

Transformation-designed optical elements

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Abstract: We describe transformation design of optical elements which, in addition to image transfer, perform useful operations. For one class of operations, including translation, rotation, mirroring and inversion, an image can be generated that is ideal in the sense of the perfect lens (combining both near- and far-field components in a flat, unit transfer function, up to the limits imposed by material imperfection). We also describe elements that perform magnification, free from geometric aberrations, even while providing free-space working distance on both the input and output sides. These magnifying elements also operate in the near- and far-field, allowing them to transfer near field information into the far field, as with the hyper lens and other related devices, however in contrast to those devices, insertion loss can be much lower, due to the matching properties accessible with transformation design. The devices here described inherently require dispersive materials, thus chromatic aberration will be present, and the bandwidth limited.

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1. Introduction

The transformation method allows one to design continuous media that exert precise and arbitrary control over a field, particularly the electromagnetic field[1][2][3][4]. Even the challenging problem of reflectionless design in arbitrary geometries is quite straight forward. The resulting material specifications are necessarily complex, being both anisotropic and inhomogeneous with specific functional forms. Implementation of such media would be hopeless without recent advances in metamaterials. These two concepts comprise a great synergy, with metamaterials enabling new media response, and transformation design suggesting new phenomena and devices supported by these media. The recent work on the design[5][6][7][8][9] and implementation[10] of metamaterial invisibility cloaks points to the potential of these concepts. In this article we apply the transformation method to the design of imaging optics.

A fundamental issue with imaging systems is the constraint on resolution imposed by the diffraction limit. Currently, sub-diffraction images are assembled from scanning probes that can sense near-fields, but the scanning process severely limits image frame rates, and the probes require near direct contact with the object. The perfect lens began to address these issues by providing an image transfer of both near- and far-field components with a free-space working distance[11]. Shortly there after Pendry transformed the perfect lens to a cylindrical geometry to obtain magnification[12][13]. More recently implementation strategies for near-field magnification have been explored with layered media[14][15] (where the proposed devices are sometimes called hyper lenses, so named because they are composed of an indefinite medium[16] with a hyperbolic dispersion relation). Experimental realizations have also appeared using layered media[17][18], and also wire arrays[19]. For a fixed input aperture, near-field magnification actually increases the amount of image information available in the far field. One might even call this a beating of the diffraction limit. If one has both a free-space working distance and near-field magnification one can acquire high resolution, sub-diffraction images using standard resolution detectors, and without making contact with the object.

Below we show how to use the transformation method to design slab elements that have a variety of useful functions - including magnification - that operate on both the near- and far-

field, and provide free-space working distance. It is worth noting some significant differences between the transformation designed near-field magnifier and previous designs. One such difference is in the geometry of the focal surfaces. The cylindrical or spherical devices have curved object and image surfaces, whereas the transformation designed element has planar object and image planes that better match typical detector arrays, and display formats. The cylindrical or oblique-cut devices magnify in only one dimension with magnification directly related to the sample geometry, where with the transformation design the magnification can be any reasonable value in each of the two dimensions of the image plane, independent of the element's thickness. Additionally, we will show that the geometric aberrations of the transformation designed element are theoretically zero, and its substantially flat transfer function has a magnitude indicating excellent coupling to the environment. For all other recent designs, no analysis is available describing the aberrations or an image transfer function that includes coupling into and out of the environment. Since all recent designs incorporate uncompensated[20], anisotropic media, which are inherently mismatched to free space, one expects weak coupling to free space fields. On the down side, the transformation designed device requires magnetic response, making the implementation path to high frequencies, such as visible light, more problematic. However, metamaterials for magnetic response in the visible spectrum are being pursued[21][22][23], and the functionality we describe in this article may increase the incentive for such work. In the nearer term, implementations for sub-diffraction microwave and terahertz imaging may be feasible.

