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Transformation of Spherical Harmonics Under Change of Reference Frame

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Summary

The problem of transforming spherical harmonics under a linear change of reference frame is solved analytically and numerically. In contrast to other methods both translation and rotation are treated with the same background theory which is comparatively straightforward and is formulated in geophysical terms; the directions of translation and rotation are arbitrary; and generally, combinations of harmonics of the same order can be dealt with equally as readily as individual harmonics.

It is found that the expansion coefficients of a translated spherical harmonic are themselves spherical harmonics in the co-ordinates of the translation vector. As a special case, the potential of an eccentric dipole is briefly discussed.

Very useful byproducts of this paper are the ways in which the given numerical algorithms can be used to evaluate any linear combination of harmonics.

1. Introduction

Take position vector

$$\mathbf{r} = (x, y, z) = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta),$$

where θ and ϕ are the colatitude and east longitude. The aim of this paper is to indicate procedures which allow any linear transformation of co-ordinates to be rapidly performed on an nth order solid spherical harmonic

$$V_n = r^n\, Y_n(\theta, \phi), \quad \text{or} \quad V_n = r^{-n-1}\, Y_n(\theta, \phi). \tag{1}$$

Here, Y_n stands for the general nth order surface harmonic:

$$Y_n = \sum_{m=0}^{n} (a_n^m \cos m\phi + b_n^m \sin m\phi)\, P_n^m(\mu), \tag{2}$$

where $\mu = \cos\theta$ and $P_n^m(\mu)$ is the associated Legendre function of the first kind. It is important in the theory of the Earth's magnetic field, and in potential theory generally, to be able to transform expressions of type (1) with ease.

Since a linear transformation can be decomposed into (a) a rotation, and (b) a translation, there are two basic problems to be considered. In the past, completely different methods have been used to perform these operations.

Problem (a) was first solved by Schmidt (1899) and his formulae have been used in geomagnetism (Kalinin 1963). A solution involving quaternions has been given by Herglotz (unpublished notes—see references) and a group theory approach due to Wigner (1959) is often used in quantum mechanics. These three authors have given equivalent results in closed forms, but for geophysicists a more useful formulation (based on Wigner's theory) can be found in Slater (1960). In geomagnetism, a numerical method, which evaluates equation (2) and its rotated version at selected points and inverts the resulting linear equations, has also been used (Stern 1965).

Hobson (1955) has given formulae for solving problem (b) which provide for translation along the z axis only. To translate in an arbitrary direction, his formulae must be used in conjunction with several rotations.

The present paper gives both closed and recursive solutions. Problems (a) and (b) are solved from basically the same approach which is comparatively straightforward being closely related to simple representations of the harmonics, and is formulated in geophysical terms. General combinations of harmonics of the same degree are considered rather than individual harmonics, and arbitrary directions of translation and rotation are allowed, so that, for example, the auxiliary rotations mentioned in the previous paragraph are not required. A canonical form of the theory is briefly discussed later in the paper and its use in some problems may save considerable time.

2. Two basic algorithms

We first state the two algorithms which form the basis of the recursive approach.

Suppose $\mathbf{u} = (u, v, w)$ is any constant vector and let ∇ be the vector operator $(\partial/\partial x, \partial/\partial y, \partial/\partial z)$.

Algorithm I

It can be shown (James 1967) that

$$\left.\begin{aligned} -(\mathbf{u}.\nabla)\left[r^{-n-1}\sum_{m=0}^{n}(a_n^m\cos m\phi+b_n^m\sin m\phi)\,P_n^m(\mu)\right] \\ = r^{-n-2}\sum_{m=0}^{n+1}(a_{n+1}^m\cos m\phi+b_{n+1}^m\sin m\phi)\,P_{n+1}^m(\mu),\end{aligned}\right\} \tag{3}$$

where for $m = 0, 1, 2, \ldots, n+1$:

$$\left.\begin{aligned} 2a_{n+1}^m &= u\,[(1+\delta_{m1})\,\alpha_{n+1,\,i}^m a_n^{m-1}-\beta_{n+1,\,i}^m a_n^{m+1}] \\ &\qquad -v[\alpha_{n+1,\,i}^m b_n^{m-1}+\beta_{n+1,\,i}^m b_n^{m+1}]+2w\,\gamma_{n+1,\,i}^m a_n^m, \\ 2b_{n+1}^m &= u[\alpha_{n+1,\,i}^m b_n^{m-1}-\beta_{n+1,\,i}^m b_n^{m+1}] \\ &\qquad +v[(1+\delta_{m1})\,\alpha_{n+1,\,i}^m a_n^{m-1}+\beta_{n+1,\,i}^m a_n^{m+1}]+2w\,\gamma_{n+1,\,i}^m b_n^m. \end{aligned}\right\} \tag{4}$$

The conventions

$$a_n^m = 0 \text{ when either } m < 0 \text{ or } m > n,$$

$$b_n^m = 0 \text{ when either } m \leqslant 0 \text{ or } m > n,$$

are to be understood; δ_{mp} is the Kronecker delta.

