TRANSFORMATIONAL TECHNIQUES IN ATONAL AND OTHER MUSIC THEORIES

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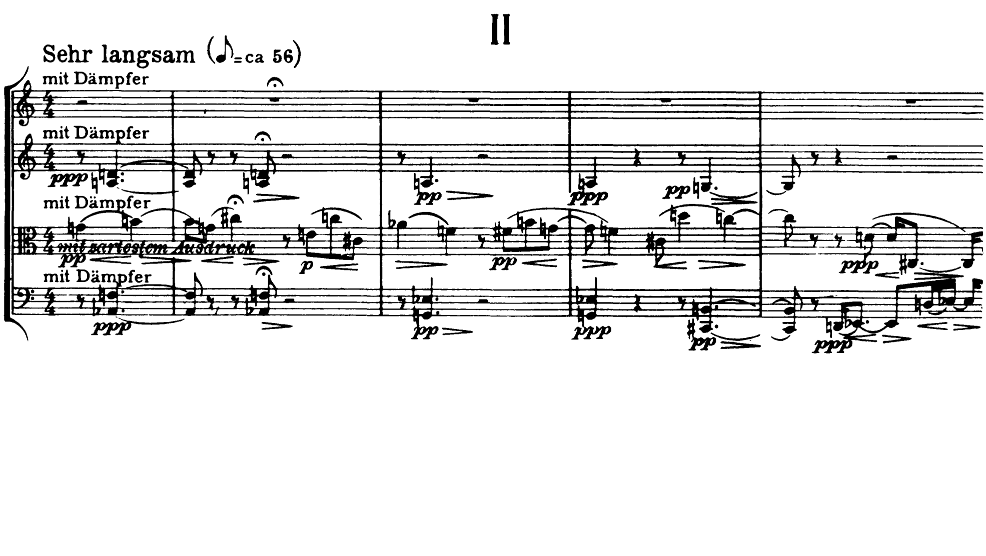
Let us consider the opening of Anton Webern's *Piece for String Quartet,* opus 5 number 2, concentrating on the roles that the pitch class set X=(G,B,C♯) and various of its forms play in this context. Figure 1 reproduces the music at issue.

The passage opens with a melodic representation of X as a *Kopfmotiv;* measure 3 cadences with a chordal representation of X in the accompaniment. The accompaniment chord of measure 2, repeated in measure 3, also projects a form of X, specifically T_8(X). The first chord in the accompaniment does not constitute, nor even include, any form of X. Rather it projects strongly the structuring force of pitch-class inversion about D.[1] Under this inversion the pitch-class D mirrors itself, the dyad (A,F) of the chord mirrors the (G,B) of the viola melody, and the A♭ as a pitch-class mirrors itself. We shall call the inversion "I", writing I(D)=D, I((A,F))=(G,B), I((G,B))=(A,F), and I(A♭)=A♭.

In measure 1 the dyad (G,B) becomes extended to X=(G,B,C♯) and the melody lingers on the high point C♯. The questioning effect of this gesture is interrelated with a search in the music for E♭=I(C♯) and (A,F,E♭)=I(X), a search which is to some extent resolved by the chord of measure 2, where the C♯ and the X trichord that embeds it find their I-partners. The 'cello E♭ of measure 2 mirrors the high C♯,

Figure 1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and that E♭ is embedded in a representation of I(X) which controls aspects of the voice-leading within the accompaniment: (A,F) within the first chord becomes (A,E♭) within the second. Figure 2 diagrams the inversional structure under discussion.

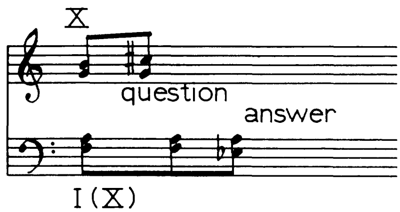
The pairing of E♭ with C♯, as an I-related partnership about D, is elaborated with even greater *Entspannung* in the extension of the phrase through measure 4. The interrelation of I(X), an accompaniment formation, with events of the viola melody, as displayed by Figure 2, is confirmed by the "half-cadence" of the viola on the F in measure 2, while the accompaniment (A,E♭) sounds beneath. This binds all of I(X) into a simultaneity, perhaps preparing the complementary appearance of all-of-X as a simultaneity in measure 3.

By "complementary", above, I mean "I-related". Still, it is significant that I(X) is also the pitch-class complement of X within the whole-tone scale that embeds them both. All the forms of X that we shall discuss over this passage are in fact subsets of that one whole-tone scale, and various pairs of X-forms combine to articulate various four-note and five-note subsets of that scale. The terminology of Allen Forte is well suited to examining the patterns of abstract inclusion-relationships which involve these and other pitch-class sets over the piece.[2] For the nonce, though, we are focusing on the ways in which X itself is transformed, rather than such patterns of inclusion.

At the four-note sonority halfway through measure 2, the disposition of pitches in register supports the establishment of a new inversional symmetry which we shall denote by "J", that is inversion about F♯. The cited four-note sonority is J-symmetrical in the registration of its pitch classes. J also relates the two X-forms discussed so far which are embedded in the sonority. That is, $T_8(X)$, represented by the three lowest notes of the sonority, and I(X), represented by the three highest notes of the sonority, are J-transforms each of the other. Algebraically, $T_8(X)=J(I(X))$ and $I(X)=J(T_8(X))$.

The structuring force of J continues into the viola melody over measures 2½-3½. The pentachord (F♯,B,G,F,C♯) is its own J-inversion,

Figure 2



a feature which is strongly projected by the registral disposition of the pitch-classes involved. Within this melodic fragment two X-forms are projected as serial segments; these are the inverted form K(X)=(B,G,F) and the transposed form T_6(X)=(G,F,C♯). These two forms are J-associates each of the other, as were I(X) and T_8(X) earlier. That is, T_6(X)=J(K(X)) and K(X)=J(T_6(X)). K is that inversion which transforms the pitch classes G and B each into the other. The compositional use of this transformation was suggested by the melodic G-B-G of the viola at the beginning of the piece, an event recalled by the melodic B-G at the end of measure 2, where K(X) appears. T_6, as well as K, leaves invariant two members of X: specifically, K(X) and X share the pitch-classes G and B; T_6(X) and X share the pitch-classes G and C♯. Later on, we shall explore more systematically the relevance of such common-tone relationships, in another context.

Other X-forms can also be extracted from the melodic pentachord (F♯,B,G,F,C♯), but I shall consider such forms as secondary here because they are not as well articulated, either serially or registrally, or by any other criterion, as are K(X) and T_6(X). Still, it is interesting to note that these secondary forms are X itself, (B,G,C♯), and its J-associate J(X)=(B,F,C♯). This relation makes one aware that J(X) too, as well as K(X) and T_6(X), has a common dyad with X, specifically the third dyad (B,C♯) of X.

As the viola moves through its F in measure 3, the accompanying T_8(X) form in the 'cello and second violin enters once more into the relations with that F which we discussed earlier: T_8(X)=(G,E♭,A) is heard in the lowest three notes against I(X)=(E♭,A,F) in the highest three; T_8(X)=J(I(X)) and I(X)=J(T_8(X)) are J-associates, each of the other. When now the viola moves on to C♯, on the fourth eighth of measure 3, a new form of X is thereby implicitly paired with or against T_8(X). T_8(X)=(G,E♭,A) remains implicitly below, but the highest three notes now implicitly form (E♭,A,C♯)=T_2(X). T_8(X)=T_6(T_2(X)) and T_2(X)=T_6(T_8(X)) are T_6-associates, each of the other.

We have now devoted special consideration to exactly three properly transposed forms of X in this passage, namely T_8(X), T_6(X) and T_2(X). The three intervals of transposition, that is 8, 6 and 2, represent the three-interval classes of X itself. For us, however, the intervals of transposition will mean more than interval-classes (types of two-note chords) alone: "transpose by 8" e.g., is not the same operation as "transpose by 4." So we shall think, e.g., not "the dyad (G,B) instances interval-class 4 (a chord of type 2-4)," but rather "T_8 transforms the pitch-class B into the pitch-class G." In the same sense, T_6 transforms G into C♯, and T_2 transforms B into C♯. As these transformations operate on various individual pitch classes (or one-note pcsets) of the viola *Kopfmotiv* itself, we hear *first* among them the effect of T_8, taking B to G, *second* the effect of T_6, taking G to C♯, and *third* the effect of T_2 on a larger level, taking B to C♯. This order corresponds suggestively to the order in which we have heard the correspondingly transposed forms of X during the passage: *first* $T_8(X)$, *second* $T_6(X)$, and *third* $T_2(X)$. We are also aware, in the "second" place above, of a T_6 relation between $T_8(X)$ and $T_2(X)$. Note how the transformational setting engages time: things happen before and after; the process is of necessity rhythmic, and the transformational networks involving T_8, T_6 and T_2 function rhythmically on both smaller and larger levels. Figure 3 provides a format which I find suggestive in portraying these ideas.

The implicit four-note chord (G,E♭,A,C♯) in measure 3 embeds forms of X other than $T_8(X)$ and $T_2(X)$, but those forms will be considered secondary in this context. The situation recalls the earlier analysis of X-forms within the viola tetrachord (B,G,F,C♯), and it is fruitful to contemplate other relations involving the two tetrachords. The earlier tetrachord was embedded in the melodic pentachord (F♯,B,G,F,C♯). That pentachord, as we noted, was J-symmetrical, having F♯ as a center of inversion. The tetrachord (G,E♭,A,C♯) implicit within measure 3 is followed directly by the climactic high D of the viola, extending the tetrachord to the unordered pentachord (D,G,E♭,C♯,A). This pentachord is I-symmetrical, having D as a center of inversion. In this respect it harks back to the inversional symmetry characteristic of the opening of the piece.

In this context we can analyze the pentachord (D,G,E♭,C♯,A) as an 8-transpose (rather than an inversion) of the pentachord (F♯,B,G, F,C♯), because the pattern of inversional symmetries suggests a T_8

Figure 3

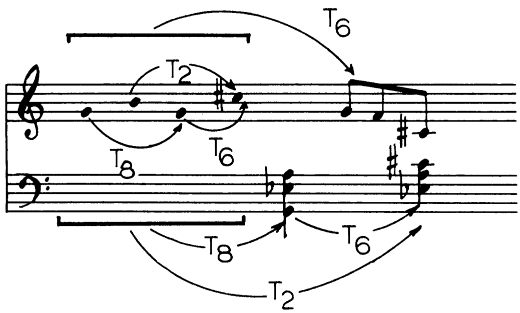


Figure 4

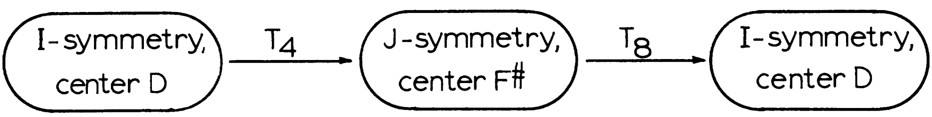
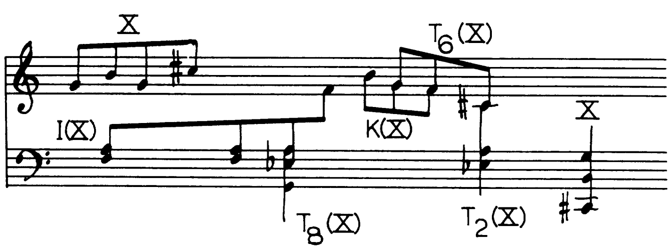


Figure 5



relation: F♯ as center of inversion for J returns to D as center of inversion for I, a center whose power persists through the middle of measure 4. This T_8 gesture retrogrades the initial transition from I-symmetry to J-symmetry; originally the centers of inversion progressed via T_4 from D to F♯. The large composite gesture can be graphed as in Figure 4.

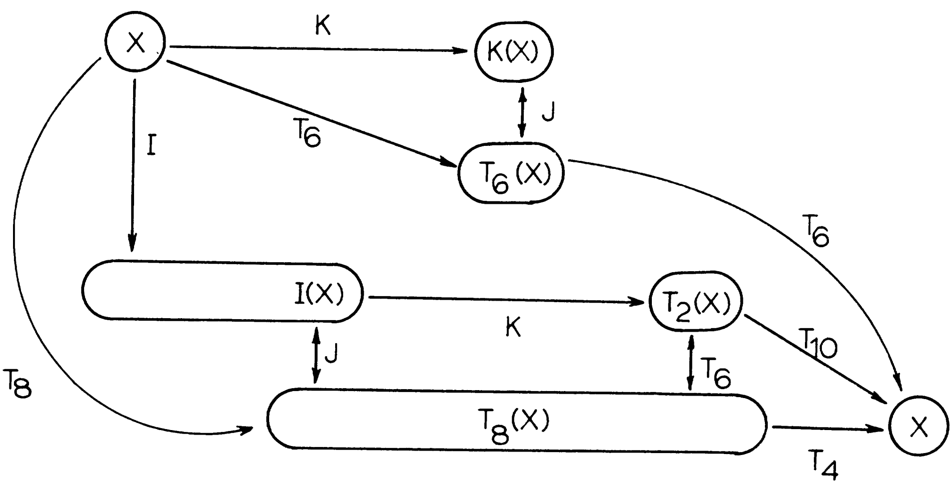
In Figure 4 one recognizes a rhythmic expansion of the opening "transformational gesture" in the pitch-class structure of the *Kopfmotiv:* G goes to B via T_4, and then returns again, via T_8, to G.

