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Transform Group of Monotonic Functions with the Same Monotonicity on $[-1, 1]$ and Operations of Fuzzy Numbers

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Abstract Operations of fuzzy numbers are the main content of the fuzzy mathematical analysis. This paper defines the transformation of monotonic bounded functions with same monotonicity on the symmetric interval $[-1, 1]$, and the four fundamental operations of fuzzy numbers based on the fuzzy structured element. It not only make operations of fuzzy numbers easier, but also start a new way for studying on the theory and application of fuzzy analysis.

Keywords Fuzzy numbers · Fuzzy structured element · Monotonic functions · Operations of fuzzy numbers

1 Introduction

The capability of a university decides its competitive advantage and management performance in the essence [1]. Therefore, it does make sense for universities to accumulate, develop, evaluate and utilize their capabilities. Operations of fuzzy numbers are the most basic content of the fuzzy mathematical analysis. Many scholars have studied the issue since 1970s in [1–3], and considered fuzzy numbers as the generalization of interval numbers. The cut sets of fuzzy numbers are interval numbers, so

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the operation on interval numbers is generalized to the operation on fuzzy numbers by extension principle. However, as the ergodicity of $\lambda \in [0,\ 1]$ in the extension principle, the operation on fuzzy numbers is very complex and difficult in [4,5].

Fuzzy numbers are the convex and normal fuzzy sets on real line R. It can be easy verified by extension principle presented by Zadeh that if fuzzy set is convex and normal, and f is a monotonic function on $[-1, 1]$, then $f(A)$ must be the convex and normal fuzzy set. This shows that the convexity and normality is the constant structured feature for monotonic transformations. We put forward the concept of fuzzy structured element and change the research of fuzzy numbers to the monotonic functions with the same monotonicity in [6,7], and prove the homeomorphism of the fuzzy real number space and monotonic functions with the same monotonicity in [8]. This article focuses on the four fundamental operations and the determination of the membership functions of fuzzy numbers where fuzzy numbers are represented by fuzzy structured element.

2 Fuzzy Structured Element and Fuzzy Numbers

Definition 1 Let E be a fuzzy set on the real numbers field R, $E(x)$ is the membership function of E, $\forall x \in R$. Then, E is called a fuzzy structured element, if $E(x)$ satisfies the following properties:

(1) $E(0) = 1,\ E(1 + 0) = E(-1 - 0) = 0$;
(2) $E(x)$ is monotonic increasing and right continuous on $[-1,\ 0]$, $E(x)$ is monotonic decreasing and left continuous on $(0,\ 1]$;
(3) $E(x) = 0(-\infty < x < -1 \text{ or } 1 < x < +\infty)$.

E is called a normal fuzzy structured element, if (1) $\forall x \in (-1,\ 1)$, $E(x) > 0$; (2) $E(x)$ is increasing and continuous on $[-1,\ 0]$, strictly monotonic decreasing and continuous on $(0,\ 1]$. E is called a symmetrical fuzzy structured element, if $E(x) = E(-x)$.

Theorem 1 *Let E be any fuzzy structured element on the real number field R, and $E(x)$ is its membership function, the function $f(x)$ is continuous and monotonic on $[-1,\ 1]$, $\hat{f}(x)$ is a set-valued extensional function from $f(x)$, then $\hat{f}(E)$ is a bounded closed fuzzy number on R, and the membership function of $\hat{f}(E)$ is $E(f^{-1}(x))$, where $f^{-1}(x)$ is a rotational symmetric function with respect to variables x and y (if $f(x)$ is a strictly monotonic function, then $f^{-1}(x)$ is the inverse function of $f(x)$).*

Proof Let $\mu_{\hat{f}(E)}(y)$ be the membership function of $\hat{f}(E)$, $f(x)$ is a monotonic increasing and bounded function on $[-1,\ 1]$. By the extension principle, we have

$$\mu_{\widehat{f}(E)}(y) = \bigvee_{y=\widehat{f}(x)} E(x), \quad \widehat{f}(E) = \bigcup_{x \in R} E(x) * \{\widehat{f}(x)\},$$

where

$$\mu_{E(x)*\{\hat{f}(x)\}}(y) = \begin{cases} E(x), & y \in \{\hat{f}(x)\}, \\ 0, & \text{Others.} \end{cases}$$

Since $\hat{f}(x)$ is an interval number at discontinuity x, then the mapping $\hat{f}$ is surjective, that is, $f^{-1}(x)$ is a mapping from $[-1,\ 1]$ onto $[f(-1),\ f(1)]$. When $y^- = f(-1-0)$, then $\mu_{\hat{f}(E)}(y^-) = E(-1-0) = 0$. When $y^+ = f(1+0)$, then $\mu_{\hat{f}(E)}(y^+) = E(1+0) = 0$. When $y < y^-$ or $y > y^+$, then $E(x) = 0$. When $y^0 = f(0)$, then $\mu_{f(E)} = E(0) = 1$.