For Schmidt quasi-normalization of the Legendre functions as is usual in geomagnetism, the parameters α, β, γ are

$$\alpha_{n+1,\,i}^m = [\tfrac{1}{2}(2-\delta_{m1})\,(n+m)(n+m+1)]^{\frac{1}{2}},$$

$$\beta_{n+1,\,i}^m = [(1+\delta_{m0})(n-m)(n-m+1)]^{\frac{1}{2}},$$

$$\gamma_{n+1,\,i}^m = [(n-m+1)(n+m+1)]^{\frac{1}{2}}.$$

The subscript i denotes that the harmonics in equation (3) represent a potential due to sources internal to the sphere $r = 1$.

Algorithm II

Winch (1968a) has shown that for a potential due to external sources

$$\left.\begin{aligned} -(\mathbf{u}.\nabla)\left[r^n \sum_{m=0}^{n} (a_n^m \cos m\phi + b_n^m \sin m\phi)\, P_n^m(\mu)\right] \\ = r^{n-1} \sum_{m=0}^{n-1} (a_{n-1}^m \cos m\phi + b_{n-1}^m \sin m\phi)\, P_{n-1}^m(\mu). \end{aligned}\right\} \quad (5)$$

where for $m = 0, 1, 2, \ldots, n-1$:

$$\left.\begin{aligned} 2a_{n-1}^m &= u[(1+\delta_{m1})\,\alpha_{n-1,e}^m a_n^{m-1} - \beta_{n-1,e}^m a_n^{m+1}] \\ &\quad - v[\alpha_{n-1,e}^m b_n^{m-1} + \beta_{n-1,e}^m b_n^{m+1}] - 2w\,\gamma_{n-1,e}^m a_n^m, \\ 2b_{n-1}^m &= u[\alpha_{n-1,e}^m b_n^{m-1} - \beta_{n-1,e}^m b_n^{m+1}] \\ &\quad + v[1+\delta_{m1})\,\alpha_{n-1,e}^m a_n^{m-1} + \beta_{n-1,e}^m a_n^{m+1}] - 2w\,\gamma_{n-1,e}^m b_n^m. \end{aligned}\right\} \quad (6)$$

With conventions on a_n^m and b_n^m as in Algorithm I, here

$$\alpha_{n-1,e}^m = [\tfrac{1}{2}(2-\delta_{m1})(n-m)(n-m+1)]^{\frac{1}{2}},$$

$$\beta_{n-1,e}^m = [(1+\delta_{m0})(n+m)(n+m+1)]^{\frac{1}{2}},$$

$$\gamma_{n-1,e}^m = [(n-m)(n+m)]^{\frac{1}{2}}.$$

According to Algorithms I and II, if V_n is the spherical harmonic given by equation (1) and if $\mathbf{u}_i (i = 1, 2, \ldots, k)$ are any k vector constants, then

$$U = (\mathbf{u}_1.\nabla)(\mathbf{u}_2.\nabla)\ldots(\mathbf{u}_k.\nabla)\, V_n$$

is also a spherical harmonic—of order $n+k$ for internal V_n and $n-k$ for external V_n—and may be written in a form similar to equation (1). The importance of the recurrence relations (4) and (6) is that they rapidly generate the harmonic coefficients of U from those of V_n. If V_n is an external harmonic and $k > n$ then $U = 0$.

Algorithm I also provides a very useful way of evaluating the individual harmonics $P_n^m(\mu) \cos m\phi$ and $P_n^m(\mu) \sin m\phi$ at any point (θ_0, ϕ_0). According to Hobson (1955) or Winch (1968b), if $\mathbf{u} = (\sin\theta_0 \cos\phi_0, \sin\theta_0 \sin\phi_0, \cos\theta_0)$, then

$$\left.\begin{aligned} (-)^n(\mathbf{u}.\nabla)^n \frac{1}{r} = r^{-n-1} \sum_{m=0}^{n} (a_n^m \cos m\phi + b_n^m \sin m\phi)\, P_n^m(\mu) \\ \text{where} \qquad \begin{pmatrix} a_n^m \\ b_n^m \end{pmatrix} = n!\, P_n^m(\mu_0) \begin{pmatrix} \cos m\phi_0 \\ \sin m\phi_0 \end{pmatrix}. \end{aligned}\right\} \quad (7)$$

Thus taking $a_0^{\,0} = 1$ and applying Algorithm I n times with $\mathbf{u}$ as above, produces (apart from the factor $n!$) the $2n+1$ nth order surface harmonics at (θ_0, θ_0).

In a similar fashion Algorithm II provides a means of evaluating any combination of harmonics like Y_n in equation (2). Since the method is not relevant to the theme of the present paper it is deferred to Appendix I.

3. Maxwell's expressions for the harmonics

It will be appropriate to use the representations of spherical harmonics favoured by Maxwell (1892). The following theory is an independent formulation.

