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TRANSFORMATION GROUPS IN NET SPACES

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We consider formal groups of transformations on the space of differential and net (finite-difference) variables. We show that preservation of meaning of difference derivatives under transformations necessarily leads to Lie–Bäcklund group. We derive formulas for extension to net variables and formulate criteria for preservation of uniformity and invariance of differences of the network and a test for the invariance of difference equations. With the help of formal Newton series we construct the ideal of the algebra of all Lie–Bäcklund operators on a uniform network which is used to derive tests for the conservatism of difference equations on the basis of a discrete analog of Noether's identity.

INTRODUCTION

In this paper we make an attempt to adapt the ideas of group analysis to the study of finite-difference equations. It is known that one and the same system of differential equations can be approximated (to given order) with the help of an infinite number of difference schemes. Hence in finite-difference modeling there is always a question about the choice of schemes preferred from some point of view. As criteria of choice must frequently fundamental physical principles present in the original model appear, such as certain conservation laws holding, variational principles, etc. In connection with this qualitative considerations acquire great value in the construction of numerical algorithms permitted one to introduce "physical content" of the object studied into the numerical method of studying its mathematical model. Such a view led to the creation of methods of construction of conservation and completely conservative difference schemes, to the integrointerpolational approach, to variational methods of construction of schemes, and other methods [8].

Invariance of differential equations with respect to a continuous group of transformations is undoubtedly a deep property of these models and reflects homogeneity and isotropicity of space–time, the validity of Galileo's principle, and other symmetry properties of physical models intuitively (or on the basis of experiments) established by their creators. Hence the reflection of a symmetry property in finite-difference models adequately reflecting symmetry of the original differential model is an important problem of the theory of difference schemes and can serve as a criterion for the choice of which we spoke above.

The attractiveness of the group-theoretic approach to the creation and study of difference schemes is that group analysis has powerful infinitesimal criteria of invariance of manifolds. This becomes apparent in that the problem of finding (or studying) a continuous group of transformations reduces to the solution (study) of a linear system of equations independent of the linearity or nonlinearity of the original model.

Yanenko and Shokin (cf. [11-13]), the first to pay attention to the possibility of using group consideration in the study of difference schemes, proposed using the first differential approximation (FDA) of difference schemes for this purpose. The FDA of difference schemes is a differential equation which occupies an intermediate position (in the sense of approximation) between the original differential model and its finite-difference analog. The FDA carries some information about the difference scheme (in the form of the coefficients of the scheme and the difference steps of the net for example) and being differential equations is

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entirely suitable for classical group analysis. With such an approach one chooses those difference schemes whose FDA admits the same group as the original continuous model. Many authors developed approaches to group properties of difference schemes based on the analysis of differential-difference equations (cf., e.g., [12]). In all the indicated cases one considers local objects.

In contrast with the papers cited, in the present paper we make the first step to the analysis of group properties of finite-difference objects directly. We find out how a continuous group of transformations acts on the difference derivatives and equations after which we impose the requirement on the finite-difference equations that they admit the same group as the original differential equations, i.e., the group acting on the net space but isomorphic to the original.

Finite-difference operators, in contrast with differential ones are defined on a finite collection of points (on the difference template) from the countable number of all points (difference net) on which we are interested in solving the problem. Such (nonlocality) of operators (from the physical point of view, the presence in the problem of characteristic scale sizes) leads to the presence of specific properties of difference operators missing in the local differential model. This appears in particular in the presence of "right" and "left" differentiation and the corresponding translations, in the existence of uniform and uniform and nonuniform nets, in the specifics of the Leibniz difference rule. The nonlocality of difference operators leads to the fact that a group of transformations may distort the proportions of the difference template. Hence in a test for invariance of difference equations it is necessary to also include the invariance of the difference template (or net) on which they are written.

We also note that finite-difference equations being a discrete model of the original differential model can have specific discrete ("crystalline") symmetries. One such symmetry, the group of reflections in the case of uniform nets, is easily discovered (cf. Sec. 7). However the search for such symmetries lies beyond the limits of our studies, since we are aiming to preserve in finite-difference equations invariance with respect to the continuous group of transformations isomorphic to the group of the original differential equations.

1. Formal Power Series and Formal Groups

We consider the space Z of sequences $(x, u, u_1, u_2, \ldots)$, where x is a simple (independent) variable, $u, u_1, u_2, \ldots$ are differential variables (we shall call u_s the derivative of s-th order). By z we shall mean any finite number of coordinates of the vector $(x, u, u_1, u_2, \ldots)$ by z^i its i-th coordinate.

In the space Z we define a map $\mathscr{D}$ (differentiation), acting according to the rule: $\mathscr{D}(x)=1$, $\mathscr{D}(u)=u_1$. $\mathscr{D}(u_s)=u_{s+1}$, $s=1, 2, \ldots$. Let $\mathscr{A}$ be the space of analytic functions $\mathscr{F}(z)$ of a finite number of variables z. Identifying $\mathscr{D}$ with the action of the first order linear differential operator

$$\mathscr{D}=\frac{\partial}{\partial x}+u_1\frac{\partial}{\partial u}+u_2\frac{\partial}{\partial u_1}+\ldots+u_{s+1}\frac{\partial}{\partial u_s}+\ldots,$$

we extend the differential $\mathscr{D}$ to functions from $\mathscr{A}$, where $\mathscr{D}(\mathscr{F}(z))\in\mathscr{A}$.

We consider sequences of formal power series in one symbol a (the parameter):

$$f^i(z, a)=\sum_{k=0}^{\infty} A_k^i(z)\,a^k, i=1,2\ldots, \tag{1}$$

where $A_k^i(z)\in\mathscr{A}$ while $A_0^i \equiv z^i$, z^i being the i-th coordinate of a vector from Z.

We denote the space of sequences of formal power series (1):

$$(f^1(z, a),\ f^2(z, a),\ \ldots,\ f^s(z, a),\ \ldots)$$

by $\tilde{Z}$. Sequences $(x, u, u_1, u_2, \ldots)$ are a special case of sequences of series (1), $Z \subset \tilde{Z}$. In $\tilde{Z}$ by definition we define addition, multiplication by a number, and product of formal series which coincide with the corresponding operations for convergent series, and also differentiation of series (1):

$$\mathscr{D}\left(\sum_{k=0}^{\infty} A_k^i(z)\,a^k\right)=\sum_{k=0}^{\infty} \mathscr{D}\left(A_k^i(z)\right)a^k,$$

$$\frac{\partial}{\partial a}\left(\sum_{k=0}^{\infty} A_k^i(z)\, a^k\right)=\sum_{k=1}^{\infty} k A_k^i(z)\, a^{k-1},$$

$$\frac{\partial}{\partial a}\left(\sum_{k=0}^{\infty} A_k^i(z)\, a^k\right)\Bigg|_{a=0}=A_1^i(z),\ i=1,\ 2,\ \ldots.$$

In $\tilde{Z}$ we consider transformation

$$z^{i*}=f^i(z,\ a),\quad i=1,\ 2,\ \ldots, \tag{2}$$

carrying the sequence z^i into the sequence z^{i*}.

The operations on series (1) introduced let us consider composition of transformations of the form (2):

$$z^{i**}=f^i(z^*,\ b)=\sum_{k=0}^{\infty} A_k^i(z^*)\, b^k=\sum_{k=0}^{\infty} A_k^i(f(z,\ a))\, b^k,\ i=1,\ 2,\ \ldots.$$

Such composition generally leads out of one-parameter series (1) from $\tilde{Z}$. We shall consider only those series (1) [and the corresponding transformations (2)], the structure of whose coefficients assures the closedness in $\tilde{Z}$ of the transformations (2):

$$f^i(f(z,a),b)=f^i(z,(a+b)),\ i=1,\ 2,\ldots\ . \tag{3}$$

The property (3) of formal series (1) means that the transformations (2) form a <u>formal one-parameter group in $\tilde{Z}$.</u>

Property (3) is equivalent to each of the following two relations (cf. [2, 4]):

1)
$$A_k^i(f(z,\ a))=\sum_{l=0}^{\infty}\frac{(k+l)!}{k!\ l!}A_{k+l}^i(z)\, a^l, \tag{4}$$
$$i=1,\ 2,\ \ldots,\ k=0,\ 1,\ 2,\ \ldots,$$

2)
$$z^{i*}=e^{aX}(z^i)\equiv\sum_{s=0}^{\infty}\frac{a^s}{s!}X^{(s)}(z^i),\ i=1,\ 2,\ \ldots, \tag{5}$$

where X is the infinitesimal operator (generator) of the group.

$$X=\sum_{i=1}^{\infty}\xi^i(z)\frac{\partial}{\partial z^i}, \tag{6}$$

$$\xi^i(z)=\frac{\partial f^i(z,\ a)}{\partial a}\Bigg|_{a=0},\ \ \xi^i(z)\in\mathscr{A},\ i=1,\ 2,\ \ldots.$$

2. Taylor Group, Introduction of Net Variables

We consider the group of transformations (2) in the space $\tilde{Z}$ of formal series with tangent field

$$\mathscr{D}=\frac{\partial}{\partial x}+u_1\frac{\partial}{\partial u}+u_2\frac{\partial}{\partial u_1}+\ldots. \tag{7}$$

The transformations (2) of this group are defined by the action of the operator $T_a\equiv e^{a\mathscr{D}}$:

$$z^{i*}=e^{a\mathscr{D}}(z^i)\equiv\sum_{s=0}^{\infty}\frac{a^s}{s!}\mathscr{D}^s(z^i).$$

The point $z^*\in\tilde{Z}$ has the following coordinates:

$$\begin{aligned} x^*&=T_a(x)=x+a,\\ u^*&=T_a(u)=\sum_{s\geqslant 0}\frac{a^s}{s!}u_s,\\ u_1^*&=T_a(u_1)=\sum_{s\geqslant 0}\frac{a^s}{s!}u_{s+1},\\ &\ldots\ldots\ldots\ldots\\ u_k^*&=T_a(u_k)=\sum_{s\geqslant 1}\frac{a^s}{s!}u_{k+s},\\ &\ldots\ldots\ldots\ldots \end{aligned} \tag{8}$$

The transformations (8) are the expansion in formal Taylor series of the function u = u(x) at the point x + a so the group with operator (7) was called the Taylor group [2].

The Taylor group is a nontrivial Lie-Bäcklund group, i.e., is not the extension to $\tilde{Z}$ of a group of pointwise or tangent transformations [4].

The Taylor group is a convenient instrument for the study of invariant properties of finite-difference objects.

We fix an arbitrary value of the parameter a = h > 0 and with the help of the tangent field (7) of the Taylor group we form a pair of operators which we shall call, respectively, the operators of discrete translation to the right and left:

$$\underset{+h}{S}=e^{h\mathscr{D}}\equiv\sum_{s=0}^{\infty}\frac{h^s}{s!}\mathscr{D}^s,\quad \underset{-h}{S}=e^{-h\mathscr{D}}\equiv\sum_{s=0}^{\infty}\frac{(-h)^s}{s!}\mathscr{D}^s.$$

The operators $\underset{+h}{S}$ and $\underset{-h}{S}$ commute with one another and T_a while $\underset{+h}{S}\cdot\underset{-h}{S}=\underset{-h}{S}\cdot\underset{+h}{S}=1$. Moreover,

$$(\underset{\pm h}{S})^n=T_a|_{a=\pm nh},\quad n=0,1,2,\ldots.$$

With the help of $\underset{+h}{S}$, $\underset{-h}{S}$ we form a pair of operators of discrete (finite-difference) differentiation to the right and left:

$$\underset{+h}{\mathscr{D}}=\frac{\underset{+h}{S}-1}{h}\equiv\sum_{s=1}^{\infty}\frac{h^{(s-1)}}{s!}\mathscr{D}^s,$$

$$\underset{-h}{\mathscr{D}}=\frac{1-\underset{-h}{S}}{h}\equiv\sum_{s=1}^{\infty}\frac{(-h)^{(s-1)}}{s!}\mathscr{D}^s.$$

The operators $\underset{+h}{S}$, $\underset{-h}{S}$, $\underset{+h}{\mathscr{D}}$, $\underset{-h}{\mathscr{D}}$, T_a commute in any combination, while

$$\underset{+h}{\mathscr{D}}=\underset{-h}{\mathscr{D}}\cdot\underset{+h}{S},\quad \underset{-h}{\mathscr{D}}=\underset{+h}{\mathscr{D}}\cdot\underset{-h}{S}.$$

We introduce formal power series of special form:

$$\underset{h}{u_1}=\underset{+h}{\mathscr{D}}(u),\ \underset{h}{u_2}=\underset{-h+h}{\mathscr{D}\mathscr{D}}(u),\ \underset{h}{u_3}=\underset{+h-h+h}{\mathscr{D}\mathscr{D}\mathscr{D}}(u),\ldots,$$

we shall call $\underset{h}{u_s}$ the finite-difference (discrete, net) derivative of s-th order. In the odd case we shall call the formal series $\underset{h}{u_{2k+1}}$ the right difference derivative. If necessary one can introduce left difference derivatives of order by the formula

$$\underset{h}{u_{\overline{2k+1}}}=\underset{h}{u_{2k+1}}-h\underset{h}{u_{2k+2}},$$

or the derivative with weight

$$\underset{h}{u^{(\sigma)}_{2k+1}}=\sigma\underset{h}{u_{2k+1}}+(1-\sigma)\underset{h}{u_{\overline{2k+1}}}=\underset{h}{u_{2k+1}}+h(1-\sigma)\underset{h}{u_{2k+2}}.$$

We shall denote the set of points $x^\alpha=\underset{+h}{S}^\alpha(x)$, $\alpha=0,\pm1,\pm2,\ldots$ by $\underset{h}{\omega}$ and call it a uniform distance net. We shall denote sequences of formal series $(\underset{h}{u_1},\underset{h}{u_2},\underset{h}{u_3},\ldots)$ by $\underset{h}{Z}$, the product of the spaces $\underset{h}{Z}$ and $\tilde{Z}-\underset{h}{\tilde{Z}}$ by:

$$\underset{h}{\tilde{Z}}=(x,u,u_1,u_2,\ldots,\underset{h}{u_1},\underset{h}{u_2},\ldots).$$

If the series $\underset{h}{u_s}$ converges, then we shall call it the continuous representation of the distance derivative $\underset{h}{u_s}$. We note that formal series $\underset{h}{u_s}$ are not represented in exponential form (5), so they do not form a group with parameter h and it is impossible to write down tangent fields.

We extend the action of the discrete translation $\underset{\pm h}{S}$ (by definition) to functions from $\mathscr{A}$:

$$\underset{\pm h}{S}\mathscr{F}(z)=\mathscr{F}(\underset{\pm h}{S}(z)),$$

this lets us find the difference derivatives of $\mathcal{F}\in\mathcal{A}$

$$\underset{\pm h}{\mathcal{D}}(\mathcal{F}(z)) = \pm \frac{\mathcal{F}(\underset{\pm h}{S}(z)) - \mathcal{F}(z)}{h}.$$

Starting from this definition it is easy to establish the discrete (difference) Leibniz rule:

$$\underset{+h}{\mathcal{D}}(F\cdot G) = \underset{+h}{\mathcal{D}}(F)G + F\underset{+h}{\mathcal{D}}(G) + h\underset{+h}{\mathcal{D}}(F)\underset{+h}{\mathcal{D}}(G),$$
$$\underset{-h}{\mathcal{D}}(F\cdot G) = \underset{-h}{\mathcal{D}}(F)G + F\underset{-h}{\mathcal{D}}(G) - h\underset{-h}{\mathcal{D}}(F)\underset{-h}{\mathcal{D}}(G),$$

where F, $G\in\mathcal{A}$.