Figure 5 collates the various forms of X we have so far discussed, in a convenient format. The figure omits forms analyzed as "secondary."

Figure 6 abstracts a transformational network from Figure 5. The arrows connecting forms of X indicate the transformations one applies, to get from one form of X to another. So, for example, the J-arrow leading from I(X) to T_8(X) on Figure 6 indicates that one "gets" from the pitch-class set I(X) to the pitch-class set T_8(X) by applying the transformation J: the latter pcset is the J-transform of the former, i.e., $T_8(X)=J(I(X))$.

To make as few analytic assertions as possible, I have included in Figure 6 only arrows that reflect my own hearing of a strong compositional connection in time, register, etc. Most of the arrows thus reflect earlier discussion. The arrow from X to T_2(X), visible earlier in Figure 3, is omitted in Figure 6: I feel that this arrow reflects a structural idea on Figure 3, rather than an immediately audible compositional connection. On Figure 6, there is a new K-arrow, from I(X) to $T_2(X)=K(I(X))$. K, it will be recalled, is the inversion operation that exchanges G with B; K thus leaves A invariant, leaves E♭ invariant, and exchanges F for C♯. The idea of leaving A and E♭ invariant, while substituting C♯ for F, was covered

Figure 6



in the earlier text, in connection with the progression of the viola melody in measure 3 from F to C♯, while the chord in the accompaniment implicitly remained.

Also new on Figure 6 are the three arrows I have drawn aiming at the cadential X on the right of the Figure. The arrow from $T_6(X)$ to X reflects my hearing the melodic G-F-C♯ of the viola in measure 3 move to the following chord (C♯,B,G) in the accompaniment: the dyad (G,C♯) is preserved; the linear order G-F-C♯, presenting F between G and C♯, is transformed into the registral order C♯-B-G, presenting B between C♯ and G. The sense of a local T_6 function is thus clarified. The arrow on Figure 6 from $T_8(X)$ to X connects the second and third accompaniment chords (a serial relation) in the low register (a registral relation). Traditional listening will emphasize with some justice the thematic and structural tritone G-to-C♯ in the bass line of this progression, somewhat at the expense of the T_4 arrow-relation. T_4 can be "heard" nonetheless by focusing the ear upon the various intervals of 4 that occur between notes of (G,E♭,A) and notes of (C♯,B,G) when the former chord moves to the latter. One can hear the G of the first chord move to the B of the second; this motion is as thematic as the tritone G-C♯, and the motion G-B occurs in close registral position. One can also hear the "voice-leading" possibility of E♭-to-G; this gesture also appears in close registral position, though with a color change. The A of the first chord at issue is far from the C♯ of the second, but one can hear the T_4 relation nonetheless, with some concentration, via the low C♯ of the viola in measure 3, which is in close registral position to that A. (The pitch class C♯ has been pre-eminently mobile in register through the entire passage.) Having heard so many intervals of 4 spanned in this way, one hears the progression as "highly T_4-ish" even if one does not work out aurally the arithmetic which shows that three intervals of 4 spanned between two three-note chords logically entail a T_4 relation between the chords. The psychology just discussed will later on be reflected by an appropriate theoretical apparatus.

It is interesting to note that the "actual" voice-leading between the two chords just discussed projects a T_{10} relation ($G=T_{10}(A)$ in the violin), a T_8 relation ($B=T_8(E♭)$ in the 'cello), and a T_6 relation ($C♯=T_6(G)$ in the 'cello). All three interval-classes of X are thus represented by the "actual" voice-leading here, as they are also by the three rightmost arrows on Figure 6; in both cases we are considering methods of "approach" to the cadential X sonority. These considerations might be of interest to a 'cellist deciding whether to play the (G,E♭) on strings IV and III, or on strings III and II, and whether to do so the same for both appearances of the double-stop, or to do so differently at its two appearances.

The T_{10} arrow leading to the final X on Figure 6 is not very audible to my ear in connection with X-forms alone. However, as we shall soon see, there is a convincing link between the T_{10}-related X-forms here and the cadential T_{10} gesture in the viola melody lying over: $C=T_{10}(D)$. To explore that link let us first return to the high C♯ of measure 1. We discussed in connection with Figure 2 how that C♯ is answered by the E♭ of measure 2; we can also hear how the C♯, during the course of the passage, moves down to middle C♯ and eventually low C♯. Neither of these observations, though, comes to grips with our common-sense hearing of a connection in register between the high C♯ of measure 1 and the subsequent high C of that measure. The high C picks up in register not just the C♯ but also the preceding B, thereby projecting the three-note cluster (B,C,C♯) in register. And the three-note cluster is clearly an important constructive unit of this piece. One hears it completely governing the middle of measure 4; one also hears it at the melodic cadence C♯-D-C of the viola line, within the pentachord (F♯,B,G,F,C♯), etc.

Since the registral connection of the C♯ to C in measure 1 is so strong, one is tempted at first to assert a transformational relation $C=T_{11}(C\sharp)$; the three-note cluster could be derived from such a T_{11} idea. However T_{11} clearly does not function over the passage in anything like the ways our other transformations do.[3] In the context of this opening section alone, the C-embedding function of the three-note cluster is much clearer, as a constructive idea. But it seems awkward to analyze this cluster as a harmonic element independent of X. Forte's theory suggests that we listen for a pitch-class set that embeds both (a form of) X and (a form of) the three-note cluster. Our pentachord will do, but it is needlessly large for the purpose; the most economical candidate is Y=(G,B,C,C♯). And, once having tuned our ears to Y, we can hear that every form of X portrayed in

Figure 5 is in fact embedded in an appropriate form of Y. Figure 7 diagrams the expansion.

The figure shows how the move down in register from C♯ to C within Y, in the viola over measure 1, is mirrored by the I-associated move up in register from E to F within I(Y), a move which is spanned by the first and last notes of the melodic sub-phrase in measures 1-2. The accompaniment chord $T_8(X)$ of measure 2 sounds beneath the A♭ of the viola; the two elements combine to form $T_8(Y)$. K(X) and $T_6(X)$ extend respectively to K(Y) and $T_6(Y)$ by adjoining the F♯ that combines with them to form the J-symmetrical pentachord. The implicit $T_2(X)=(E♭,A,C♯)$ in measure 3 combines with the immediately following high D to form $T_2(Y)$. And finally, the C that originally extended melodic X to melodic Y reappears as the melodic cadence tone in measure 3, where it sounds simultaneously with harmonic X to form harmonic Y. The web of Figure 7 catches every pitch-class event of measures 0-3, except for the I-centers D and A♭ within the first chord.

Because Figure 7 expands Figure 5 perfectly, the network of Figure 6, which applied to Figure 5, will also apply perfectly to Figure 7 if we simply replace the symbol "X" on Figure 6 by the symbol "Y" passim. When we make that substitution, we will be reassured about the T_{10} arrow that leads to the final Y on the graph: the asserted T_{10} relation in fact leads the high D of the viola in measure 3, part of $T_2(Y)$, to the subsequent high C, the corresponding part of Y. The relation $C=T_{10}(D)$ is, of course, amply audible here.

We have seen that the network of Figure 6 is valid in our analysis for the forms of X and for the forms of Y indifferently. An analogous network would also be valid if we substituted, for X-forms or Y-forms, the corresponding forms of the three-note cluster. The latter network, one notes, would be analytically "valid" but not "complete", for it would not catch, e.g., the C♯-D-C of the viola melody in measure 3, which is not embedded in a Y-form of Figure 7. The point is that, leaving questions of analytic pertinence aside, the conceptual structure of Figure 6, so far as it describes a network of relationships

Figure 7

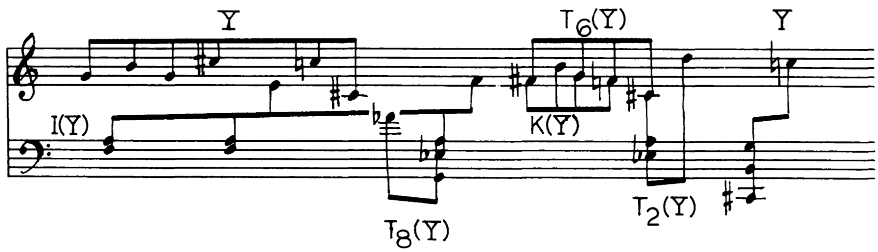
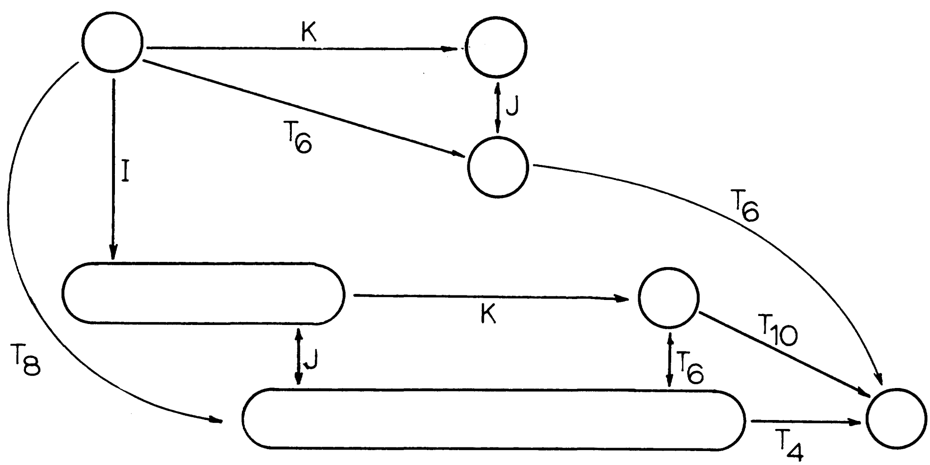


Figure 8



among transformations, is completely independent of the objects being transformed. The purely transformational aspect of Figure 6 remains formally determinate if we ignore completely the contents of its circled nodes, and graph only the transformational "gestures" as in Figure 8.

If we input any suitable thing as an operand into the upper left-hand node of the Figure and follow the transformational arrows, the various other nodes will automatically be filled by appropriate forms of that thing: I(thing), T_8(thing)=J(I(thing)), K(thing), T_6(thing)= J(K(thing)), etc. One will automatically return to the original form of the thing at the lower right-hand corner of the graph. In this connection, it is crucial that the inversional transformations I, J and K be understood as meaning "inversion about D", "inversion about F♯" and "inversion about A" respectively; they must not be defined by any internal property of any single "thing" that might chance to get sent through the mill. Then the transformational relations appropriate to the graph will always obtain regardless of the operand being transformed: the J-associate of the K-associate will necessarily be its 8-transpose; etc. It is of course also understood that the operations of transposition and inversion will be applicable in the familiar pitch-class sense to the operand and its various (transposed and inverted) forms. Under these conditions, the graph of Figure 8 is well-formed no matter whether the operand to be input is X, Y, (B,C,C♯), the pitch-class C♯, a serial motif, a 12-tone row, etc.

At this point, a little formalism will be helpful. Figure 8 instances a generic form which we shall call a "transformation-graph". A transformation-graph, in general, comprises a family of "nodes", together with an "arrow-relationship" obtaining among certain ordered pairs of nodes. For each ordered pair of nodes in the arrow-relationship, a certain transformation is specified. To be well-formed, such a graph must observe certain restrictions. For instance, if nodes 1-and-2, nodes 2-and-3, and nodes 1-and-3 are all in the arrow relationship, then the transformation associated with 1-to-2, followed by the transformation associated with 2-to-3, must produce the transformation associated with 1-to-3. (Thus, in Figure 8, the J-associate of the I-associate of any suitable thing will be the 8-transpose of that thing.) More general restrictions of this sort must be observed; the formalities will be covered in an appendix.

The graph of Figure 8 is "connected": given any two nodes, there is a path connecting them if we follow a suitable sequence of arrows forwards and/or (if necessary) backwards. This property of the graph enables us to infer the operand-content of all nodes, once we know the content of any one node. In this connection, we are implicitly invoking the fact that the family of operations at issue forms a mathematical "group". The appendix will make the matter formal for those who wish to pursue it.

Figure 8 also has a certain "chronological" structure: intuitively, some arrows go from left to right, others go up, others go down, but none goes from right to left. We thus have some sense of what happens "earlier" and what "later" on the graph that is consistent with our naive sense of time.[4] The appendix will also formalize the notion of "chronological" here, a rather tricky matter. Note that the chronological structure of Figure 8 has necessitated the extension of certain nodes from circles to ovals in the visual picture; this aspect of the structure, even leaving the contents of the nodes aside, embeds the graph in an implicit context of rhythm and relative duration, somewhat capturing certain rhythmic implications of Figures 5 and 7 in an abstract graphic structure.[5]

Figure 8, as a particular transformation-graph, can be given a name. Let us call it BIGSCHEME. We can give other names to other graphs; let us define by Figure 9, for instance, another graph which we shall call FIRST4THEN8.

This graph, as well as BIGSCHEME, was pertinent to our analysis of the Webern passage. If we input into FIRST4THEN8 the operand "I-symmetry, center D", we will obtain the network displayed earlier in Figure 4, which showed the progressive shifting of inversional

Figure 9

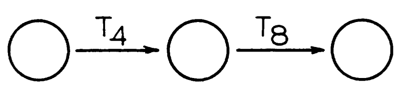
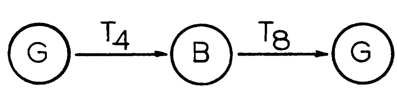


Figure 10



symmetries and centers over the passage. If, on the other hand, we input the single pitch-class G into FIRST4THEN8, we obtain the network of pitch-classes displayed in Figure 10, which models the opening melodic gesture of the viola.