For any $x_1,\ x_2 \in [-1,\ 0]$, $y_1 = f(x_1)$, $y_2 = f(x_2)$. Since $f(x)$ is increasing on $[-1,\ 1]$, then $y^+ > y^-$. We also have $y_1 \le y_2$ as $x_1 \le x_2$. Since $\mu_{\hat{f}(E)}(y_1) = E(x_1)$, $\mu_{\hat{f}(E)}(y_2) = E(x_2)$ and $E(x_1) \le E(x_2)$, then $\mu_{\hat{f}(E)}(y_1) \le \mu_{\hat{f}(E)}(y_2)$, that is, $\mu_{\hat{f}(E)}(y)$ is monotonic increasing on $[y^-,\ y^0]$. Similarly, we can prove that $\mu_{\hat{f}(E)}(y)$ is monotonic decreasing on $[y^0,\ y^+]$. Thus, $\hat{f}(E)$ is a normal and convex set on R (that is, a fuzzy number).

Since $\mu_{\hat{f}(E)}(y) = E(x)$ as $y = f(x)$, so $x = f^{-1}(y)$. Next, taking $x = f^{-1}(y)$ into $\mu_{\hat{f}(E)}(y) = E(x)$, we immediately have

$$\mu_{\hat{f}(E)}(y) = E\left(\hat{f}^{-1}(y)\right),$$

or

$$\mu_{\hat{f}(E)}(x) = E\left(\hat{f}^{-1}(x)\right).$$

If $f(x)$ is a monotonic decreasing function on $[-1,\ 1]$, the proof can be shown in a similar manner. □

Theorem 2 *For a given normal fuzzy structured element E and any finite fuzzy number A, there always exists a monotonic bounded function f on* $[-1,\ 1]$ *such that* $A = f(E)$.

Proof Denote $E(x) = E^L(x)$ on $[-1,\ 0)$ and $E(x) = E^R(x)$ on $(0,\ 1]$. It is easy to understand $E^L(x)$ is a strictly monotonic increasing function on $[-1,\ 0)$, and $E^R(x)$ is a strictly monotonic decreasing function. Conveniently, suppose that $A(x)$ is the membership function of the fuzzy number A. Let

$$a_1 = Inf\{x|A(x) > 0\},\quad b_1 = Inf\{x|A(x) = 1\},$$

and

$$a_2 = Sup\{x|A(x) > 0\},\quad b_1 = Sup\{x|A(x) = 1\}.$$

Denote $A(x) = A^L(x)$ on $[a_1,\ b_1)$ and $A(x) = A^R(x)$ on $(b_2,\ a_2]$. It is easy to see that $a_1 \le b_1 \le b_2 \le a_2$, and $A^L(x)$ is a nondecreasing function on $[a_1,\ b_1)$, and $A^R(x)$ is a nonincreasing function on $(b_2,\ a_2]$. Let

$$R_E = \{(x,\ E(x))|x \in R,\ E(x) \in [-1,\ 1]\},$$
$$R_A = \{(y,\ A(y))|y \in R,\ A(y) \in [-1,\ 1]\},$$

be the graph of the fuzzy structured element E and the fuzzy number A, respectively.

Since E is a normal fuzzy structured element, then there exists a unique $x \in [-1, 0)$ such that $E^L(x) = A^L(y)$ for all $y \in [a_1, b_1)$. Similarly, there also exists a unique $x \in [-1, 0)$ such that $E^R(x) = A^R(y)$ for all $y \in (b_2, a_2]$. And there exists $x = 0$ such that $E(x) = A(y)$ for all $y \in [b_1, b_2]$. Therefore, $x = g(y)$ is the mapping from $[a_1, a_2]$ onto $[-1, 1]$.

For all $y_1, y_2 \in [a_1, b_1)$, we have $x_1 = g(y_1)$ and $x_2 = g(y_2)$. If $y_1 < y_2$, then $A(y_1) \leq A(y_2)$. Since $E^L(x_1) = A^L(y_1)$ and $E^L(x_2) = A^L(y_2)$, we have $E(x_1) \leq E(x_2)$. Since $E^L(x)$ is monotonic increasing, then $x_1 \leq x_2$. For all $y_1, y_2 \in (b_2, a_2]$, if $y_1 < y_2$, then $A(y_1) \geq A(y_2)$. Let $x_1 = g(y_1)$ and $x_2 = g(y_2)$, since $E^R(x_1) = A^R(y_1)$ and $E^R(x_2) = A^R(y_2)$, we have $E(x_1) \geq E(x_2)$. Since $E^R(x)$ is monotonic decreasing, then $x_1 \leq x_2$. Hence, $x = g(y)$ is a monotonic nondecreasing function from $[a_1, a_2]$ onto $[-1, 1]$.