We consider how to extend the Taylor group to $\underset{h}{Z}$. In $\underset{h}{\tilde{Z}}$ we define a transformation of the variables $\underset{h}{u_s}$ as follows:

$$\begin{cases} \underset{h}{u_1^*} = \underset{+h}{\mathcal{D}}(u^*) = u_1^* + \frac{h}{2!}u_2^* + \frac{h^2}{3!}u_3^* + \dots, \\ \underset{h}{u_2^*} = \underset{-h+h}{\mathcal{D}\mathcal{D}}(u^*) = u_2^* + \frac{h^2}{12}u_4^* + \dots, \\ \dots\dots\dots\dots\dots\dots \end{cases}$$

where u_s^* are formal series of the form (1) with tangent field of the Taylor group: $\zeta^s = \left.\frac{\partial u_s^*}{\partial a}\right|_{a=0} = u_{s+1}$, $s=1, 2, \dots$. The tangent field for the variables $\underset{h}{u_s^*}$ will be as follows:

$$\underset{h}{\zeta^1} = \left.\frac{\partial \underset{h}{u_1^*}}{\partial a}\right|_{a=0} = \zeta^1 + \frac{h}{2!}\zeta^2 + \dots = u_2 + \frac{h}{2!}u_3 + \dots = \mathcal{D}(\underset{h}{u_1}), \tag{9}$$
$$\underset{h}{\zeta^2} = \mathcal{D}(\underset{h}{u_2}), \dots, \underset{h}{\zeta^s} = \mathcal{D}(\underset{h}{u_s}), \dots.$$

Thus, the tangent field of the Taylor group extended to $\underset{h}{\tilde{Z}}$ can be identified with the operator

$$D = \sum_{i=1}^{\infty} \mathcal{D}(z^i)\frac{\partial}{\partial z^i},$$

where z^i are the cooordinates of the vector $(x, u, u_1, u_2, \dots, \underset{h}{u_1}, \underset{h}{u_2}, \dots)$.

We note that the coordinates (9) are formal power series in h so the series $\underset{h}{u_s^*} = \underset{h}{u_s^*}(h, a)$ are formal power series in two symbols, "group" in a and "nongroup" in h.

Now we consider the result of action of the operator of discrete translation $\underset{\pm h}{S}$ on the net space $\underset{h}{Z} = (x, u, \underset{h}{u_1}, \underset{h}{u_2}, \dots)$:

$$\underset{\pm h}{S}(x) = x \pm h,$$
$$\underset{+h}{S}(u) = u + h\left(\sum_{s\geq 1}\frac{h^{s-1}}{s!}\mathcal{D}^s(u)\right) \equiv u + h\underset{h}{u_1}.$$

Analogously we get

$$\underset{-h}{S}(u) = u - h\underset{-h}{S}(\underset{h}{u_1}) = u - h\underset{h}{u_1} + h^2\underset{h}{u_2}.$$

In exactly the same way as a result of the action of $\underset{\pm h}{S}$ on $\underset{h}{u_k}$, k = 1, 2,....we find the formal series $\underset{h}{u_s}$. As a result we get the table of action of the discrete translation operator on the net space $\underset{h}{Z}$, cf. Table 1.

From Table 1 one easily gets the table of the action of the discrete differentiation operator $\underset{\pm h}{\mathcal{D}} = \pm\frac{\underset{\pm h}{S} - 1}{h}$ on the point $(x, u, \underset{h}{u_1}, \underset{h}{u_2}, \dots)$, cf. Table 2.

TABLE 1. Action of the Discrete Translation Operator $\underset{\pm h}{S}$ $(x, u, \underset{h}{u}_1, \underset{h}{u}_2, \ldots)\in \underset{h}{Z}$

$\underset{-h}{S}$ left translation	$\underset{+h}{S}$ right translation
$\underset{-h}{S}(x)=x-h$	$\underset{+h}{S}(x)=x+h$
$\underset{-h}{S}(u)=u-h\underset{h}{u}_1+h^2\underset{h}{u}_2$	$\underset{+h}{S}(u)=u+h\underset{h}{u}_1$
$\underset{-h}{S}(\underset{h}{u}_1)=\underset{h}{u}_1-h\underset{h}{u}_2$	$\underset{+h}{S}(\underset{h}{u}_1)=\underset{h}{u}_1+h\underset{h}{u}_2+h^2\underset{h}{u}_3$
$\underset{-h}{S}(\underset{h}{u}_2)=\underset{h}{u}_2-h\underset{h}{u}_3+h^2\underset{h}{u}_4$	$\underset{+h}{S}(\underset{h}{u}_2)=\underset{h}{u}_2+h\underset{h}{u}_3$
. . . .	. . . .
$\underset{-h}{S}(\underset{h}{u}_{2k+1})=\underset{h}{u}_{2k+1}-h\underset{h}{u}_{2k+2}$	$\underset{+h}{S}(\underset{h}{u}_{2k+1})=\underset{h}{u}_{2k+1}+h\underset{h}{u}_{2k+2}+h^2\underset{h}{u}_{2k+3}$
$\underset{-h}{S}(\underset{h}{u}_{2k+2})=\underset{h}{u}_{2k+2}-h\underset{h}{u}_{2k+3}+h^2\underset{h}{u}_{2k+4}$	$\underset{+h}{S}(\underset{h}{u}_{2k+2})=\underset{h}{u}_{2k+2}+h\underset{h}{u}_{2k+3}$
. . . .	. . . .

TABLE 2. Action of the Discrete Differentiation Operator on Coordinates of a Point $(x, u, \underset{h}{u}_1, \underset{h}{u}_2, \ldots)\in \underset{h}{Z}$

$\underset{-h}{\mathscr{D}}$ left differentiation	$\underset{+h}{\mathscr{D}}$ right differentiation
$\underset{-h}{\mathscr{D}}(x)=1$	$\underset{+h}{\mathscr{D}}(x)=1$
$\underset{-h}{\mathscr{D}}(u)=\underset{h}{u}_1-h\underset{h}{u}_2$	$\underset{+h}{\mathscr{D}}(u)=\underset{h}{u}_1$
$\underset{-h}{\mathscr{D}}(\underset{h}{u}_1)=\underset{h}{u}_2$	$\underset{+h}{\mathscr{D}}(\underset{h}{u}_1)=\underset{h}{u}_2+h\underset{h}{u}_3$
$\underset{-h}{\mathscr{D}}(\underset{h}{u}_2)=\underset{h}{u}_3-h\underset{h}{u}_4$	$\underset{+h}{\mathscr{D}}(\underset{h}{u}_2)=\underset{h}{u}_3$
. . . .	. . . .
$\underset{-h}{\mathscr{D}}(\underset{h}{u}_{2k+1})=\underset{h}{u}_{2k+2}$	$\underset{+h}{\mathscr{D}}(\underset{h}{u}_{2k+1})=\underset{h}{u}_{2k+2}+h\underset{h}{u}_{2k+3}$
$\underset{-h}{\mathscr{D}}(\underset{h}{u}_{2k+2})=\underset{h}{u}_{2k+3}-h\underset{h}{u}_{2k+4}$	$\underset{+h}{\mathscr{D}}(\underset{h}{u}_{2k+2})=\underset{h}{u}_{2k+3}$
. . . .	. . . .

Under the action of the Taylor group on $\underset{h}{\tilde{Z}}$ the point $z=(x, u, u_1, u_2, \ldots, \underset{h}{u}_1, \underset{h}{u}_2, \ldots)$ as a grows describes a one-parameter curve, the orbit of the point z.

Since $(\underset{\pm h}{S})^n = T_a|_{a=\pm nh}$, $n=0, \pm 1, \pm 2, \ldots$ an orbit of the Taylor group is a "continuous translation" drawn through the "discrete translation" $(\underset{\pm h}{S})^n$. Under the condition of convergence of the formal series considered one can speak of the geometrical meaning of the net variables $\underset{h}{u}_s$. In particular, $\underset{h}{u}_1$ is the tangent of the angle of inclination of the chord joining the points u and $\underset{+h}{S}(u)$ in the (x, u) plane where the orbit of the point z of the Taylor group is projected.

3. A Criterion for the Invariance of a Net

A uniform distance net $\underset{h}{\omega}$ is the most wide-spread method of discretization of the space of independent variables. Transformations of the formal one-parameter group varying x can distort the net depriving it of uniformity which one can tell from the finite-difference equations written in terms of $\underset{h}{\omega}$. Hence we single out the class of admissible transformations preserving the uniformity of the net.

Suppose given on $\underset{h}{\tilde{Z}}$ a formal group of transformations G:

$$x^* = f(z, a), \quad u_2^* = \varphi_2(z, a),$$

. . . .

$$u^* = \varphi(z, a), \quad \underset{h}{u_1^*} = \psi_1(z, a),$$
$$u_1^* = \varphi_1(z, a), \quad \underset{h}{u_2^*} = \psi_2(z, a).$$
$$\cdots\cdots\cdot$$

The group G is completely defined by the operator

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \sum_{s \geq 1} \zeta^s \frac{\partial}{\partial u_s} + \sum_{l \geq 1} \underset{h}{\zeta^l} \frac{\partial}{\partial \underset{h}{u_l}}. \tag{10}$$

We supplement the space $\underset{h}{\tilde{Z}}$ with new variables, right step h_+ and let step h_- at the point z. h_+ and h_- transform (by definition) as follows:

$$h_+^* = \underset{+h}{S}(x^*) - x^* = (\underset{+h}{S} - 1) f(z, a),$$
$$h_-^* = (-\underset{-h}{S} + 1) x^* = (1 - \underset{-h}{S}) f(z, a), \quad h_+^*|_{a=0} = h_-^*|_{a=0} = h.$$

The supplementary coordinates of the operator (10) will be:

$$\left.\frac{\partial h_+^*}{\partial a}\right|_{a=0} = (\underset{+h}{S} - 1)\xi(z) = \xi(\underset{+h}{S}(z)) - \xi(z) = h\underset{+h}{\mathcal{D}}(\xi),$$
$$\left.\frac{\partial h_-^*}{\partial a}\right|_{a=0} = (1 - \underset{-h}{S})\xi(z) = \xi(z) - \xi(\underset{-h}{S}(\xi)) = h\underset{-h}{\mathcal{D}}(\xi).$$

Knowledge of the tangent field for h_+, h_- easily lets us derive the following fact (cf. [2]).

THEOREM 1. In order that under the transformations of the group G the net $\underset{h}{\omega}$ remain uniform, i.e., $h_+^* = h_-^*$ it is necessary and sufficient that at each point $z \in \underset{h}{\tilde{Z}}$

$$\underset{+h}{\mathcal{D}} \underset{-h}{\mathcal{D}}(\xi(z)) = 0. \tag{11}$$

Remark. The requirement of (11) means the preservation of uniformity of an arbitrary net in the whole space $\underset{h}{\tilde{Z}}$. In considering a concrete difference equation $\mathcal{F}(z) = 0$ on a uniform net $\underset{h}{\omega}$ one can relax (11) replacing it by the condition:

$$\underset{+h}{\mathcal{D}} \underset{-h}{\mathcal{D}}(\xi(z))|_{\mathcal{F}(z)=0}^{=0}. \tag{12}$$

(An example of such a relaxation will be given below.)

Examples of groups satisfying (11).

1. (11) is satisfied in particular by a group G with ξ = const, i.e., transformations under which translation along the x coordinate occurs. For example, for the Taylor group $\xi = 1$, $x^* = x + a$, $h^* = h$ = const.
2. A solution of (11) in particular will $\xi = Ax$, A = const, i.e., transformations under which dilatation of the x axis occurs. Here $h_+^* = h_-^* = e^{aA}h$, a being the parameter of the group.
3. (11) is satisfied by a group G for which $\xi(x + h) = \xi(x)$ is a periodic function with period h.
4. In the more general case $\xi(\underset{\pm h}{S}(z)) = \xi(z)$, i.e., $\xi(z)$ is invariant with respect to the discrete translation $\underset{\pm h}{S}$.
5. (11) is satisfied by the function $\xi(z) = A(z)x + B(z)$, where A(z) and B(z) are arbitrary invariants of the discrete translation $\underset{\pm h}{S}$.

Example of a group (of projective transformations) not satisfying (11):

$$X = x^2 \frac{\partial}{\partial x} + \ldots.$$

Theorem 1 shows that one can only preserve the uniformity of a net for a rather small class of transformations. Hence if we want to preserve in a net space all groups present in the "continuous" space $\tilde{Z}$ then we must necessarily consider nonuniform nets.

Now suppose given in $\tilde{Z}_h$ a nonuniform net ω. At each point $z \in \tilde{Z}_h$ there is given a pair of numbers, a right step h_+ and a left step h_-; the operators $\underset{+h}{S}$, $\underset{-h}{S}$, $\underset{+h}{\mathscr{D}}$, $\underset{-h}{\mathscr{D}}$ lose commutativity and become "local," i.e., depending on the point z.

Suppose we are given a right step h_+ as a function of x: $h_+=\varphi(x)$. The left step is the right step at the point displaced by h_- to the left: $h_-=\varphi(x-h_-)$ hence one can restrict oneself to consideration of h_+ only. Conversely, if there is given a function $\varphi(x)$ and a point x_0 with which discretization of the x axis starts, then one can uniquely recover the angles of the net $\underset{h}{\omega}$. Under the action of the group G the size of x will change and with it the variables h_+, h_- too. After transformations of G the new step h_+ will be expressed generally with the help of another function of x^*:

$$h_+^*=\tilde{\varphi}(x^*).$$

We shall say that a given uniform net ω is invariant with respect to transformations of G in the space $\tilde{Z}_h=(x, u, u_1, u_2, \ldots, \underset{h}{u_1}, \underset{h}{u_2}, \ldots, h_+, h_-)$ if

$$h_+=\varphi(x) \tag{13}$$

is an invariant manifold, i.e., in the new variables it will still be valid that $h_+^*=\varphi(x^*)$. The following theorem which is easily obtained with the help of the operator (10) gives a test for the invariance of the manifold (13).

THEOREM 2. In order that the difference net $\underset{h}{\omega}$ defined by (13) be invariant with respect to transformations of G with the operator (10) it is necessary and sufficient that the following condition hold:

$$\xi(\underset{+h}{S}(z))-\xi(z)\left(1+\frac{\partial\varphi}{\partial x}\right)\Big|_{(13)}=0. \tag{14}$$

Example. We consider projective transformations defined by the operator $X=x^2\frac{\partial}{\partial x}+\ldots$. The criterion for preserving uniformity does not hold: $\underset{-h}{\mathscr{D}}\underset{+h}{\mathscr{D}}(x^2)=2$ so it is necessary to consider a nonuniform net $h_+=\varphi(x)$. We extend the operator by h_+ and h_-

$$X=x^2\frac{\partial}{\partial x}+h_+(2x+h_+)\frac{\partial}{\partial h_+}+h_-(2x-h_-)\frac{\partial}{\partial h_-}. \tag{15}$$

It is easy to construct an invariant net starting from the invariants of the group G: $\mathscr{I}_1=x+\frac{x^2}{h_+}$, $\mathscr{I}_2=x-\frac{x^2}{h_-}$. We construct, for example, an invariant nonuniform net on the interval $(0, L_0)$. Taking $L_0=\mathscr{I}_1$ we get the equality

$$h_+=\frac{x^2}{L_0-x}, \tag{16}$$

the left step is determined from the equation $h_-=\varphi(x-h_-)$:

$$h_-=\frac{x^2}{L_0+x}. \tag{17}$$

It is easy to verify that (16) and (17) give an invariant manifold with respect to the operator (15). One can see that the ratio $\frac{h_+}{h_-}=\frac{L_0+x}{L_0-x}$ also gives an invariant manifold, i.e., the transformations of G preserving the invariance of the net preserve the proportions of the difference pattern.

