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ORIGINAL ARTICLE

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Transformations for temperature flux in multiscale models of the tropics

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Abstract How much of the observed planetary-scale heating in the tropics is due to eddy flux convergence? A mathematical framework to address this important practical issue is developed here. We describe a pair of velocity transformations that remove components of the upscale temperature flux in the multiscale intraseasonal, planetary, equatorial synoptic-scale dynamics (IPESD) framework derived by Majda and Klein [J. Atmos. Sci. **60**: 393–408, (2003)]. Using examples from the models of the Madden-Julian Oscillation of Biello and Majda [Proc. Natl. Acad. Sci. **101**: 4736–4741, (2004); J. Atmos. Sci. **62**: 1694–1721, (2005); Dyn. Oceans Atmos., in press] we demonstrate that the transformation for the meridional temperature flux convergence is possible with any restrictions on the heating profile, we show under which conditions the transformation for the vertical temperature flux convergence exists and, further, that the meridional transformation leads to a reinterpretation of lower troposphere Ekman dissipation as active heating plus zonal momentum drag. The meridional temperature flux transformation and induced meridional circulation is a new, tropical wave example of the transformed Eulerian mean theory in the case of strong vertical stratification of potential temperature. The asymptotic ordering of the flows means that the removal of the meridional temperature flux convergence has implications for how planetary-scale heating rates are inferred from velocity convergence measurements.

Keywords Multiscale models · Tropical meteorology · Transformed Eulerian mean

1 Introduction

The interaction across multiple scales in the equatorial troposphere has become increasingly recognized as the correct framework in which to understand large-scale organization. In particular, the upscale transport of temperature and momentum is evident within the Madden–Julian oscillation (MJO, also called tropical intraseasonal oscillation) [16,14,19] and models have attempted to capture this interaction both numerically [8] and phenomenologically [13]. Recent models by Biello and Majda [10,5,7] use the multiscale framework derived by Majda and Klein [11] to attain realistic planetary-scale flows using plausible synoptic-scale heating profiles which are inferred from latent heating from convective cloud systems.

The systematic asymptotic framework derived in [11], called the intraseasonal, planetary, equatorial synoptic-scale dynamics (IPESD) models, calculates the flow response to synoptic- and planetary-scale heating fluctuations. In the IPESD framework, the troposphere is forced by latent heating with a variability of 10 K/day

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on the synoptic scales and a weaker variability ($\sim$ 1.5 K/day) on the planetary scales. The resultant flow splits into two components. The first is the synoptic-scale fluctuating component, which is governed by balanced dynamics and forced directly by the synoptic-scale fluctuating heating. The second is the planetary-scale mean component, which is governed by dynamics in meridional geostrophic balance. These planetary dynamics are forced by both the planetary-scale mean heating fluctuations ($\sim$ 1.5 K/day) and the nonlinear upscale flux convergence of zonal momentum and potential temperature. Therefore the planetary-scale flow is a nonlinear function of the heating. In particular, since the planetary-scale heating and upscale heat flux combine in the equations for the planetary scale flow, there is no straightforward way of inferring the actual latent heating given a planetary-scale flow profile. Furthermore, to use multiscale models carefully, any specification of the heating rate must be accompanied by the scale over which the measurements are filtered.

It has been recognized in the past [17] that momentum flux convergence forcing the linear equatorial long wave equations can be interpreted as direct heating driving these equations, in addition to meridional/vertical balanced flow. In this paper, we make a similar, but more general observation in the context of the IPESD multiscale models for the tropical atmosphere [11], and apply these ideas to recent multiscale models of the MJO [10,5,7]. We show that the meridional component of any temperature flux convergence on the planetary scales can always be removed in favor of a momentum flux convergence, at the expense of introducing a meridional/vertical residual circulation localized to the region containing synoptic-scale heating fluctuations. Since the meridional/vertical residual flow on the planetary scales is weaker than the balanced synoptic-scale fluctuating flow, this transformation essentially removes a very weak circulation whose signal is overwhelmed by the other components of the flow.

In cases where there is some zonal symmetry in the synoptic-scale heating fluctuations, the vertical component of the temperature flux convergence on the planetary scales can also be removed in favor of a momentum forcing and at the expense of introducing a zonal/vertical flow (Walker-like cell) localized to the heating region. Unlike the case of the meridional component of the temperature flux convergence, the zonal/vertical flow induced by removing the vertical component of the temperature flux convergence is of the same order of magnitude as the synoptic-scale fluctuations. The symmetry required of the heating is not so stringent as to be unrealistic, and, in fact, the MJO models in [5] contained exactly these symmetries. Furthermore, the solvability condition for such solutions to exist requires that convective envelopes do not move westward, consistent with the eastward motion of the MJO envelope. In the case where the zonal symmetries are not exact, we show that a portion of the heating can, nonetheless, be transformed into momentum forcing.