2. Optical design geometry

As is traditional in optical design, we will describe optical elements with reference to an optical axis, which we will take to be the z -axis in our Cartesian coordinate system. Optical elements are considered to extend in the transverse (x, y) directions sufficiently far that the electromagnetic fields at the transverse edges of an element are assumed to be negligible. To verify this condition is a matter of practical importance, but an encumbrance to be shed in a first analysis. This assumption provides a particular freedom when using the transformation method to design perfectly matched, (i.e. zero reflecting for all angles of incidence), elements. With this method one must use a coordinate transformation that is everywhere continuous to achieve perfectly-matched behavior. Since a finite sized material object is by definition completely surrounded by free space, and free space is described by a "flat" Cartesian coordinate system, a finite-sized, perfectly-matched material object must be described by a coordinate transformation that is continuous with flat Cartesian space, everywhere along its outer boundary. We will relax this condition in the transverse direction of our optical elements, and design elements that are perfectly matched for electromagnetic fields that are near the optic axis and sufficiently far from the transverse edges, using coordinate transformations that cannot be continuous to flat space on their transverse edges. This extra freedom allows for some new functionality in manipulating fields along the optic axis.

3. Transformation design

As described elsewhere[1][2], the transformation design method employs a coordinate transformation to design a material object that can manipulate electromagnetic fields in a desired manner. One imagines manipulating space and thus the electromagnetic fields that reside in it. A coordinate transformation quantitatively describes this manipulation

$$x^{i'} = f^{i'}(x^i) \quad (1)$$

where x^i are the reference coordinates, (usually representing flat space), and $x^{i'}$ are the coordinates of the manipulated space. In these new coordinates Maxwell's equations are identical

in form, but the linear constitutive equations describing the medium are changed. An alternative viewpoint of these changes is that one is manipulating the material properties in an unchanged, flat space. The two viewpoints can account for the same electromagnetic behavior. The manipulation-of-space viewpoint is a useful design picture and the manipulation-of-material viewpoint leads to an implementable medium specification. These material properties are given by

$$\varepsilon^{i'j'} = \mu^{i'j'} = \det(\Lambda_{i'}^{i'})^{-1} \Lambda_{i'}^{i'} \Lambda_{j'}^{j'} \delta^{ij} \quad (2)$$

where the reference material properties are those of free space, and the transformation matrix is given by

$$\Lambda_{j'}^{i'} = \frac{\partial x^{i'}}{\partial x^j} \quad (3)$$

4. Perfect lens

Understanding the perfect lens in terms of transformation design[24] is a great basis for understanding all the optical elements we will discuss in this article. This is particularly true, since here we will consider only slab elements, (though transformation designed elements with non-parallel and non-planar surfaces may also prove useful.) The perfect lens is interesting in that it corresponds to a non-one-to-one coordinate transformation. This is clear, since the perfect lens has triplets of planes on which the field distributions are identical: the object plane, the internal image plane and the external image plane. The corresponding coordinate transformation maps these three planes from a single plane in the reference space (Figs.1A and B). The coordinate transformation is given by

$$x' = x \quad (4a)$$

$$y' = y \quad (4b)$$

$$z' = \begin{cases} z-d & z' < d/2 \\ -z & -d/2 < z' < d/2 \\ z+d & d/2 < z' \end{cases} \quad (4c)$$

where d is the thickness of the lens. The z -component of this transform is shown as the solid blue line in Fig. 2. Using Eq. 3, we calculate the transformation matrix inside the lens, $-d/2 < z' < d/2$, where the material properties differ from free space

$$(\Lambda_{j'}^{i'}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (5)$$

and from this the material properties, (Eq. 2),

$$(\varepsilon^{i'j'}) = (\mu^{i'j'}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (6)$$

For simplicity, here we use a slope, $\partial z'/\partial z = -1$, inside the lens, but this need not be the case. Slopes differing from ± 1 lead to anisotropic material properties, but conveniently allow the combined focal distance to differ from being equal the lens thickness. In fact, the mapping need not be linear; any continuous relation not multi-valued in z will lead to finite and unique material properties. The key design parameters of this relation are the values of z' that map to the same z value. For example, in Eq. 4c, $z' = -d, 0, d$ all map to $z = 0$. These points