Theorem 2 admits generalization to the case when the net $\underset{h}{\omega}$ depends on a solution.

Let a net be given by the equation $h_+=\varphi(z)$, where $\varphi(z)\in\mathscr{A}$. Then the criterion for its invariance will look as follows:

$$\xi(\underset{+h}{S}(z))-\xi(z)-X(\varphi(z))\,|_{h_+=\varphi(z)}=0, \tag{18}$$

where X is an operator of the form (10).

4. Transformations Preserving the Meaning of the Finite-Difference Derivatives

We consider a formal one-parameter group G of transformations in $\tilde{Z}_h$

$$\begin{array}{ll} x^*=f(z,\ a) & \underset{h}{u_1^*}=\varphi_1(z,\ a) \\ u^*=g(z,\ a) & \ldots\ldots \\ u_1^*=g_1(z,\ a) & \underset{h}{u_2^*}=\varphi_2(z,\ a) \\ u_2^*=g_2(z,\ a) & \ldots\ldots \\ \ldots\ldots & \end{array} \tag{19}$$

We define the transformation of the difference variables $\underset{h}{u_s}$ as follows:

$$\underset{h}{u_1^*}=\sum_{s\geq 1}\frac{(h_+^*)^{s-1}}{s!}g_s(z,\ a)=u_1^*+\frac{h_+^*}{2!}u_2^*+\ldots,$$

$$\underset{h}{u_2^*}=\sum_{l\geq 1}\sum_{s\geq 1}\frac{(-h_-^*)^{l-1}}{l!}\,\frac{(h_+^*)^{s-1}}{s!}g_{s+l}(z,\ a),\ \ldots,$$

where

$$h_+^*=(\underset{+h}{S}-1)f(z,\ a),\quad h_-^*=(1-\underset{-h}{S})f(z,\ a),$$

while the operators $\underset{+h}{S}$, $\underset{-h}{S}$ are "local," i.e., represent the corresponding translations with steps h_+ and h_-.

In order to apply the formal group G (19) to finite-difference objects we also need to choose f, g, g_1,... for which preservation of meaning of the finite-difference derivatives occurs. As the definition of difference derivatives we take Table 1, for example, the first difference derivative preserves its meaning if the equality

$$\underset{h}{u_1}h_+=\underset{+h}{S}(u)-u \tag{20}$$

is an invariant manifold of the group G (19). By analogy with groups of tangent transformations (cf. [4]) we call (20) the first order discrete tangency codnition and the following rows of Table 2, respectively, the second, third, etc. order discrete tangency conditions.

Let the group G correspond to the operator:

$$\underset{h}{\tilde{X}}=\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial u}+\sum_{s\geq 1}\zeta_s\frac{\partial}{\partial u_s}+\sum_{l\geq 1}\underset{h}{\zeta_l}\frac{\partial}{\partial \underset{h}{u_l}}+(\xi(\underset{+h}{S}(z))-\xi(z))\frac{\partial}{\partial h_+}+(\xi(z)-\xi(\underset{-h}{S}(z)))\frac{\partial}{\partial h_-}, \tag{21}$$

where $\xi, \eta \in \mathcal{A}$ and ζ_s and $\underset{h}{\zeta_l}$ are power series in h with coefficients from $\mathcal{A}$.

THEOREM 3. Let G be a formal one-parameter group with operator (21). At each point of $\tilde{Z}_h$ let (20) represent an invariant manifold of G. Then for the coordinates of the operator (21) one has the following chain of relations:

$$\zeta_s=\mathcal{D}^s(\eta-\xi u_1)+\xi u_{s+1}),\ s=1,\ 2,\ldots, \tag{22}$$

i.e., the formula of Lie–Bäcklund groups (cf. [4]).

The invariance of (20) guarantees the preservation of meaning of the first difference derivative under transformations of G (19) and as (22) which is obtained in Theorem 3 shows, the preservation of the meaning of all "ordinary" derivatives u_1, u_2,... . Does such a group preserve the meaning of the second, third, etc. difference derivatives? The next theorem gives the answer.

THEOREM 4. Suppose given a formal one-parameter group G (19) with operator (21) and suppose at each point of $\tilde{Z}_h$ (at each node of the net ω) the relation (20) be invariant. Then G preserves discrete tangency of any finite order (cf. [2]).

Thus, a formal group G preserving the meaning of the first difference derivative is a Lie–Bäcklund group in $\tilde{Z}$ and extends to Z_h while preserving the meaning of all difference derivatives of finite order. We note the nonlocality of the given interpretation of Lie–Bäcklund groups: two points on a smooth curve located at a small but finite distance from one another go into two points on the image of this curve (in the multidimensional case Lie–Bäcklund transformations carry a neighborhood of the point z of the locally analytic manifold Φ into a neighborhood of the manifold Φ*).

We give extension formulas for finite-difference derivatives obtained by successive action of the operator $\underset{h}{\tilde{X}}$ (21) on the rows of Table 1:

$$\begin{gathered}\underset{h}{\zeta_1}=\underset{+h}{\mathscr{D}}(\eta)-\underset{h}{u_1}\underset{+h}{\mathscr{D}}(\xi),\\ \underset{h}{\zeta_2}=\underset{-h}{\mathscr{D}}(\underset{h}{\zeta_1})-\underset{h}{u_2}\underset{-h}{\mathscr{D}}(\xi),\\ \cdots\cdots\cdots\cdots\cdots\\ \underset{h}{\zeta_{2k}}=\underset{-h}{\mathscr{D}}(\underset{h}{\zeta_{2k-1}})-\underset{h}{u_{2k}}\underset{-h}{\mathscr{D}}(\xi),\\ \underset{h}{\zeta_{2k+1}}=\underset{+h}{\mathscr{D}}(\underset{h}{\zeta_{2k}})-\underset{h}{u_{2k+1}}\underset{+h}{\mathscr{D}}(\xi),\quad k=1,\ 2,\ \ldots.\end{gathered}\tag{23}$$

We note that the recurrent chain of formulas (23) goes formally into the Lie–Bäcklund groups formulas as $h \to 0$.

5. Two-Dimensional Case: Extension Formulas; Invariant Nets

Let Z be the space of sequences $(x, u, \underset{1}{u}, \underset{2}{u}, \ldots)$, $x=\{x^i\}$, $i=1, \ldots, n$; $u=\{u^k\}$, $k=1, 2, \ldots, m$; $\underset{1}{u}=\{u^k_i\}$ be the collection of mn partial derivatives of first order, $\underset{2}{u}$ be the collection of second order derivatives, etc.

The extension formulas obtained earlier generalize easily to the case of several dependent variables u^k: for this it suffices to treat them as the component-wise description of the vector u^k.

Essential changes occur in passing to several variables x^i. In order to avoid involved formulas we restrict ourselves to the case $n = 2$: x^1, x^2. We shall omit the index $k = 1,\ldots, m$ for u^k.

We consider two forms differentiation:

$$\begin{gathered}\mathscr{D}_1=\frac{\partial}{\partial x^1}+u_1\frac{\partial}{\partial u}+u_{11}\frac{\partial}{\partial u_1}+u_{21}\frac{\partial}{\partial u_2}+\ldots,\\ \mathscr{D}_2=\frac{\partial}{\partial x^2}+u_2\frac{\partial}{\partial u}+u_{12}\frac{\partial}{\partial u_1}+u_{22}\frac{\partial}{\partial u_2}+\ldots,\end{gathered}\tag{24}$$

where

$$u_1=\frac{\partial u}{\partial x^1},\quad u_{11}=\frac{\partial^2 u}{(\partial x^1)^2},\quad u_{21}=\frac{\partial^2 u}{\partial x^2\partial x^1},\ \ldots,$$

while in (24) summation over the missing index k is understood.

The operators $\mathscr{D}_1$, $\mathscr{D}_2$ generate two commuting Taylor groups whose finite transformations are determined by the action of $T^1_a=e^{a\mathscr{D}_1}$ and $T^2_a=e^{a\mathscr{D}_2}$. Fixing two arbitrary values of the two parameters $h_1 > 0$ and $h_2 > 0$ we form two kinds of discrete translation operators:

$$\begin{gathered}\underset{\pm h_1}{S}=e^{\pm h_1\mathscr{D}_1}\equiv\sum_{s>0}\frac{(\pm h_1)^s}{s!}\mathscr{D}_1^s,\\ \underset{\pm h_2}{S}=e^{\pm h_2\mathscr{D}_2}\equiv\sum_{s>0}\frac{(\pm h_2)^s}{s!}\mathscr{D}_2^s.\end{gathered}\tag{25}$$

Correspondingly we shall have two pairs of difference differentiation operators:

$$\underset{\pm h_i}{\mathscr{D}}=\pm\frac{\underset{\pm h_i}{S}-1}{h_i},\quad i=1,2.\tag{26}$$

The operators $\underset{\pm h_1}{S}$, $\underset{\pm h_2}{S}$, $\underset{\pm h_1}{\mathscr{D}}$, $\underset{\pm h_2}{\mathscr{D}}$ commute in any combination.

The set of points in the (x^1, x^2) plane:

$$\{\underset{\pm h_1}{S^\alpha}(x^1),\ \underset{\pm h_2}{S^\beta}(x^2)\},\quad \alpha, \beta=0, 1, 2, \ldots,$$

will be called a uniform difference net and denote by $\underset{h}{\omega}$.

Analogously to the one-dimensional case we introduce the difference derivatives

$$\underset{h}{u_1}=\underset{+h_1}{\mathscr{D}}(u),\quad \underset{h}{u_2}=\underset{+h_2}{\mathscr{D}}(u),\quad \underset{h}{u_{1\bar{1}}}=\underset{-h_1}{\mathscr{D}}\ \underset{+h_1}{\mathscr{D}}(u),$$
$$\underset{h}{u_{12}}=\underset{+h_2}{\mathscr{D}}\ \underset{+h_1}{\mathscr{D}}(u),\quad \underset{h}{u_{1\bar{2}}}=\underset{-h_2}{\mathscr{D}}\ \underset{+h_1}{\mathscr{D}}(u),\ \text{and t.d.}$$

Correspondingly one generalizes Tables 1 and 2 of discrete translations and differentiations.

Let $\underset{h}{\tilde{Z}}$ be a sequence of formal power series with analytic coefficients:

$$z^{j*}=\sum_{s=0}^{\infty}A_s^j(z)\,a^s,\quad A_0^j=z^j, \tag{27}$$

z^j are the coordinates of the vector $(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{h}{u_1}, \underset{h}{u_2}, \underset{h}{u_{12}}, \ldots)$. We shall treat the sequence of series (27) as a transformation in $\underset{h}{\tilde{Z}}$. Among the sereis of the form (27) as before we shall be interested only in those which form a formal one-parameter group and are described by infinitesimal operators

$$\underset{h}{\tilde{X}}=\xi^1\frac{\partial}{\partial x^1}+\xi^2\frac{\partial}{\partial x^2}+\eta\frac{\partial}{\partial u}+\sum_{s=1}^{\infty}\zeta_{i_1\ldots i_s}\frac{\partial}{\partial u_{i_1,\ldots,i_s}}+\sum_{l=1}^{\infty}\underset{h}{\zeta_{i_1\ldots i_l}}\frac{\partial}{\partial \underset{h}{u_{i_1\ldots i_l}}},\quad i_s,\ i_l=\{1,2\}. \tag{28}$$

Supplementing $\underset{h}{\tilde{Z}}$ with the variables h_1, h_2 we extend the operator (28):

$$\underset{h}{\tilde{X}}=\ldots+h_1\underset{+h_1}{\mathscr{D}}(\xi^1)\frac{\partial}{\partial h_1}+h_2\underset{+h_2}{\mathscr{D}}(\xi^2)\frac{\partial}{\partial h_2}.$$

We calculate the coordinates of the operator (28) for the difference derivatives.

For this we consider in $\underset{h}{\tilde{Z}}$ a two-dimensional surface (as before, we omit the index k in u^k):

$$u=\psi(x^1, x^2). \tag{29}$$

Let the formal group G of transformations

$$x^1=f^1(z,a),\quad x^2=f^2(z,a),\quad u=\varphi(z,a),\ \ldots,$$

whose tangent field is defined by the operator (28) act on $\underset{h}{\tilde{Z}}$. Under the action of the group G the manifold (29) goes into $u^*=\psi^*(x^{1*}, x^{2*})$ or

$$\varphi(z,a)=\psi^*(f^1(z,a),\ f^2(z,a)). \tag{30}$$

We act on (30) by the operator $(\underset{+h_1}{S}-1)$:

$$(\underset{+h_1}{S}-1)\varphi(z,a)=\psi^*(\underset{+h_1}{S}(f^1),\ \underset{+h_1}{S}(f^2))-\psi^*(f^1, f^2)=$$
$$=\left\{\frac{\psi^*(\underset{+h_1}{S}(f^1),\ \underset{+h_1}{S}(f^2))-\psi^*(\underset{+h_1}{S}(f^1),\ f^2)}{(\underset{+h_1}{S}-1)f^2}\right\}(\underset{+h_1}{S}-1)f^2+\psi^*(\underset{+h_1}{S}(f^1),\ f^2)-\psi^*(f^1, f^2).$$

Applying the operation $\left.\frac{\partial}{\partial a}\right|_{a=0}$ to the equality obtained, we get

$$(\underset{+h_1}{S}-1)(\eta)=\underset{+h_1}{S}(u_2)(\underset{+h_1}{S}-1)(\xi^2)+\underset{h}{\zeta_1}h_1+\underset{h}{u_1}(\underset{+h_1}{S}-1)(\xi^1),$$

whence

$$\underset{h}{\zeta_1}=\underset{+h_1}{\mathscr{D}}(\eta)-\underset{h}{u_1}\underset{+h_1}{\mathscr{D}}(\xi^1)-\underset{+h_1}{S}(u_2)\underset{+h_1}{\mathscr{D}}(\xi^2), \tag{31}$$

where $\underset{+h_1}{S}(u_2)$ is the "continuous" derivative $u_2=\frac{\partial u}{\partial x^2}$ at the point shifted to the right by the step h_1 along the x^1 axis (on the discrete representation in $\underset{h}{\tilde{Z}}$ of the "continuous" derivatives, cf. below).

Analogously we get the following extension formulas:

$$\underset{h}{\zeta_2}=\underset{+h_2}{\mathscr{D}}(\eta)-\underset{+h_2}{S}(u_1)\underset{+h_2}{\mathscr{D}}(\xi^1)-\underset{h}{u_2}\underset{+h_2}{\mathscr{D}}(\xi^2), \tag{32}$$

$$\zeta_{\substack{1\bar{1}\\h}} = \underset{-h_1+h_1}{\mathcal{D}\,\mathcal{D}}(\eta) - 2\underset{h}{u_{1\bar{1}}}\underset{+h_1}{\mathcal{D}}(\xi^1) - \frac{1}{h_1}\underset{+h_1}{S}(u_2)\underset{+h_1}{\mathcal{D}}(\xi^2) + \frac{1}{h_1}\underset{-h_1}{S}(u_2)\underset{-h_1}{\mathcal{D}}(\xi^2), \tag{33}$$

$$\zeta_{\substack{2\bar{2}\\h}} = \underset{-h_2+h_2}{\mathcal{D}\,\mathcal{D}}(\eta) - 2\underset{h}{u_{2\bar{2}}}\underset{+h_2}{\mathcal{D}}(\xi^2) - \frac{1}{h_2}\underset{+h_2}{S}(u_1)\underset{+h_2}{\mathcal{D}}(\xi^1) + \frac{1}{h_2}\underset{-h_2}{S}(u_1)\underset{-h_2}{\mathcal{D}}(\xi^1), \tag{34}$$

$$\zeta_{\substack{12\\h}} = \underset{+h_2+h_1}{\mathcal{D}\,\mathcal{D}}(\eta) - \underset{h}{u_{12}}(\underset{+h_1}{\mathcal{D}}(\xi^1) + \underset{+h_2}{\mathcal{D}}(\xi^2)) + \frac{1}{h_1}\underset{+h_2}{S}(u_1)\underset{+h_2}{\mathcal{D}}(\xi^1) + \frac{1}{h_2}\underset{+h_1}{S}(u_2)\underset{+h_1}{\mathcal{D}}(\xi^2), \tag{35}$$

We note that the extension formulas (32)-(35) can be obtained with the help of the Lie–Bäcklund groups formulas just as in the one-dimensional case (cf. above).