We also use this induced circulation to describe explicitly the effects of Ekman pumping from a barotropic boundary layer at the base of the free troposphere. Examples of the induced momentum flux convergence for the MJO models of [7] are considered.

The transformations that remove components of the temperature flux convergence are closely related to the concept of the transformed Eulerian mean first discussed by Andrews and McIntyre [2–4] in the context of wave/mean flow interactions and since used to understand atmospheric [1,9] and ocean mixing [12,18]. The specific connection between the meridional/vertical flow transformation presented here and the transformed Eulerian-mean theory arises through the strong vertical mean stratification of the potential temperature, or equivalently the constancy of the buoyancy frequency.

This paper is structured as follows. The IPESD models are summarized in the following subsection, along with a discussion of the strengths of the dependent variables at the two scales in the asymptotic theory. In Sect. 2 the transformation which removes the meridional temperature flux convergence is presented, along with two examples which relate to the MJO models in [5]. In Sect. 3 the necessary conditions to affect a transformation to remove the vertical temperature flux convergence are discussed. The transformation is then presented and the defining equations for the zonal/vertical circulation are solved in general in Sect. 3.1. A simple example of this transformation is provided in Sect. 3.2. The transformation of the Ekman flux is provided in Sect. 4 along with an illustrative example. The paper concludes with a discussion of the implications of these results for inferring the magnitude of the heating rate from tropospheric velocity measurements.

1.1 The IPESD models for the MJO

The MJO model outlined in [10,5,7] used the IPESD model [11] for the Boussinesq equations, where both the density and buoyancy frequency are constant as a function of height. The full specification of the synoptic- and planetary-scale flows is presented in [7] and is summarized in the following. The total flow consists of planetary-scale mean plus synoptic-scalefluctuations

$$
\begin{aligned}
\theta &= \theta'(\epsilon x, x, y, z, t) + \overline{\Theta}(\epsilon x, y, z, t) + O(\epsilon)\\
p &= p'(\epsilon x, x, y, z, t) + \overline{P}(\epsilon x, y, z, t) + O(\epsilon)\\
u &= u'(\epsilon x, x, y, z, t) + \overline{U}(\epsilon x, y, z, t) + O(\epsilon)\\
v &= v'(\epsilon x, x, y, z, t) + \epsilon \overline{V}(\epsilon x, y, z, t)\\
w &= w'(\epsilon x, x, y, z, t,) + \epsilon \overline{W}(\epsilon x, y, z, t).
\end{aligned}
\tag{1}
$$

which vary on synoptic scales x and y, in the vertical, z on long zonal planetary scale, $X = \epsilon x$ and on intraseasonal time scales, t. The variables are scaled so that $-10/3 \leq y \leq 10/3$, $0 \leq z \leq \pi$ and $-4/3 \leq X \leq 4/3$ describes the whole zonal extent of the equatorial troposphere from ± 5000 km meridionally and 16 km vertically. The unit of time is 3.3 days, which is useful in describing intraseasonal variations. Faster variations have been disregarded but can be included without significant differences. The velocity is measured in units of 5 m/s in the horizontal and 0.016 m/s in the vertical direction and the potential temperature is measured in units of 3.3° K.

The separation of the forcing into a planetary-scale mean plus synoptic-scale variations is anisotropic on the large scales: the planetary-scale mean meridional and vertical velocity are $O(\epsilon)$ smaller than their synoptic-scale fluctuations. This is a consequence of the fact that the heating anomaly is separated into its zonal synoptic-scale fluctuating component and a weaker zonal planetary-scale mean

$$
S_\theta = S'_\theta(X, x, y, z, t) + \epsilon \overline{S_\theta}(X, y, z, t) \tag{2}
$$

and is measured in units of 10 K/day. The MJO models discussed in [10,5] consider forcing only through latent heat release, upper troposphere drag dissipation, thermal dissipation and lower troposphere drag dissipation through coupling to a barotropic boundary layer at $z = 0$ [7].