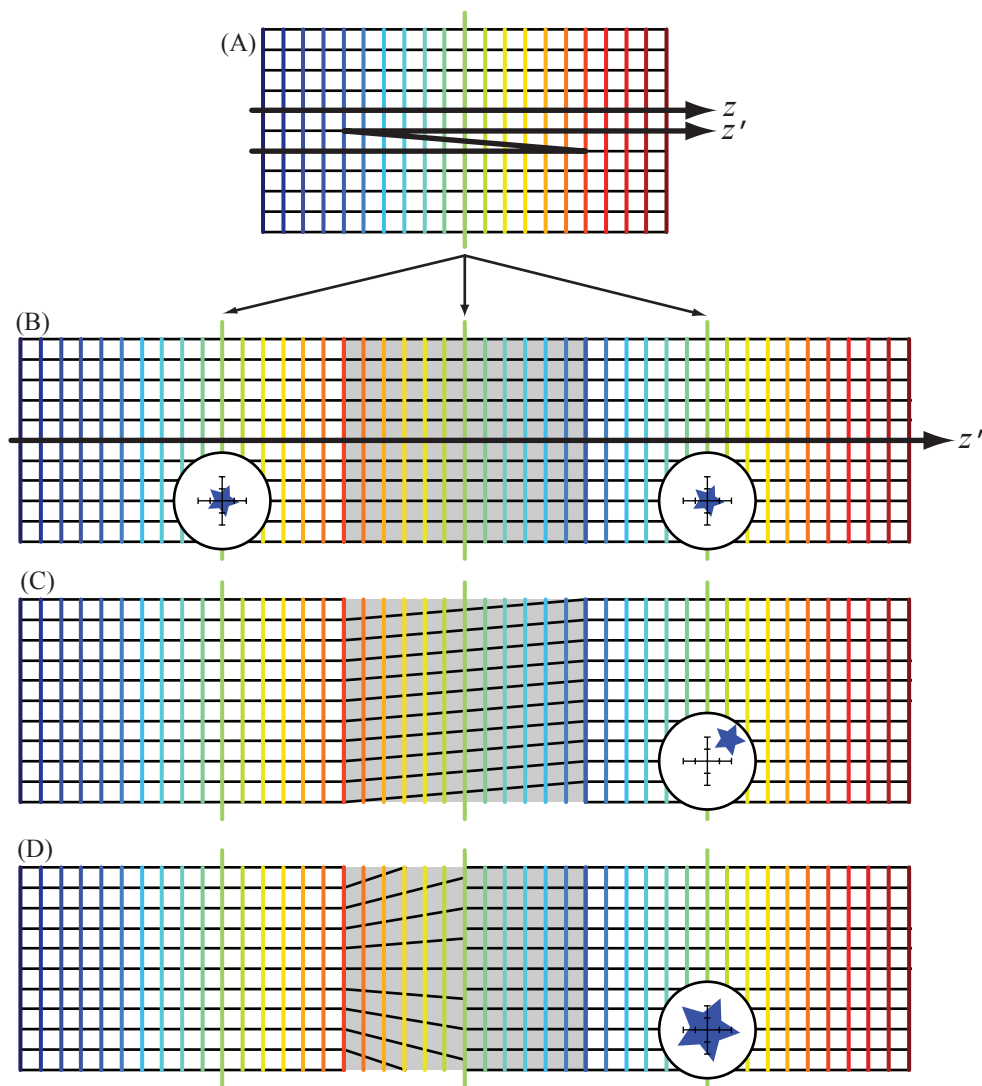


Fig. 1. Coordinate transformations from the reference space (A) to a space corresponding to a perfect lens (B), an image translator (C) and a magnifier (D). The folded representation of the z' -axis in the reference space, and the color coding of the vertical coordinate lines display the multi-valued mapping. The gray shading indicates the region where the coordinate transformation differs from "flat" Cartesian space, or equivalently, where the material properties differ from free space.