Let j be a running index over the integers which satisfy $0 < |j| \leqslant \frac{1}{2}m$ and include $j = 0$ when m is odd. Let $\mathbf{t}_m{}^j$ be unit vectors in the x–y plane separated by angles π/m and placed symmetrically about the positive x axis ($\phi = 0$) so that $\mathbf{t}_m{}^{-j}$ is the reflection of $\mathbf{t}_m{}^j$. When $\mathbf{t}_m{}^0$ exists (i.e. for odd m), it coincides with the x axis. Let $\mathbf{s}_m{}^j$ be the unit vector obtained by rotating $\mathbf{t}_m{}^j$ through $\pi/2m$ to the east (i.e. in the direction of increasing ϕ) and let $\hat{\mathbf{z}}$ be the unit vector in the z direction. Define the operators T_n^m and S_n^m by

$$\left.\begin{aligned} T_n^m &= (-2)^m(1+\delta_{m0})\left[(\hat{\mathbf{z}}.\nabla)^{n-m}\prod_j(\mathbf{t}_m{}^j.\nabla)\right],\\ S_n^m &= (-2)^m(1-\delta_{m0})\left[(\hat{\mathbf{z}}.\nabla)^{n-m}\prod_j(\mathbf{s}_m{}^j.\nabla)\right].\end{aligned}\right\} \quad (8)$$

Then it may be shown (Appendix II) that

$$\left.\begin{aligned} r^{-n-1}\,P_n^m(\mu)\cos m\phi &= A_{nm}\,T_n^m\left[\frac{1}{r}\right],\\ r^{-n-1}\,P_n^m(\mu)\sin m\phi &= A_{nm}\,S_n^m\left[\frac{1}{r}\right].\end{aligned}\right\} \quad (9)$$

For Schmidt quasi-normalization,

$$A_{nm} = (-)^{n+m}[2(1+\delta_{m0})(n-m)!\,(n+m)!]^{-\frac{1}{2}}.$$

So any surface harmonic (2) may be expressed in the form

$$Y_n = r^{n+1}\,D_n\left[\frac{1}{r}\right], \quad (10)$$

where D_n is the differential operator given by

$$D_n = \sum_{m=0}^{n} A_{nm}(a_n^m\,T_n^m + b_n^m\,S_n^m). \quad (11)$$

Equation (10) deserves emphasis since it describes Y_n as the combination of two parts: D_n, which is invariant under change of origin; and r, which is invariant under rotation.

If we adopt the conventions

$$T_n{}^{-m} = (-)^m\,T_n^m, \quad S_n{}^{-m} = (-)^{m+1}\,S_n^m,$$
$$A_{nm} = 0 \quad \text{for} \quad n < |m|,$$

then equations (9) are valid for all integers m and are consistent with the usual conventions

$$P_n{}^{-m}(\mu) = (-)^m\,P_n^m(\mu) \quad \text{and} \quad P_n^m(\mu) = 0 \quad \text{for} \quad n < |m|.$$

The geometrical arrangement of the vectors $\mathbf{t}_m{}^j$ and $\mathbf{s}_m{}^j$ suggests that T_n^m and S_n^m may be simply expressed in terms of complex variables. Let $\xi = x+iy$, $\eta = x-iy$. Then

$$2\frac{\partial}{\partial\xi} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}, \quad 2\frac{\partial}{\partial\eta} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y},$$

and when acting on harmonic functions

$$4\frac{\partial^2}{\partial\xi\partial\eta} = -\frac{\partial^2}{\partial z^2}. \tag{12}$$

As shown in Appendix II (or in Hobson 1955),

$$\left.\begin{aligned} T_n^m &= (-2)^m \left(\frac{\partial}{\partial z}\right)^{n-m}\left[\left(\frac{\partial}{\partial \xi}\right)^m + \left(\frac{\partial}{\partial \eta}\right)^m\right], \\ S_n^m &= i(-2)^m \left(\frac{\partial}{\partial z}\right)^{n-m}\left[\left(\frac{\partial}{\partial \xi}\right)^m - \left(\frac{\partial}{\partial \eta}\right)^m\right]. \end{aligned}\right\} \tag{13}$$

The complex representations (13) allow straightforward proofs of the following relations (14), (15) and (16), which are needed to derive closed forms for the transformation coefficients.

First, by direct substitution, noting equation (12), we obtain

$$\left.\begin{aligned} T_n^m T_k^l &= T_{k+n}^{l+m} + (-)^m T_{k+n}^{l-m}, \\ T_n^m S_k^l &= S_{k+n}^{l+m} + (-)^m S_{k+n}^{l-m}, \\ S_n^m S_n^l &= -T_{k+n}^{l+m} + (-)^m T_{k+n}^{l-m}; \end{aligned}\right\} \tag{14}$$

and for $n \geqslant l \geqslant p \geqslant 0$

$$\left(\frac{\partial}{\partial z}\right)^{n-l}\left(\frac{\partial}{\partial \xi}\right)^{l-p}\left(\frac{\partial}{\partial \eta}\right)^{p} = (-)^p 2^{-l-1}(T_n^{2p-l} + iS_n^{2p-l}). \tag{15}$$