Suppose that $y = g^{-1}(x)$, then $g^{-1}(x)$ is a monotonic increasing bounded function on $[-1, 1]$, and $f(x)$ is the extension of $g^{-1}(x)$ on R, that is, $f(x) = g^{-1}(x)$ on $[-1, 1]$. Hence, we obtain the corresponding membership function $A(y) = A(f(x)) = E(x)$ for $y = f(x)$. Then the graph of the fuzzy number A is

$$R_A = \{(y, A(y))|y \in R, A(y) \in [-1, 1]\} = \{(f(x), E(x)t)|x \in R, E(x) \in [-1, 1]\}.$$

By the extension principle, we have

$$A(y) = \bigcup_{x \in R} E(x) * \{f(x)\} = f(E).$$

□

According to Definition 1, E is a given fuzzy structured element on R, for any $\lambda \in (0, 1]$, the λ-level set of E is denoted as $E_\lambda = \{x|E(x) \geq \lambda\} = [e_\lambda^-, e_\lambda^+]$, then we have $e_\lambda^- \in [-1, 0]$, $e_\lambda^+ \in [0, 1]$.

Theorem 3 *Suppose that f is a monotonic bounded function on $[-1, 1]$, E is a given fuzzy structured element on R, a fuzzy number $A = f(E)$. If f is a monotonic increasing function on $[-1, 1]$, then the λ-level set of A is a closed interval on R, and it is denoted as $A_\lambda = [f(E)]_\lambda = f[e_\lambda^-, e_\lambda^+] = [f(e_\lambda^-), f(e_\lambda^+)]$. If f is a monotonic decreasing function on $[-1, 1]$, then $A_\lambda = [f(e_\lambda^+), f(e_\lambda^-)]$.*

Proof Since the function f satisfies the properties expressed by Theorem 1, then we have $[f(E)]_\lambda = f(E_\lambda)$ for all $\lambda \in (0, 1]$.

Since f is monotonic on the closed interval $E_\lambda = [e_\lambda^-, e_\lambda^+] \subseteq [-1, 1]$, if f is a monotonic increasing function, then we have $f(e_\lambda^-) \leq f(x) \leq f(e_\lambda^+)$ for all $x \in [e_\lambda^-, e_\lambda^+]$. Therefore, $A_\lambda = f[e_\lambda^-, e_\lambda^+] = [f(e_\lambda^-), f(e_\lambda^+)]$. If f is a monotonic decreasing function, then we have $f(e_\lambda^-) \geq f(x) \geq f(e_\lambda^+)$ for all $x \in [e_\lambda^-, e_\lambda^+]$. Hence, $A_\lambda = f[e_\lambda^-, e_\lambda^+] = [f(e_\lambda^+), f(e_\lambda^-)]$. □

3 Transformation of Monotonic Functions with Same Monotonicity

It is easy to know that if $f(x)$ is a monotonic function on the symmetrical internal, then both $f(-x)$ and $-f(x)$ are monotonic functions. If $f(x)$ satisfies $f(x) > 0$ or

$f(x) < 0$, then $\frac{1}{f(x)}$ is also a monotonic function, and it has opposite monotonicity with $f(-x)$. Therefore, we can obtain that $-f(-x)$, $\frac{1}{f(-x)}$ and $-\frac{1}{f(x)}$ have same sequential monotonicity with $f(-x)$ on $[-1, 1]$.

Let $D[-1, 1]$ denote the family of monotonic bounded functions with the same monotonicity. we define the several transformations on $D[-1, 1]$,

$$\tau_i : D[-1, 1] \to D[-1, 1], \quad i = 0, 1, 2, 3,$$

then

$$\tau_0(f) = f, \quad \tau_1(f) = f^{\tau_1}, \quad \tau_2(f) = f^{\tau_2}, \quad \tau_3(f) = f^{\tau_3},$$

where $f^{\tau_1}(x) = -f(-x)$, $f^{\tau_2}(x) = \frac{1}{f(-x)}(f(-x) \neq 0)$, $f^{\tau_2}(x) = \frac{1}{-f(x)}(f(x) \neq 0)$ for all $x \in [-1, 1]$.

Denote $T = \tau_0, \tau_1, \tau_2, \tau_3$, and introduce the multiplication on T:

$$\tau_i \tau_j(f) = f^{\tau_i \tau_j} = \left(f^{\tau_j}\right)^{\tau_i}.$$

It is easy to prove the defined multiplication is closed, satisfies the commutative principle and has the following properties: (1) $\tau_1\tau_2 = \tau_2\tau_1 = \tau_3$, $\tau_2\tau_3 = \tau_3\tau_2 = \tau_1$, $\tau_1\tau_3 = \tau_3\tau_1 = \tau_2$, (2) for any $k = 0, 1, 2, 3$, $\tau_0\tau_k = \tau_k\tau_0 = \tau_k$ that is, τ_0 is the identity element, and (3) for any $k = 0, 1, 2, 3$, $\tau_k\tau_k = \tau_0$, that is, there exists the inverse element.

In conclusion, sets of all transformations on the symmetric interval $[-1, 1]$ with the above mentioned multiplication operations is a commutative group.