The relation of Figure 4 to Figure 9, Figure 10 to Figure 9, and Figure 6 to Figure 8 can be generalized by the notion of a "transformation-network". This construct comprises a transformation-graph, a certain family of operands for the transformations, and a function CTN associating each node with the particular operand it is to "contain". The statement "$T_6(X)=CTN(\text{node } 5)$" can thus be read as stating "the operand $T_6(X)$ is the contents of node 5." A transformation-network is formally restricted by stipulating that if node 1 and node 2 are in the arrow relation, and if the transformation F is associated with that arrow, then F(operand 1)=operand 2, where operand 1 and operand 2 are the respective contents of nodes 1 and 2. The appendix will take all this up again, yet more formally.

Because of the various restrictions we have built into these constructs, it is possible to generate an entire connected transformation-network by stipulating (1) its graph, (2) any specific node, and (3) a specific operand which that node is to contain. We can then follow arrows forward and/or backward on the graph, filling in the appropriate operand-forms for the contents of the various nodes according to the indicated chains of transformations. In this way, for instance, the network of Figure 10 can be generated from the graph FIRST4-THEN8 of Figure 9, by stipulating that the left-hand node of that graph is to contain the pitch class G. If we call the left-hand node "lhn", we can then refer to the network of Figure 10 as (FIRST4-THEN8,lhn,G). In a similar spirit we can call the network of Figure 4 (FIRST4THEN8,lhn,(I-symmetry,center D)). If we denote by "uln" the upper left-hand node on Figure 8, we can call the network of Figure 6 (BIGSCHEME,uln,X). If we want to refer to the analogous network which expands each form of X to the corresponding form of Y, modeling Figure 7, we can call that network (BIGSCHEME,uln,Y).

By using the formalism of transformation-graphs and networks, we can assert a very specific formal relationship between the G-B-G of the viola melody and the progression of I-and-J-symmetries over the entire cited passage: each aspect of the music instances a transformation network whose graph is FIRST4THEN8. We shall call the two networks "isographic", to reflect that relationship.

The networks (BIGSCHEME,uln,X) and (BIGSCHEME,uln,Y) are also isographic. The relation between the compositional representations of the two networks is quite different, however, from the relation between the G-B-G network and the inversional symmetry network. Specifically, each X-form that represents the contents of a node of (BIGSCHEME,uln,X) is musically embedded, node by node, in the compositional representation of the second. This enables us to assert a very much stronger relation of X to Y in this piece than the mere embedding of X itself in Y itself, either abstractly or in the music of measures 0-1. And the transformation-graphs and networks are essential to a precise formulation of that strong relationship.

Such graphs and networks are well-suited to illustrating points of contact between atonal theory and transformational theories of tonal functions. To make this clear, we need only conceive "dominant of", "parallel of", etc. as transformations which, when applied to a given chord/root/key, produce another chord/root/key. The transformational outlook introduces an attractive kinetic component into theories which suffer from a static character when "dominant" et al. are conceived merely as labels for particular chords/roots/keys in particular contexts.[6]

To illustrate the point, let us define a tonal transformation S^{-1}(ess-inverse): the statement "$S^{-1}(C)=G$" asserts that $C=S(G)$, i.e. C is the subdominant of G. We shall use S^{-1}, rather than S, because in the work at hand we shall want to associate S^{-1}, not S, with forward-pointing arrows on chronological transformation-graphs, asserting thereby that "C becomes subdominant of G (via S^{-1}) rather than "G progresses to its subdominant (via S)." In like spirit, D^{-1} will be the inverse dominant transformation. A D^{-1} arrow from C to F on a network will mean "C becomes the dominant of F." Better yet, and more in the spirit of Riemann, we can think and speak of "functions" rather than chords, roots, and/or keys; we can make statements like "the harmony of measure 1 functions as the subdominant of the harmony in measure 2." In this spirit, I have constructed the

Figure 11

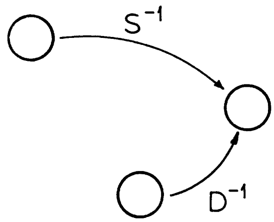
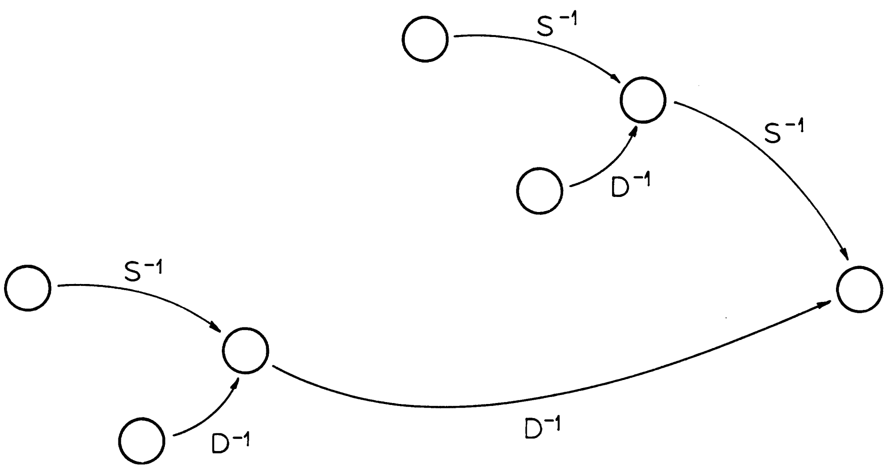


Figure 12



transformation-graphs of Figures 11-12, and have attempted to demonstrate how these graphs apply to the networks suggested by Figures 13 and 14.

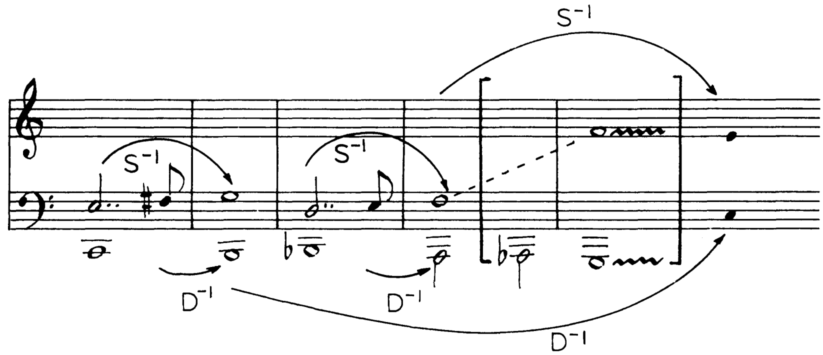
Note the way in which Figure 12 expands Figure 11. The former graph uses the latter to "diminute" a mirror image of itself. The abstract structure can be defined formally by our machinery, of course. We might say that the relation of the G^6 and F^6 functions to the final C function, in each of Figures 13 and 14, is "anti-isographic" to the progression that governs measures 1-2, and the progressions that govern measures 3-4, in each Figure.

One sees in Figures 13-14 the logic of using S^{-1} and D^{-1} rather than S and D, to associate with pertinent graphs. If one used S and D, then all the arrows on Figures 11 and 12 would lead right-to-left, rather than left-to-right. Alternatively, one could construct the retrograde graphs of Figures 11-12, using S and D arrows left-to-right, but those alternate graphs would analyze the music backwards. The implications for Riemann-like tonal theories are important, but this is not the place to develop them.

Riemannian theory would, I think, best analyze the dominant seventh harmonies, bracketed in Figures 13 and 14, as combined D-and-S functions: they prolong the dominant function of the G harmony from measure 2 of each Figure; they also prolong the subdominant function, since the F root of measure 4 in each Figure becomes the seventh of the G^7 harmony. From this point of view, it is logical that the G^7 harmony does not fill any node in the graph of Figure 12; it rather prolongs the functions of the earlier G^6 and F^8 harmonies, which do fill nodes.

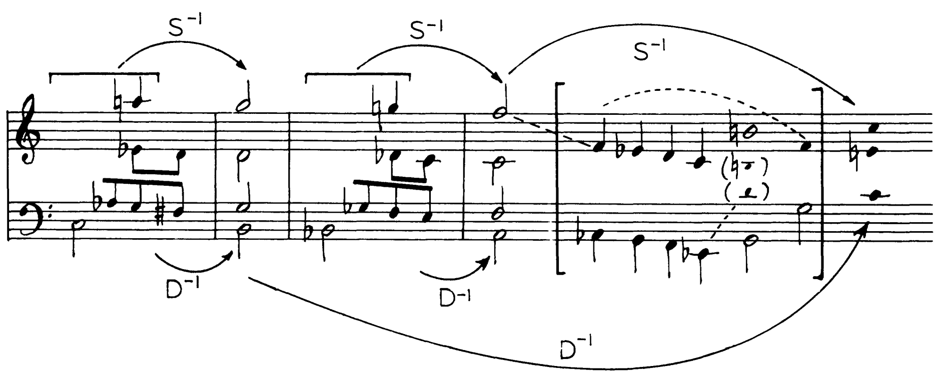
This analysis, while "logical", is of course still far from an adequate discussion of the musical passages; any such discussion would have to engage the structural voice-leading, and any voice-leading analysis will have to attribute more autonomy to the G^7 harmonies, whose bass notes are the goals of activity in the bass lines initiated at the very openings of the pieces. Voice-leading interpretations can, never-

Figure 13



After Beethoven,
Sonata Op. 53
opening

Figure 14



After Mozart,
Quartet K. 465,
opening

theless, be strongly influenced by the kinesis of Figures 12 through 14: the arrow patterns specifically support hearing Figure 15(a) as the voice-leading paradigm underlying both the Beethoven and the Mozart passages, rather than say Figure 15(b) or (c) for the Beethoven, or Figure 15(d) for the Mozart passage. Unfortunately, it would be out of place here to pursue farther the implications and problems of Figures 15(a)-(d). Very crucial matters of tempo, texture and register, e.g., become involved at once; these require a larger context within the music itself for adequate exploration.[7]

Despite the strong prompting of Figure 12 it is not necessary, I think, to go so far as Figure 15(a) in denying large tonic function to the opening sonorities of the piece at issue, particularly the bass notes which after all *are* "the opening sonorities" in a very literal sense. Still, I find Figure 15(a), in conjunction with Figures 12-13-14, a useful reading to contemplate because of the attention it focuses on the functional distinction between the opening harmonies and the tonic downbeats that occur at the ends of Figures 13-14, distinctions which are reflected by striking changes in texture and register in both cases (and also tempo and mode in the Mozart piece). In this connection Figure 12 provides a good model for the functional distinction: a tonic filling the left-hand node of a graph has a formal "input" function, while a tonic filling the right-hand node of the graph has a formal "output" function.

In sum, the mechanics of transformation-graphs and networks provide a useful conceptual connection between Riemann-like tonal theories and aspects of traditional atonal theory.[8] In these contexts, ideas of isography (anti-isography, isographic diminution, etc.) come naturally into play. The use of isographic techniques, while far from ubiquitous in the atonal literature, is quite common in Webern's practice, and not uncommon in other atonal music. Figure 16, for example, demonstrates Schoenberg's use of isographic networks in the song opus 15, number 11. As in earlier examples, the intervals of the *Kopfmotiv* are represented as transpositional relations between pitch-classes (or one-note pcsets); this formalism enables us to relate the intervals by which the *Motiv* is transposed to the internal intervallic

Figure 15

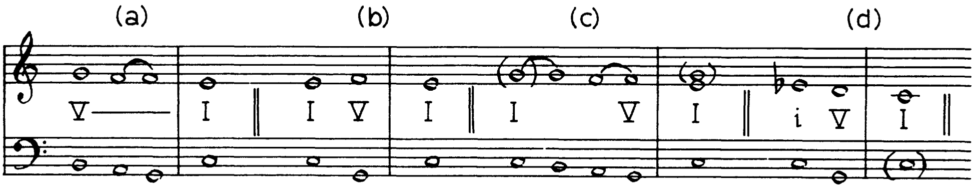
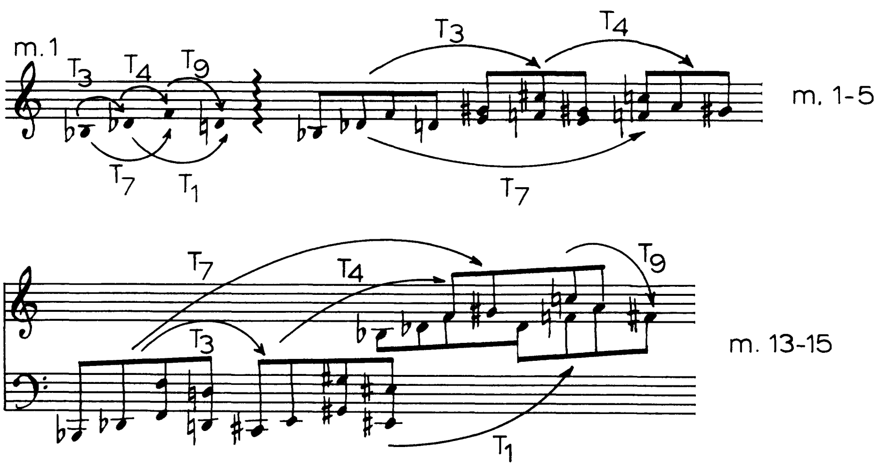


Figure 16



network within the *Motiv* itself. In this case, the procedure enables us to avoid attributing "roots" to the "major-minor triads" involved, thinking of the first "triad" as being "prolonged" harmonically, etc. (Some analysts may not wish to avoid such discourse; still, it is nice to be able to adopt it by choice, and not *faute de mieux.*)

Figure 17 illustrates a more subtle use of isography in Schoenberg's Piano Piece opus 19, number 6. Certain intervals of the opening chord, regarded as transpositional relations among its constituent pitch-classes, also govern the pattern through which the melodic motif of the "falling minor 9th" (or 16th) is developed through the piece. The pitch-class intervals of the opening chord at issue here, NB, are also represented by "falling" pitch-relationships, as indicated by the downward-pointing arrows on Figure 17. The compositional presentation of those structural arrows thus joins many other "falling" gestures of this piece which, according to Willi Reich, "is said to have been sketched out immediately after Schoenberg returned home from Mahler's funeral."[9]

The height of Schoenberg's involvement with such techniques is no doubt the Passacaglia, number 8 from *Pierrot Lunaire* opus 21. The graph associated with the succession E-G-E♭ of the *Kopfmotiv,* a graph we might call 3PLUS8EQUALS11, is ubiquitous over the piece, and generates a host of other graphs as well.[10] The piece is exceptional in the extent to which these techniques are manifest in the very forefront of the listening experience.