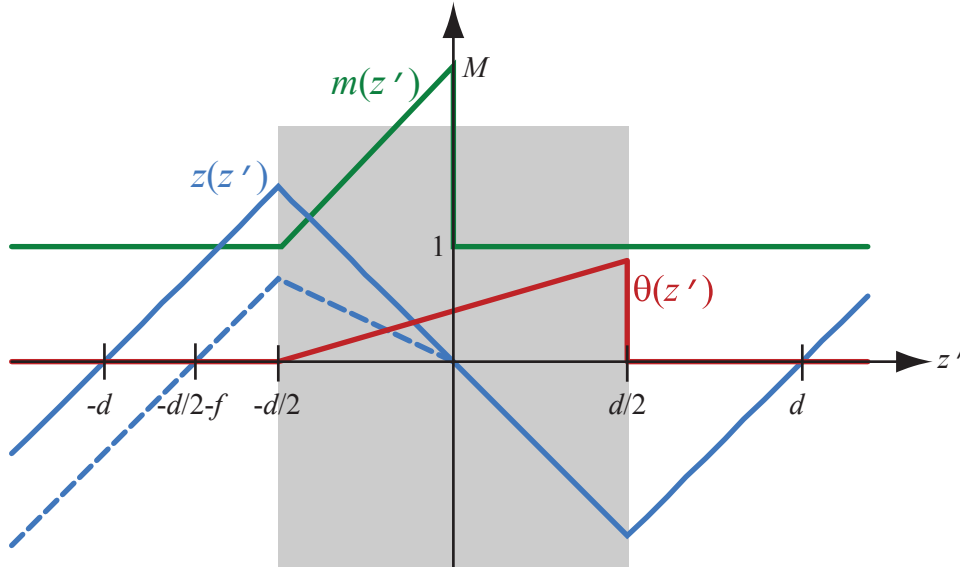


Fig. 2. z -coordinate transformation corresponding to a perfect lens of thickness d (solid blue), and modified to shorten the front focal length from $d/2$ to f (dashed blue). Transverse function used in coordinate transformation corresponding to an image rotating element (red) and a magnifying element (green). The gray shaded region indicates where these functions correspond to material properties that differ from free space.

indicate the locations of one of an infinite set of object, and image planes. Below it will be necessary to align an internal image plane with a discontinuity in the transformation of the x - and y -coordinates.

5. Invariant operations

Without diminishing the imaging fidelity from that of the perfect lens, one can also perform a variety of image processing operations. Any operation from the group of operations for which two dimensional free space is invariant can be performed without introducing reflection or aberration. These operations include: translation, rotation, inversion and mirroring. The operation must be varied continuously from the identity to the desired value across the element. For example, the transformation for a rotation is

$$x' = x \cos \theta(z') - y \sin \theta(z') \quad (7a)$$

$$y' = x \sin \theta(z') + y \cos \theta(z') \quad (7b)$$

with the transformation of the z coordinate as above, and $\theta(z')$ is a continuous function with $\theta(-d/2) = 0$ and $\theta(d/2)$ is the desired rotation, (Fig. 2). Note that the coordinate transformation is continuous on the left edge of the element, $z' = -d/2$, but not on the right, $z' = d/2$. However, since the properties of the space are invariant for this rotation, this discontinuity is a property of the choice of coordinates and not the space itself, thus no physical manifestation, such as reflection or refraction, can result. The mapping for a translation is shown in Fig. 1C. The operations of mirroring and inversion may present some practical problems since the most obvious transformation results in all the field energy incident on the element being mapped through a line or point respectively.

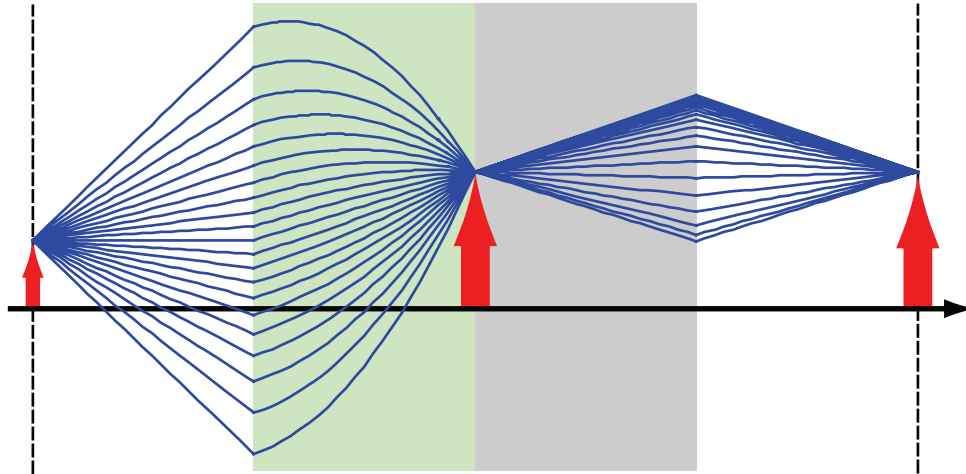


Fig. 3. Ray diagram for the magnifying element. Both the internal and external images (large red arrows) are magnified relative to the object (smaller red arrow). All the magnification occurs in the left half of the element. A non-aberrating refraction occurs at the element center where the internal image is formed.