A uniform rectangular difference net $\underset{h}{\omega}$ is characterized by the pair of relations

$$h_1^+ = h_1^-, \quad h_2^+ = h_2^-. \tag{36}$$

The requirement of invariance of the relations (36) leads to

$$\begin{gathered} \xi^1(\underset{+h_1}{S}(z)) - 2\xi^1(z) + \xi^1(\underset{-h_1}{S}(z)) = 0, \\ \xi^2(\underset{+h_2}{S}(z)) - 2\xi^2(z) + \xi^2(\underset{-h_2}{S}(z)) = 0. \end{gathered} \tag{37}$$

In the case of a nonuniform net rectangular net ω

$$\underset{h}{\omega}\begin{cases} h_1^+ = \varphi_1(x^1), \\ h_2^+ = \varphi_2(x^2), \end{cases} \tag{38}$$

the requirement of its invariance leads to the following relations:

$$\begin{gathered} \xi^1(\underset{+h_1}{S}(z)) - \xi^1(z)\left(1 + \frac{\partial\varphi_1}{\partial x^1}\right)\Big|_{\underset{h}{\omega}} = 0, \\ \xi^2(\underset{+h_2}{S}(z)) - \xi^2(z)\left(1 + \frac{\partial\varphi_2}{\partial x^2}\right)\Big|_{\underset{h}{\omega}} = 0. \end{gathered} \tag{39}$$

Analogously one can also consider invariant curvilinear difference nets.

We consider an example of an invariant net. Suppose given in the (x, y, u) space the operator of projective transformations

$$X = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}. \tag{40}$$

This operator does not satisfy (37) so the uniform rectangular net will be noninvariant. We construct an invariant rectangular nonuniform net.

We extend the operator (40) by h_1^+ and h_2^+

$$\widetilde{X} = \ldots + (2x + h_1^+)\,h_1^+\frac{\partial}{\partial h_1^+} + xh_2^+\frac{\partial}{\partial h_2^+}.$$

In (x, y, u, h_1^+, h_2^+) space there are four invariants:

$$\mathcal{I}_1 = \frac{x}{y}, \quad \mathcal{I}_2 = \frac{h_2^+}{x}, \quad \mathcal{I}_3 = \frac{x\,(h_1^+ + x)}{h_1^+}, \quad \mathcal{I}_4 = u.$$

We need to write down an invariant manifold of the form (38). One easily get an invariant representation of the manifold (38) in terms of the invariants $\mathcal{I}_1 - \mathcal{I}_4$. For example, in the form:

$$\underset{h}{\omega}\begin{cases} h_1^+ = \dfrac{x^2}{L_0 - x}, \quad 0 < x < L_0, \\ h_2^+ = y, \quad 0 < y < +\infty. \end{cases}$$

6. Invariance of Difference Equations

Let $\underset{h}{Z}$ be the space of sequences of net variables $(x, u, \underset{h}{u_1}, \underset{h}{u_2}, \ldots)$, $\underset{h}{\mathcal{A}}$ be the space of analytic functions of a finite number of variables z from Z. Then a finite-difference equation (or system of equations when $u:u^k$, $k = 1,\dots,m$) can be written as follows:

$$\mathscr{F}(z)|_{\underset{h}{\omega}}=0, \quad \mathscr{F}(z)\in\underset{h}{\mathscr{A}}. \tag{41}$$

This equation is written on a countable number of points of the difference net $\underset{h}{\omega}$ (uniform or nonuniform).

Suppose given in $\underset{h}{Z}$ extended by the variable h (one or several), a formal one-parameter group G, a Lie–Bäcklund group with operator X, which is extended to h by the formulas obtained previously.

We shall say that (41) is a homogeneous difference equation if at each point of $\underset{h}{\omega}$ it admits translation to a neighboring node (cf. [8]):

$$\underset{\pm h}{S}\mathscr{F}(z)|_{\underset{h}{\omega}}=0.$$

In this condition we do not include invariance with respect to discrete translation of the net $\underset{h}{\omega}$ since it itself is obtained by discretization of the space of independent variables, i.e., by the action of $\underset{\pm h}{S}^{\alpha}$ on some "starting" point. However an arbitrary net $\underset{h}{\omega}$ will not necessarily be an invariant manifold of the group G. Let $\underset{h}{\omega}$ be given in $\underset{h}{Z}$ by the equation $\underset{h}{\Omega}(z, h)=0$, $\Omega\in\underset{h}{\mathscr{A}}$. Then we shall say that the difference equation (41) admits the group G if

$$\begin{cases}\mathscr{F}(z)=0,\\ \underset{h}{\Omega}(z, h)=0,\end{cases} \tag{42}$$

is an invariant manifold of the group G. A criterion for invariance of the manifold (42) looks as follows:

$$X\mathscr{F}|_{(42)}=0, \quad X\underset{h}{\Omega}|_{(42)}=0. \tag{43}$$

The proof of the validity of the criterion (43) is a complete repetition of the proof of the corresponding criterion of [4] for the invariance of a manifold with respect to Lie–Bäcklund groups.

The process of creating a difference scheme is a change of variables in $\underset{h}{\tilde{Z}}$ from $\tilde{Z}$ to $\underset{h}{Z}$ in which one imposes requirements of approximation to the necessary order (cf. [8]) and preservation of the original group on the change of variables.

We consider examples of such transformations.

1. The ordinary differential equation

$$u_2=e^u \tag{44}$$

admits the two-parameter pointwise group defined by the operators

$$X_1=\frac{\partial}{\partial x}, \quad X_2=x\frac{\partial}{\partial x}-2\frac{\partial}{\partial u}.$$

We make the approximating change of variables

$$x\to y, \quad u\to v, \quad u_1\to \underset{h}{v_1}, \quad u_2\to \underset{h}{v_{1\bar{1}}}, \dots$$

Equation (44) becomes

$$\underset{h}{v_{1\bar{1}}}=e^v, \tag{45}$$

and the corresponding operators which we extend by h^+ and h^- will look as follows:

$$\underset{h}{X_1}=\frac{\partial}{\partial y}, \quad \underset{h}{X_2}=y\frac{\partial}{\partial y}-2\frac{\partial}{\partial v}-\underset{h}{v_1}\frac{\partial}{\partial \underset{h}{u_1}}-2\underset{h}{v_{1\bar{1}}}\frac{\partial}{\partial \underset{h}{u_{1\bar{1}}}}+h^+\frac{\partial}{\partial h^+}+h^-\frac{\partial}{\partial h^-}, \tag{46}$$

Both operators satisfy the criterion for preservation of uniformity of a net (11) so the uniform net which one can write as $h^+ = h^-$ will be invariant.

The invariance criterion (45) on the uniform net $\underset{h}{\omega}$ gives

$$\underset{h}{X_1}(\underset{h}{v_{1\bar{1}}}-e^v)\Big|_{(45),\underset{h}{\omega}}\equiv 0,$$

$$\underset{h}{X}_2(\underset{h}{v}_{1\bar{1}}-e^v)\Big|_{(45),\underset{h}{\omega}}=-2(\underset{h}{v}_{1\bar{1}}-e^v)\Big|_{(45),\underset{h}{\omega}}=0.$$

We note that one can construct invariant difference equations in $\underset{h}{Z}$ starting from a complete collection of invariants and finite-difference invariants (discrete analogs of the differential invariants) of the group G in the net space. For the operators $\underset{h}{X}_1$, $\underset{h}{X}_2$ in $(y, v, \underset{h}{v}_1, \underset{h}{v}_{1\bar{1}}, h)$ space we have three invariants

$$\mathcal{J}_1=\underset{h}{v}_1 e^{-\frac{v}{2}},\quad \mathcal{J}_2=\underset{h}{v}_{1\bar{1}}e^{-v},\quad \mathcal{J}_3=h^2e^v.$$

Equation (45) is equivalent to the invariant representation $\mathcal{J}_2=1$. From these invariants it is also possible to compose other invariants by approximating (44) to the second order while guaranteeing the invariance property. For example, the equation

$$\underset{h}{v}_{1\bar{1}}=e^v+h^2e^{2v},$$

equivalent to $\mathcal{J}_2=1+\mathcal{J}_3$, will admit the operators (46) and the equation

$$\underset{h}{v}_{1\bar{1}}=e^v+h^2e^v$$

is not invariant with respect to the operator $\underset{h}{X}_2$.

2. We consider the simplest second order linear equation

$$u_2=0. \tag{47}$$

It is known (cf. [7]) that it admits an 8-parameter pointwise group among whose operators in particular there is the following one:

$$X=u\frac{\partial}{\partial x}.$$

In $\underset{h}{Z}$ we consider the finite-difference equation $\underset{h}{v}_{1\bar{1}}=0$ which approximates (47) to the second order. The corresponding operator

$$\underset{h}{X}=v\frac{\partial}{\partial y}-\underset{h}{v}_1^2\frac{\partial}{\partial \underset{h}{v}_1}+\underset{h}{v}_{1\bar{1}}(2h^+\underset{h}{v}_{1\bar{1}}-3\underset{h}{v}_1)\frac{\partial}{\partial \underset{h}{v}_{1\bar{1}}} \tag{48}$$

does not satisfy the criterion for preserving of uniformity of the net: $\underset{-h+h}{\mathcal{D}\mathcal{D}}(\xi)\neq 0$. However, one can all the same use a uniform net. Indeed, we write our manifold in the form of two equalities:

$$\underset{h}{v}_{1\bar{1}}=0,\quad h^+=h^-, \tag{49}$$

and we extend the operator (48) by h^+ and h^-

$$\underset{h}{X}=\ldots+h^+\underset{h}{v}_1\frac{\partial}{\partial h^+}+h^-\underset{h}{v}_{\bar{1}}\frac{\partial}{\partial h^-}, \tag{50}$$

where $\underset{h}{v}_{\bar{1}}$ is the left difference derivative of first order, $\underset{h}{v}_{\bar{1}}=\underset{h}{v}_1-h^-\underset{h}{v}_{1\bar{1}}$. We act by the operator (50) on the manifold (49):

$$(2h^+\underset{h}{v}_{11}-3\underset{h}{v}_1)\underset{h}{v}_{1\bar{1}}\Big|_{(49)}=0,$$
$$h^+\underset{h}{v}_1-h^-\underset{h}{v}_{\bar{1}}\Big|_{(49)}=0.$$

The invariance of the second equation follows from (49) and from the equation

$$h^-\underset{h}{v}_{1\bar{1}}=\underset{h}{v}_1-\underset{h}{v}_{\bar{1}}.$$

Thus, the uniform net is not invariant in the entire space $\underset{h}{Z}$ but admits the operator (50) on the manifold (49).

3. The equation of nonlinear thermal conductivity

$$u_t=(u^\sigma\ u_x)_x \tag{51}$$

admits (cf. [6]) the four-parameter group defined by the operators:

$$X_1=\frac{\partial}{\partial t},\quad X_2=\frac{\partial}{\partial x},\quad X_3=2t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x},\quad X_4=\sigma t\frac{\partial}{\partial t}-u\frac{\partial}{\partial u}. \tag{52}$$

We make the change of variables in $\underset{h}{\tilde{Z}}$: $x\to\bar{x}$, $t\to\bar{t}$, $u\to v$, $u_x\to \underset{h}{v_1}$, $u_t\to\underset{\tau}{v_1}$, $u_{xx}\to\underset{h}{v_{\bar{1}1}},\dots$ The operators (52) in $\underset{h}{Z}$ in extended form will look as follows:

$$\underset{h}{X_1}=\frac{\partial}{\partial\bar{t}};\quad \underset{h}{X_2}=\frac{\partial}{\partial\bar{x}};$$

$$\underset{h}{X_3}=2\bar{t}\frac{\partial}{\partial\bar{t}}+\bar{x}\frac{\partial}{\partial\bar{x}}-\underset{h}{v_1}\frac{\partial}{\partial\underset{h}{v_1}}-2\underset{\tau}{v_1}\frac{\partial}{\partial\underset{\tau}{v_1}}-2\underset{h}{v_{\bar{1}1}}\frac{\partial}{\partial\underset{h}{v_{\bar{1}1}}}+2\tau\frac{\partial}{\partial\tau}+h\frac{\partial}{\partial h};$$

$$\underset{h}{X_4}=\sigma\bar{t}\frac{\partial}{\partial\bar{t}}-v\frac{\partial}{\partial v}-(\sigma+1)\underset{\tau}{v_1}\frac{\partial}{\partial\underset{\tau}{v_1}}-\underset{h}{v_1}\frac{\partial}{\partial\underset{h}{v_1}}-\underset{h}{v_{\bar{1}1}}\frac{\partial}{\partial\underset{h}{v_{\bar{1}1}}}+\sigma\tau\frac{\partial}{\partial\tau}. \tag{53}$$

We note that all four operators preserve uniformity of the net $\underset{h,\tau}{\omega}$.