In fact, it is correct to view transformation-graphs and networks in general not as ubiquitous features of atonal music, but rather as paradigms that are only sometimes completely fulfilled in any given piece. Such paradigms are important for two reasons: first, they exemplify in a pure form the tendency of transformational gestures to exfoliate over phrases and complete pieces; second, they interrelate on the one hand transformations affecting pitch-class sets, centers of inversion, etc., and on the other hand transformations affecting individual pitch classes (or one-note pcsets). Networks of the latter transformations take over the role played in conventional atonal theories by such notions as "motive structure", "interval content", etc.

Let the reader recall in this connection my earlier discussion of the T_4 relation between the second and third chord-formations in the Webern passage: (C♯,B,G)=T_4(G,E♭,A). Despite the "actual" voice-leading, I claimed that the T_4 relation could be heard by focusing on the quantity of intervals 4 between notes of (G,E♭,A) and notes of (C♯,B,G): (C♯=T_4(A), B=T_4(G), and G=T_4(E♭). See Figure 18.

Two ways of reading Figure 18 are of interest here. The first reading says, "there are three ways of spanning the interval 4 between notes of the respective chords." This reading refers specifically to a directed pitch-class interval "4", and models the pertinent relation between the chords by invoking my "interval function" as a pertinent construct: INTF(chord 1, chord 2, 4)=3.[11] The second reading says, "if we apply the transformation T_4 to the first chord, it will then have three common tones with the second chord." This reading does not involve any specific reference to an "interval"; rather it uses the idea of transformation (T_4) and common-note relationship. An implicit question is posed: "If I apply T_4 to chord 1, how much like chord 2 will it be?" In this case, the answer is, "completely," as far as pitch-class content is concerned. Given the cardinalities of the chords, that answer reflects the formal equation CMNF(T_4, chord 1, chord 1)=3, an equation which uses Regener's "common-note function."[12] A virtue of the common-note model is that it can ask analogous questions about inversional transformations (and cycle-of-fifths transformations, etc.) as well as transpositions. In general, I can ask, "if I operate on chord 1 with the operation OP, how much will it be like chord 2, i.e. how many common tones will OP(chord 1) have with chord 2?" The answer is given by the number CMNF(OP, chord 1, chord 2).

To appreciate the suggestiveness of the transformational approach here, let us examine certain aspects of Schoenberg's Piano Piece opus 19, number 6, that grow out of relations between the two

Figure 17

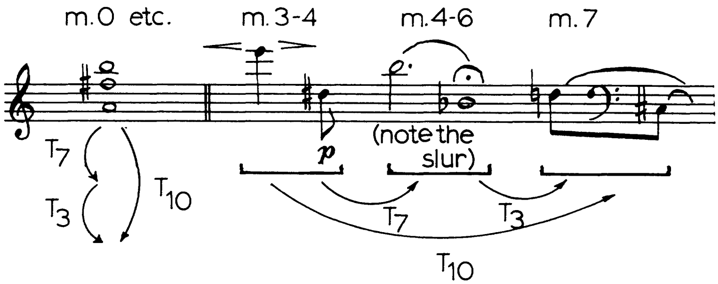
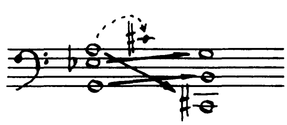


Figure 18



opening chords.[13] If we call the first chord "rh" and the second chord "lh", Figure 19 collates diagrams involving all transpositional and inversional operations OP that satisfy CMNF(OP, rh, lh)=2. (No such operation satisfies CMNF(OP,rh,lh)=3.)

The notes of rh and lh are written on the Figure using open note-heads; the meaning of the solid note-heads will be explained presently. Figure 19(a) illustrates the relation CMNF(T_6,rh,lh)=2. If we transpose rh by 6, then the F# of rh is transformed to the C of lh and the B of rh is transformed to the F of lh; the lines on Figure 19(a) depict those relations. The progression from rh to lh thus sounds "fairly T_6-ish." In order for it to sound "completely T_6-ish", one of two things would have to happen. Either the A of rh would have to be replaced by C#, or the G of lh would have to be replaced by D#. The pertinent C# for rh and D# for lh are portrayed by the solid note-heads on Figure 19(a).

Figure 19(b) illustrates in analogous fashion how the progression from rh to lh sound "fairly T_1-ish." For it to sound "completely T_1-ish", the A of rh must be replaced by E, or the F of lh by B♭. Figure 19(c) illustrates how the progression sounds "fairly I-ish", where I is inversion about A. For the progression to sound "completely I-ish", the A of RH must be replaced by C# or the F of lh by A. In the latter respect, note how Figure 19(c) resembles 19(a): in each case, the indicated substitution in rh is C#-for-A. In an exactly analogous respect, Figure 19(d) illustrates how the progression sounds "fairly J-ish", where J is inversion exchanging F# for F and B for C. Intuitively, all four of Figures 19(a)-(d) try to match the fourth (F#,B) of rh with one of the two fourths of lh, either by an appropriate transposition or by an appropriate inversion. The solid notes of rh in those figures try to extend (F#,B) to an appropriate form of lh in each case, that form being of necessity either (F#,B,E) or (C#,F#,B); this is why the solid notes of rh are all either E or C# in 19(a)-(d). The solid notes of lh, in 19(a)-(d), reflect various ways of selecting one of the two fourths of lh and adjoining a solid note to create a form of rh. Note how all this formalism handles both transpositions and inversions in absolutely analogous fashion.

Figure 19

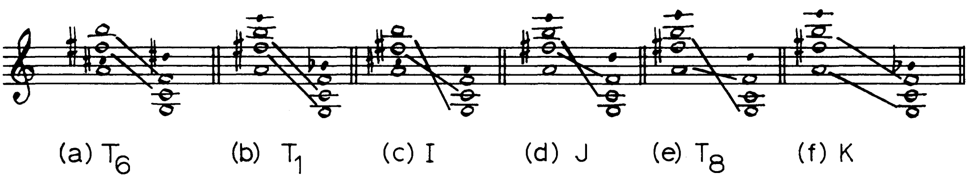
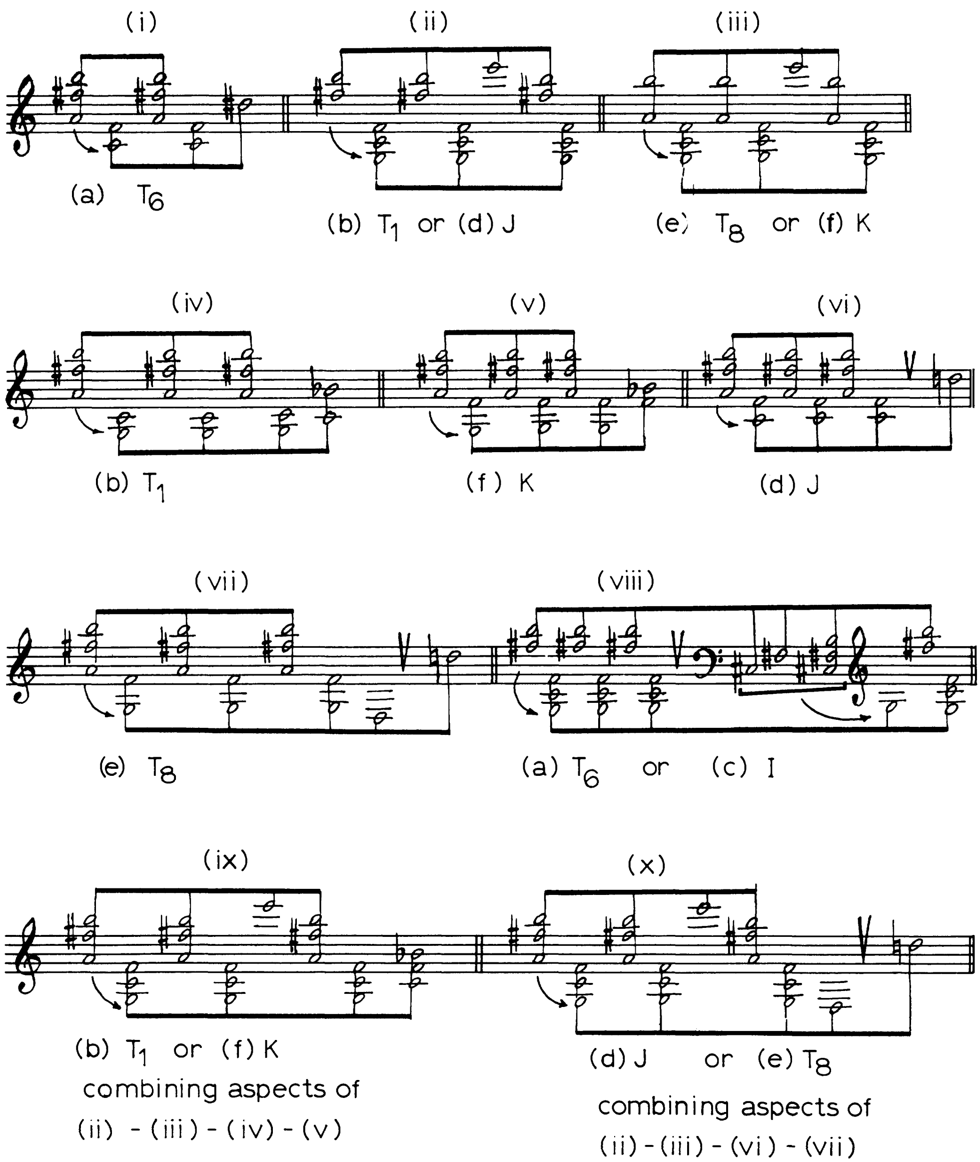


Figure 20


(i)
(ii)
(iii)
(a) T_6
(b) T_1 or (d) J
(e) T_8 or (f) K
(iv)
(v)
(vi)
(b) T_1
(f) K
(d) J
(vii)
(viii)
(e) T_8
(a) T_6 or (c) I
(ix)
(x)
(b) T_1 or (f) K
combining aspects of
(ii) - (iii) - (iv) - (v)
(d) J or (e) T_8
combining aspects of
(ii) - (iii) - (vi) - (vii)

Figures 19(e)-(f) match the dyad (A,B) of rh with the dyad (G,F) of lh, by transposition (19(e)) or inversion (19(f)). The operation K of 19(f) is the appropriate inversion operation, exchanging A with G and B with F. K thus exchanges F♯ with B♭ and E with C. E continues to appear as the solid note of rh in 19(e)-(f), since the only form of lh that includes (A,B) must perforce be (A,B,E). The solid notes of lh in Figures 19(e)-(f) are interesting since they reinforce as it were the lust of B♭ and D to be generated from the progression rh-to-lh, that lust which was already portrayed in 19(b) and 19(d).

In fact, one sees that 19(e) pairs off "naturally" with 19(d) in a certain formal respect here, as regards the solid notes involved. In this context, that is, we can meaningfully assert a formal similarity between the "fairly T_8-ish" quality of the progression rh-to-lh and the "fairly J-ish" quality of the progression: both qualities urge rh to generate an E, and/or lh to generate a D. (The virtue of the theoretical apparatus is of course that it enables us to translate these "qualities" and "urges" into demonstrable features of Figure 19.) Just as 19(e) pairs off with 19(d), so 19(f) pairs off with 19(b). That is, the "fairly K-ish" and "fairly T_1-ish" qualities of the progression both urge rh to generate an E, and/or lh to generate a B♭. All four of the above "qualities" urge rh to generate an E; 19(a) and 19(c), in distinction, urge rh to generate a C♯.

I find it suggestive to think of these generative lusts as musical tensions and/or potentialities which later events of the piece will resolve and/or realize to greater or lesser extents. The reader will recall, from the discussion on the Webern piece about Figure 2 earlier, that we spoke of the urge of the viola's high C♯, and the urge of the completed X-form, to find their inversional partners in that context. Figure 2 showed how those urges were satisfied, and the event sent into motion a train of events leading to the eventual formation of a complex transformation-network. Just so, Figure 20 attempts to show, in connection with the present Schoenberg piece, how the various lusts and urges reflected by the solid notes of Figure 19 eventually become satisfied during the subsequent music (q.v.), as the various concomitant transformations of Figure 19 jockey one with another for priority in potential network-formation.