Secondly—proofs being outlined in Appendix II—for m and $l \geqslant 0$:

$$\left.\begin{aligned} T_n^m[r^k P_k^l(\mu)\, e^{il\phi}] &= r^{k-n}(\alpha P_{k-n}^{l-m}(\mu)\, e^{i(l-m)\phi} + \beta P_{k-n}^{l+m}(\mu)\, e^{i(l+m)\phi}), \\ S_n^m[r^k P_k^l(\mu)\, e^{il\phi}] &= ir^{k-n}(\alpha P_{k-n}^{l-m}(\mu)\, e^{i(l-m)\phi} - \beta P_{k-n}^{l+m}(\mu)\, e^{i(l+m)\phi}), \end{aligned}\right\} \tag{16}$$

where

$$\alpha = (-)^n(1+\delta_{lm})\, A_{k-n,\, l-m}/A_{kl},$$

$$\beta = (-)^{n+m}(1-\delta_{l0})\, A_{k-n,\, l+m}/A_{kl}.$$

Equations (16) represent four real equations which hold except in the trivial cases when the left-hand sides are zero. For the special case $k = n$, they reduce to the ' orthogonality ' relations

$$\left.\begin{aligned} T_n^m[r^n P_n^l(\mu)\, e^{il\phi}] &= (-)^n \delta_{lm}/A_{nm}, \\ S_n^m[r^n P_n^l(\mu)\, e^{il\phi}] &= (-)^n i\delta_{lm}/A_{nm}. \end{aligned}\right\} \tag{17}$$

Any operator D_n, defined by equation (11), may be applied to the solid harmonic V_n given by equation (1), either by using Algorithms I or II or by using relations (14) or (16). The choice between these two alternatives leads to the basic distinction between the recursive and closed solutions to be presented in this paper.

4. Transformation techniques

4.1 *Rotation*

In this section asterisks ' * ' distinguish co-ordinates and operators etc. in the rotated frame from corresponding ones in the initial frame.

For the simple case of polar rotation to the east through angle ε, replacement of ϕ in equation (2) by $\phi^* + \varepsilon$ indicates the rotated form of Y_n. More generally, suppose the polar axis to which colatitude θ^* is referred has direction cosines given by $\mathbf{U} = (U, V, W) \neq \hat{\mathbf{z}}$, and let longitude ϕ^* be measured from the meridian containing a reference point G whose position vector has direction cosines $\mathbf{U}_G = (U_G, V_G, W_G)$. From spherical trigonometry, the components of a vector $\mathbf{u} = (u, v, w)$ are, after rotation,

$$\left.\begin{aligned} u^* &= [(w - w^* W)\cos\varepsilon + (uV - vU)\sin\varepsilon][1 - W^2]^{-\frac{1}{2}}, \\ v^* &= [(uV - vU)\cos\varepsilon - (w - w^* W)\sin\varepsilon][1 - W^2]^{-\frac{1}{2}}, \\ w^* &= \mathbf{u}.\mathbf{U}. \end{aligned}\right\} \tag{18}$$

In equation (18), ε is the angle between $\mathbf{U} \wedge \hat{\mathbf{z}}$ and $\mathbf{U} \wedge \mathbf{U}_G$, and is obtained from

$$R\cos\varepsilon = \mathbf{U} \wedge \hat{\mathbf{z}}.\mathbf{U} \wedge \mathbf{U}_G,$$

$$R\sin\varepsilon = \mathbf{U}_G \wedge \mathbf{U}.\hat{\mathbf{z}}.$$

Now the rotated form of Y_n is

$$Y_n^*(\theta^*, \phi^*) = r^{n+1} \overset{*}{D}_n \left[\frac{1}{r}\right]$$

$$= \sum_{m=0}^{n} (\overset{*}{a}{}_n^m \cos m\phi^* + \overset{*}{b}{}_n^m \sin m\phi^*)\, P_n^m(\mu^*), \tag{19}$$

and the coefficients $\overset{*}{a}{}_n^m$ and $\overset{*}{b}{}_n^m$ can be obtained recursively in the manner described below.

(i) Find the operators $\overset{*}{T}{}_n^m$ and $\overset{*}{S}{}_n^m$ by replacing the direction gradients $(\mathbf{u}.\nabla)$ in equations (8) by their rotated forms $(\mathbf{u}^*.\nabla^*)$, where $\mathbf{u}^*$ comes straight from equation (18).

(ii) Apply $\overset{*}{D}_n$ to $1/r$ by using Algorithm I. To do this, apply $\overset{*}{T}{}_n^m$ and $\overset{*}{S}{}_n^m$ separately and take the appropriate combinations (as defined by equation (11)) of the resulting coefficients.