6. Magnification

A space representing an electromagnetic medium is not however invariant to expansion or contraction. These operations (performed on an isotropic medium) are equivalent to varying the scalar refractive index. This lack of invariance is not a problem if one can accept the requirement that the image and object planes lie in media of differing refractive index. For magnification, M , one can have perfect lens image fidelity, when the image plane lies in a medium of refractive index n/M , where n is the refractive index of the medium containing the object. This constraint is not convenient, as one often wishes to have the object and image planes both in free space. There is an intrinsic media mismatch when projecting a magnified image into the same medium as the object. One can however, still perform this function with no aberrations and minimal reflection by judicious positioning of the mismatch discontinuity. The best position, for reasons described below, is at a focal plane. However, choosing the object or *external* image plane precludes having any free space working distance on the input or output side respectively. Fortunately, internal image planes can be created when negative material properties are employed. The transformation for a magnification centered on $x = y = 0$ is given by

$$x' = m(z')x \quad (8a)$$

$$y' = m(z')y \quad (8b)$$

where the magnification function, $m(z')$, varies continuously from 1 to the desired magnification, M , as shown in Fig. 2, (green line). This configuration results in an image at $z = d/2$ that is magnified by a factor M relative to the object at $z = -d/2$ (Fig. 1D).

The reason to locate the discontinuity at a focal plane is to eliminate aberrations. In general a material interface will introduce aberrations since a point source viewed through an interface appears not only displaced from its actual position, but also no longer exactly point-like. The refraction at the interface does not preserve the spherical phase fronts exactly. This deviation from spherical phase fronts is characterized by the aberration coefficients (the lowest order of which, the Seidel aberrations, are: spherical, coma, astigmatism, field curvature and

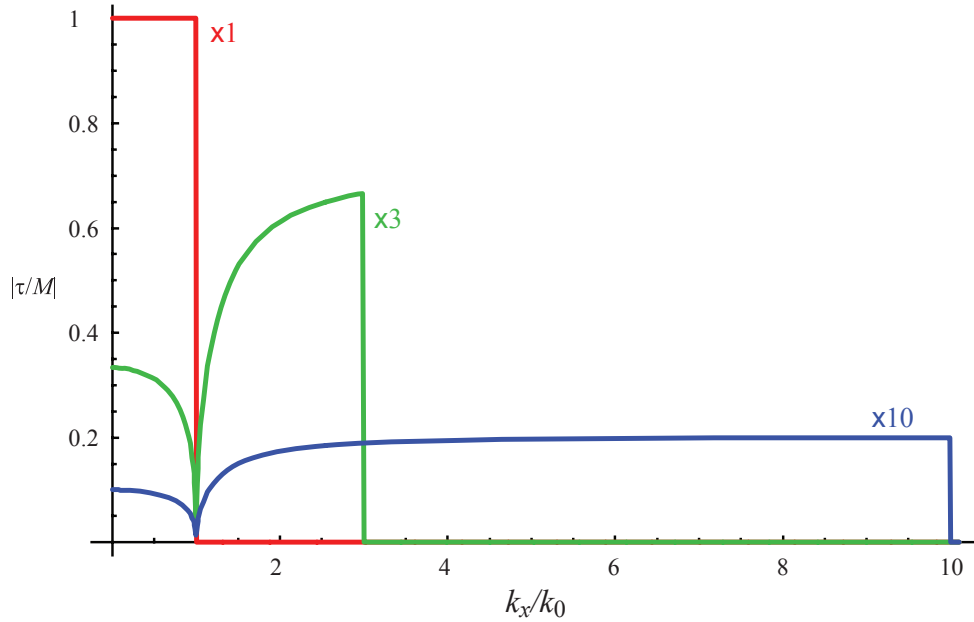


Fig. 4. Magnitude of the spatial transfer function for the magnifying element, with magnification, M , of: one (red), three (green) and ten (blue). The dependent variable is the transverse component of the wave vector at the input. (The transverse component of the wave vector at the output is reduced by the magnification factor.)

distortion[25][26]). If however the point source is located *at* the interface, it appears to be a point source from both sides - no geometric aberrations are introduced of any order (Fig. 3).

In reality the unmodified perfect lens has an infinity of focal plane sets. Every z -oriented plane within the lens is an internal image plane, corresponding to an external image and object plane. By positioning the discontinuity of the x - y coordinate transformation at a particular location (in our case $z' = 0$), one chooses which one of these focal plane sets is going to be aberration free.

7. Transfer function

Zero geometric aberration does not imply perfect imaging. To characterize the deviation from perfect imaging, we calculate the spatial transfer function. This can be done quite easily, and without reference to the complex material specifications that arise in all but the simplest transformation designed media.

