

By Proposition 4.1, $G^{\times}G$ and, in view of (4.13), the functions

$$(I - G^{\times}G)\tilde{Y} \qquad \text{and} \qquad \tilde{Y}(I - HH^{\times}) \tag{4.15}$$

are proper. Hence

$$Y = (I - G^{\times}G)\tilde{Y} + (G^{\times}G)\tilde{Y}(I - HH^{\times}). \tag{4.16}$$

is proper, and

$$\begin{aligned} Z &= G^\times R H^\times + Y \\ &= G^\times R H^\times + Y - G^\times G Y H H^\times. \end{aligned}$$

Thus, (4.12) with Y proper is the general proper solution of equation (2.9′).

Theorem 4.2 *Let G, H, and R be rational matrix functions. Pick a point $\alpha \in \mathbf{C}$ which is neither a pole nor a zero of G or H, and compute realizations $(A_G, B_G, C_G, D_G, \alpha)$ and $(A_H, B_H, C_H, D_H, \alpha)$ of G and H. Compute generalized inverses $D_G^\ddagger$ and $D_H^\ddagger$ of D_G and D_H such that the row span of $D_H^\ddagger$ intersects trivially with $H^{\circ\ell}(\infty)$ and the column span of $D_G^\ddagger$ intersects trivially with $G^{\circ r}(\infty)$. Let*

$$G^\times(z) = D_G^\ddagger - D_G^\ddagger C_G \left(\frac{1}{z-\alpha} - A_G^\times \right)^{-1} B_G D_G^\ddagger,$$

and let

$$H^\times(z) = D_H^\ddagger - D_H^\ddagger C_H \left(\frac{1}{z-\alpha} - A_H^\times \right)^{-1} B_H D_H^\ddagger$$

where $A_G^\times = A_G - B_G D_G^\ddagger C_G$ and $A_H^\times = A_H - B_H D_H^\ddagger C_H$. Then R admits a factorization $R = GXH$ with the factor X causal if and only if $GG^\times R H^\times H = R$ and the function $G^\times R H^\times$ is proper. Moreover, if the condition is satisfied, X is a causal factor if and only if

$$X = G^\times R H^\times + Y - G^\times G Y H H^\times$$

for some proper rational matrix function Y.

Theorem 4.2 yields the following parametrization of disturbance decoupling controllers for the plant (2.1).

Theorem 4.3 *Pick a point $\alpha \in \mathbf{C}\backslash\sigma(A)$ which is not a zero of $T_{y\leftarrow u_c}(s) = C(s-A)^{-1}B$ or $T_{y_m\leftarrow u_d}(s) = C_m(s-A)^{-1}B$. Let $A_\alpha = (A-\alpha)^{-1}$, and let*

$$\begin{aligned} B_G &= -(\alpha - A)^{-2} B, & C_G &= C, & D_G &= C(\alpha - A)^{-1} B \\ B_H &= -(\alpha - A)^{-2} B_d, & C_H &= C_m, & D_H &= C(\alpha - A)^{-1} B_d \\ B_R &= -(\alpha - A)^{-2} B, & C_R &= C, & D_R &= C(\alpha - A)^{-1} B_d. \end{aligned}$$

Choose a generalized inverses $D_G^\ddagger$ and $D_H^\ddagger$ of D_G and D_H such that the column span of $D_G^\ddagger$ intersects trivially with $T^{\circ r}_{y\leftarrow u_c}(\infty)$ and the row span of $D_H^\ddagger$ intersects trivially with $T^{\circ\ell}_{y_m\leftarrow u_d}(\infty)$. Then there exists a disturbance decoupling compensator if and only if

$$GG^\times R H^\times H = R$$

and the function $G^\times R H^\times$ is proper, where

$$G(z) = D_G + C_G \left(\frac{1}{z-\alpha} - A_\alpha \right)^{-1} B_G,$$

$$G^\times(z) = D_G^\ddagger - D_G^\ddagger C_G \left(\frac{1}{z-\alpha} - A_\alpha + B_G D_G^\ddagger C_G \right)^{-1} B_G D_G^\ddagger,$$

$$H(z) = D_H + C_H \left(\frac{1}{z-\alpha} - A_\alpha \right)^{-1} B_H,$$

$$H^\times(z) = D_H^\ddagger - D_H^\ddagger C_H \left(\frac{1}{z-\alpha} - A_\alpha + B_H D_H^\ddagger C_H \right)^{-1} B_H D_H^\ddagger,$$

and

$$R(z) = D_R + C_R \left(\frac{1}{z-\alpha} - A_\alpha \right)^{-1} B_R.$$

Moreover, suppose there exists a disturbance decoupling compensator. Then T_c is the transfer function of a disturbance decoupling compensator if and only if

$$T_c = \left(G^\times R H^\times + Y - G^\times G Y H H^\times\right)\left(T_{y_m \leftarrow u_c}(G^\times R H^\times + Y - G^\times G Y H H^\times) - I\right)^{-1}$$

for some proper rational matrix function Y where $T_{y_m \leftarrow u_c}(s) = C_m(s-A)^{-1}B$.

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