In $\underset{h}{Z}$ we consider the difference scheme approximating (51) to order $(\tau + h^2)$:

$$\underset{\tau}{v_1}=\underset{+h}{\mathscr{D}}(k(v)\underset{h}{\hat{v}_{\bar{1}}})=k(v)\underset{h}{\hat{v}_{\bar{1}1}}+\underset{+h}{\mathscr{D}}(k(v))\underset{h}{\hat{v}_1}, \tag{54}$$

where $\hat{z}\equiv\underset{+\tau}{S}(z)$ is the variable z on the "upper layer" with respect to $\bar{t}$, k(v) is the difference approximation of the coefficient of thermal conductivity u^σ. The scheme (54) of divergent type uses a six-point template (cf. [8]). We choose the following approximation of the coefficient $k = u^\sigma$

$$k(v)=\frac{1}{2}[v^\sigma+\underset{-h}{S}(v^\sigma)],$$

so $\underset{+h}{\mathscr{D}}(k(v))=\frac{1}{2h}(\underset{+h}{S}(v^\sigma)-\underset{-h}{S}(v^\sigma))$ and the scheme (54) looks as follows:

$$\underset{\tau}{v_1}=1/2[v^\sigma+\underset{-h}{S}(v^\sigma)]\underset{h}{\hat{v}_{\bar{1}1}}+\frac{1}{2h}[\underset{+h}{S}(v^\sigma)-\underset{-h}{S}(v^\sigma)]\underset{h}{\hat{v}_1}. \tag{54*}$$

We extend the operators (53) by the variables $\underset{h}{\hat{v}_1}$, $\underset{h}{\hat{v}_{\bar{1}1}}$, $\underset{+h}{S}(v^\sigma)$, $\underset{-h}{S}(v^\sigma)$:

$$\underset{h}{X_3}=\dots-\underset{h}{\hat{v}_1}\frac{\partial}{\partial\underset{h}{\hat{v}_1}}-2\underset{h}{\hat{v}_{\bar{1}1}}\frac{\partial}{\partial\underset{h}{\hat{v}_{\bar{1}1}}};$$

$$X_4=-\underset{h+h}{S}(v)\frac{\partial}{\partial(\underset{+h}{S}(v))}-\underset{-h}{S}(v)\frac{\partial}{\partial(\underset{-h}{S}(v))}-\underset{h}{\hat{v}_1}\frac{\partial}{\partial\underset{h}{\hat{v}_1}}-\underset{h}{\hat{v}_{\bar{1}1}}\frac{\partial}{\partial\underset{h}{\hat{v}_{\bar{1}1}}};$$

($\underset{h}{X_1}$, $\underset{h}{X_2}$ are not extendable). The difference equation (54*) obviously admits $\underset{h}{X_1}$ and $\underset{h}{X_2}$. We verify its invariance in relation to $\underset{h}{X_3}$ and $\underset{h}{X_4}$:

$$\underset{h}{X_3}(\underset{\tau}{v_1}-0.5[v^\sigma+\underset{-h}{S}(v^\sigma)]\underset{h}{\hat{v}_{\bar{1}1}}-0.5(h)^{-1}[\underset{+h}{S}(v^\sigma)-\underset{-h}{S}(v^\sigma)]\underset{h}{\hat{v}_1})|_{(54*)}=-2(\underset{\tau}{v_1}-0.5[v^\sigma+\underset{-h}{S}(v^\sigma)]\underset{h}{\hat{v}_{\bar{1}1}}-$$

$$-0.5(h)^{-1}[\underset{+h}{S}(v^\sigma)-\underset{-h}{S}(v^\sigma)]\underset{h}{\hat{v}_1})|_{(54*)}=0.$$

$$\underset{h}{X_4}(\underset{\tau}{v_1}-0.5[v^\sigma+\underset{-h}{S}(v^\sigma)]\underset{h}{\hat{v}_{\bar{1}1}}-\frac{1}{2h}[\underset{+h}{S}(v^\sigma)-\underset{-h}{S}(v^\sigma)]\underset{h}{\hat{v}_1}|_{(54*)}=$$

$$=-(\sigma+1)(\underset{\tau}{v_1}-0.5[v^\sigma+\underset{-h}{S}(v^\sigma)]\underset{h}{\hat{v}_{\bar{1}1}}-\frac{1}{2h}[\underset{+h}{S}(v^\sigma)-\underset{-h}{S}(v^\sigma)]\underset{h}{\hat{v}_1}|_{(54*)}=0.$$

Here we have used the fact that $\underset{\pm h}{S}$ commutes with any function from $\underset{h}{\mathscr{A}}$.

The invariance of the finite-difference equation (54*) lets us, just as in the differential case, construct its invariant solutions. However the nonlocality of the discrete equations leads to some singularities of this procedure. The difference equation is given on a finite collection of points of the difference net, on the difference template, so under a map into the space of invariants of the group it is necessary that the difference net and template agree in the space of invariants.

In $(\bar{t}, \bar{x}, v)$ space the operator X_1 has two invariants: $\bar{x}$, v. An invariant solution $v(\bar{x}, \bar{t}) = \tilde{v}(\bar{x})$ is a stationary solution defined by the equation

$$k(\tilde{v})\underset{h}{\tilde{v}_{\bar{1}1}} + \underset{+h}{\mathscr{D}}(k(\tilde{v}))\underset{h}{\tilde{v}_1} = 0,$$

while the step of the net h must remain the same as in the original space.

Analogously one gets a homogeneous solution by the operator $\underset{h}{X_2}$. A solution which is invariant with respect to the operator

$$X_1 + \alpha X_2 = \frac{\partial}{\partial \bar{t}} + \alpha\frac{\partial}{\partial \bar{x}},$$

is the difference running wave: $v(\bar{x}, \bar{t}) = \tilde{v}(\lambda)$, $\lambda = \bar{x} - \alpha\bar{t}$. The step of the difference net $\Delta\lambda$ along the λ axis should be consistent with the original steps h, τ and the velocity of the wave α:

$$\alpha = \frac{h}{\tau}, \quad \Delta\lambda = h.$$

These relations mean that the lines λ = const pass through the nodes of the original $(\bar{x}, \bar{t})$ plane. The compatibility of the difference templates can be achieved for $\alpha = k(h/\tau)$, k = 1, 2,..., i.e., when the velocity of the running wave is a multiple of the difference velocity h/τ. Difference running waves of the equation of thermal conductivity were first considered in [9].

We consider a self-similar solution of (54*) which is invariant with respect to the one-parameter group of dilatations corresponding to X_4. The invariants will be: $\bar{x}$, $v^\sigma/\bar{t}$ and we seek a solution in the form $v(\bar{x}, \bar{t}) = \tilde{v}(\bar{x})/(\bar{t})^{1/\sigma}$.

Substitution of the invariant representation of $v(\bar{x}, \bar{t})$ into (54*) leads to an equation for $\tilde{v}(\bar{x})$ on the (n + 1)-st layer with respect to $n = \bar{t}/\tau$

$$(\tilde{v}^\sigma(\bar{x}+h) + \tilde{v}^\sigma(\bar{x}))\underset{h}{\tilde{v}_1}(\bar{x}) - (v^\sigma(\bar{x}) + v^\sigma(\bar{x}-h))\underset{h}{\tilde{v}_{\bar{1}}}(\bar{x}) + 2hn(\sqrt{n^2+n}-1)\tilde{v}(\bar{x}) = 0. \quad (55)$$

In this equation the step of the net h coincides with the original one. Solving (55) we find a solution of the original equation from the formula

$$v(\bar{x}, \bar{t}) = \tilde{v}(\bar{x})((n+1)\tau)^{-1/\sigma}, \quad n = 0, 1, 2, \ldots.$$

7. Newton Group, Commutation Properties and Factorization of Lie–Bäcklund Operators in Net Space

The Taylor group defined in $\underset{h}{\tilde{Z}}$ by the operator $\mathscr{D}$ allowed us to extend the action of a formal group to the net variables $(\underset{h}{u_1}, \underset{h}{u_2}, \ldots)$. In the theory of Lie–Bäcklund groups [4] the Taylor group also plays an essential role. With the help of a generalization of it, groups defined on $\underset{h}{\tilde{Z}}$ by the operators $\xi^i\mathscr{D}_i$, $\xi^i(z)\in\mathscr{A}$ admitted by the differential equations one makes a transition to the quotient-algebra of Lie–Bäcklund operators. The representatives of this quotient-algebra have independent variables as invariants and the extension formulas for them have simple and convenient form.

In this section in the simplest case of one independent variable x and uniform net $\underset{h}{\omega}$ we consider the difference analog of this construction.

One constructs a group of transformations on the net space, the Newton group, which is isomorphic to the Taylor group. With the help of the Newton group one constructs an ideal of the algebra of all Lie–Bäcklund operators on the net space. The ideal constructed is used for factorization of the set of operators admitted by the finite-difference equations.

An orbit of the Taylor group, i.e., a one-parameter curve in $\tilde{Z}$ obtained as the trajectory of an arbitrary point $(x, u, u_1, \ldots)$ under the action of the operator $T_a = e^{a\mathscr{D}}$ at the points $a = \pm nh$ (n = 0, 1, 2,...) coincides with the points obtained by the action of the discrete translation $(\underset{\pm h}{S})^n$. In other words, an orbit of the Taylor group is a "continuous translation" drawn through a "discrete translation." The question arises of the inversion of this procedure, obtaining a continuous translation through the discrete, or in other words, the question of representation of the Taylor group on the net space $\underset{h}{Z}$. The following heuristic considerations let us find out which power series should be used for such a representation.

If translation along an orbit of the Taylor group for a = h gives a discrete translation $\underset{+h}{S}$ then in order to get translation by a ≠ ±nh we act on a point in $\tilde{Z}$ by the operator $\underset{+h}{S}$ "a nonintegral number of times," i.e., we introduce fractional powers of the operator $\underset{+h}{S}$. Leaving aside the question of convergence of the operator series which arise, we decompose $(\underset{+h}{S})^{a/h}$ in a power series:

$$(\underset{+h}{S})^{a/h}\equiv(1+h\underset{+h}{\mathscr{D}})^{a/h}=1+\frac{a}{h}h\underset{+h}{\mathscr{D}}+\frac{a}{h}\left(\frac{a}{h}-1\right)\frac{h^2}{2!}\underset{+h}{\mathscr{D}}^2+\ldots=$$
$$=1+a\underset{+h}{\mathscr{D}}+\frac{a(a-h)}{2!}\underset{+h}{\mathscr{D}}^2+\frac{a(a-h)(a-2h)}{3!}\underset{+h}{\mathscr{D}}^3+\ldots=1+\sum_{s=1}^{\infty}\left\{\prod_{k=0}^{s-1}(a-kh)\right\}\frac{\underset{+h}{\mathscr{D}}^s}{s!}. \tag{56}$$

The quantities $a^{[s]}=\prod_{k=0}^{s-1}(a-kh)$ figuring in (56) are called generalized powers of a (cf. [1, 5]).

Under the action of the operator series (56) the x coordinate goes into x* = x + a, the u coordinate into

$$u^*=u\mp a\underset{h}{u_1}+\frac{a(a-h)}{2!}(\underset{h}{u_2}+h\underset{h}{u_3})+\ldots,$$

i.e., is the Newton series decomposition of the function u = u(x) at the point (x + a) on the uniform net of nodes x, x + h, x + 2h,... .

Analogously we get a decomposition into a series of the fractional power of the operator of discrete translation to the left (a > 0)

$$(\underset{-h}{S})^{\frac{a}{h}}\equiv(1-h\underset{-h}{\mathscr{D}})^{\frac{a}{h}}=1+\sum_{s=1}^{\infty}\left\{\prod_{k=0}^{s-1}(kh-a)\right\}\frac{\underset{-h}{\mathscr{D}}^s}{s!}. \tag{57}$$

The action of the series (57) on the u coordinate gives the decomposition into a Newton series of the function u = u(x) at the point (x − a) on the uniform net of nodes x, x − h, x − 2h,... .

The action of the operator series (56)-(57) on the point (x, u, u_1, u_2,...) coincides with the action of the Taylor group at the points **a** = ±nh. (We note incidentally that at these points the series (56)-(57) break off, have a finite number of terms.)

$$\underset{a}{N^+}|_{a=nh}=1+\sum_{s=1}^{\infty}\left\{\prod_{k=0}^{s-1}(a-kh)\right\}\frac{\underset{+h}{\mathscr{D}}^s}{s!}\Bigg|_{a=nh}=(\underset{+h}{S})^n,$$
$$\underset{a}{N^-}|_{a=nh}=1+\sum_{s=1}^{\infty}\left\{\prod_{k=0}^{s-1}(kh-a)\right\}\frac{\underset{-h}{\mathscr{D}}^s}{s!}\Bigg|_{a=nh}=(\underset{-h}{S})^n.$$

We regroup the formal operators of the series (56)-(57) in powers of the parameter a:

$$\underset{a}{N^+}=\sum_{s=0}^{\infty}\frac{a^s}{s!}\left(\sum_{n=1}^{\infty}\frac{(-h)^{n-1}}{n}\underset{+h}{\mathscr{D}}^n\right)^s,$$
$$\underset{a}{N^-}=\sum_{s=0}^{\infty}\frac{(-a)^s}{s!}\left(\sum_{n=1}^{\infty}\frac{h^{n-1}}{n}\underset{-h}{\mathscr{D}}^n\right)^s. \tag{58}$$

Thus, the operators (58) are defined in $\underset{h}{Z}=(x, u, \underset{h}{u_1}, \underset{h}{u_2},\ldots)$ and can be represented in essential form (5):

$$\underset{a}{N^+}=e^{a\underset{+h}{\tilde{D}}},\quad \underset{a}{N^-}=e^{-a\underset{-h}{\tilde{D}}}, \tag{59}$$

where

$$\underset{+h}{\widetilde{\mathscr{D}}} = \sum_{n=1}^{\infty} \frac{(-h)^{n-1}}{n} \underset{+h}{\mathscr{D}^n}, \quad \underset{-h}{\widetilde{\mathscr{D}}} = \sum_{n=1}^{\infty} \frac{h^{n-1}}{n} \underset{-h}{\mathscr{D}^n}. \tag{60}$$

The representation (59)-(60) means that the action of the operators $\underset{a}{N^+}$, $\underset{a}{N^-}$ on the point $(x, u, \underset{h}{u_1}, \underset{h}{u_2}, \ldots)$ forms a pair of formal groups of transformations on $\underset{h}{Z}$:

$$\begin{cases} x^* = x \pm a, \\ u^* = u \pm a \underset{\pm h}{\widetilde{\mathscr{D}}}(u) + \frac{a^2}{2!} \underset{\pm h}{\widetilde{\mathscr{D}}^2}(u) \pm \ldots, \\ \underset{h}{u_1^*} = \underset{h}{u_1} \pm a \underset{\pm h}{\widetilde{\mathscr{D}}}(\underset{h}{u_1}) + \frac{a^2}{2!} \underset{\pm h}{\widetilde{\mathscr{D}}^2}(\underset{h}{u_1}) \pm \ldots, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases} \tag{61}$$

The second row of these transformations is the formal expansion (to the right and left) of the function u = u(x) at the point (x = ±a) in a Newton series. The remaining rows can be obtained by term-by-term difference differentiation, since the operators $\underset{+h}{\mathscr{D}}$, $\underset{-h}{\mathscr{D}}$ and $\underset{+h}{\widetilde{\mathscr{D}}}$, $\underset{-h}{\widetilde{\mathscr{D}}}$ commute. We shall call the group (61) the Newton group.

One can treat the action of $\underset{a}{N^+}$ and $\underset{a}{N^-}$ for a > 0 as formal Newton interpolation (respectively, to the right and left) on an infinite number of equidistant nodes; for $a<0$ $\underset{a}{N^+}$ and $\underset{a}{N^-}$ give, respectively, extrapolation to the left and right. We calculate the tangent field of the pair of formal groups (61), the Newton group:

$$\xi^{\pm} = \frac{\partial x^*}{\partial a}\Big|_{a=0} = \pm 1,$$

$$\eta^{\pm} = \frac{\partial u^*}{\partial a}\Big|_{a=0} = \pm \underset{\pm h}{\widetilde{\mathscr{D}}}(u), \tag{62}$$

$$\underset{h}{\zeta_1^{\pm}} = \frac{\partial \underset{h}{u_1^*}}{\partial a}\Big|_{a=0} = \pm \underset{\pm h}{\widetilde{\mathscr{D}}}(\underset{h}{u_1}), \ldots.$$

Instead of the pair of tangent fields (62) we shall consider the infinitesimal operators of the Newton group:

$$\underset{h}{\mathscr{D}^+} = \frac{\partial}{\partial x} + \underset{+h}{\widetilde{\mathscr{D}}}(u) \frac{\partial}{\partial u} + \underset{+h}{\widetilde{\mathscr{D}}}(\underset{h}{u_1}) \frac{\partial}{\partial \underset{h}{u_1}} + \ldots,$$

$$\underset{h}{\mathscr{D}^-} = -\frac{\partial}{\partial x} - \underset{-h}{\widetilde{\mathscr{D}}}(u) \frac{\partial}{\partial u} - \underset{-h}{\widetilde{\mathscr{D}}}(\underset{h}{u_1}) \frac{\partial}{\partial \underset{h}{u_1}} - \ldots, \tag{63}$$

$$\underset{\pm h}{\widetilde{\mathscr{D}}} = \sum_{n=1}^{\infty} \frac{(\mp h)^{n-1}}{n} \underset{\pm h}{\mathscr{D}^n}.$$

In the operator $\underset{h}{\mathscr{D}^-}$ we retained the sign "−" since $\underset{h}{\mathscr{D}^-}$ defines translation to the left for a positive value of the parameter a. Thus, with the help of heuristic considerations we have constructed a formal group on $\underset{h}{Z}$, the Newton group. Its orbit coincides with the orbit of the Taylor group at the points a = kh.