In the IPESD models, the synoptic scales are described by balanced dynamics forced by the synoptic-scale heating fluctuations

$$
\begin{aligned}
-y\, v' + p'_x &= 0\\
y\, u' + p'_y &= 0\\
w' &= S'_\theta, \qquad \overline{S'_\theta} = 0\\
p'_z &= \theta'\\
u'_x + v'_y + w'_z &= 0
\end{aligned}
\tag{3}
$$

with no-penetration top and bottom boundary conditions

$$
w' = 0 \quad \text{at} \quad z = 0,\ \pi. \tag{4}
$$

The planetary scales are driven by the upscale momentum and thermal fluxes and by the mean heating on planetary scales

$$
\begin{aligned}
\overline{U}_t - y\overline{V} + \overline{P}_X &= F^U - d_0\, \overline{U}\\
y\overline{U} + \overline{P}_y &= 0\\
\overline{\Theta}_t + \overline{W} &= F^\theta - d_\theta\, \overline{\Theta} + \overline{S}_\theta\\
\overline{P}_z &= \overline{\Theta}\\
\overline{U}_X + \overline{V}_y + \overline{W}_z &= 0
\end{aligned}
\tag{5}
$$

where the flux convergences are

$$
\begin{aligned}
F^U &= -\overline{(v'\, u')}_y - \overline{(w'\, u')}_z\\
F^\theta &= -\overline{(v'\, \theta')}_y - \overline{(w'\, \theta')}_z
\end{aligned}
\tag{6}
$$

and the overbars denote zonal means with respect to the small, synoptic scale. The top of the troposphere acts like a rigid lid for the planetary-scale flows in this model

$$
\overline{W} = 0 \quad \text{at} \quad z = \pi. \tag{7}
$$

However, the bottom boundary is coupled to the flow in the barotropic boundary layer below $z = 0$ yielding a boundary condition

$$\overline{W} = -\pi\,\Delta\, v_y^B \tag{8}$$

where v^B is the meridional component of the flow in the boundary layer, which is induced by pressure anomalies in the overlying free troposphere [6,15]. The coupling parameter, Δ, is proportional to the ratio of the thickness of the barotropic boundary layer to the depth of the free troposphere; for a 0.5-km-thick boundary layer $\pi\,\Delta \approx 1$. It is shown in [5] that the flow in the boundary layer is determined by the mean zonal momentum at the base of the free troposphere by

$$v^B = \frac{d\,y}{d^2 + y^2}\,\overline{U}\Big|_{z=0} \tag{9}$$

where d is the drag dissipation rate in the barotropic boundary layer and $d \approx 0.33$ for a dissipation rate of 1 day.

The MJO models in [10,5,7] consider synoptic-scale heating fluctuations confined to a moving convective envelope with a zonal extent of 10,000 km. The convection is chosen to model lower troposphere congestus heating in the eastern portion of the envelope and upper troposphere westward-tilted superclusters in the western portion of the envelope. It was shown in [10,5] that such a pattern induces an upscale momentum and thermal flux which is of the same magnitude but opposite sign in the western portion of the convective envelope compared to that in the eastern portion of the envelope. In particular, the basic MJO models have the property that the zonal mean of each component of the upscale fluxes is equal to zero.

2 Removing the meridional temperature flux convergence with a meridional circulation

By introducing an explicit meridional/vertical residual circulation within the convective envelope and at the expense of modifying the zonal momentum forcing, we can remove the meridional component of the temperature flux from the forcing of the long wave equations. The transformation proceeds by first considering the long wave equations (5) forced by mean heating and eddy flux convergence (6). Define the new vertical and meridional velocities, W^{**}, V^{**} by

$$\begin{aligned}\overline{W} &= W^{**} + W_H = W^{**} - \overline{(v'\,\theta')}_y \\ \overline{V} &= V^{**} + V_H = V^{**} + \overline{(v'\,\theta')}_z\end{aligned} \tag{10}$$

which is clearly a stream function form, i.e.

$$\begin{bmatrix} V_H \\ W_H \end{bmatrix} = \begin{bmatrix} \Psi_z \\ -\Psi_y \end{bmatrix} \quad \text{where} \quad \Psi = \overline{(v'\,\theta')}, \tag{11}$$

meaning that the meridional/vertical flow given by V_H and W_H is incompressible and constitutes a meridional/vertical residual circulation akin to a Hadley cell.