Alternatively, closed forms for $\overset{*}{a}{}_n^m$ and $\overset{*}{b}{}_n^m$ may be derived as follows: with $\mathbf{u} = \mathbf{r}$, equations (18) are, in matrix notation, $\mathbf{r}^* = \mathbf{O}\mathbf{r}$, which can be inverted, since rotation matrices are orthogonal, to $\mathbf{r} = \mathbf{O}^T \mathbf{r}^*$. The partial derivatives in the rotated frame are then readily obtained in terms of the partial derivatives in the initial frame. Write

$$\left.\begin{aligned} \frac{\partial}{\partial z^*} &= a\frac{\partial}{\partial z} + b\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}, \\ \frac{\partial}{\partial \eta^*} &= f\frac{\partial}{\partial z} + g\frac{\partial}{\partial \xi} + h\frac{\partial}{\partial \eta}, \end{aligned}\right\} \tag{20}$$

where a, b, c, f, g and h are constants, possibly complex but easily calculated in terms of U, V, W and ε, or whatever other parameters are used to describe $\mathbf{O}$. From equation (13),

$$\overset{*}{T}{}_n^m + i\overset{*}{S}{}_n^m = (-)^m 2^{m+1} \left(\frac{\partial}{\partial z^*}\right)^{n-m} \left(\frac{\partial}{\partial \eta^*}\right)^m.$$

Substituting from equation (20), expanding by the binomial theorem, and observing equation (15), we obtain

$$\overset{*}{T}{}_n^m + i\,\overset{*}{S}{}_n^m = \sum_{lpqt} \Gamma_{lpqt}^{nm}\, (T_n^\lambda + i\, S_n^\lambda), \tag{21}$$

where $\lambda = 2p+2t-l-q$, and

$$\Gamma_{lpqt}^{nm} = (-)^{m+p+t} 2^{m-l-q} \binom{n-m}{l}\binom{l}{p}\binom{m}{q}\binom{q}{t} a^{n-m-l} b^{l-p} c^{p} f^{m-q} g^{q-t} h^{t}.$$

The summation $\sum_{lpqt}$ is over all non-negative integral values of l, p, q and t—the binomial coefficients like $\binom{l}{p}$ being zero for $l < p$.

Consider the equation arrived at by equating (19) to equation (2), and operate on both sides by $\overset{*}{T}_n^m + i\overset{*}{S}_n^m$. According to formulae (17), the result is

$$\overset{*}{a}_n^m + i\overset{*}{b}_n^m = (-)^n A_{nm}(\overset{*}{T}_n^m + i\overset{*}{S}_n^m)\,[r^n\, Y_n(\theta, \phi)]$$

which becomes, using equations (21) and (17) again,

$$\overset{*}{a}_n^m + i\overset{*}{b}_n^m = \sum_{lpqt} \chi_{lpqt}^{nm}\,(a_n^{\lambda} + i\,b_n^{\lambda}), \qquad (22)$$

where $\chi_{lpqt}^{nm} = A_{nm}\,\Gamma_{lpqt}^{nm}/A_{n\lambda}$, and where for negative λ we adopt the conventions $a_n^{\lambda} = (-)^{\lambda} a_n^{-\lambda}$ and $b_n^{\lambda} = (-)^{\lambda+1} b_n^{-\lambda}$.

It remains only to equate real and imaginary parts in equation (22). In that equation (22) is valid for a general rotation matrix **O**, it is remarkable for its simplicity, which tends to make this method preferable to the recursive approach. On the other hand, since Algorithms I and II allow direct methods for performing translations—see next section—the recursive solution may be favoured on the grounds that design of a computer program for carrying out translations and/or rotations is slightly simplified.

4.2 *Translation*

Consider a change in origin from O' to O where

$$\overrightarrow{OO'} = \mathbf{R} = R(\sin\theta_0 \cos\phi_0, \sin\theta_0 \sin\phi_0, \cos\theta_0).$$

A point with position vector $\mathbf{r}' = \mathbf{r} - \mathbf{R}$ before translation will have position vector $\mathbf{r}$ after translation. Let ∇_0 denote the gradient in the space defined by R, θ_0, ϕ_0 and let $\mathring{D}_n$, $\mathring{T}_n^m$ and $\mathring{S}_n^m$ be the corresponding forms of D_n, T_n^m and S_n^m. It is necessary to treat internal and external harmonics separately but first the following preliminary remarks should be noted.

It is an elementary result that, with $r' = |\mathbf{r} - \mathbf{R}|$,

$$\nabla_0 \left[\frac{1}{r'}\right] = -\nabla \left[\frac{1}{r'}\right];$$

hence

$$\mathring{D}_n \left[\frac{1}{r'}\right] = (-)^n D_n \left[\frac{1}{r'}\right]. \qquad (23)$$

Also, if

$$e^{-\mathbf{R}.\nabla} = \sum_{k=0}^{\infty} \frac{(-)^k (\mathbf{R}.\nabla)^k}{k!},$$

then from equation (7),

$$e^{-\mathbf{R}.\nabla} = \sum_{k=0}^{\infty} R^k \sum_{l=0}^{k} A_{kl}\, P_k^{\,l}(\mu_0)[\cos l\phi_0\; T_k^{\,l} + \sin l\phi_0\; S_k^{\,l}] \qquad (24)$$

when operating on harmonic functions.