For our magnification element, we use a coordinate transformation that is continuous everywhere except at the internal image plane, ($z = 0$), thus reflection can only occur at this location. One can use the following theorem to easily calculate the reflection and transmission at this interface. If material A is perfectly matched to material B , and material C is perfectly matched to material D , then the reflection and transmission properties of the interface A - C are identical to those of the interface B - D . The material on the left side of the interface (at $z = 0^-$) is perfectly matched to the isotropic, homogenous medium with $\epsilon = \mu = 1/M$, and the material on the right side of the interface (at $z = 0^+$) is perfectly matched to free space, $\epsilon = \mu = 1$. From standard

boundary matching the Fresnel formula for the transmission is

$$\tau = \frac{2Mk_{z-}}{Mk_{z-} + k_{z+}} \quad (9)$$

where the z -component of the wave vector on the right side, k_{z-} , and left side, k_{z+} , are given by

$$Mk_{z-} = \sqrt{k_0^2 - k_x^2} \quad (10a)$$

$$k_{z+} = \sqrt{k_0^2 - \frac{k_x^2}{M^2}} \quad (10b)$$

and, k_x , refers to the transverse wave vector at the object plane, (not the transverse wave vector at the interface and image plane, which is a factor of M smaller), and $k_0 = \omega/c$ is the free space wave vector magnitude. Note that standard Fresnel reflection and transmission formulas are based on uniform plane wave solutions that occur only in homogenous media. If we allow the magnification function, $m(z')$, to vary right up to the interface as shown in Fig. 2, these formulas do not strictly apply. However, if this leads to a degradation in the transfer function one can always make this function constant near the interface, thus specifying a homogenous layer there.

If ideal materials could be used to implement the coordinate transformations, the discontinuity at the internal image plane would account for the only degradation of the transfer function, and it would be flat out to arbitrarily large transverse wave vectors. As with the perfect lens, it will be true that a small amount of material lossiness combined with non-negligible lens thickness will significantly limit the transfer of these deep near field components. However, like the hyper lens, the magnification applied here converts some near field components into far-field components. In the plot of the transfer function (Fig. 4), the spatial bandwidth is only shown extending as far as is supported by the near-field to far-field conversion, namely, $k_x = Mk_0$. The total theoretical transfer function is a combination of Eq. 9 and the factor $1/M$. This later factor results from the reflectionless magnification that occurs prior to the interface, and is derivable from conservation of energy.

8. Material properties

When calculating the material properties for the magnifying element one quickly finds that a design not carefully constrained will require values of the permittivity and permeability with unrealizable large magnitudes. This problem can be mitigated by two means. One is to generalize the above design to allow for the front focal length, f , to differ from the half-thickness, $d/2$ [20]. This is easily accomplished by changing the slope of z - z' coordinate map, (Fig. 2, dashed line.) The second means is a restriction of the input aperture, D , of the element, Fig. 5; the material properties are most extreme at the front surface of the element, at the aperture boundary. One can minimize the required material properties at this boundary for a given magnification, M , and f-number, $F = f/D$. Performed analytically, this minimization is not particularly tractable, but can be handled numerically quite easily. One finds that for small values of F/M , the minimizing f-number, f_{\min} , is given by

$$f_{\min} \approx \frac{F}{M}d \quad (11)$$

Using this minimization, one finds that useful magnifications and low f-numbers can be obtained with realizable material property magnitudes, i.e. magnitudes less than ten (Fig. 5B).