The transformed long wave equations are

$$\begin{aligned}\overline{U}_t - yV^{**} + \overline{P}_X &= F^{U**} - d_0\,\overline{U} \\ y\overline{U} + \overline{P}_y &= 0 \\ \overline{\Theta}_t + W^{**} &= F^{\theta**} - d_\theta\,\overline{\Theta} + \overline{S}_\theta \\ \overline{P}_z &= \overline{\Theta} \\ \overline{U}_X + V_y^{**} + W_z^{**} &= 0\end{aligned} \tag{12}$$

which look exactly like the original long wave equations but describing the transformed meridional/vertical velocities, and the transformed eddy flux convergences,

$$\begin{aligned}F^{U\,**} &= -\overline{(v'\,u')}_y - \overline{(w'\,u')}_z + y\,\overline{(v'\,\theta')}_z \\ F^{\theta\,**} &= -\overline{(w'\,\theta')}_z.\end{aligned} \tag{13}$$

The residual circulation that arises through this transformation, (V_H, W_H), has many interesting properties and those which are unique to the MJO models will be discussed in the examples below. In general, though, it is zonally confined to the region of synoptic-scale heating fluctuations. Furthermore, this residual flow is of the same strength as the planetary-scale mean meridional/vertical flow, meaning that it is much weaker than the synoptic-scale fluctuating component of the meridional/vertical flow, which is also confined to the forcing region; such a flow is difficult to pick out above the fluctuations.

2.1 Examples of induced meridional flows in the MJO models

We consider the effect of this meridional/vertical velocity transformation on the temperature and momentum flux convergences due to synoptic-scale heating used in the MJO models [10,5]. Motivated by observations the synoptic-scale heating considered in these models was chosen to resemble westward-tilted superclusters in the western portion of a moving convective envelope and lower troposphere congestus heating in the eastern portion of a moving convective envelope. The MJO models considered a zonally localized envelope of heating (described by the first positive range of a cosine with a width of 10,000 km). whose convective activity varied between lower troposphere congestus in the east and westward-tilted upper troposphere superclusters in the west.

The momentum and temperature flux convergences are shown in Fig. 1 in the case of equatorially symmetric synoptic-scale heating fluctuations, and Fig. 3 for synoptic-scale heating fluctuations centered at 900 km south (see [5] for details). The fluxes and their vertical and meridional components are shown as functions of height and latitude in the troposphere in the upward/westward-tilted supercluster portion of the convective envelope; in the lower troposphere, congestus portion of the envelope, the signs of all of the components of the forcing are reversed.

The meridional temperature flux is shown in panel (e) of Figs. 1 and 3; these are precisely the contours of vertical velocity for the induced residual circulation given by Eq. (11). Panel (a) of Figs. 2 and 4 show the resultant momentum flux convergence from Eq. (13) and panel (b) shows the stream function from Eq. (11).

It is clear that the transformation makes an exceedingly small change to the momentum flux convergence for the equatorially symmetric heating example since the temperature flux convergence is always small at the equator. For the example with heating at 900 km south, there is a stronger residual circulation in the southern hemisphere with no flow in the northern hemisphere. In this supercluster portion of the convective envelope, the circulation flows downward above the heating region; in the congestus portion of the convective envelope the situation is reversed, flowing upward above the heating region. The momentum flux convergence in Fig. 4 is significantly different than its unmodified counterpart in Fig. 3, in particular, upper tropospheric westerlies are forced far southward of the center of heating.

After making this transformation, the planetary-scale flow consists of this Hadley-like, residual circulation plus the planetary-scale flow driven by the total momentum forcing [panel (a) of Figs. 2 and 4] and the vertical component of the temperature flux convergence [panel (f) of Figs. 1 and 3].

3 Removing the vertical temperature flux convergence with a zonal circulation

There is no general transformation to remove the vertical flux of temperature from $F^{\theta\,**}$, but notice that the reason the meridional flux convergence is removable is that, since it is a meridional derivative, its meridional mean is zero. If the vertical flux can be expressed as an X-derivative of a function which vanishes outside of the heating region, then it can also be removed by introducing a zonal/vertical circulation: a localized Walker-like cell. Unlike the meridional flow of the previous section, such a circulation would have the property that its zonal component is the same order of magnitude as both the synoptic-scale fluctuating zonal flow and the planetary mean zonal flow making it a measurable signal on the planetary scales; therefore it is in no sense a secondary circulation.

In the congestus/supercluster model of synoptic-scale fluctuations [5], the zonal average of the upscale flux is zero (as is the zonal average of the second baroclinic planetary mean heating) and its vertical component can be removed by introducing this zonal/vertical circulation. A careful reader will notice that the meridional component can also be removed with the zonal circulation instead of the meridional secondary circulation introduced above. Since this zonal circulation has a larger zonal flow, is more computationally cumbersome and is less general, it seems more appropriate to consistently remove the meridional temperature flux employing the secondary meridional/vertical circulation above.