Figure 20(i) shows how the urge depicted in Figure 19(a), for the A of rh to find its T_6-transform D♯, is satisfied by the D♯ that is in fact the next pitch-class event of the piece after the repeated statement of rh-lh. And the next event is the E depicted in Figure 20(ii) and (iii); as the Figures indicate, this event expands, prolongs and confirms the T_1-ish, J-ish, T_8-ish and K-ish qualities of the rh-lh progression according to the schemata of Figures 19(a),(d),(e) and (f) respectively. And so on, through Figure 20(viii). In the latter Figure we see the natural pairing-off of T_6 and I discussed earlier in connection with Figures 19(a) and (c).

Figure 20(ix) shows an analogous natural pairing-off of T_1 and K, as per Figures 19(b) and (f), when aspects of Figures 20(ii) through (v) are combined to indicate certain possibilities for a transformation-network involving 4-note chords. Figure 20(x) does the same for the natural pairing-off of J with T_8.

The combination of Figures 20(ii) through (v) into Figure 20(ix) analyzes the generation of the new chord (C,F,B♭) as the outcome of a complex system of potentialities and urges induced by the kinetic character of transformations characterizing a progression, namely rh-to-lh. It is also legitimate, indeed necessary, to point out that the new chord (C,F,B♭) can be generated more simply, just by transposing lh via one of its own most common internal intervals. The T_5 operation which transposes lh into (C,F,B♭) is not, however, a "kinetic" or "progressive" operation in the sense of Figures 19-20. That is, it does not arise from any aspect of the *progression* rh-to-lh. Rather, it arises from the prominence of 5 as a static vertical interval within lh itself. (The interval 5 also appears as a static verticality within rh.) This is not at all to argue that T_5 and T_7 are "unimportant", or even of subordinate importance to the progressive transformations of Figures 19-20. Indeed, T_5 and T_7 do a good deal of work in the music, especially in extending chains of fourths up and down.[14] What I wish to emphasize is that our theoretical format enables us to distinguish a formally different function for T_5 and T_7, than for T_6, T_1, T_8 etc. in this context. T_5 and T_7 we can call "internal", "spatial", "prolongational" etc. These transformations have to do with static, internal features of various events. The importance of T_5 has to do with the fact, e.g., that T_5(lh) has two common tones with lh itself; this does not involve any relationship in time between lh and something else that comes before or after. In contrast, the "kinetic", "dynamic" or "progressive" transformations T_6, T_1, T_8, I, J and K assume importance because of the relative similarities between one thing (rh), when suitably transformed, and a *subsequent* and *different* thing (lh) to which it is moving.

This distinction between "spatial" or "internal" and "dynamic" or "progressive" transformations is methodologically attractive. It is also strikingly modeled and reflected by certain algebraic relations among the abstract transformations themselves. To pursue this matter, let us first isolate all the strongest "internal" transformations associated with rh and/or lh in our present example. Two such transformations are T_5 and T_7, as already noted. The interval-class 5 appears maximally often both in lh and in rh, and 5 is a unique maximal interval class in lh. To put this in our terms, T_5(rh) and T_7(rh) each have as many tones in common with rh as does any other transposed form of rh; T_5(lh) and T_7(lh) each have two tones in common with lh, strictly more than does any other transposed form of lh.

Interval-class 2 is also characteristic of the internal structure of both rh and lh; further, it is the only interval-class other than 5 contained in lh. So T_2 and T_{10} are also transformations with a special "internal" character here.

There are of course quite a few inversions preserving one note or one dyad of rh, and quite a few preserving one note or one dyad of lh. Two inversions do more than this, as regards the internal structures of rh and lh. Inversion about C, which we shall call operations L, preserves all of lh: L(lh)=lh. And M, the inversion which exchanges F and B, also exchanges F and C; thus M(rh) has two common tones with rh, and also M(lh) has two common tones with lh.

In sum, we can collate the transpositions T_5 and T_2, their inverses T_7 and T_{10}, and the two inversions L and M, as the transformations of "maximally internal" character here.[15] They contrast with the "maximally progressive" transformations T_1, T_6, T_8, I, J and K. The "internal" transformations make a thing or each of two things (rh and/or lh) very like itself; the "progressive" transformations make an earlier thing (rh) very like a later, different thing (lh). Now if I first transform rh to be very like lh, and then transform the result to be very like itself, it is likely that the final result will still and again be very like lh. We can thus reasonably expect an-internal-transformation-following-a-progressive-transformation to be itself a progressive transformation. Figure 21 tabulates some algebraic equations that instance just such features of our situation.

The table can be read: column-heading operation, followed by table-entry operation, results in row-head operation. Thus T_6, the second column-head, followed by M, the table entry in the fourth row of that second column, results in J, the operation at the head of the fourth row. If we transpose something by a tritone, and then take the M-associate of that tritone-transpose, we will have at hand the J-associate of the original operand. Algebraically, we express this as $M(T_6(\text{operand}))=J(\text{operand})$, or even more simply $MT_6=J$. Again, the latter equation can be read, "the M-associate of the tritone-transpose is the J-associate." In similar fashion, inspecting the third column and the fifth row, we note the relation $LT_8=K$, which we can read, "the L-associate of the 8-transpose (of any operand) is the K-associate (of that operand)." One sees from the table in how many various ways the progressive transformations, at the heads of columns and rows, are related in this fashion by internal transformations entered onto the table. The progressive transformation T_1, for example, can be expressed in four different ways as the composition of an internal with a progressive transformation: $T_1=T_7T_6$, $T_1=T_5T_8$, $T_1=LJ$, and $T_1=MI$. The progressive transformation J can also be expressed in four different such ways: $J=LT_1$, $J=MT_6$, $J=T_7K$, and $J=T_5I$.

Figure 21

	T_1	T_6	T_8	J	K	I
T_1		T_7	T_5	L		M
T_6	T_5		T_{10}	M		L
T_8	T_7	T_2			L	
J	L	M			T_7	T_5
K			L	T_5		T_{10}
I	M	L		T_7	T_2	

If we isolate the three progressive transpositions, we can arrange them in a transformation-network by invoking such equations. (See Figure 22.)

The T_5 arrow from the bottom to the middle node of the figure means $T_5T_8=T_1$; the T_2 arrow from the top to the bottom node means $T_2T_6=T_8$. Here (all) the three progressive transpositions are the contents of nodes, and (all) the internal transpositions act as transformations associated with arrows on the graph. Figure 23 shows that (all) the three progressive inversions can be arranged in a network isographic with Figure 22. The analogous T_5 and T_2 arrows here mean $J=T_5I$, $I=T_2K$.

We can highlight the "internal" character of the graph-transformations very effectively by pointing to yet another isographic network, namely that one which can be read directly from (all) the internal intervallic relations within lh itself (Figure 24).

One can thus show formally how the internal structure of lh itself is thematically "prolonged" via certain interrelations among the progressive transformations, as those transformations jockey for power in the manner of Figure 20. The isography of Figures 22-23-24 is amusing but not really very startling (compared say to that of Figure 17); it is really close to implicit in the very logic of "internal" and "progressive" transformational categories. What is actually being exhibited powerfully here is the utility of the transformational outlook, and its organizing force in connection with common-note considerations.

Figure 22

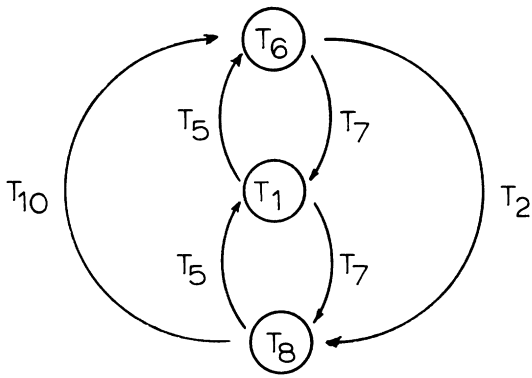


Figure 23

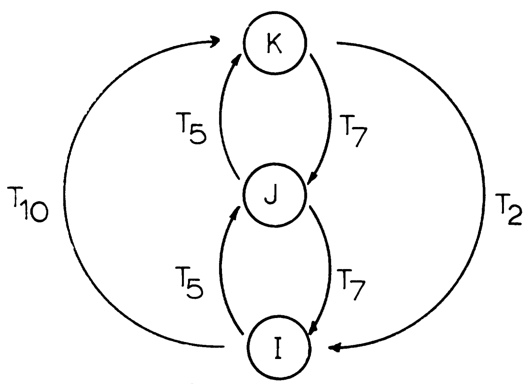
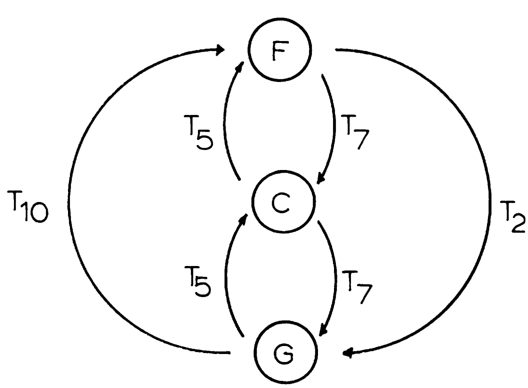


Figure 24



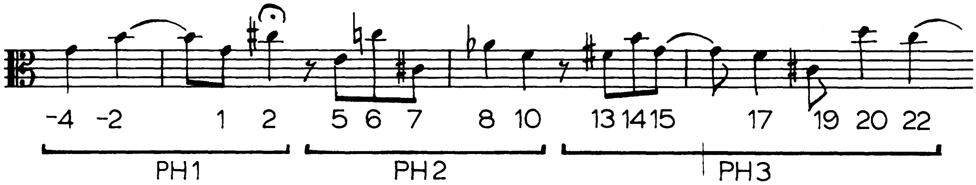
Let me hasten to caution any readers who may be unfamiliar with this piece that the foregoing discussion does not constitute an analysis. It is purely an elaboration of Figure 19, to support an assertion that such theoretical material is relevant to analysis.[16]

To exemplify further the analytic possibilities inherent in transformational theory, we shall now examine certain rhythmic aspects of the viola melody from the Webern piece studied earlier. Figure 25 transcribes the pitches and durations of that melody.

Underneath the note-heads of the Figure I have written numbers measuring how many written eighths, before or after the first bar line of the score, each note is attacked. The opening G is attacked 4 eighths before the bar line; the subsequent B is attacked 2 eighths before; the subsequent G is attacked one eighth after; the subsequent C# 2 eighths after. And so on. I ignore the fermata for reasons that will become clear presently.

By this means one obtains three sets of time points, PH1, PH2 and PH3, corresponding as indicated on the Figure to the attack points of notes within the three "phrases" of the melody. PH1 is the time-point set (-4, -2,1,2), PH2=(5,6,7,8,10), and PH3=(13,14,15,17,19, 20,22). We shall be examining certain "progressive" transformational relations between PH1 and PH2, between PH2 and PH3, and between PH1 and PH3. In this situation, we can easily ignore the fermata because any translation of PH2 and PH3 forward in time, due to the fermata, will simply be reflected by a corresponding algebraic translation ("transposition") of the transformations at issue between PH1 and PH2, or PH1 and PH3. Further observations on problems of modeling are developed in a footnote.[17] The numerical labels for time-points make it easy to compute standard transformations of the sets PH1, PH2 and PH3, and to count common-element relations among transformed sets. For instance, given any integer i we can ask this question: suppose PH2 were performed i eighths later; how many common attack-points would it share with PH3? That is, how many common things are there between T_i(PH2), the i-transpose of PH2, and PH3? That is, what is the numerical value of

Figure 25



CMNF(T_i, PH2, PH3)? A convenient format for answering this question is provided by the matrix of Figure 26.

The time-points of PH3 are listed horizontally along the top of the Figure; the time-points of PH2 are listed vertically at its left. Each numerical entry on the table is the difference between the number that heads the column, and the number which heads the row, in which that entry appears. For instance, the number entered in the column headed "19" and the row headed "7" is the number $12 = 19-7$.

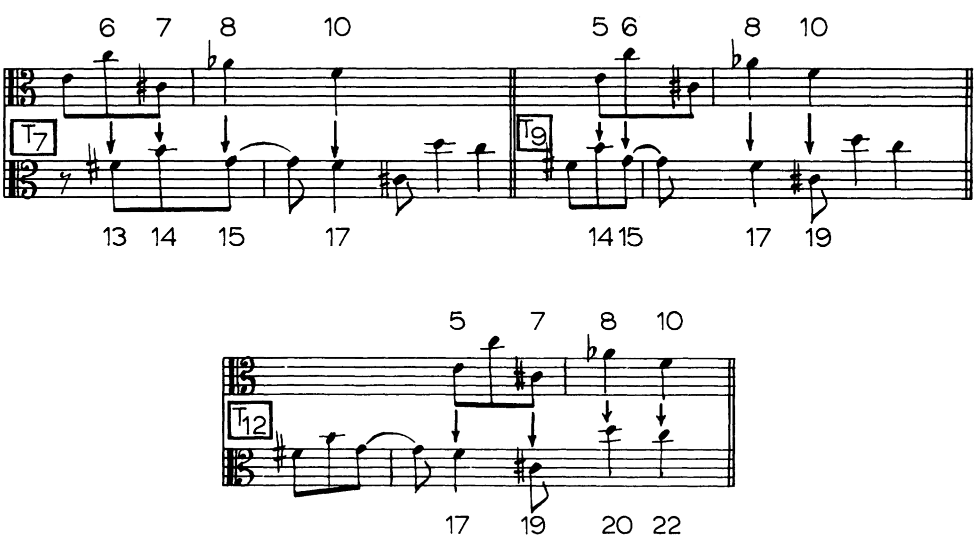
If the number i appears N times as an entry on the table, then there are N different ways of writing $i=y-x$ subject to the constraint that y and x be members of PH3 and PH2 respectively. Conversely, if there are N different ways of writing $i=y-x$ subject to that constraint, then there will be N entries of i on the table. Subject to the constraint, $i=y-x$ if and only if $y=i+x$, which is so if and only if y is the i-transpose of x, that is $y=T_i(x)$. So i appears N times on the table if and only if N members of PH2 have their i-transposes lying within PH3. And that is so if and only if T_i(PH2) has N common members with PH3. Thus, finally, the number of times that i appears on the table is N=CMNF(T_i, PH2, PH3).