4 Operations of Fuzzy Numbers Based on the Fuzzy Structured Element

Let A and B be two fuzzy numbers, $\otimes$ be a binary operation on the real field, if $f(A, B) = A \otimes B$, by the multivariate expansion principle, we have

$$A \otimes B = \bigcup_{\lambda \in [0, 1]} \lambda \wedge (A_\lambda \otimes B_\lambda), \quad \forall \lambda \in [0, 1],$$

where $A_\lambda = [a_\lambda^-, a_\lambda^+]$, $B_\lambda = [b_\lambda^-, b_\lambda^+]$ are the λ-level sets of A and B, respectively.

Theorem 4 *Suppose that E is a given symmetrical fuzzy structured element, g is a monotonic and bounded function on* $[-1, 1]$, *a fuzzy number $B = g(E)$, then we have*

$$(1) -B = g^{\tau_1}(E); \quad (2)\, 1/B = g^{\tau_2}(E); \quad (3)\, 1/B = g^{\tau_1\tau_2}(E) = g^{\tau_3}(E).$$

Proof Suppose that g is a monotonic increasing function, and we denote the λ-level set of E as $E_\lambda = [e_\lambda^-, e_\lambda^+]$.

(1) For all $\lambda \in [0, 1]$, $B_\lambda = [g(E)]_\lambda = [g(e_\lambda^-), g(e_\lambda^+)]$, by the definition of interval numbers, we have,

$$-B_\lambda = \left[-g\left(e_\lambda^+\right), -g\left(e_\lambda^-\right)\right].$$

Since E is a symmetrical fuzzy structured element, then

$$-B_\lambda = \left[-g\left(-e_\lambda^-\right),\ -g\left(-e_\lambda^+\right)\right] = g^{\tau_1}\left(E_\lambda\right) = \left[g^{\tau_1}(E)\right]_\lambda.$$

By the extension principle, we have

$$-B = \bigcup_{\lambda\in[0,\,1]} \lambda * (-B_\lambda) = \bigcup_{\lambda\in[0,\,1]} \lambda \wedge \left[g^{\tau_1}(E)\right]_\lambda = g^{\tau_1}(E).$$

(2) For all $\lambda \in [0,\ 1]$, $B_\lambda = [g(E)]_\lambda = [g(e_\lambda^-),\ g(e_\lambda^+)]$, according to the definition of interval numbers, we have

$$\left(\frac{1}{B}\right)_\lambda = \frac{1}{B_\lambda} = \left[\frac{1}{g(e_\lambda^+)},\ \frac{1}{g(e_\lambda^-)}\right] = \left[\frac{1}{g(-e_\lambda^-)},\ \frac{1}{g(-e_\lambda^+)}\right] = g^{\tau_2}\left(E_\lambda\right) = \left[g^{\tau_2}(E)\right]_\lambda.$$

By the extension principle, the conclusion is obtained.

(3) Since $\frac{1}{B_\lambda} = \left[\frac{1}{g(-e_\lambda^-)},\ \frac{1}{g(-e_\lambda^+)}\right]$, then we have

$$-\frac{1}{B_\lambda} = \left[-\frac{1}{g(-e_\lambda^+)},\ -\frac{1}{g(-e_\lambda^-)}\right] = \left[-\frac{1}{g(e_\lambda^-)},\ -\frac{1}{g(e_\lambda^+)}\right] = g^{\tau_3}\left(E_\lambda\right) = \left[g^{\tau_3}(E)\right]_\lambda.$$

□

Theorem 5 *Let E be a symmetrical fuzzy structured element, then* $f(e_\lambda^-) = -f^{\tau_1}(e_\lambda^+),\ f(e_\lambda^+) = -f^{\tau_1}(e_\lambda^-)$.

Proof Since $f^{\tau_1}(x) = -f(-x)$, then $f^{\tau_1}(e_\lambda^-) = -f(-e_\lambda^-)$, and since E is a symmetrical fuzzy structured element, then we have $e_\lambda^- = -e_\lambda^+$, so we obtain that $f(e_\lambda^-) = -f^{\tau_1}(e_\lambda^+)$. Similarly, we can prove that $f(e_\lambda^+) = -f^{\tau_1}(e_\lambda^-)$. □

Theorem 6 *Let E be a symmetrical fuzzy structured element, $f,\ g \in D[-1,\ 1]$, (suppose that they are all monotonic increasing functions), fuzzy numbers $A = f(E)$ and $B = g(E)$, for all $\lambda \in (0,\ 1]$, then we have $f_\lambda(E) = [f(e_\lambda^-),\ f(e_\lambda^+)]$, and $f^{\tau_k}(x)\ (k = 1,\ 2,\ 3)$ are the same sequential monotonic transformations of $f(x)$, then*