Applying Taylor's theorem for $r > R$,

$$\frac{1}{r'} = e^{-\mathbf{R}\cdot\nabla}\left[\frac{1}{r}\right].$$

Hence, from equations (7) or (24),

$$\frac{1}{r'} = \sum_{k=0}^{\infty} R^k r^{-k-1} \sum_{m=0}^{k} P_k^m(\mu)\, P_k^m(\mu_0) \cos m(\phi-\phi_0) \tag{25a}$$

—an absolutely convergent series. When $r < R$, the convergent representation of $1/r'$ is equation (25a) with r and R interchanged—equation (25b) say. The reader may recognize that application of the addition theorem for Legendre functions will reduce series (25a) and (25b) to precisely those series which are often used to define the Legendre polynomials P_n^0.

Suppose the potential to be transformed is

$$V_n(\mathbf{r}') = f_n(r')\, Y_n(\theta', \phi'),$$

where $f_n(r)$ stands for r^n or r^{-n-1}, and Y_n, as in equation (2), has harmonic coefficients a_n^m, b_n^m $(m = 0, 1, \ldots, n)$. The potential at $\mathbf{r}$ after translation is

$$V(\mathbf{r}) = V_n(\mathbf{r}-\mathbf{R}) \tag{26a}$$

$$= (-)^n V_n(\mathbf{R}-\mathbf{r}). \tag{26b}$$

The equivalence of equations (26a) and (26b) is a 'symmetry' property of spherical harmonics (already seen in equation (23)).

(i) *External harmonics.* When $f_n(r) = r^n$, the Taylor expansion of equation (26a) is:

$$V(\mathbf{r}) = e^{-\mathbf{R}\cdot\nabla}\, V_n(\mathbf{r}) \tag{27}$$

$$= \sum_{k=0}^{n} \frac{R^k r^{n-k}}{k!} \sum_{m=0}^{n-k} (a_{n-k}^m \cos m\phi + b_{n-k}^m \sin m\phi)\, P_{n-k}^m(\mu), \tag{28}$$

where k applications of Algorithm II, starting from the known coefficients a_n^m, b_n^m $(m = 0, 1, \ldots, n)$ and taking $\mathbf{u} = \mathbf{R}/R$, rapidly generate the coefficients a_{n-k}^m, $b_{n-k}^m (m = 0, \ldots, n-k)$. This is the recursive solution. Series (28) shows that after translation nth order external harmonics become a superposition of external harmonics with orders less than and equal to n, the Taylor series containing $n+1$ terms. On the other hand, from equation (26b),

$$V(\mathbf{r}) = (-)^n e^{-\mathbf{r}\cdot\nabla_0}\, V_n(\mathbf{R}), \tag{29}$$

which, by its similarity to equation (27), shows that $V(\mathbf{r})$ is also a spherical harmonic in the co-ordinates R, θ_0, ϕ_0. Indeed, use of the appropriate form of equation (24) in equation (29) and a comparison of the result with equation (28) leads to the closed forms

$$\begin{pmatrix} a_{n-k}^m \\ b_{n-k}^m \end{pmatrix} = (-)^n k!\, A_{n-k,m} \begin{pmatrix} \mathring{T}_{n-k}^m \\ \mathring{S}_{n-k}^m \end{pmatrix} [R^n\, Y_n(\theta_0, \phi_0)]\,. \tag{30}$$

By using rules (16), equations (30) may be written explicitly in terms of the kth degree external harmonics at (R, θ_0, ϕ_0).

(ii) *Internal harmonics.* When $f_n(r) = r^{-n-1}$, the regions $r > R$ and $r < R$ must be considered separately.

For $r > R$ the Taylor expansion of equation (26a) is:

$$V(\mathbf{r}) = e^{-\mathbf{R}\cdot\nabla} V_n(\mathbf{r})$$

$$= \sum_{k=n}^{\infty} \frac{R^{k-n} r^{-k-1}}{(k-n)!} \sum_{m=0}^{k} (a_k^m \cos m\phi + b_k^m \sin m\phi) P_k^m(\mu). \quad (31)$$

The coefficients a_k^m, b_k^m ($m = 0, 1, \ldots, k$) can be rapidly generated by $k-n$ applications of Algorithm I, taking $\mathbf{u} = \mathbf{R}/R$ and starting from the known a_n^m, b_n^m ($m = 0, 1, \ldots, n$). Equation (31) shows that for $r > R$, nth order internal harmonics translate to a superposition of internal harmonics with orders greater than and equal to n. On the other hand, closed forms of a_k^m and b_k^m may be obtained by using equations (10), (23) and (25a) to write

$$V(\mathbf{r}) = D_n \left[\frac{1}{r'}\right]$$

$$= (-)^n \mathring{D}_n \left[\sum_{k=n}^{\infty} R^k r^{-k-1} \sum_{m=0}^{k} P_k^m(\mu) P_k^m(\mu_0) \cos m(\phi-\phi_0)\right],$$

which, on comparing with equation (31), leads to

$$R^{k-n} \begin{pmatrix} a_k^m \\ b_k^m \end{pmatrix} = (-)^n (k-n)!\, \mathring{D}_n \left[R^k P_k^m(\mu_0) \begin{pmatrix} \cos m\phi_0 \\ \sin m\phi_0 \end{pmatrix}\right]. \quad (32)$$

Relations (11) and (16) allow equation (32) to be written explicitly in terms of the $(k-n)$th degree external harmonics at (R, θ_0, ϕ_0).