Now we show that the Newton group (61) with tangent field (62) is really a "discrete" representation of the Taylor group on $\underset{h}{Z}$.

It is known that the finite transformations of a continuous group are related in a one-to-one fashion with infinitesimal (infinitely small) transformations. In the case of pointwise groups this relation is expressed with the help of a finite system of Lie equations. In the case of Lie–Bäcklund groups the corresponding relation is expressed by an infinite chain of Lie equations whose solution is given by the unique recurrent sequence of coefficients of formal series (cf. [4]). In both cases the solution of this system can be represented in the form of an exponential map. In the case of $\underset{h}{\tilde{Z}_n}$ a finite transformation of any coordinate z^i is given by (5):

$$z^{i*}=\underset{a}{S}(z^i)\equiv e^{aX}(z^i)\equiv\sum_{s=0}^{\infty}\frac{a^s}{s!}X^{(s)}(z^i). \tag{64}$$

One can invert the series (64), i.e., establish an infinitesimal transformation aX(z) from the finite transformation $\underset{a}{S}(z)$ in the form of the logarithmic series (cf., e.g., [10]):

$$aX(z^i)=\ln[1+(\underset{a}{S}-1)](z^i)\equiv\left[(\underset{a}{S}-1)-\frac{(\underset{a}{S}-1)^2}{2}+\right.$$
$$\left.\dots+(-1)^{n-1}\frac{(\underset{a}{S}-1)^n}{n}+\dots\right](z^i)=\sum_{s=1}^{\infty}(-1)^{s-1}\frac{(\underset{a}{S}-1)^s}{s}(z^i). \tag{65}$$

We apply the process of reconstruction of the tangent field X from the finite transformations to the Taylor group, taking as value of the parameter a = h:

$$e^{a\mathscr{D}}|_{a=h}=\underset{+h}{S}=1+h\underset{+h}{\mathscr{D}},$$
$$h\mathscr{D}=h\underset{+h}{\mathscr{D}}-\frac{h^2\underset{+h}{\mathscr{D}^2}}{2}+\dots(-1)^{n-1}\frac{(h\underset{+h}{\mathscr{D}})^n}{n}+\dots,$$

whence

$$\mathscr{D}=\sum_{n=1}^{\infty}\frac{(-h)^{n-1}}{n}\underset{+h}{\mathscr{D}^n}, \tag{66}$$

i.e., we get an expression which coincides with the operator $\underset{+h}{\tilde{\mathscr{D}}}$. In (66) we have omitted argument z^i on which the corresponding operator acts. If $z^i\in\underset{h}{Z}$ then it is understood that on the left side of (66) the difference derivatives are expressed by series; if $z^i\in\tilde{Z}$ then it is necessary to express the operator $\underset{+h}{\mathscr{D}}$ in terms of $e^{h\mathscr{D}}$. Equation (66) gives the action of the tangent field of the Taylor group on the coordinate z^i. The infinitesimal operator of the Taylor group on $\tilde{Z}$ can be written as follows (we note that $\underset{+h}{\tilde{\mathscr{D}}}(x)=1$)

$$\underset{h}{\mathscr{D}^+}=\frac{\partial}{\partial x}+\underset{+h}{\tilde{\mathscr{D}}}(u)\frac{\partial}{\partial u}+\underset{+h}{\tilde{\mathscr{D}}}(\underset{h}{u_1})\frac{\partial}{\partial \underset{h}{u_1}}+\dots+\underset{+h}{\tilde{\mathscr{D}}}(\underset{h}{u_s})\frac{\partial}{\partial \underset{h}{u_s}}+\dots. \tag{67}$$

Analogously for a = −h we get

$$\mathscr{D}=\sum_{n=1}^{\infty}\frac{h^{(n-1)}}{n}\underset{-h}{\mathscr{D}^n}=\underset{-h}{\tilde{\mathscr{D}}},$$
$$\underset{h}{\mathscr{D}^-}=\frac{\partial}{\partial x}+\underset{-h}{\tilde{\mathscr{D}}}(u)\frac{\partial}{\partial u}+\underset{-h}{\tilde{\mathscr{D}}}(\underset{h}{u_1})\frac{\partial}{\partial \underset{h}{u_1}}+\dots+\underset{-h}{\tilde{\mathscr{D}}}(\underset{h}{u_s})\frac{\partial}{\partial \underset{h}{u_s}}+\dots. \tag{68}$$

Thus, a Taylor group having the tangent field

$$\mathscr{D}=\frac{\partial}{\partial x}+u_1\frac{\partial}{\partial u}+u_2\frac{\partial}{\partial u_1}+\dots+u_{s+1}\frac{\partial}{\partial u_s}+\dots,$$

on $\tilde{Z}$ can be represented in the $\underset{h}{\tilde{Z}}$ Newton group with a pair of tangent fields (62)-(63), i.e., <u>the Taylor group and the Newton group are isomorphic</u>. If we are given a transformation from $\tilde{Z}$ to $\underset{h}{Z}$ then the coordinates of the infinitesimal operator of the Taylor group is replaced with the help of the operator series $\underset{\pm h}{\tilde{\mathscr{D}}}$ which can be written as a relation:

$$\mathscr{D}\Leftrightarrow\begin{cases}\displaystyle\sum_{n=1}^{\infty}\frac{(-h)^{n-1}}{n}\underset{+h}{\mathscr{D}^n},\\ \displaystyle\sum_{n=1}^{\infty}\frac{h^{n-1}}{n}\underset{-h}{\mathscr{D}^n}.\end{cases} \tag{69}$$

The upper part of the formula uses Newton series to the right, the lower to the left.

Equation (69) has been known for a long time (cf., e.g., [5]). It was apparently first obtained by Lagrange [14]. Clearly the fact that (69) gives a connection between the coordinates of infinitesimal operators of the corresponding groups was not known since at that time the concept of group had not even been formlated.

Now we consider some commutation properties of operators on net spaces.

Suppose given two Lie–Bäcklund operators on the same uniform net $\underset{h}{\omega}$

$$X_i = \xi^i \frac{\partial}{\partial x} + \eta^i \frac{\partial}{\partial u} + [\underset{+h}{\mathcal{D}}(\eta^i) - \underset{h}{u_1}\underset{+h}{\mathcal{D}}(\xi^i)] \frac{\partial}{\partial \underset{h}{u_1}} + \ldots + h\underset{+h}{\mathcal{D}}(\xi^i)\frac{\partial}{\partial h},$$

$$i = 1, 2.$$

For any two operators X_1, X_2 we introduce an operation of multiplication (commutation) by the usual formula: $[X_1, X_2] = X_1X_2 - X_2X_1$. The commutator $[X_1, X_2]$ does not contain differentiation of higher than the first order and hence is an operator of the formal group:

$$[X_1, X_2] = (X_1(\xi^2) - X_2(\xi^1))\frac{\partial}{\partial x} + (X_1(\eta^2) - X_2(\eta^1))\frac{\partial}{\partial u} +$$

$$+ [X_1(\underset{+h}{\mathcal{D}}(\eta^2) - \underset{h}{u_1}\underset{+h}{\mathcal{D}}(\xi^2)) - X_2(\underset{+h}{\mathcal{D}}(\eta^1) - \underset{h}{u_1}\underset{+h}{\mathcal{D}}(\xi^1))]\frac{\partial}{\partial \underset{h}{u_1}} + \ldots + [X_1(h\underset{+h}{\mathcal{D}}(\xi^2)) - X_2(h\underset{+h}{\mathcal{D}}(\xi^1))]\frac{\partial}{\partial h}. \quad (70)$$

Is the commutator $[X_1, X_2]$ a Lie–Bäcklund operator? For this it suffices to verify whether it preserves "discrete tangency" of first order (i.e., the meaning of the first difference derivative at each point of $\underset{h}{\omega}$)

$$\underset{h}{d}u = \underset{h}{u_1}h, \quad (71)$$

where $\underset{h}{d} \equiv \underset{+h}{S} - 1$.

Extending the operator (70) by the variable $\underset{h}{d}u$ according to the formula (cf. [2]):

$$\left.\frac{\partial(\underset{h}{d}u^*)}{\partial a}\right|_{a=0} = h\underset{+h}{\mathcal{D}}(X_1(\eta^2) - X_2(\eta^1)),$$

we act by it on (71) which gives the following condition:

$$\underset{+h}{\mathcal{D}}X_1(\eta^2) - \underset{+h}{\mathcal{D}}X_2(\eta^1) - X_1\underset{+h}{\mathcal{D}}(\eta^2) + X_2\underset{+h}{\mathcal{D}}(\eta^1) - \underset{+h}{\mathcal{D}}(\xi^1)\underset{+h}{\mathcal{D}}(\eta^2) + \underset{+h}{\mathcal{D}}(\xi^2)\underset{+h}{\mathcal{D}}(\eta^1) = 0. \quad (72)$$

To prove the validity of (72) we need to calculate $[X, \underset{+h}{\mathcal{D}}] \equiv X\underset{+h}{\mathcal{D}} - \underset{+h}{\mathcal{D}}X$. The expression $X(\eta)$ is a function from $\underset{h}{\mathcal{A}}$, i.e., an analytic function of a finite number of variables from $\underset{h}{Z}$ if $\eta \in \underset{h}{\mathcal{A}}$. By the definition of discrete differentiation of a function from $\underset{h}{\mathcal{A}}$ we have:

$$\underset{+h}{\mathcal{D}}X(\eta) = \frac{1}{h}(X(\underset{+h}{S}(\eta)) - X(\eta)),$$

hence

$$[X, \underset{+h}{\mathcal{D}}] = -\underset{+h}{\mathcal{D}}(\xi)\underset{+h}{\mathcal{D}}. \quad (73)$$

Substitution of (73) into (72) turns the latter into an identity. Thus, the commutator (70) preserves discrete tangency of the first order. Since an operator of a formal group which preserves at each point of $\underset{h}{\omega}$ discrete tangency of the first order also preserves tangency of any finite order, $[X_1, X_2]$ preserves any finite discrete tangency and is thus a Lie–Bäcklund operator. Thus the following theorem is true.

THEOREM 5. The set of Lie–Bäcklund operators defined on the same uniform net $\underset{h}{\omega}$ forms a Lie algebra with multiplication

$$[X_1, X_2] = X_1X_2 - X_2X_1.$$

Now we consider the tangent field of the Newton group, i.e., the pair of operators

$$\underset{h}{\mathcal{D}^+} = \frac{\partial}{\partial x} + \underset{+h}{\widetilde{\mathcal{D}}}(u)\frac{\partial}{\partial u} + \underset{+h}{\widetilde{\mathcal{D}}}(\underset{h}{u_1})\frac{\partial}{\partial \underset{h}{u_1}} + \ldots,$$

$$\underset{h}{\mathcal{D}^-} = -\frac{\partial}{\partial x} - \underset{+h}{\widetilde{\mathcal{D}}}(u)\frac{\partial}{\partial u} - \underset{-h}{\widetilde{\mathcal{D}}}(\underset{h}{u_1})\frac{\partial}{\partial \underset{h}{u_1}} - \ldots, \quad (74)$$

where

$$\tilde{\underset{\pm h}{\mathscr{D}}}=\sum_{n=1}^{\infty}\frac{(\mp h)^{n-1}}{n}\underset{\pm h}{\mathscr{D}}^{n},$$

We note that to one tangent field of the Taylor group in $\tilde{Z}$

$$\mathscr{D}=\frac{\partial}{\partial x}+u_1\frac{\partial}{\partial u}+u_2\frac{\partial}{\partial u_1}+\ldots$$

corresponds the pair of fields (74) in the net space $\underset{h}{Z}$. This doubling of objects, the emergence of "right" and "left," is a characteristic feature of net spaces and concerns not only the operators of the Newton group (74), but also the discrete translation $\underset{\pm h}{S}=e^{\pm h\mathscr{D}}$ discrete differentiation $\underset{\pm h}{\mathscr{D}}$ etc. This doubling is connected with the presence of a specific discrete group, the group of reflections: $x \to -x$ leading to a change of sign of the step of the net h: $h \to -h$. Hence, instead of a pair of Newton groups with operators (74) one can consider one, which means factorization by the group of reflections. Thus, in the one-dimensional case for a uniform net $\underset{h}{\omega}$ passage from the Taylor group in $\tilde{Z}$ to the Newton group in $\underset{h}{\tilde{Z}}$ means passage to an isomorphic continuous group with the addition of the group of reflections.

We also note that the Newton group being a representation of the Taylor group is a nontrivial Lie–Bäcklund group (cf. [4]).

We consider the Lie–Bäcklund operator

$$X=\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial u}+\underset{h}{\zeta_1}\frac{\partial}{\partial \underset{h}{u_1}}+\underset{h}{\zeta_2}\frac{\partial}{\partial \underset{h}{u_2}}+\ldots+h\underset{+h}{\mathscr{D}}(\xi)\frac{\partial}{\partial h},$$

where

$$\underset{h}{\zeta_1}=\underset{+h}{\mathscr{D}}(\eta)-\underset{h}{u_1}\underset{+h}{\mathscr{D}}(\xi),\quad \underset{h}{\zeta_2}=\underset{-h}{\mathscr{D}}(\underset{h}{\zeta_1})-\underset{h}{u_2}\underset{-h}{\mathscr{D}}(\xi),\ \ldots\ . \tag{75}$$

The formulas (75) $\underset{h}{\zeta_1}$, $\underset{h}{\zeta_2}$, ... guarantee the preservation of meaning of the finite-difference derivatives under transformations of the group.

It is easy to see that the operators (74) satisfy the constraints (75), i.e., <u>the Newton group preserves the meaning of the finite-difference derivatives of any finite order</u>.

Multiplying a Lie–Bäcklund operator on the left by a function $\tilde{\xi}(z)\in\underset{h}{\mathscr{A}}$ generally we leave the set of Lie–Bäcklund operators. We introduce a special operation of multiplication on the left of a Lie–Bäcklund operator by an analytic function $\tilde{\xi}(z)\in\underset{h}{\mathscr{A}}:\tilde{\xi}*X$. In the operator $\tilde{\xi}*X$ the first coordinates are multiplied by $\tilde{\xi}$

$$\tilde{\xi}*X=\tilde{\xi}\xi\frac{\partial}{\partial x}+\tilde{\xi}\eta\frac{\partial}{\partial u}+\ldots,$$

and the remaining coordinates are constructed so that they define a group which preserves the first order finite-difference derivatives (and hence also any difference derivative of finite order). Thus, $\tilde{\xi}*X$ must satisfy the formulas (75):

$$\tilde{\xi}*X=\tilde{\xi}\xi\frac{\partial}{\partial x}+\tilde{\xi}\eta\frac{\partial}{\partial u}+[\underset{+h}{\mathscr{D}}(\tilde{\xi}\eta)-\underset{h}{u_1}\underset{+h}{\mathscr{D}}(\tilde{\xi}\xi)]\frac{\partial}{\partial \underset{h}{u_1}}+\ldots+\underset{+h}{\mathscr{D}}(\tilde{\xi}\xi)\frac{\partial}{\partial h}, \tag{76}$$

and not coincide with the operator $\tilde{\xi}\cdot X$. The same operation of multiplication on the left by $\tilde{\xi}(z)$ can be introduced into the "continuous" space Z requiring preservation of tangency of infinite order. Suppose given in $\tilde{Z}=(x^*, u^*, u_1^*, u_2^*, \ldots)$ a Lie–Bäcklund operator

$$X=\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial u}+\sum_{s>1}\zeta_s\frac{\partial}{\partial u_s},$$

where

$$\zeta_s=\mathscr{D}^s(\eta-\xi u_1)+\xi u_{s+1}=\mathscr{D}(\zeta_{s-1})-u_s\mathscr{D}(\xi),$$
$$\mathscr{D}=\frac{\partial}{\partial x}+u_1\frac{\partial}{\partial u}+u_2\frac{\partial}{\partial u_1}+\ldots.$$

Then the operation of multiplication (*) gives:

$$\tilde{\xi}*X=\tilde{\xi}\xi\frac{\partial}{\partial x}+\tilde{\xi}\eta\frac{\partial}{\partial u}+[\mathscr{D}(\tilde{\xi}\eta)-u_1\mathscr{D}(\tilde{\xi}\xi)]\frac{\partial}{\partial u_1}+\ldots.$$

It is easy to see that

$$\tilde{\xi}*X=\tilde{\xi}\cdot X+\sum_{s\geqslant 1}\sum_{n=1}^{s}C_s^n\mathscr{D}^n(\tilde{\xi})\,\mathscr{D}^{(s-n)}(\eta-\xi u_1)\frac{\partial}{\partial u_s},$$

i.e., in order that the Lie–Bäcklund operator can be multiplied on the left by an arbitrary function $\tilde{\xi}\in\mathscr{A}$ without leaving the set of operators preserving the condition of tangency of infinite order, it is necessary and sufficient that $\xi = \xi u_1$. The coordinates of the operator $\mathscr{D}$ of the Taylor group satisfy this condition. Thus, the operator $\mathscr{D}$ is the unique Lie–Bäcklund operator which can "with impunity" be multiplied on the left by $\tilde{\xi}(z)\in\mathscr{A}$.