Since PH2 has five members, that number N can never be greater than 5. Inspecting Figure 26, we see that no entry actually appears five times; this means that no transposed form of PH2 is completely embedded in PH3. However, each of the numbers $i=7,9$ and 12 appears four times on the table; thus T_7(PH2), T_9(PH2) and T_{12}(PH2) each have four common attack-points with PH3. In our earlier parlance, the relation between the attack structures of PH2 and PH3 sounds "fairly T_7-ish," "fairly T_9-ish," and "fairly T_{12}-ish." Or, to put it yet another way, the attack structure of PH3 includes something like a variation of the attack structure of PH2, played 7 eighths later, something like a variation of that attack structure played 9 eighths later, and something like a variation played 12 eighths later. Figure 27 illustrates these relationships.

Figure 26

	13	14	15	17	19	20	22
5	8	9	10	12	14	15	17
6	7	8	9	11	13	14	16
7	6	7	8	10	12	13	15
8	5	6	7	9	11	12	14
10	3	4	5	7	9	10	12

Figure 27



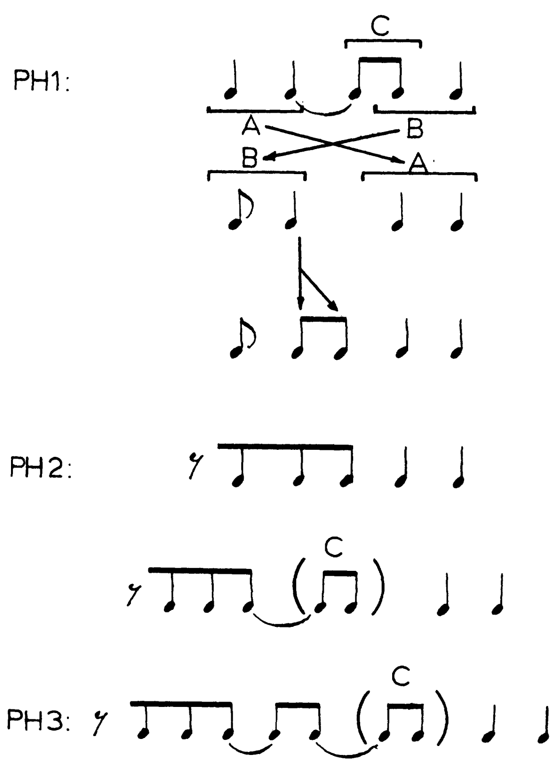
The T_{12} relation in Figure 27 is easy to hear because it is powerfully supported by pitch and contour associations within the melody: the re-attained low C#, the leaps up from C# with agogic accents, the rhythmic match-up of the final cadential gestures within PH2 and PH3, etc. The T_{12} relation is also supported by the "strong" and "weak" implications of the written meter, but it is dubious to invoke that criterion in this context. One does hear the written meter, I would say, but not on the basis of the attack patterns alone. Crucial, e.g., are the eighth rests at time-points 4 and 12, each followed by attacks on three successive eighths. The rests are not picked up by our attack-point analysis; there are only three entries of 8 on Figure 26, as opposed to four entries of 7 and four entries of 9. Thus Figure 26 cannot model our urge to associate the three pickup eighths at the end of measure 2 (following a rest) with the three pickup eighths at the end of measure 1 (following a rest). It offers us not T_8, which the latter hearing suggests, but T_7 and T_9.

I do not mean to deny interesting functions for T_7 and T_9 as "progressive" transformations here. (And the "internal" T_2 that relates them is indeed important; we shall pick it up later.) Still I do not think that attack-point transposition is an optimal way to think of transforming PH1 "progressively" into PH2, or either into PH3. Attack-and-release analysis, or attack-and-duration analysis could go farther. An even better model, I suspect, might eventually be provided by some scheme of "grammatical" transformations, which I have tried to suggest intuitively by the format of Figure 28.

However, it seems a lengthy and difficult project to try to work out a pertinent formal grammar, for such a scheme, rigorously and with general applicability to other pieces.

More revealing than attack-point transpositions here are various "progressive" attack-point inversions (i.e. retrogrades). This is not surprising if one recalls Webern's fondness for palindromes, manifest in his later work and, as we shall see, already latent here. Common-member relations involving inverted forms of our time-point sets can be tabulated by using a method of "sums" analogous to the method

Figure 28



of "differences" reflected in Figure 26. Figure 29 below gives tables of sums (a) for PH1 and PH2, (b) for PH2 and PH3, and (c) for PH1 and PH3.

Figure 29(a) tells us that there are three instances of $y+x=6$, or $y=I_6(x)$, such that y is a member of PH2 and x is a member of PH1. There are also three such instances of $y=I_8(x)$. Here the numbers 6 and 8 are used to label inversion operations for obvious algebraic reasons. We should remember, though, that the *sum* of the numbers y and x, unlike their difference, depends on the distances of the corresponding events from the event labeled as time-point zero. Time-point zero, and the distances of the y and x events therefrom, may be irrelevant to the musical effect of the inversion. In particular, we should not expect to find any pertinent structural event at time-point "6", when we examine the influence of "I_6" in relating PH1 to PH2. Rather, the important structuring moment in connection with I_6 is the center of inversion, time-point 3. If $y=2I_3x)$, then $y=6-x$ and $y-3=3-x$; hence the event at y comes just as far after time-point 3 as the event at x came before time-point 3.[18] Figure 30(a) shows the structuring influence of I_6 in relating PH1 to PH2; the dotted line demarcates the center of inversion at time-point 3, which labels the fermata on the C . Attack-points paired by the inversion (whose time numbers add to 6) are connected by slurs on the Figure.

Figure 30(b) shows in analogous fashion the structuring influence of I_8 in relating PH1 to PH2. The dotted line demarcates the center of inversion at time-point 4, where the rest begins.

The idea that some temporal inversion relates PH1 to PH2 is attractive because of the general sense of contour retrograde affecting the corresponding melodic gestures. The ambivalence between Figure 30(a) and Figure 30(b), as to just how the quasi-palindrome works itself out, is also attractive in modeling a certain floating character in the rhythmic effect of the music. The fermata and the rest assume interesting formal functions in this connection.

Figure 29

	5	6	7	8	10
−4	1	2	3	4	6
−2	3	4	5	6	8
1	6	7	8	9	11
2	7	8	9	10	12

(a) PH1 and PH2
3 entries of 6,8

	13	14	15	17	19	20	22
5	18	19	20	22	24	25	27
6	19	20	21	23	25	26	28
7	20	21	22	24	26	27	29
8	21	22	23	25	27	28	30
10	23	24	25	27	29	30	32

(b) PH2 and PH3
4 entries of 25, 27

	13	14	15	17	19	20	22
−4	9	10	11	13	15	16	18
−2	11	12	13	15	17	18	20
1	14	15	16	18	20	21	23
2	15	16	17	19	21	22	24

(c) PH1 and PH3
4 entries of 15

The two progressive inversions that relate PH1 to PH2, I_6 and I_8, are themselves related by the "internal" transposition T_2. That is, $I_8=T_2I_6$. The latter equation can be verified algebraically: if y is the I_8-associate of x and z is the I_6-associate of x, then $y=8-x=2+(6-x)=2+z=T_2(z)$; hence $I_8(x)=y=T_2(z)=T_2I_6(x)$. The musical effect of this relation can be seen in Figure 30: for example, the I_6-associate of time-point 1 is time-point 5 (Figure 30(a)); the I_8-associate of time-point 1, time-point 7, arrives 2 eighths later (Figure 30(b)).

T_2 is an "internal" transformation specifically in connection with the internal structure of PH2. PH2 contains three temporal intervals of 2, as many as it does intervals of 1; it contains fewer than three instances of all other temporal intervals. Equivalently, if we displace PH2 one or two eighths later, the displaced form will have three common attack-points with PH2 itself; other forward-displaced forms of PH2 will have fewer than three common attack-points with PH2. This "internal T_2-ishness"of PH2 is arguably the first indication anywhere in the music that the written eighth notes group functionally in pairs, rather than in threes, not at all, etc. And that metric function for T_2 is also displayed in its relating the two quasi-palindromes we have just examined, via the equation $I_8=T_2I_6$.

In contrast to PH2, PH1 has no "internal T_2-ishness." Indeed, it has no "internal T_i-ishness" for any temporal interval i. Its four time-points span one interval of 1, one interval of 2, one interval of 3, one interval of 4, one interval of 5 and one interval of 6. PH1, then, is a sort of rhythmic all-interval or equal-interval set. The way in which PH1, PH2 and PH3 each "unfold" their internal rhythmic structures, as they are exposed in time, time-point by time-point, can conveniently be studied using machinery I have developed elsewhere.[19]

Let us return to Figure 29(b). It shows that I_{25} and I_{27} will have maximal structural influence among inversions, in relating PH2 to PH3. The internal transformation T_2 is again involved:$I_{27}=T_2I_{25}$. Figures 31(a) and (b) indicate how the influences of the new inversion operations are manifest in the music. Since 25 and 27 are odd numbers, the centers of inversion now lie halfway between eighth-beats in the score. As noted earlier, our simple attack-point model cannot adequately engage the crucial rests that, along with the attacks which follow them, suggest a T_8 relation between PH2 and PH3.

Figure 30

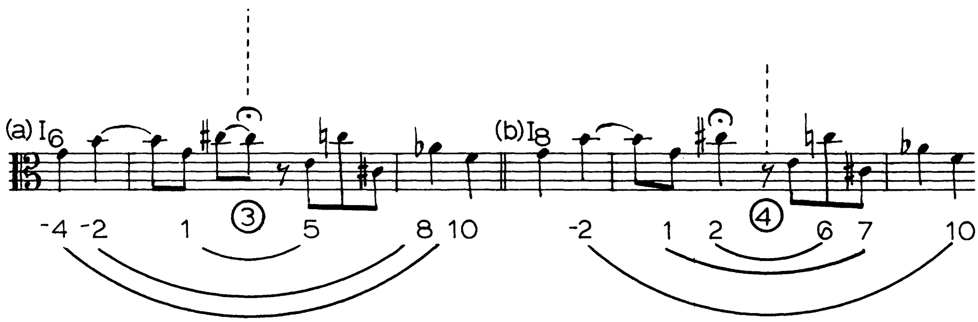


Figure 31

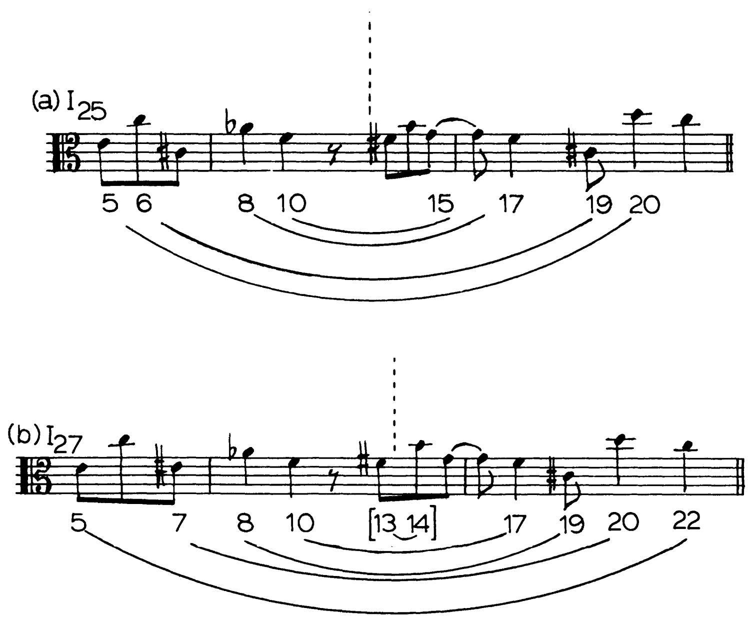
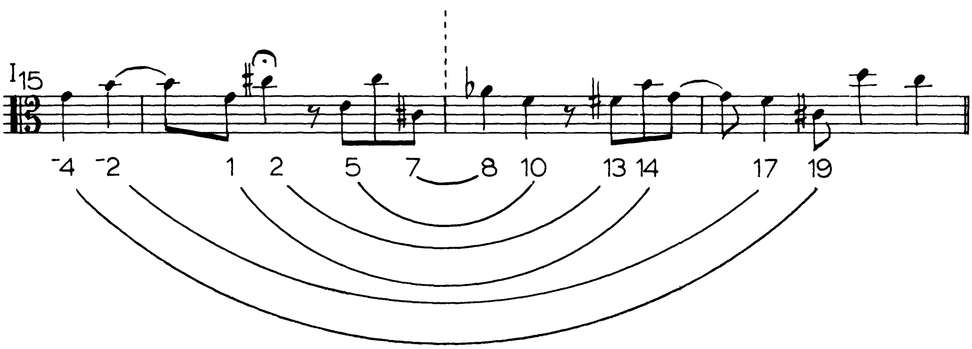


Figure 29(c), finally, shows us that I_{15} is the unique inversion that completely embeds an inverted form of PH1 in PH3. (In fact, I_{15}(PH1) is a completely unique form of PH1 embedded inside PH3; examination of a difference-table for PH1 and PH3 will reveal that no entry appears as many as four times, so no transposed form of PH1 can be embedded in PH3.) Investigating farther, we can note that I_{15} is also a strong "internal" transformation with respect to PH2. Specifically, I_{15} is a unique proper-transposition-or-inversion operation such that the corresponding form of PH2 has as many as four common time-points with PH2 itself. So, among all proper transpositions and inversions, I_{15} transforms PH2 into something most like itself. That aspect of I_{15}, as well as the complete embedding of I_{15}(PH1) in PH3, can be observed in Figure 32.