For $r < R$, equation (25b) implies

$$V(\mathbf{r}) = D_n \left[\sum_{k=0}^{\infty} r^k R^{-k-1} \sum_{m=0}^{k} P_k^m(\mu) P_k^m(\mu_0) \cos m(\phi-\phi_0)\right], \quad (33)$$

$$= \sum_{k=0}^{\infty} r^k R^{-n-k-1} \sum_{m=0}^{k} (c_k^m \cos m\phi + d_k^m \sin m\phi) P_k^m(\mu) \quad (34)$$

say (noting that the terms corresponding to $k = 0, 1, \ldots, n-1$ in equation (33) reduce to zero when operated on by D_n). The values of c_k^m and d_k^m ($m = 0, 1, \ldots, k$) may be obtained recursively by first using Algorithm I and equations (7) to evaluate the surface harmonics at (θ_0, ϕ_0), and then applying D_n in equation (33) by using Algorithm II. Compared to the previous two translation cases this is not a very convenient way of finding the coefficients; the closed forms indicated below provide a faster evaluation.

According to equation (23), D_n in equation (33) may be replaced by $(-)^n \mathring{D}_n$. Thus

$$R^{-n-k-1} \begin{pmatrix} c_k^m \\ d_k^m \end{pmatrix} = (-)^n \mathring{D}_n \left[R^{-k-1} P_k^m(\mu_0) \begin{pmatrix} \cos m\phi_0 \\ \sin m\phi_0 \end{pmatrix}\right], \quad (35)$$

which, by equations (11) and (14), may be expanded as quite a simple combination of $(n+k)$th order internal harmonics at (R, θ_0, ϕ_0). Evaluation of c_k^m and d_k^m then, awaits only evaluation of the surface harmonics at (θ_0, ϕ_0), which is rapidly accomplished through Algorithm I and equations (7). Equation (34) shows that for $r < R$, nth order internal harmonics translate to a superposition of external harmonics of all orders.

5. A canonical representation of D_n

It is relevant to mention that the use of a factorized form of D_n may result in increasing the speed with which some of the techniques of this paper may be applied.

Sylvester (1909) showed that when acting on harmonic functions, D_n can be written in the canonical form:

$$D_n = (-)^n M_n(\mathbf{u}_1 . \nabla)(\mathbf{u}_2 . \nabla) \dots (\mathbf{u}_n . \nabla). \tag{36}$$

M_n is called the multipole strength and the n unit vectors $\mathbf{u}_i$ ($i = 1, 2, \dots, n$) are called the multipole axes (of D_n, Y_n or V_n). This factorization of D_n is unique except for the sense of the vectors and the sign of M_n. The simplification which the multipole representation (36) of D_n allows is apparent from comparison with equations (11) and (8). The time needed to apply D_n to any harmonic, through Algorithm I or II, would be reduced by a factor of $2n+1$. For problems where D_n is only used once, this saving is counteracted by the time required to factorize D_n. When D_n is to be used at least several times, considerable advantage results from evaluating the multipole parameters M_n and $\mathbf{u}_i$ ($i = 1, 2, \dots, n$) first. It is useful, for example, to have available the multipole parameters of the geomagnetic field; however, until recently, a satisfactory technique for evaluating these parameters from the harmonic coefficients had not been published. Algorithm I may be used to overcome the non-linearity problems associated with a reduction to canonical form and in this way the geomagnetic multipoles up to the eighth order have been found (James 1968) and used with equation (18) to effect a rotation from geographic to geomagnetic co-ordinates.

6. Some remarks on the eccentric dipole

The simplest multipole is the dipole, represented by $D_1 = -M(\mathbf{u}.\nabla)$ where $M\mathbf{u}$ is the dipole moment; an eccentric dipole is one displaced off centre, at the point $\mathbf{R}$ say. Attempts are often made to choose M, $\mathbf{u}$, $\mathbf{R}$ (i.e. six parameters) so that the eccentric dipole potential in the region $r > R$ will reproduce, in a least-square sense, the eight leading terms in the harmonic expansion of the internal part of the geomagnetic field. To find the eccentric dipole potential various approaches have been used (Schmidt 1934; Elsasser 1941; Hurwitz 1960; Kalinin 1963), but it is clear that this problem is equivalent to the translation problem of Section 4.2 (ii). The relevant coefficients may be easily obtained from equation (32) (or from equation (35) if the region $r < R$ is to be considered).

According to Algorithms I and II, the coefficients in equations (32) and (35) with $n = 1$ will only contain Legendre functions like $P^p_{k\pm1}(\mu_0)$ where p is non-negative and taken from $m-1$, m and $m+1$. Since $P_n{}^p(1) = \delta_{p0}$, it follows that the eccentric dipole expansion is considerably simplified if $\overrightarrow{OO'}$ is taken as polar axis. The only surviving terms are those containing $P_n^m(\mu)$ where $m = 0$ or 1.