This situation does not hold in the net space Z_h; it is impossible to multiply the tangent field of the Newton group $\underset{h}{\mathscr{D}}^{\pm}$ on the left by $\tilde{\xi}(z)\in\underset{h}{\mathscr{A}}$ which is connected with the specifics of the Leibniz difference rule. Hence it is necessary to use (75), that is, to construct an operator of the form

$$\begin{aligned}\tilde{\xi}*\underset{h}{\mathscr{D}}^{\pm}&=\pm\tilde{\xi}\frac{\partial}{\partial x}\pm\tilde{\xi}\underset{\pm h}{\tilde{\mathscr{D}}}(u)\frac{\partial}{\partial u}\pm\\ &\pm[\underset{+h}{\mathscr{D}}(\tilde{\xi}\underset{\pm h}{\mathscr{D}}(u))-u_1\underset{h}{}\underset{+h}{\mathscr{D}}(\tilde{\xi})]\frac{\partial}{\partial \underset{h}{u_1}}+\ldots\pm h\underset{+h}{\mathscr{D}}(\tilde{\xi})\frac{\partial}{\partial h}.\end{aligned}\tag{77}$$

We note that in (77) the coordinate of $h\underset{+h}{\mathscr{D}}(\tilde{\xi})$ defining the deformation of the step of the net ω_h which is equal to zero in the operator $\underset{h}{\mathscr{D}}^{\pm}$ of the Newton group, appears.

We consider the commutation properties of the Lie–Bäcklund operators X, $\underset{h}{\mathscr{D}}^{\pm}$, $\xi*\underset{h}{\mathscr{D}}^{\pm}$ in the net space Z_h.

<u>LEMMA 1.</u> For the Lie–Bäcklund operators X and $\underset{h}{\mathscr{D}}^{\pm}$ defined on the same uniform net ω_h one has the following relation:

$$[X,\underset{h}{\mathscr{D}}^{\pm}]=-[\underset{h}{\mathscr{D}}^{\pm}(\xi)]*\underset{h}{\mathscr{D}}^{\pm}.\tag{78}$$

<u>LEMMA 2.</u> For any Lie–Bäcklund operators X, $\tilde{\xi}*\underset{h}{\mathscr{D}}^{\pm}$, $\tilde{\xi}(z)\in\underset{h}{\mathscr{A}}$ defined on the same uniform net ω_h one has the following commutation relation:

$$[\tilde{\xi}*\underset{h}{\mathscr{D}}^{\pm},X]=(\tilde{\xi}*\underset{h}{\mathscr{D}}^{\pm}(\xi)-X(\tilde{\xi}))*\underset{h}{\mathscr{D}}^{\pm}.\tag{79}$$

The validity of (78) and (79) indicated in Lemmas 1 and 2 is established by direct calculation of coefficients of $\partial/\partial x$ and $\partial/\partial u$; the coincidence of the remaining coefficients is guaranteed by Theorem 5, since Lie–Bäcklund operators stand on the left and right in the indicated equalities.

The multiplication (*) introduced above and Lemma 2 let us make the following assertion.

<u>THEOREM 6.</u> The set of operators of the form

$$X_*=\tilde{\xi}*\underset{h}{\mathscr{D}}^{\pm}=\tilde{\xi}\frac{\partial}{\partial x}+\tilde{\xi}\underset{\pm h}{\tilde{\mathscr{D}}}(u)\frac{\partial}{\partial u}+\ldots\tag{80}$$

with arbitrary coefficients $\tilde{\xi}(z)\in\underset{h}{\mathscr{A}}$ form an ideal in the Lie algebra of all Lie–Bäcklund operators {X} on the net ω_h.

Hence instead of the Lie algebra of operators

$$X=\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial u}+[\underset{+h}{\mathscr{D}}(\eta)-\underset{h}{u_1}\underset{+h}{\mathscr{D}}(\xi)]\frac{\partial}{\partial \underset{h}{u_1}}+\ldots$$

one can consider the quotient algebra by the ideal (80).

As representatives of the indicated quotient algebra we shall consider operators for which the coordinate $\xi = 0$

$$\bar{X}=\bar{\eta}\frac{\partial}{\partial u}+\ldots,\tag{81}$$

where the coordinate $\bar{\eta}=\eta-\xi\underset{+h}{\tilde{\mathscr{D}}}(u)$. We shall call the operators (81), just as in the continuous case (cf. [4]), canonical operators. For them the extension formula have simple form:

$$\underset{h}{\zeta_1}=\underset{+h}{\mathscr{D}}(\bar{\eta}),\quad \underset{h}{\zeta_2}=\underset{-h}{\mathscr{D}}\underset{+h}{\mathscr{D}}(\bar{\eta}),\ \ldots .$$

We note that the independent variable for the canonical operator $\bar{X}$ is invariant, so the step of the net (the coordinate of $\partial/\partial h$ in the operator $\bar{X}$ is equal to zero) is also invariant. By virtue of Lemma 1, the canonical operators $\bar{X}$ commute with the operators $\underset{h}{\mathscr{D}}^{+}$, $\underset{h}{\mathscr{D}}^{-}$.

The canonical operators $\bar{X}$ will be used in the following section in formulating a criterion for conservation of difference equations admitting a group of transformations preserving the uniformity of the net $\underset{h}{\omega}$.

8. Discrete Analog of Noether's Theorem for a Class of Transformations

We consider conservative properties of difference equations in the simplest case, one independent variable and uniform net $\underset{h}{\omega}$.

The Eulier operator in $\tilde{Z} = (x, u, u_1, u_2, \ldots)$

$$\frac{\delta}{\delta u}=\frac{\partial}{\partial u}+\sum_{s=1}^{\infty}(-1)^{s}\mathscr{D}^{s}\left(\frac{\partial}{\partial u_s}\right),$$

where

$$\mathscr{D}=\frac{\partial}{\partial x}+u_1\frac{\partial}{\partial u}+\ldots+u_{s+1}\frac{\partial}{\partial u_s}+\ldots,$$

can be represented in $\underset{h}{Z}$. We shall assume that it is applied to functions of the form $\mathscr{L}(x, u, \underset{h}{u_1})\in\underset{h}{\mathscr{A}}$ defined on the net $\underset{h}{\omega}$. Noting that

$$\frac{\partial}{\partial u_s}=\frac{\partial \underset{h}{u_1}}{\partial u_s}\frac{\partial}{\partial \underset{h}{u_1}}=\frac{h^{s-1}}{s!}\frac{\partial}{\partial \underset{h}{u_1}},\quad s=1,\ 2,\ldots,$$

we get

$$\frac{\delta}{\delta u}=\frac{\partial}{\partial u}+\sum_{s=1}^{\infty}\frac{(-1)^{s}h^{s-1}}{s!}\mathscr{D}^{s}\left(\frac{\partial}{\partial \underset{h}{u_1}}\right)=\frac{\partial}{\partial u}-\underset{-h}{\mathscr{D}}\left(\frac{\partial}{\partial \underset{h}{u_1}}\right), \tag{82}$$

where $\underset{-h}{\mathscr{D}}$ is the operator of difference differentiation to the left. We note that in (82) "continuous" partial differentiation with respect to $\underset{h}{u_1}$ is applied first and afterwards "discrete" differentiation to the left.

We shall call the finite-difference equation

$$\frac{\delta\mathscr{L}}{\delta u}=\frac{\partial\mathscr{L}}{\partial u}-\underset{-h}{\mathscr{D}}\left(\frac{\partial\mathscr{L}}{\partial \underset{h}{u_1}}\right)=0 \tag{83}$$

the discrete Euler equation, the function $\mathscr{L}=\mathscr{L}(x, u, \underset{h}{u_1})$ the net (discrete, finite-difference) Lagrange function, any solution of (83) an extremal.

Example. Let $\mathscr{L}=\frac{1}{2}\underset{h}{u_1^2}+e^{u}$ so the Euler equation (83) will be as follows:

$$\underset{h}{u_{1\bar{1}}}-e^{u}=0.$$

Suppose given on the net $\underset{h}{\omega}$ a finite-difference functional

$$\underset{h}{L}=\sum_{\underset{h}{\Omega}}\mathscr{L}(x,\ u,\ \underset{h}{u_1})\,h, \tag{84}$$

where summation is over a finite or infinite domain $\Omega\subset\underset{h}{\omega}$.

Suppose given in $\underset{h}{Z}=(x, u, \underset{h}{u_1}, \underset{h}{u_2},\ldots, h)$ a one-parameter group of Lie–Bäcklund transformations G defined by the operator

$$X=\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial u}+\underset{h}{\zeta_1}\frac{\partial}{\partial \underset{h}{u_1}}+\ldots+h\underset{+h}{\mathscr{D}}(\xi)\frac{\partial}{\partial h}, \tag{85}$$

We shall call the sum

$$\underset{h}{L^*}=\sum_{\underset{h}{\Omega^*}}\mathscr{L}(x^*,\ u^*,\ \underset{h}{u_1^*})\,h^*, \tag{86}$$

where the domain of summation $\underset{h}{\Omega^*}$ is obtained from the domain $\underset{h}{\Omega}$ by transformations of the group G, the transformed value of the net functional (84).

We shall call the net functional $\underset{h}{L}$ <u>invariant with respect to the group G</u> if for all transformations of the group G and any domains of summation $\underset{h}{\Omega}$ one has:

$$\sum_{\underset{h}{\Omega}}\mathscr{L}(x,\ u,\ \underset{h}{u_1})\,h=\sum_{\underset{h}{\Omega^*}}\mathscr{L}(x^*,\ u^*,\ \underset{h}{u_1^*})\,h^*. \tag{87}$$

We find out under what conditions on the discrete Lagrange function $\mathscr{L}(x,\ u,\ \underset{h}{u_1})$ (87) holds. We make a change of variables in (87) under which on the right one will have summation over the "old" domain $\underset{h}{\Omega}$

$$\sum_{\underset{h}{\Omega}}\mathscr{L}(x,\ u,\ \underset{h}{u_1})\,h=\sum_{\underset{h}{\Omega}}\mathscr{L}(e^{aX}(x),\ e^{aX}(u),\ e^{aX}(\underset{h}{u_1}))\,e^{aX}(h).$$

Since the domain of summation $\underset{h}{\Omega}$ is arbitrary, the last equality is equivalent to the following:

$$\mathscr{L}(x,\ u,\ \underset{h}{u_1})\,h=\mathscr{L}(x^*,\ u^*,\ \underset{h}{u_1^*})\,h^*. \tag{88}$$

Equation (88) means that the function $\mathscr{L}(x,\ u,\ \underset{h}{u_1})\,h$ is an invariant of the group of transformation G in the space $\underset{h}{Z}=(x,u,\ \underset{h}{u_1},\ \underset{h}{u_2},\ldots,\ h)$.

We write down a <u>necessary and sufficient condition for the invariance</u> of this function with the help of (85):

$$X(\mathscr{L}(x,\ u,\ \underset{h}{u_1}),\ h)=0$$

or

$$\xi\frac{\partial\mathscr{L}}{\partial x}+\eta\frac{\partial\mathscr{L}}{\partial u}+[\underset{+h}{\mathscr{D}}(\eta)-\underset{h}{u_1}\underset{+h}{\mathscr{D}}(\xi)]\frac{\partial\mathscr{L}}{\partial\underset{h}{u_1}}+\mathscr{L}\underset{+h}{\mathscr{D}}(\xi)=0. \tag{89}$$

Thus we have the following theorem.

<u>THEOREM 7.</u> In order that the net functional (84) be invariant with respect to the formal one-parameter group G with operator (85) it is necessary and sufficient that (89) hold.

Equation (89) is analogous to the corresponding condition in the "continuous" case [4] and tends to it formally as $h\to 0$. If necessary (89) is easily generalized to "left" difference derivatives $\underset{h}{u_{\bar{1}}}=\underset{h}{u_1}-h\underset{h}{u_2}$, half-sums of steps h, etc.