The pitch-structure of the melody supports the palindromic aspects of the Figure nicely. Within the music for PH2, the "minor third" E-C♯ at time-points 5 and 7 is answered by the "minor third" A♭-F at time-points 8 and 10, suggesting retrograde-inversion of the pitch succession. The opening "major third" G-B at time points -4 and -2 is similarly answered, now in transposed retrograde, by the "major third" F-C♯ at time-points 17 and 19. The closure of the quasi-palindrome on the low C♯ at time-point 19 is attractive: it interacts well with other surprising aspects of the following D-C in the melody, and it brings out a large-scale relation of the low C♯ to the opening G which fits well with our earlier pitch-class analysis of the passage.

The palindromic aspects of Figure 32 are important because they indicate a rationale for asserting a structural closure of some sort toward the end of PH3 in the rhythmic context as a whole. The large palindrome embedding I_{15}(PH1) in PH3 and emphasizing the closeness of I_{15}(PH2) to PH2 itself is intrinsically of strong effect. It also can be thought of as "resolving" smaller-scale ambiguities about weaker quasi-palindromes, ambiguities manifest in Figure 30(a)-(b) and Figure 31(a)-(b). The ability of our system to model structural closure here is particularly significant because nothing else at hand provides such a model. That includes specifically the attractive

Figure 32



"transformational grammar" of Figure 28, which did not model any such closure. If the grammar could be made rigorous in an intuitively plausible way, it seems to me that the issue of closure would have to be confronted and resolved. (And there is a lot more that makes the "if" a big "if".)

We have explored transformational techniques in an atonal pitch-class context. We have noted points of contact with Riemann-like tonal theories, and we have applied our machinery, *mutatis mutandis,* to an atonal rhythmic context. The notions exposed can easily be applied to yet other contexts. Transformational graphs and networks are obviously relevant to the disposition of row-forms in a classical twelve-tone piece.[20] The ideas of transformational "urges", and of various transformations jockeying for network-forming priority, that came up in connection with some of our work, might be particularly suggestive in approaching the study of open-form pieces. Do such considerations, for instance, influence a performer in constructing an intuitively convincing performance from the score of Stockhausen's *Klavierstueck XI*? Here one would want to consider at least rhythm, heavily influenced by the composer's directions, as well as pitch. *Plus Minus* might also repay such investigation; heavy computer assistance would be welcome and perhaps even essential, to cope with all the potential transformations involved.

APPENDIX

Here I shall develop more rigorously the formalities of transformation-graphs, etc. I have tried to make the discourse as accessible as possible to a determined reader, but here and there I have had to assume a certain degree of familiarity with mathematical set-theory and group theory.

There is a wide variety of formal definitions possible for the constructs involved. I have selected my definitions here on the basis of three criteria. (1) The defined constructs are optimally applicable for the purposes to which I have put them in this paper. (2) The definitions are easy to modify, should other purposes suggest any of a number of conceivable modifications. (3) The constructs are easy to define and manipulate on computers using LISP-based languages; they fit easily into the sorts of contexts typically considered and manipulated by Artificial Intelligence.[21]

DEFINITION 1. A *node-arrow system* is an ordered pair (NODES, ARROW) comprising a finite family NODES of undefined "nodes" N, and an unspecified relation ARROW on the family of nodes.

The relation ARROW is considered to be a collection (any collection) of ordered pairs of nodes. Nodes N_1 and N_2 are "in the arrow relation" if the ordered pair (N_1,N_2) is a member of the collection ARROW.

DEFINITION 2. A node-arrow system is *disconnected* if NODES can be partitioned into two proper subsets NODES1 and NODES2 in such fashion that, for every node N_1 in NODES1 and every node N_2 in NODES2, neither (N_1,N_2) nor (N_2,N_1) is in the arrow relation. A node-arrow system is *connected* if it is not disconnected.

It is easily shown that any disconnected node-arrow system can be partitioned into a finite number of "connected components", none of which communicates with any other via ARROW. Hence, for practical purposes, we can restrict our attention to connected systems.

DEFINITION 3. In a node-arrow system, nodes N and N′ are *coextensive* if $N=N'$, or if both (N,N') and (N',N) are in the arrow relation.

For example, on Figure 6 earlier the node containing $T_8(X)$ is coextensive with the node containing $I(X)$; it is also coextensive with the node containing $T_2(X)$. Although the $I(X)$ node and the $T_2(X)$ node are both coextensive with the $T_8(X)$ node, they are not coextensive with each other. Simply by inspecting the graph, without reference to the music under analysis, we can see that the $I(X)$ node is "earlier than" the $T_2(X)$ node, in some sense. We can formalize that intuition by a definition.

DEFINITION 4. In a node-arrow system, node N is *earlier than* node N′, and N′ is *later than* N, if there exists any sequence of nodes $N_0,N_1,\ldots,N_K$ such that $N_0=N$ and $N_K=N'$, and also such that

(N_{k-1},N_k) is in the arrow relation for each $k=1,\ldots,K$, and also such that not every (N_k,N_{k-1}) is in the arrow relation.

Intuitively, we demand *some* path from N to N′ within the system that involves traversing *at least one* one-way arrow. The definition makes the I(X) node on Figure 6 "earlier than" the $T_2(X)$ node. It makes the left hand X node, on that Figure, "earlier than" the node containing $T_8(X)$ (because of the one-way T_8 arrow). Note that the $T_8(X)$ node is *both* coextensive with *and* earlier than the $T_2(X)$ node.

DEFINITION 5. A node-arrow system is *chronological* if no pair of nodes (N,N') exists, such that N is both earlier than and later than N′.

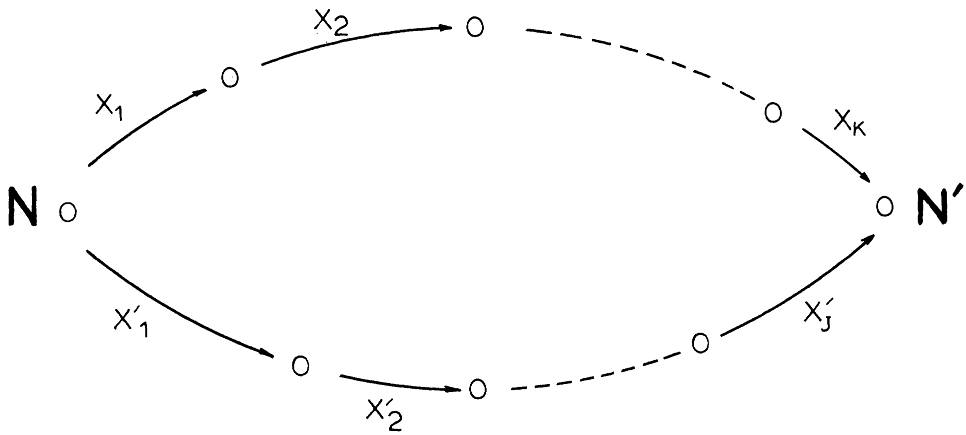
As mentioned earlier in the text, non-chronological systems are by no means uninteresting for musical applications. Still, it is nice to have Definition 5 available for formal purposes. Alternate definitions for "earlier than" etc. are possible, leading to alternate definitions of "chronological" systems.

DEFINITION 6. A *transformation-graph* is an ordered triple ((NODES,ARROW),G,tr), where (NODES,ARROW) is a node-arrow system, G is a mathematical group, and tr, the "transition function", is a function assigning to each member (N_1,N_2) of ARROW a value $tr(N_1,N_2)$ in the group G, subject to the following condition. Condition: suppose $N=N_0$, (N_0,N_1) and (N_1,N_2) and...and (N_{K-1},N_K) are all in the arrow relation, and $N_k=N'$. Suppose also $N=N'_0$, (N'_0,N'_1) and (N'_1,N'_2) and...and (N'_{J-1},N'_J) are all in the arrow relation, and $N'_J=N'$. For each k between 1 and K, set $x_k=tr(N_{k-1},N_k)$. For each j between 1 and J, set $x'_j=tr(N'_{j-1},N'_j)$. Then the group products $x_k\ldots x_2x_1$ and $x'_J\ldots x'_2x'_1$ are equal.

The point of the condition is to ensure that different arrow paths from N to N′ will not lead to different overall implicit transformations relating the potential contents of N to the potential contents of N′. Figure 33 should clarify the context to which the condition applies.

Definition 6 could be considerably broadened if we replaced the group G by a general semigroup. Much of the following work could

Figure 33



be carried through. Broadening the definition would also allow more musical applications. But it would make our model, already quite cumbersome, subject to even further mathematical niceties and qualifications. And sticking to groups has definite advantages in connection with connected graphs. A later remark will pick this issue up.

DEFINITION 7. Given transformation-graphs
((NODES,ARROW),G,tr) and ((NODES′,ARROW′),G′,tr′),
an *isomorphism* of the first onto the second is an ordered pair (f,g) satisfying conditions (1) through (3) following. (1) f is a 1-to-1 map of NODES onto NODES′ such that (N_1,N_2) is a member of ARROW if and only if $(f(N_1),f(N_2))$ is a member of ARROW′. (2) g is an isomorphism of G onto G′. (3) For every member (N_1,N_2) of ARROW, $tr'(f(N_1),f(N_2))=g(tr(N_1,N_2))$.

For an example of such isomorphism, let us first consider Figure 8, which diagrams a transformation-graph ((NODES,ARROW),G,tr) pertinent to the Webern piece. G is the abstract group of "transpositions and inversions". In this graph, I,J and K mean respectively "invert about D", "invert about F♯", and "invert about A, so as to exchange G and B". Now imagine the Webern piece transposed up a half-step, producing a piece Webern&. An analogous graph ((NODES&,ARROW&,G&,tr&) would evidently apply to the analysis of Webern&. We shall see that the intuitive "analogy" here is a formal isomorphism in the sense of Definition 7. The nodes and arrows of the new graph would evidently be the same as the nodes and arrows of Figure 8. Also, all the transpositions attached to the arrows of Figure 8 would evidently obtain as well in the new graph. But the centers of inversion pertinent to Webern& would all shift up a half-step along with the rest of the music. So in place of I,J and K, the new graph would have I&, "invert about E♭", J&, "invert about G", and K&, "invert about B♭ so as to exchange A♭ and C."

Now we are ready to demonstrate the formal isomorphism. Take NODES&=NODES and ARROW&=ARROW; take the identity map as the function f of Definition 7. Take G&=G; the new group, like the old, is the abstract group of all transpositions and inversions. The isomorphism g of Definition 7 works as follows: $g(T_j)=T_j$ for any transposition operation T_j; if L is any inversion operation, then g(L)=L&, where, if L exchanges pitch classes u and v, L& exchanges pitch classes $T_1(u)$ and $T_1(v)$, each a half-step "higher". Using the latter relation, we can set $w=T_1(u)$ and compute $L\&(w)=T_1(v)=T_1L(u)=T_1LT_1^{-1}(w)$. So $L\&=T_1LT_1^{-1}$. Thus $g(L)=T_1LT_1^{-1}$ for any inversion L. Also $g(T_j)=T_j=T_1T_jT_1^{-1}$ for any transposition T_j. In sum, then, $g(OP)=T_1\ OP\ T_1^{-1}$ for any member OP of G, and g is in fact an inner automorphism of G=G&, *a fortiori* an isomorphism as demanded by Definition 7. Condition 3 of the Definition is clearly satisfied, and hence the intuitive "analogy" of the graphs is indeed a formal isomorphism.

Definition 7 can be extended in the obvious ways to cover "homomorphisms", "automorphisms" etc. Several kinds of "homomorphisms" can be studied. If the mapping of NODES into NODES′ is not 1-to-1, many finesses become involved. The embedding of graphs within other graphs, as instanced in Figures 11-12 earlier, is also worth some study.

DEFINITION 8. A *transformation-network* is an ordered quadruple (((NODES,ARROW),G,tr),S,CTN,F), where ((NODES,ARROW),G,tr) is a transformation-graph, S is a set of "operands", CTN is a function from NODES into S, and F is a faithful representation of G as a group of operations F_x on S, all subject to this condition: for every member (N_1,N_2) of ARROW, if $x=tr(N_1,N_2)$ and $s_1=CTN(N_1)$ (the "contents" of N_1) and $s_2=CTN(N_2)$ (the "contents" of N_2), then $F_x(s_1)=s_2$.

In a connected transformation-network, given the transition-graph, the set S, the representation F, and the contents $CTN(N_0)$ of any one node N_0, one can uniquely infer the contents CTN(N) of every other node N. This inference relies upon the fact that G (or at least its representation on S) is a group, not just a semigroup.