(1) *If A and B are any bounded fuzzy numbers, then $A + B = (f + g)(E)$ with the membership function $\mu_{A+B}(x) = E((f + g)^{-1}(x))$.*
(2) *If A and B are any bounded fuzzy numbers, then $A - B = (f + g^{\tau_1})(E)$ with the membership function $\mu_{A-B}(x) = E((f + g^{\tau_1})^{-1}(x))$.*
(3) *If A and B are positive fuzzy numbers, then $A \cdot B = (f \cdot g)(E)$ with the membership function $\mu_{A\cdot B}(x) = E((f \cdot g)(x))$.*
(4) *If A and B are negative fuzzy numbers, then $A \cdot B = [f^{\tau_1} \cdot g^{\tau_1}](E)$ with the membership function $\mu_{A\cdot B}(x) = E[(f^{\tau_1} \cdot g^{\tau_1})^{-1}(x)]$.*
(5) *If A is a negative fuzzy number, B is a positive fuzzy number, then $A \cdot B = [f^{\tau_1} \cdot g]^{\tau_1}(E) = [-f \cdot g^{\tau_1}](E)$ with the membership function $\mu_{A\cdot B}(x) = E[(-f \cdot g^{\tau_1})^{-1}(x)]$.*

(6) *If A and B are positive fuzzy numbers, and 0 is not included in the support of B, that is* $0 \notin \text{supp}A$, *then* $A \div B = A \cdot \frac{1}{B} = f(E) \cdot g^{\tau_2}(E) = (f \cdot g^{\tau_2})(E)$ *with the membership function* $\mu_{A \div B}(x) = E[(f \cdot g^{\tau_2})^{-1}(x)]$.
(7) *If A and B are negative fuzzy numbers, and 0 is not included in the support of B, then* $A \div B = A \cdot \frac{1}{B} = f^{\tau_1}(E) \cdot g^{\tau_3}(E) = (f^{\tau_1} \cdot g^{\tau_3})(E)$ *with the membership function* $\mu_{A \div B}(x) = E[(f^{\tau_1} \cdot g^{\tau_3})^{-1}(x)]$.
(8) *If A is a negative fuzzy number, B is a positive fuzzy number, and 0 is not included in the support of B, then* $A \div B = (-f \cdot g^{\tau_3})(E)$ *with the membership function* $\mu_{A \div B}(x) = E[(-f \cdot g^{\tau_3})^{-1}(x)]$.

Proof Suppose that $f, g \in D[-1, 1]$ and they are all monotonic increasing functions, for any $\lambda \in [0, 1]$, we have

$$[f(E)]_\lambda = \left[f\left(e_\lambda^-\right), f\left(e_\lambda^+\right)\right], \quad [g(E)]_\lambda = \left[g\left(e_\lambda^-\right), g\left(e_\lambda^+\right)\right].$$

(1) Let $H = A + B$, according to the definition of the operations on interval numbers, we have

$$[h(E)]_\lambda = [f(E)]_\lambda + [g(E)]_\lambda = \left[h\left(e_\lambda^-\right), h\left(e_\lambda^+\right)\right] = [(f+g)(E)]_\lambda.$$

Therefore, by decomposition theorem, we have

$$[H]_\lambda = [f(E)]_\lambda - [g(E)]_\lambda = \left[f\left(e_\lambda^-\right) - g\left(e_\lambda^+\right), f\left(e_\lambda^+\right) - g\left(e_\lambda^-\right)\right].$$

That is, $A + B = f(E) + g(E) = h(E) = (f+g)(E)$.

(2) Let $H = A - B$, according to the definition of the operations on interval numbers, we have

$$[H(E)]_\lambda = [f(E)]_\lambda - [g(E)]_\lambda = \left[f\left(e_\lambda^-\right) - g\left(e_\lambda^+\right), f\left(e_\lambda^+\right) - g\left(e_\lambda^-\right)\right].$$

Since E is a symmetric fuzzy structured element, from Theorem 4, we have

$$\begin{aligned}[H]_\lambda &= \left[f\left(e_\lambda^-\right) - g\left(e_\lambda^+\right), f\left(e_\lambda^+\right) - g\left(e_\lambda^-\right)\right] \\ &= \left[f\left(e_\lambda^-\right) + g^{\tau_1}\left(e_\lambda^-\right), f\left(e_\lambda^+\right) + g^{\tau_1}\left(e_\lambda^+\right)\right] \\ &= [f(E)]_\lambda + [g(E)]_\lambda,\end{aligned}$$

then $A - B = f(E) + g^{\tau_1}(E)$. Since $f(x) + g^{\tau_1}(x)$ is a monotonic bounded function on $[-1, 1]$, from Theorem 6, we have $\mu_{A-B}(x) = \mu_{f(E)+g^{\tau_1}(E)}(x) = E[(f + g^{\tau_1})^{-1}(x)]$.