<u>Example.</u> Suppose given a one-parameter dilatation group G with operator $X = x(\partial/\partial x)$. We extend it to net variables $\underset{h}{u_1},\ \underset{h}{u_{1\bar{1}}},\ h$ (we take the net $\underset{h}{\omega}$ uniform)

$$X=x\frac{\partial}{\partial x}-\underset{h}{u_1}\frac{\partial}{\partial\underset{h}{u_1}}-2\underset{h}{u_{1\bar{1}}}\frac{\partial}{\partial\underset{h}{u_{1\bar{1}}}}+h\frac{\partial}{\partial h}.$$

We consider the Lagrange function $\mathscr{L}=(x+h)\underset{h}{u_1}^2$. We verify (89):

$$x\underset{h}{u_1}^2-2u_1^2(x+h)+h\underset{h}{u_1}^2+(x+h)\underset{h}{u_1}^2=0,$$

i.e. the net functional with indicated function $\mathscr{L}$ will be invariant. We calculate the difference variational derivative of L

$$\frac{\sigma\mathcal{L}}{\sigma \underset{h}{u}} = -2\underset{-h}{\mathcal{D}}\,\delta\,(x+h)\,\underset{h}{u_1}) = -2\,(x\underset{h}{u_{1\bar{1}}} + \underset{h}{u_1}).$$

Thus, the Euler equation for $\mathcal{L} = (x+h)\,\underset{h}{u_1}^2$ will be the second order finite-difference equation

$$x\underset{h}{u_{1\bar{1}}} + \underset{h}{u_1} = 0.$$

This equation naturally admits the same group G:

$$X\,(x\underset{h}{u_{1\bar{1}}} + \underset{h}{u_1})\Big|_{x\underset{h}{u_{1\bar{1}}} + \underset{h}{u_1}=0} = -(x\underset{h}{u_{1\bar{1}}} + \underset{h}{u_1})\Big|_{x\underset{h}{u_{1\bar{1}}} + \underset{h}{u_1}=0}.$$

Suppose given on the net $\underset{h}{\omega}$ the finite-difference equation

$$\mathcal{F}\,(x,\ u,\ \underset{h}{u_1},\ \underset{h}{u_{1\bar{1}}}) = 0. \tag{90}$$

We shall say that for Eq. (89) there exists a conservation law if there exists a function $A(x,\ u,\ \underset{h}{u_1}) \in \underset{h}{\mathcal{A}}$ such that on any solution of (90) one has:

$$\underset{h}{\mathcal{D}}^{\pm}(A)\Big|_{\mathcal{F}=0} = 0, \tag{91}$$

where $\underset{h}{\mathcal{D}}^{\pm}$ is the tangent vector of the Newton group:

$$\underset{h}{\mathcal{D}}^{\pm} = \frac{\partial}{\partial x} + \underset{\pm h}{\tilde{\mathcal{D}}}(u)\frac{\partial}{du} + \underset{\pm h}{\tilde{\mathcal{D}}}(\underset{h}{u_1})\frac{\partial}{\partial \underset{h}{u_1}} + \ldots,$$

$\underset{\pm h}{\tilde{\mathcal{D}}}$ is "Lagrangian differentiation":

$$\underset{\pm h}{\tilde{\mathcal{D}}} = \sum_{n=1}^{\infty} \frac{\mp h^{n-1}}{n}\ \underset{\pm h}{\mathcal{D}}^{n}$$

We note that in the one-dimensional case which we consider (91) is equivalent to the fact that $A(x,\ u,\ u_1)$ is an invariant of the Newton group on Eqs. (90). One can rewrite this fact in finite form:

$$\underset{\pm h}{S^{\alpha}}(A\,(x,\ u,\ \underset{h}{u_1}))\Big|_{\mathcal{F}=0} = A\,(x,\ u,\ \underset{h}{u_1})\Big|_{\mathcal{F}=0},\quad \alpha = 1, 2, \ldots, \tag{93}$$

where $\underset{\pm h}{S}$ are the discrete translation operators. Considering that $\underset{\pm h}{\mathcal{D}} = \pm(\underset{\pm h}{S} - 1)/h$ one can also rewrite this equation in another way:

$$\underset{\pm h}{\mathcal{D}}(A\,(x,\ u,\ \underset{h}{u_1})\Big|_{\mathcal{F}=0} = 0. \tag{94}$$

A conservation law in the form (94) is a finite algebraic expression on the net $\underset{h}{\omega}$. If it holds on $\mathcal{F}=0$ this means that it is an algebraic consequence of the latter.

We consider the special case of a one-parameter group G on the net space $\underset{h}{Z}$ whose transformations do not affect the independent variable. The corresponding operator has the form

$$X = \eta\frac{\partial}{\partial u} + \underset{+h}{\mathcal{D}}(\eta)\frac{\partial}{\partial \underset{h}{u_1}} + \ldots, \tag{95}$$

where $\eta(z)$ is a function from $\underset{h}{\mathcal{A}}$.

We shall consider, as before, Lagrange net functions depending on a difference derivative of at most first order.

LEMMA 3. The following operator identity holds:

$$\eta\frac{\partial}{\partial u} + \underset{+h}{\mathcal{D}}(\eta)\frac{\partial}{\partial \underset{h}{u_1}} \equiv \eta\left[\frac{\partial}{\partial u} - \underset{-h}{\mathcal{D}}\left(\frac{\partial}{\partial \underset{h}{u_1}}\right)\right] + \underset{+h}{\mathcal{D}}\left\{\eta \underset{-h}{S}\left(\frac{\partial}{\partial \underset{h}{u_1}}\right)\right\}. \tag{96}$$

The validity of (96) is established by direct calculation.

We note that as $h \to 0$ (96) formally coincides with the corresponding Noether identity since as $h \to 0$ the operators of discrete translation tend to the identity operator.

Equation (96) lets us relate the conservativeness of a difference equation with the invariance of the corresponding dicrete functional.

THEOREM 8. Let the Euler equation $\frac{\delta \mathscr{L}}{\delta \underset{h}{u}}=0$ be invariant with respect to the group G with operator (95). The functions $A(x, u, \underset{h}{u_1})=\eta \underset{-h}{S}\left(\frac{\partial \mathscr{L}}{\partial \underset{h}{u_1}}\right)$ satisfy a conservation law if and only if the variational functional (84) with $\mathscr{L}=\mathscr{L}(x, u, \underset{h}{u_1})$ is invariant on extremals (i.e., on solutions of $\frac{\delta \mathscr{L}}{\delta \underset{h}{u}}=0\Big)$.

Example. The finite-difference equation $\underset{h}{u_{1\bar{1}}}=0$ admits in particular the operators

$$X_1=\frac{\partial}{\partial u};\quad X_2=u\frac{\partial}{\partial u}+\underset{h}{u_1}\frac{\partial}{\partial \underset{h}{u_1}}+\underset{h}{u_{1\bar{1}}}\frac{\partial}{\partial \underset{h}{u_{1\bar{1}}}}.$$

We take $\mathscr{L}=\frac{1}{2}\underset{h}{u_1^2}$. The corresponding Euler equation $\underset{h}{u_{1\bar{1}}}=0$ also admits X_1, X_2. Calculations from the formula $A=\eta \underset{-h}{S}\left(\frac{\partial \mathscr{L}}{\partial \underset{h}{u_1}}\right)$ for X_1 gives $A=\underset{h}{u_{\bar{1}}}=\underset{h}{u_1}-h\underset{h}{u_{1\bar{1}}}$ where on extremals the condition (89) of invariance of $\mathscr{L}h$ holds so $\underset{+h}{\mathscr{D}}(\underset{h}{u_{\bar{1}}})|_{\underset{h}{u_{1\bar{1}}}}=0$. For X_2 $A=u\underset{h}{u_{\bar{1}}}$ however the variational functional with $\mathscr{L}=\frac{1}{2}\underset{h}{u_1^2}$ does not satisfy the invariance condition (89) on solutions of $\underset{h}{u_{1\bar{1}}}=0$ so $u\underset{h}{u_{\bar{1}}}$ does not generate a conservation law.

It was shown above that any formal one-parameter group of transformations on $\underset{h}{Z}$ with operator

$$X=\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial u}+[\underset{+h}{\mathscr{D}}(\eta)-\underset{h}{u_1}\underset{+h}{\mathscr{D}}(\xi)]\frac{\partial}{\partial \underset{h}{u_1}}+\ldots+h\underset{+h}{\mathscr{D}}(\xi)\frac{\partial}{\partial h}$$

under the condition

$$\underset{-h}{D}\,\underset{+h}{D}(\xi)=0$$

can be adequately described by factored operators of the form

$$\bar{X}=\bar{\eta}\frac{\partial}{\partial u}+\underset{+h}{\mathscr{D}}(\bar{\eta})\frac{\partial}{\partial \underset{h}{u_1}}+\underset{-h}{\mathscr{D}}\,\underset{+h}{\mathscr{D}}(\bar{\eta})\frac{\partial}{\partial \underset{h}{u_{1\bar{1}}}}+\ldots, \tag{97}$$

where $\bar{\eta}=\eta-\xi\underset{+h}{\tilde{\mathscr{D}}}(u)$, $\underset{+h}{\tilde{\mathscr{D}}}=\sum_{n\geq 1}\frac{(-h)^{n-1}}{n}\underset{h}{\mathscr{D}^n}$.

We rewrite (96) for the operators (97):

$$(\eta-\xi\underset{+h}{\tilde{\mathscr{D}}}(u))\frac{\partial}{\partial u}+\underset{+h}{\mathscr{D}}(\eta-\xi\underset{+h}{\tilde{\mathscr{D}}}(u))\frac{\partial}{\partial \underset{h}{u_1}}\equiv(\eta-\xi\underset{+h}{\tilde{\mathscr{D}}}(u))\left[\frac{\partial}{\partial u}-\underset{-h}{\mathscr{D}}\left(\frac{\partial}{\partial \underset{h}{u_1}}\right)\right]+\underset{+h}{\mathscr{D}}\left[(\eta-\xi\underset{+h}{\tilde{\mathscr{D}}}(u))\underset{-h}{S}\left(\frac{\partial}{\partial \underset{h}{u_1}}\right)\right]. \tag{98}$$

The identity (98) lets us reformulate Theorem 8 for a group G of transformations $\underset{h}{Z}$ preserving uniformity of the net $\underset{h}{\omega}$.

THEOREM 8*. Let the Euler equation $\frac{\delta \mathscr{L}}{\delta u}=0$ be invariant with respect to the group G with operator

$$X=\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial u}+[\underset{+h}{\mathscr{D}}(\eta)-\underset{h}{u_1}\underset{+h}{\mathscr{D}}(\xi)]\frac{\partial}{\partial \underset{h}{u_1}}+h\underset{+h}{\mathscr{D}}(\xi)\frac{\partial}{\partial h},$$

while

$$\underset{-h}{\mathscr{D}}\,\underset{+h}{\mathscr{D}}(\xi)|_{\frac{\delta \mathscr{L}}{\delta u}=0}=0.$$

The functions $A(x, u, \underset{h}{u_1})=(\eta=\xi\underset{+h}{\tilde{\mathscr{D}}}(u))\underset{-h}{S}\left(\frac{\partial \mathscr{L}}{\partial \underset{h}{u_1}}\right)$ satisfy the conservation law $\underset{+h}{\mathscr{D}}(A)|_{\frac{\delta \mathscr{L}}{\delta u}=0}=0$ if and only if the net functional (84) with $\mathscr{L}=\mathscr{L}(x, u, \underset{h}{u_1})$ is invariant on extremals with respect to the operator

$$\bar{X}=\bar{\eta}\frac{\partial}{\partial u}+\underset{+h}{\mathscr{D}}(\bar{\eta})\frac{\partial}{\partial \underset{h}{u_1}}+\ldots,\quad \bar{\eta}=\eta-\xi\underset{+h}{\tilde{\mathscr{D}}}(u).$$

Example. We consider on the uniform net ω_h the finite-difference equation

$$\underset{h}{u_2}=0.$$

This equation admits, as is easy to see, the following operators:

$$X_1=\frac{\partial}{\partial x};\ X_2=x\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}-\underset{h}{u_2}\frac{\partial}{\partial \underset{h}{u_2}}+h\frac{\partial}{\partial h};$$

which preserve uniformity of the net ω_h. As Lagrange net function we take $\mathscr{L}=\frac{1}{2}\underset{h}{u_1^2}$.

In using Theorem 8* we need the extended manifold [F] obtained from $\underset{h}{u_2}=0$ by successive difference differentiation $\underset{\pm h}{\mathscr{D}}$:

$$[F]:\underset{h}{u_2}=0,\ \underset{h}{u_3}=0,\ \underset{h}{u_4}=0,\ \ldots.$$

The canonical operator $\overline{X}_1$ is obtained by factorization of X_1 by the operator of the Newton group $\underset{h}{\mathscr{D}^+}$:

$$\overline{X}_1=\widetilde{u}_1\frac{\partial}{\partial u}+\underset{+h}{\mathscr{D}}(\widetilde{u}_1)\frac{\partial}{\partial \underset{h}{u_1}}+\underset{-h}{\mathscr{D}}\underset{+h}{\mathscr{D}}(\widetilde{u}_1)\frac{\partial}{\partial \underset{h}{u_2}}+\ldots,$$

where $\widetilde{u}_1\equiv\underset{+h}{\widetilde{\mathscr{D}}}(u)$. We show that $\mathscr{L}=\frac{1}{2}\underset{h}{u_1^2}$ is invariant with respect to $\overline{X}_1$

$$\overline{X}_1(\mathscr{L})\Big|_{[F]}=\underset{h}{u_1}\underset{+h}{\widetilde{\mathscr{D}}}(\underset{h}{u_1})\Big|_{[F]}=0,$$

since

$$\underset{+h}{\widetilde{\mathscr{D}}}(\underset{h}{u_1})=\underset{+h}{\mathscr{D}}(\underset{h}{u_1})\Big|_{[F]}-\frac{h}{2}\underset{+h}{\mathscr{D}^2}(\underset{h}{u_1})\Big|_{[F]}+\frac{h^2}{3}\underset{+h}{\mathscr{D}^3}(\underset{h}{u_1})\Big|_{[F]}+\ldots, \tag{99}$$

Each summand on the right side of (99) is equal to zero on [F], for example: $\underset{+h}{\mathscr{D}}(\underset{h}{u_1})\Big|_{[F]}=\underset{h}{u_2}+h\underset{h}{u_3}\Big|_{[F]}=0.$

Theorem 8* gives us the density of the conservation law in the form of a series:

$$A_1=(\eta-\xi\widetilde{u}_1)\underset{-h}{S}\left(\frac{\partial\mathscr{L}}{\partial \underset{h}{u_1}}\right)=-\underset{h}{u_{\bar{1}}}\widetilde{u}_1.$$

We see that A_1 generates the conservation law:

$$\underset{+h}{\mathscr{D}}(A_1)\Big|_{[F]}=-\underset{+h}{\mathscr{D}}(\underset{h}{u_{\bar{1}}}\widetilde{u}_1)\Big|_{[F]}=-\underset{h}{u_2}\widetilde{u}_1-\underset{h}{u_1}\underset{+h}{\widetilde{\mathscr{D}}}(\underset{h}{u_1})\Big|_{[F]}=0.$$

It is necessary to factorize the operator X_2 by the operator

$$\xi*\underset{h}{\mathscr{D}^+}=x\frac{\partial}{\partial x}+x\widetilde{u}_1\frac{\partial}{\partial u}+[\underset{+h}{\mathscr{D}}(x\widetilde{u}_1)-\underset{h}{u_1}]\frac{\partial}{\partial \underset{h}{u_1}}+\ldots+h\frac{\partial}{\partial h}.$$

As a result we get the following operator:

$$\overline{X}_2=(u-x\widetilde{u}_1)\frac{\partial}{\partial u}+\underset{+h}{\mathscr{D}}(u-x\widetilde{u}_1)\frac{\partial}{\partial \underset{h}{u_1}}+\underset{-h}{\mathscr{D}}\underset{+h}{\mathscr{D}}(u-x\widetilde{u}_1)\frac{\partial}{\partial \underset{h}{u_2}}+\ldots.$$

We see the invariance of $\mathscr{L}=\frac{1}{2}\underset{h}{u_1^2}$ in relation to $\overline{X}_2$ on [F]:

$$\underset{h}{u_1}\underset{+h}{\mathscr{D}}(u-x\widetilde{u}_1)\Big|_{[F]}=\underset{h}{u_1}(\underset{h}{u_1}-\widetilde{u}_1-(x+h)\underset{+h}{\widetilde{\mathscr{D}}}(\underset{h}{u_1}))\Big|_{[F]}=0.$$

The bracket on the right side is equal to zero on [F] by virtue of (99) and the following:

$$\underset{h}{u_1}-\widetilde{u}_1\Big|_{[F]}=\frac{h}{2}\underset{+h}{\mathscr{D}}(\underset{h}{u_1})\Big|_{[F]}-\frac{h^2}{3}\underset{+h}{\mathscr{D}^2}(\underset{h}{u_1})\Big|_{[F]}+\ldots=0.$$

By Theorem 8*, a function A_2 generating a conservation law will be $A_2=(u-x\widetilde{u}_1)\underset{h}{u_{\bar{1}}}$.

Indeed,

$$\underset{+h}{\mathscr{D}}(A_2)\Big|_{[F]}=\underset{+h}{\mathscr{D}}(u-x\widetilde{u}_1)\underset{h}{u_1}+(u-x\widetilde{u}_1)\underset{h}{u_2}\Big|_{[F]}=0,$$

since $\underset{h}{u_1}\underset{+h}{\mathscr{D}}(u-x\tilde{u}_1)\Big|_{[F]}=0$ by virtue of (100).

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