One avoids a lot of mathematical grief by stipulating, as I have done in definition 8, that the representation of G on S be faithful. The stipulation could conceivably be relaxed, should any good reason arise for wishing to relax it.

At last we are in a position to define "isography" rigorously.

DEFINITION 9. Two transformation-networks are *isographic* if their constituent transformation-graphs are isomorphic.

Not, NB, if they have "the same" transformation-graph. First of all, it is not clear what "the same" is to mean, especially when we want to compare e.g., transformations operating on individual pitch-classes, with transformations operating on sets of pitch-classes, with transformations operating on centers of inversion, etc. etc. We have so far skirted the problem by thinking of "an abstract group" and various of its (faithful) representations, in this context. But in what way can we abstractly distinguish two general abstract groups that are isomorphic? Hence Definition 9 seems indicated. Note that it makes Figure 6 "isographic" with the analogous figure for Webern&, even though the latter network would involve different transformations (I&,J& and K&).

One could go on to generalize, by formal definition, the special relation noted earlier in connection with the graph BIGSCHEME, between the network involving S=forms-of-X and the network involving S′=forms-of-Y. Many other formal constructs can be explored abstractly, following the promptings of imagination, and/or analytic situations, and/or compositional ideas. In particular, one can explore a formal model for the way in which transformations can "jockey for power" in trying to establish rival provisional or potential networks. The latter study will involve not only the mechanics of the common-member function, but also notions of expectations, information, entropy et al.

NOTES

1. This aspect of the piece was noted and discussed by Bruce Archibald in his article "Some Thoughts on Symmetry in Early Webern: Op.5,No.2", *Perspectives of New Music* 10,2 (Spring-Summer 1972), 159-163. Archibald notes and discusses, too, other matters that will come up shortly in the present text.

2. Allen Forte, *The Structure of Atonal Music* (New Haven and London: Yale University Press, 1973). Another excellent approach to these matters can be found in John Rahn, *Basic Atonal Theory* (New York: Longman, Inc., 1980). Rahn's approach to atonal transformations resonates powerfully

with much of the material to be developed here. So do his remarks elsewhere on graphs and networks, as e.g. in "Relating Sets," *Perspectives of New Music* 18,2 (Spring-Summer 1980), 483-498, especially 494-497.

3. Consider the pitch-class set Z that comprises X and the accompaniment chord of measures 0-1, together with the E♭ that is to be the I-partner of C♯. The complement of Z (i.e. the set of pitch classes not yet stated) is the tetrachord (G♭,B♭,C,E). This tetrachord has a familiar form. It embeds $T_{11}(X)$=(G♭,B♭,C), together with an associated inverted form of X. On Figure 1, one sees G♭ withheld until the final sub-phrase of the melody begins, in measure 2. On Figure 1, B♭ has yet to appear. The first B♭ of the piece, in fact, appears as the twelfth pitch-class directly after Figure 1 ends. The B♭ appears over G♭ (F♯) in the bass; one can analyze the dyad as in T_{11} relation to the opening (G,B). When, after breaking off at the end of Figure 1, the melodic Hauptstimme resumes in measure 5, its sequence of pitch classes is B♭-E-C-F♯-(D)-F-E; the gesture occurs over F and F♯ in the bass. Here the Z-complementing tetrachord (G♭,B♭,C,E) is expanded into a T_{11}-form of the pentachord already discussed earlier: (F,B♭,G♭,E,C)= T_{11}(F♯,B,G,F,C♯). Thus T_{11} does eventually exert a large-scale influence over events of the piece, but not within the opening section itself.

4. By no means do I wish to imply that "non-chronological" graphs would be useless for analytic purposes. A number of recent studies speak eloquently to the contrary. See for instance Jonathan Kramer, "Multiple and Non-Linear Time in Beethoven's Opus 135," *Perspectives of New Music* 11,2 (Spring-Summer 1973), 122-145.

5. Suggestive here is the theory explored by Allen Forte in his article "Aspects of Rhythm in Webern's Atonal Music," *Music Theory Spectrum* 2(1980), 90-109. One might, for instance, associate with each node of a chronological graph a beginning time-point and an ending time-point, or permissible ranges for beginning and ending points, in such a way as to engage Forte's constructs.

6. The interested reader will find further development of general notions along such lines in my article "A Formal Theory of Generalized Tonal Functions," *Journal of Music Theory* 26.1 (Spring 1982), 23-60.

7. For instance, I would be ready to argue that a completely adequate analysis of the Beethoven passage requires its conceptual integration into contexts that include the opening of the Rondo, that include hearing E4, rather than E3, as the structural bass underlying the E major material of the second group in the first movement, etc.

8. Historically intermediary between the classical style of Figures 12-14 and the atonal idiom represented by the Webern passage, certain aspects of Wagner's motivic harmonic practice should be particularly well suited for investigation by transformation-graphs and networks.

9. Willi Reich, *Schoenberg*, trans. Leo Black (London: Longman Group Limited, 1971), page 55. I do not know any further who said what to whom on what occasions.

10. 3PLUS8EQUALS11 governs transposition networks through which the *Kopfmotiv* itself gets sent, along with various of its forms (clarinet, measure 8; piano, measure 9; piano, measure 12; piano, measure 13; etc.). The "11 arrow" of 3PLUS8EQUALS11 generates the descending chromatic line which is first heard as an "independent" countersubject to the *Kopfmotiv*. The clarinet begins to make this generation clear in measure 11; the piano makes it all too obvious through measures 19-23. And so on.

11. For an explanation of this construct in depth, the reader can consult my article "Forte's Interval Vector, my Interval Function, and Regener's Common-Note Function," *Journal of Music Theory* 1977, 194-237.

12. See note 11.

13. I am indebted to Michael Bushnell, who turned my attention to many of the following points with work leading to an unpublished research paper at the State University of New York, Stony Brook, 1981. Bushnell restricted his attention to relations involving intervals, transpositions and interval functions. Here I translate some of his work into common-note discourse, and augment the context by including inversions, as well as transpositions, in the family of germane transformations.

14. This aspect of the piece can be made the basis for an interesting analytic overview. Jonathan Kramer carried such an analysis through many years ago in an unpublished research paper at the University of California, Berkeley.

15. Systematic methods exist for being sure one has found all the "maximally internal" operations one wants to find here. Similar methods enable one to be sure one has found all the "maximally progressive" transformations relating rh to lh, etc. A good method is to label the pitch classes with numbers 0 through 11 according to some consistent convention, and then to inspect certain tables of differences and sums of such labels (modulo 12). An analogous method, using numerical labels for various time-points, will

be demonstrated in some detail later on, in connection with some rhythmic analysis. I have avoided attaching numerical labels to pitch-classes here for reasons I have expounded elsewhere. ("A Label-Free Development for 12-Pitch-Class Systems," *Journal of Music Theory* 1977, 29-48.)

16. The ability of this little piece to sustain (and withstand!) indefinite analysis of all kinds is truly remarkable. Important published analyses include Allen Forte, "Context and Continuity in an Atonal Work," *PNM* volume 1, number 2, 72-82; idem, *The Structure of Atonal Music* (New Haven and London: Yale University Press, 1973), 97-100; Robert Cogan and Pozzi Escot, *Sonic Design* (Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1976), 50-59; and Elaine Barkin, "Arnold Schoenberg's Opus 19/6," *In Theory Only* volume 4, number 8, 18-26. A detailed rhythmic analysis of the first half of the piece appears in David Lewin, "Some Investigations into Foreground Rhythmic and Metric Patterning," *Music Theory: Special Topics*, ed. Richmond Browne (New York: Academic Press, 1981), 110-117. To illustrate the point I made at the beginning of this note, let me add some observations in the way of "Romantic" analysis. The rh and lh chords can be heard as two tolling bells (at Mahler's funeral). They toll unevenly, every $7=3+4$ quarters. The events of measures $7\frac{1}{4}$ to 9 represent the ruminations of Schoenberg during one of these tolling periods, now articulated by the first attack of measure 7, the last attack of measure 8, and the first attack of measure 9, into $7=4+3$ quarters. These ruminations, that is, are "inside the bells". Note "*genau im Takt.*" The D-C♯-D-()-E♭ gesture in measure 7 might refer to the same succession of pitch-classes (and three exact pitches) in the first movement of the *Eroica* Symphony, during the opening 'cello theme. The rumination eulogizes the memory of Mahler as a hero (in a complicated poetic way, via Beethoven's inscription on the symphony); it also possibly recalls a memory of Mahler conducting the symphony. The idea of recollection is supported by numerous musical recollections from the first number of the opus. The progression rh-lh itself strongly echoes the vertical sonorities (A,B) and (G,C,F) on the first attack and at the fourth eighth of measure 1, in number 1. The D♯-E-D♯ that follows, in number 6, strongly echoes the same melodic gesture, that closed off the end of number 1. These recollections are "compressed" in number 6, as were the poetic ruminations of Schoenberg in the Romantic analysis.

17. We cannot engage even the rhythmic structure of the sustained melody adequately without also taking into account its durations or its release-points, as well as its attack-points. (Given the attack-points, the release-points can be inferred from the durations, or the durations from the release-points.) In this connection, we should attach *two* numbers to each rhythmic event in the melody: its attack and release, or its attack and duration. We

could then apply various techniques to this two-dimensional model. However, we can still legitimately study the attack-structure of the melody in itself, and it simplifies our transformational formalism a good deal to do so here.

Figure 25 assigns the time label "zero" to a point at which nothing is actually "happening" in the sound. One can ask just what "time" the number 0 is labeling, and this question will lead us into very deep philosophical waters. The safest, but not very satisfying, answer is that it is not directly labeling any time at all, but rather a certain notational feature of the score, the bar line, which we presume by various historical and cultural conventions to be significant. This throws the main burden of explication back on those conventions, which is fair but not very enlightening. In addition, it is not clear to what extent the conventions are still legitimate for music in this style.

Another answer, more satisfying but very difficult to handle, is that the number 0 is labeling an equivalence class of "real" moments in time, an equivalence class to which the bar line also refers. Each moment in that class is defined as the "real" time, within any specific competent performance of the piece, at which we feel we have gone as far beyond the attack of the B as that attack was beyond the attack of the G. This attribution of meaning to the number 0 certainly engages our listening better than the other, purely notational, meaning. But it raises hosts of new issues and complexities. How can we define what is a "competent" rhythmic performance without circularity in this situation? What is "any specific performance"? All those that I have heard? All that have ever been? All that ever could be? All that I have ever mentally imagined? That anyone has ever (competently) imagined? Could ever imagine? Etc. And, yet further, in what sense is the "real" time of these occasions "real"? Suppose that at the same moment one quartet in New York has just reached the bar line of measure 1, another quartet in New Haven has just reached the F♯ corresponding to our "attack-point 13"; in what formal sense does this "moment" coincide with *two* distinct non-equivalent "times" by our criterion, and "real" times at that? A theory adequate to the resolution of all these difficulties would probably have to define musical time by clocks based on pertinent "local activities." Such a theory might well have common features with various aspects of Bergsonian time and Special Relativity. Two times would be "equivalent" if they marked suitably similar stages in suitably similar local systems of activity, performed or imagined. It is not clear to me how "musical activity" could be defined without invoking "musical time" a priori; perhaps one would have to take "musical activity" as an undefined primitive concept here, somewhat in the spirit of John Cage. This would raise other philosophical problems, probably soluble but not to the taste of everyone interested.

I suspect that the big underlying problem is the inadequacy of the present tense in Indo-European languages to discuss matters involving the immediacies of musical time. For example, when I say, "the A in measure 9 *is* (beautiful/sustained/a passing tone/etc.)," in what time-system *is* "the A" (whatever that is) in a temporal present along with me? Few will be content with a semiotic fantasy of light rays striking my eye after reflecting off an area including a certain black circle on one of certain agreed-on pieces of paper, or with my imagining at the present moment such an event. The sense in which "the" A "is" in my present, beyond such a purely notational sense, involves in many ways just the same problems as those encountered in attributing more than notational significance to "time-point zero". To specify what I mean when I say "the A in measure 9 of this piece is. . .," I would probably find myself saying instead "this particular class of equivalent stages, within that particular class of equivalent series of competent local activities, is. . .". (Or, instead of "is", "bears such and such relations to such and such other classes of stages.")

18. The article "A Label-Free Development. . .," cited at the end of note 15 earlier, goes into the abstract modeling problems raised by attaching numerical labels to inversion operations. The discussion there, in connection with pitch classes and numbers modulo 12, applies here, *mutatis mutandis,* in connection with time points and ordinary integers.

19. "Some Investigations Into Foreground Rhythmic and Metric Patterning," *Music Theory: Special Topics,* ed. Richmond Browne (New York and London: Academic Press, 1981), chapter 5, pages 101-137.

20. For example, consider Webern's *Piano Variations* opus 27. The graph consisting of two nodes connected by one arrow, along with the transformation "pitch-class invert about A (or E♭)," is associated with many events lasting only two eighths in the second movement, and also with longer antecedent-consequent relations affecting entire row-forms at the beginning and end of the third movement. In any Schoenberg piece using a semi-combinatorial row, the local disposition of row forms throughout the piece typically involves many isographic networks of rows, based on the graph that has four nodes, all interconnected by arrows labeled "make retrograde", "combinatorially invert", and "combinatorially retrograde-invert," and subgraphs of that graph.

21. An excellent introductory text for interested uninitiates is that by Patrick Henry Winston, *Artificial Intelligence* (Reading, Massachusetts and Menlo Park, California: Addison-Wesley Publishing Company, 1977).