(3) Let $H = A \cdot B$, since both A and B are positive fuzzy numbers, then $0 \le f(e_\lambda^-) \le f(e_\lambda^+)$, $0 \le g(e_\lambda^-) \le g(e_\lambda^+)$. According to the definition of the operations on interval numbers, we have

$$\begin{aligned}A_\lambda \cdot B_\lambda = [f(E)]_\lambda \cdot [g(E)] &= \left[f\left(e_\lambda^-\right) \cdot g\left(e_\lambda^-\right), f\left(e_\lambda^+\right) \cdot g\left(e_\lambda^+\right)\right] \\ &= \left[(f \cdot g)\left(e_\lambda^-\right), (f \cdot g)\left(e_\lambda^+\right)\right] = [(f \cdot g)(E)]_\lambda = [h(E)]_\lambda,\end{aligned}$$

where $h = f \cdot g$. Therefore, $H = h(E) = (f \cdot g)(E)$, from Theorem 1,we have $\mu_H(x) = \mu_{A \cdot B}(x) = [E(f \cdot g)^{-1}(x)]$.

(4) Since both A and B are negative fuzzy numbers, then there exists functions $f(x) \leq 0$, $g(x) \leq 0$, and $0 \leq f(e_\lambda^-) \leq f(e_\lambda^+)$, $0 \leq g(e_\lambda^-) \leq g(e_\lambda^+)$. According to the definition of the operations on interval numbers, we have

$$\begin{aligned} A_\lambda \cdot B_\lambda = [f(E)]_\lambda \cdot [g(E)]_\lambda &= \left[f\left(e_\lambda^-\right), f\left(e_\lambda^+\right)\right] \cdot \left[g\left(e_\lambda^-\right), g\left(e_\lambda^+\right)\right] \\ &= \left[f\left(e_\lambda^-\right) \cdot g\left(e_\lambda^-\right), f\left(e_\lambda^+\right) \cdot g\left(e_\lambda^+\right)\right]. \end{aligned}$$

Since E is a symmetric fuzzy structured element, from Theorem 5, we have

$$\begin{aligned} A_\lambda \cdot B_\lambda &= \left[f^{\tau_1}\left(e_\lambda^-\right) \cdot g^{\tau_1}\left(e_\lambda^-\right), f^{\tau_1}\left(e_\lambda^+\right) \cdot g^{\tau_1}\left(e_\lambda^+\right)\right] \\ &= \left[\left(f^{\tau_1} \cdot g^{\tau_1}\right)\left(e_\lambda^-\right), \left(f^{\tau_1} \cdot g^{\tau_1}\right)\left(e_\lambda^+\right)\right] \\ &= \left[\left(f^{\tau_1} \cdot g^{\tau_1}\right)(E)\right]_\lambda = [h(E)]_\lambda. \end{aligned}$$

Then $[h(E)] = [(f^{\tau_1} \cdot g^{\tau_1})(E)]$, therefore, $\mu_H(x) = \mu_{A \cdot B}(x) = E((f^{\tau_1} \cdot g^{\tau_1})^{-1}(x))$.

Since $f^{\tau_1}(x) \cdot g^{\tau_1}(x) = [-f(-x)] \cdot [-g(-x)] = [(f \cdot g)(-x)]$. Let $y = (f \cdot g)(-x)$, then $(-x) = (f \cdot g)^{-1}(y)$, that is, $x = -(f \cdot g)^{-1}(y)$. So, $\mu_{A \cdot B}(x) = E(-(f \cdot g)^{-1}(x))$.

(5) Since A is a negative fuzzy number, B is a positive fuzzy number, then $f(e_\lambda^-) \leq f(e_\lambda^+) \leq 0$, $0 \leq g(e_\lambda^-) \leq g(e_\lambda^+)$. According to the definition of multiplication on interval numbers, we have $A_\lambda \cdot B_\lambda = [f(e_\lambda^-) \cdot g(e_\lambda^-),\ f(e_\lambda^+) \cdot g(e_\lambda^+)]$. From Theorem 5, we obtain that

$$\begin{aligned} A_\lambda \cdot B_\lambda &= \left[f\left(e_\lambda^-\right) \cdot \left(-g^{\tau_1}\left(e_\lambda^-\right)\right), f\left(e_\lambda^+\right) \cdot \left(-g^{\tau_1}\left(e_\lambda^+\right)\right)\right] \\ &= \left[\left(-f \cdot g^{\tau_1}\right)\left(e_\lambda^-\right), \left(-f \cdot g^{\tau_1}\right)\left(e_\lambda^+\right)\right] \\ &= \left[\left(-f \cdot g^{\tau_1}\right)(E)\right]_\lambda. \end{aligned}$$

(6) Since both A and B are positive fuzzy numbers, both f and g are monotonic increasing functions, then $0 \leq f(e_\lambda^-) \leq f(e_\lambda^+)$, $0 \leq g(e_\lambda^-) \leq g(e_\lambda^+)$. According to the definition of the operations on interval numbers, we have

$$\begin{aligned} A_\lambda \div B_\lambda = [f(E)]_\lambda \div [g(E)]_\lambda &= \left[f\left(e_\lambda^-\right), f\left(e_\lambda^+\right)\right] \div \left[g\left(e_\lambda^-\right), g\left(e_\lambda^+\right)\right] \\ &= \left[-\frac{f(e_\lambda^-)}{g(e_\lambda^-)}, -\frac{f(e_\lambda^+)}{g(e_\lambda^+)}\right] = \left[f\left(e_\lambda^-\right) \cdot g^{\tau_2}\left(e_\lambda^-\right), f\left(e_\lambda^+\right) \cdot g^{\tau_2}\left(e_\lambda^+\right)\right] \\ &= f\left(E_\lambda\right) \cdot g^{\tau_2}\left(E_\lambda\right) = \left[\left(f \cdot g^{\tau_2}\right)\left(E_\lambda\right)\right]. \end{aligned}$$

Then $A \div B = (f \cdot g^{\tau_2})(E)$, therefore, $\mu_{A \div B}(x) = E((f \cdot g^{\tau_2})^{-1}(x))$.