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APPENDIX I

The external analogue of equations (7)

Recalling equation (24) but with **R** replaced by **u** (as defined in equation (7)), we have according to equations (17):

$$(\mathbf{u}.\nabla)^n[r^n\, P_n^m(\mu)\, e^{im\phi}] = n!\, P_n^m(\mu_0)\, e^{im\phi_0},$$

or more generally, with Y_n defined by equation (2),

$$(\mathbf{u}.\nabla)^n[r^n\, Y_n(\theta, \phi)] = n!\, Y_n(\theta_0, \phi_0). \tag{A.1}$$

Formula (A.1) provides us with an elegant means of evaluating combinations of harmonics at the point (θ_0, ϕ_0). One merely applies Algorithm II n times, starting with the known coefficients a_n^m, b_n^m $(m = 0, 1, \ldots, n)$ of Y_n and taking $\mathbf{u} = (\sin\theta_0 \cos\phi_0,\ \sin\theta_0 \sin\phi_0,\ \cos\theta_0)$. According to (A.1) the resulting coefficient is $a_0{}^0 = (-)^n\, n!\, Y_n(\theta_0, \phi_0)$.

Thus two means of evaluating harmonics are available—Algorithm I and equations (7) or Algorithm II and equation (A.1). The former is preferable in general since n applications of Algorithm I produce all the individual surface harmonics up to the nth order—$n(n+2)$ in all—whereas use of (A.1) to find just the $2n+1$ nth order harmonics requires $n(2n+1)$ applications of Algorithm II. Access to the individual harmonics allows any combination to be easily calculated and these individual values may be recorded for future reference.

One of the tests applied to Algorithms I and II when checking for amplification of rounding-off errors was the evaluation of a sum of 20th order surface harmonics. With calculations holding nine significant figures the difference between the two available methods was 1 in 10^7.

APPENDIX II

Proof of equations (13)

Let $t_m{}^j$ and $s_m{}^j$ represent the vectors $\mathbf{t}_m{}^j$ and $\mathbf{s}_m{}^j$ in the complex plane. Consistent with the definition of $\mathbf{t}_m{}^j$ we have

$$t_m{}^{-j} = (t_m{}^j)^{-1};$$

hence

$$\prod_j t_m{}^j = 1. \tag{A.2}$$

The distribution of the $t_m{}^j$ in the plane is such that the m vectors $(t_m{}^j)^2$ are the mth roots of $(-1)^{m+1}$ (i.e. 1 when m is odd and -1 when m is even).
It follows that

$$\prod_j [a(t_m{}^j)^2+b] \equiv a^m+b^m \tag{A.3}$$

for all a and b. Thus, from (A.2) and (A.3),

$$\prod_j (\mathbf{t}_m{}^j.\nabla) = \prod_j \left(t_m{}^j \frac{\partial}{\partial\xi} + t_m{}^{-j}\frac{\partial}{\partial\eta}\right),$$

$$= \left(\frac{\partial}{\partial\xi}\right)^m + \left(\frac{\partial}{\partial\eta}\right)^m. \tag{A.4}$$

Similarly, consistent with the definition of $\mathbf{s}_m{}^j$ $(m \neq 0)$, $s_m{}^j = e^{i\pi/2m}\, t_m{}^j$ and so, from equation (A.4),

$$\prod_j (\mathbf{s}_m{}^j.\nabla) = i\left[\left(\frac{\partial}{\partial\xi}\right)^m - \left(\frac{\partial}{\partial\eta}\right)^m\right]. \tag{A.5}$$

Equations (13) follow from (A.4) and (A.5).

Proof of equations (9)

Assume equations (9) to be true for a particular value of n. By taking appropriate combinations of the equations constituting equation (14) we can show

$$\left.\begin{aligned} &\left.\begin{aligned} T_{n+1}^m &= (\partial/\partial z)\, T_n^m, \\ S_{n+1}^m &= (\partial/\partial z)\, S_n^m, \end{aligned}\right\} \text{ for } 0 \leqslant m \leqslant n, \\ &T_{n+1}^{n+1} = -(\partial/\partial x)\, T_n{}^n + (\partial/\partial y)\, S_n{}^n, \\ &S_{n+1}^{n+1} = -(\partial/\partial x)\, S_n{}^n - (\partial/\partial y)\, T_n{}^n. \end{aligned}\right\} \tag{A.6}$$

Applying the operators in equation (A.6) to $1/r$ and using Algorithm I shows that equations (9) are also valid for n replaced by $n+1$. The case $n = 1$ is easily verified and the general form of equations (9) follows by induction.

Proof of equations (16)

Since $\nabla = \frac{1}{2}(-T_1{}^1,\ -S_1{}^1,\ T_1{}^0)$, equations (16) with $n = 1$ and $m = 0$, 1 are equivalent to Algorithm II, and are therefore true for all values of k and l (with $0 \leqslant l \leqslant k$). The general form of equations (16) follows by induction, using equation (A.6) and Algorithm II.

For an alternative derivation, find the coefficients c_k^m and d_k^m in equation (35) explicitly by using equations (14), and substitute them into equation (34). Equations (16) follow after some manipulation and a comparison with equation (33).

A third method of proving equations (16) has been given by Hobson (1955) who found one of the four real equations corresponding to equations (16).