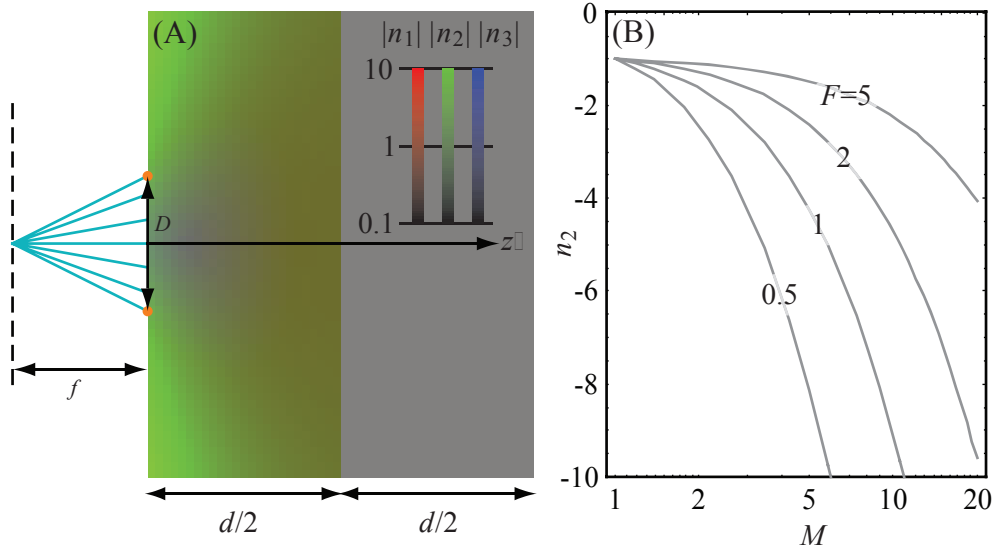


Fig. 5. Material properties of the magnifying element. The magnitudes of the three principle values of the permittivity and permeability are represented by the intensities of the three color channels: red, green and blue (A) with $M = 3$ and $F = 1$. The scale is logarithmic as shown in the inset. The principle value that assumes large magnitudes, n_2 , is most extreme at the front surface at the aperture boundary (orange dots). (B) shows the value of n_2 for various f-numbers, F , and magnifications, M .

From Fig. 2, we see that in the left half of the element we have

$$m(z') = M + \frac{M-1}{d/2} z' \quad (12a)$$

$$z = -\frac{f}{d/2} z' \quad (12b)$$

which leads to the coordinate transformation

$$x' = \left(M - \frac{M-1}{f} z \right) x \quad (13a)$$

$$y' = \left(M - \frac{M-1}{f} z \right) y \quad (13b)$$

$$z' = -\frac{d}{2f} z \quad (13c)$$

Computing the partial derivatives one finds the transformation matrix, Λ_{ij}' . Then inverting Eq. 13, to re-express it in terms of the primed coordinates (the coordinates of the material specification), one finds

$$\left(\Lambda_{ij}' \right) = M \begin{pmatrix} 1 + \alpha z' & 0 & -\frac{\beta x'}{1 + \alpha z'} \\ 0 & 1 + \alpha z' & -\frac{\beta y'}{1 + \alpha z'} \\ 0 & 0 & -\frac{\beta}{\alpha} \end{pmatrix} \quad (14)$$

where

$$\alpha = \frac{M-1}{M} \frac{1}{d/2} \quad (15a)$$

$$\beta = \frac{M-1}{M} \frac{1}{fM} \quad (15b)$$

We note that the transformation matrix is not diagonal since the transformation (Eq. 13) couples the z -coordinate with the transverse coordinates. Using Eq. 3 and 14, one can calculate the material property tensor which is complicated and the display of which is not particularly illuminating. For further analysis, (or implementation), one must diagonalize the material property tensor. Since Eq. 3 yields real, symmetric matrices (when real valued coordinate transformations are used), this diagonalization can always be accomplished and an ortho-normal basis found. In this case, only one of the eigenvalues has a simple expression,

$$n_1 = -\frac{f}{d/2} \quad (16)$$

where n_1 represents the common value of the permittivity and permeability, $n_1 = \epsilon_1 = \mu_1$. For the parameters explored, all three eigenvalues are negative, and n_1 and n_3 lie between -1 and 0 . These values are plotted in Fig. 5A, using the three color channels (red, green, blue) to indicate their magnitude. The right hand half of the element is a uniform gray with the material properties given by Eq. 6.

9. Conclusion

The optical elements discussed here have desirable properties, such as freedom from aberrations, minimal reflection, free space working distances, and near field capability, while performing useful operations, such as magnification and rotation. In particular, near field capability combined with magnification allows one to send near field information into the far-field, as in the hyper-lens and similar devices. The transformation method allows one to design these elements in a straight-forward (if unfamiliar) manner. We have described this design process, and shown, for the important case of a near-field magnifier, what performance can be expected with lossless media, and what range of material properties are required for a given magnification and f-number. In light of recent advances in the application of metamaterials to devices, (such as the invisibility cloak), implementation of these elements in the microwave and possibly millimeter-wave ranges seems feasible in the near future. Higher frequency implementations will depend on the spectral progress of the field. Of course, all the usual constraints apply including bandwidth, and absorption.