(7) Since both A and B are negative fuzzy numbers, then $0 \leq f(e_\lambda^-) \leq f(e_\lambda^+)$, $0 \leq g(e_\lambda^-) \leq g(e_\lambda^+)$. According to the definition of the operations on interval numbers, we have

$$A_\lambda \div B_\lambda = [f(E)]_\lambda \div [g(E)]_\lambda$$

$$= \left[f\left(e_\lambda^-\right), f\left(e_\lambda^+\right)\right] \div \left[g\left(e_\lambda^-\right), g\left(e_\lambda^+\right)\right]$$
$$= \left[f\left(e_\lambda^-\right), f\left(e_\lambda^+\right)\right] \cdot \left[\frac{1}{g(e_\lambda^+)}, \frac{1}{g(e_\lambda^-)}\right].$$

Since $\frac{1}{g(e_\lambda^+)} \leq \frac{1}{g(e_\lambda^-)} \leq 0$, then

$$A_\lambda \div B_\lambda = \left[\frac{f(e_\lambda^+)}{g(e_\lambda^-)}, \frac{f(e_\lambda^-)}{g(e_\lambda^+)}\right] = \left[\frac{-f^{\tau_1}(e_\lambda^-)}{g(e_\lambda^-)}, \frac{-f^{\tau_1}(e_\lambda^+)}{g(e_\lambda^+)}\right]$$
$$= \left[f^{\tau_1}\left(e_\lambda^-\right) \cdot g^{\tau_3}\left(e_\lambda^-\right), f^{\tau_1}\left(e_\lambda^+\right) \cdot g^{\tau_3}\left(e_\lambda^+\right)\right]$$
$$= \left[f^{\tau_1}\left(e_\lambda^-\right) \cdot g^{\tau_3}\left(e_\lambda^-\right), f^{\tau_1}\left(e_\lambda^+\right) \cdot g^{\tau_3}\left(e_\lambda^+\right)\right]$$
$$= \left[\left(f^{\tau_1} \cdot g^{\tau_3}\right)\left(e_\lambda^-\right), \left(f^{\tau_1} \cdot g^{\tau_3}\right)\left(e_\lambda^+\right)\right]$$
$$= \left[\left(f^{\tau_1} \cdot g^{\tau_3}\right)(E)\right]_\lambda.$$

Therefore, $A \div B = [(f^{\tau_1} \cdot g^{\tau_3})(E)]$, with the membership function $\mu_{A \div B}(x) = E((f^{\tau_1} \cdot g^{\tau_3})^{-1}(x))$.

(8) Since A is a negative fuzzy number, B is a positive fuzzy number, then $f(e_\lambda^-) \leq f(e_\lambda^+) \leq 0$, $0 \leq g(e_\lambda^-) \leq g(e_\lambda^+)$. According to the definition of multiplication on interval numbers, we have $A_\lambda \div B_\lambda = [f(E)]_\lambda \div [g(E)]_\lambda = [f(e_\lambda^-), f(e_\lambda^+)] \cdot \left[\frac{1}{g(e_\lambda^+)}, \frac{1}{g(e_\lambda^-)}\right]$, where $0 \leq \frac{1}{g(e_\lambda^+)} \leq \frac{1}{g(e_\lambda^-)}$, then

$$A_\lambda \div B_\lambda = \left[\frac{f(e_\lambda^-)}{g(e_\lambda^-)}, \frac{f(e_\lambda^+)}{g(e_\lambda^+)}\right] = \left[f\left(e_\lambda^-\right) \cdot \left(-g^{\tau_3}\left(e_\lambda^-\right)\right), f\left(e_\lambda^+\right) \cdot \left(-g^{\tau_3}\left(e_\lambda^+\right)\right)\right]$$
$$= \left[\left(-fg^{\tau_3}\right)\left(e_\lambda^-\right), \left(-fg^{\tau_3}\right)\left(e_\lambda^+\right)\right] = \left[\left(-fg^{\tau_3}\right)(E)\right]_\lambda.$$

So $A \div B = (-f \cdot g^{\tau_3})(E)$, with the membership function $\mu_{A \div B}(x) = E((-f \cdot g^{\tau_3})^{-1}(x))$.

In many situations of fuzzy number operations, the monotonic functions and are easily to be founded. Therefore, through the way in Theorem 6, the choice of the membership function after fuzzy number operations can be made easily. □

Example 1 The membership functions of fuzzy numbers A and B are as follows, respectively, $\mu_A(x) = \begin{cases} 0, & x \leq 0 \\ x, & 0 < x < 1 \\ 1, & x = 1 \\ 2 - x, & 1 < x < 2 \\ 0, & x \geq 2 \end{cases}$ and $\mu_B(x) = \begin{cases} 0, & x \leq 2 \\ x - 2, & 2 < x < 3 \\ 1, & x = 3 \\ 4 - x, & 3 < x < 4 \\ 0, & x \geq 4 \end{cases}$

Calculate $A + B$, $A - B$, $A \cdot B$ and $A \div B$ with the method of structured element.

Solve let E be the fuzzy structured element with the membership function

$$E(x) = \begin{cases} 1+x, & -1 \leq x \leq 0, \\ 1-x, & 0 < x \leq 1, \\ 0, & \text{Otherwise.} \end{cases}$$

Then the fuzzy numbers both $A = 1 + E$ and $B = 3 + E$ are positive fuzzy numbers.

Suppose $A = f(E)$ and $B = g(E)$, then $f(x) = 1 + x$ and $g(x) = 3 + x$. (1) $A + B = (1 + E) + (3 + E) = 4 + 2E$, $(f + g)(x) = 4 + 2x \Rightarrow (f + g)^{-1}(x) = \frac{x-4}{2}$. When $-1 \leq \frac{x-4}{2} \leq 0$, then $2 \leq x \leq 4$; when $0 < \frac{x-4}{2} \leq 1$, then $4 < x \leq 6$. Therefore,

$$\mu_{A+B}(x) = E\left[(f+g)^{-1}(x)\right] = E\left[\frac{x-4}{2}\right] = \begin{cases} 1+(x-4)/2, & 2 \leq x \leq 4, \\ 1-(x-4)/2, & 4 < x \leq 6, \\ 0, & \text{Otherwise.} \end{cases}$$

(2) $A - B = (1 + E) - (3 + E) = -2 + 2E$, then

$\mu_{A-B}(x) = E\left(\frac{x+2}{2}\right)$.

When $-1 \leq \frac{x+2}{2} \leq 0$, then $-4 \leq x \leq -2$; when $0 < \frac{x+2}{2} \leq 1$, then $-2 < x \leq 0$. Hence,

$$\mu_{A-B}(x) = \begin{cases} 1+(x+2)/2, & -4 \leq x \leq -2, \\ 1-(x+2)/2, & -2 < x \leq 0, \\ 0, & \text{Otherwise.} \end{cases}$$

(3) $A \cdot B = f(E) \cdot g(E) = [f \cdot g](E)$, then

$$\mu_{A \cdot B}(x) = E\left[(f \cdot g)^{-1}(x)\right].$$

And

$$(f \cdot g)(x) = (1+x)(3+x) = x^2+4x+3 = (x+2)^2-1, \quad (f \cdot g)^{-1}(x) = \sqrt{1+x}-2.$$

When $-1 \leq \sqrt{1+x} - 2 \leq 0$, then $0 \leq x \leq 3$; when $0 < \sqrt{1+x} - 2 \leq 1$, then $3 < x \leq 8$.

Therefore,

$$\mu_{A \cdot B}(x) = E\left[(f \cdot g)^{-1}(x)\right] = \begin{cases} \sqrt{1+x}-1, & 0 \leq x \leq 3, \\ 3-\sqrt{1+x}, & 3 < x \leq 8, \\ 0, & \text{Otherwise.} \end{cases}$$

(4) $A \div B = f(E) \cdot g^{\tau_2}(E)$, then

$$\mu_{A \div B}(x) = E\left[\left(f \cdot g^{\tau_2}\right)^{-1}(x)\right].$$

And

$$f(x) \cdot g^{\tau_2}(x) = \frac{f(x)}{g(-x)} = \frac{1+x}{3-x} = \frac{4}{3-x} - 1.$$

Then

$$\left(f \cdot g^{\tau_2}\right)^{-1}(x) = 3 - \frac{4}{1+x}.$$

When $-1 \leq 3 - \frac{4}{1+x} \leq 0$, then $0 \leq x \leq \frac{1}{3}$; when $0 < 3 - \frac{4}{1+x} \leq 1$, then $\frac{1}{3} < x \leq 1$. Hence,

$$\mu_{A \div B}(x) = \begin{cases} 4 - 4/(1+x), & 0 \leq x \leq 1/3, \\ -2 + 4/(1+x), & 1/3 < x \leq 1, \\ 0, & \text{Otherwise.} \end{cases}$$

5 Conclusion

This paper puts forward to the operations of fuzzy numbers and the expression of the membership function with the method of fuzzy structured element. An example is given to prove the feasibility and convenience of the operations of fuzzy numbers. The method given in the paper has a great capability and significance in the practical application.

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