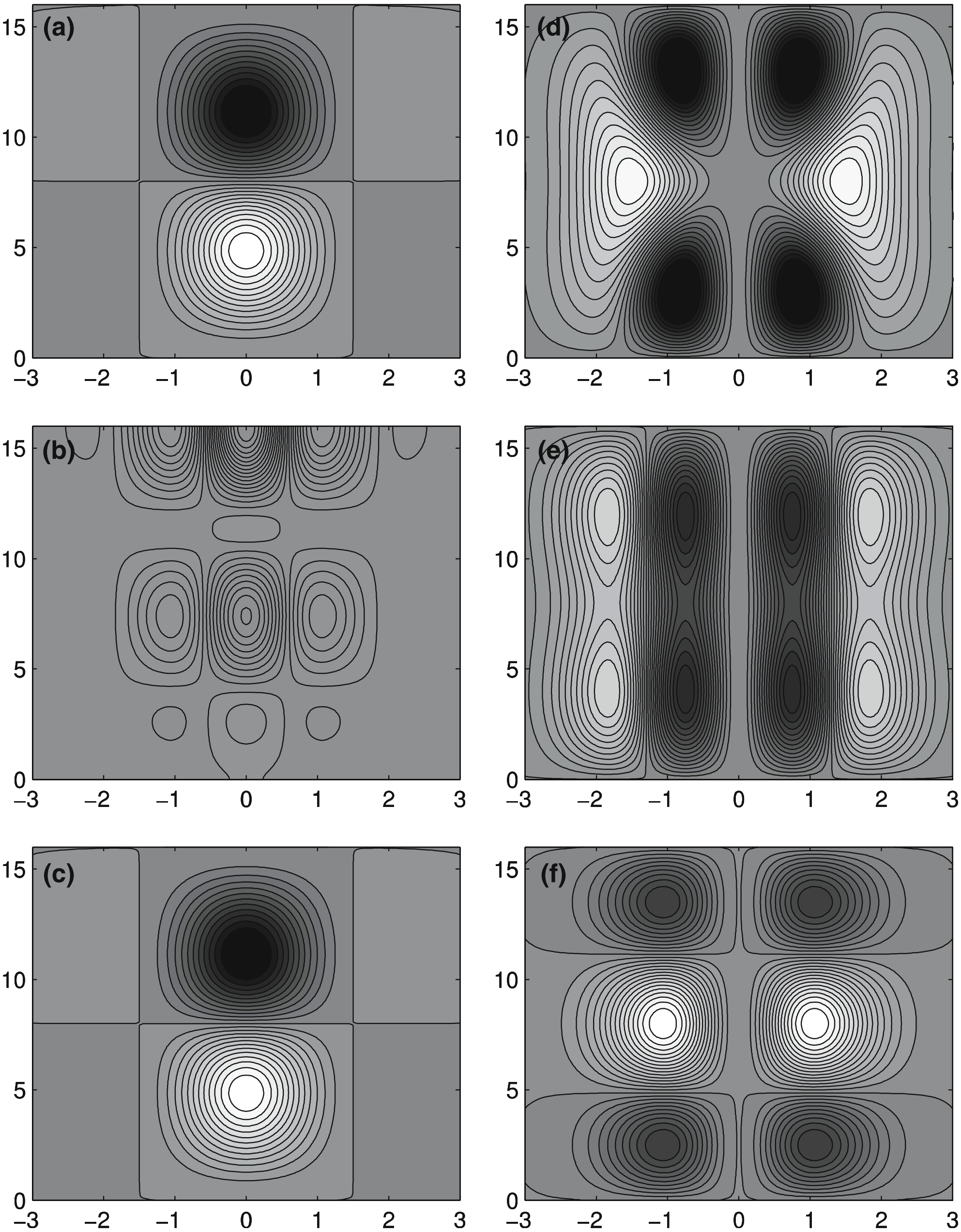

Fig. 1 The eddy flux convergences as a function of latitude (in 1000 km) and height (in km) in the troposphere for synoptic-scale heating centered at the equator in the supercluster portion of the convective envelope for the MJO models. **a** Total momentum flux convergence, **b** meridional component of momentum flux convergence, **c** vertical component of momentum flux convergence. **d-f** are the same as **a-c** except for the temperature flux convergence. The non-dimensional scale for the momentum flux convergences is larger than that of the temperature flux convergences

Beginning with Eqs. (12) and (13) and making the additional assumption that the zonal integral of the vertical component of the temperature flux is zero, it can be expressed as

$$\begin{aligned} F^{\theta\,**} &= -\overline{(w'\,\theta')}_z \\ &= -G_{Xz}(X, y, z, t) \end{aligned} \tag{14}$$

where, without loss of generality,

$$G = 0 \quad |X| > L_*, \tag{15}$$

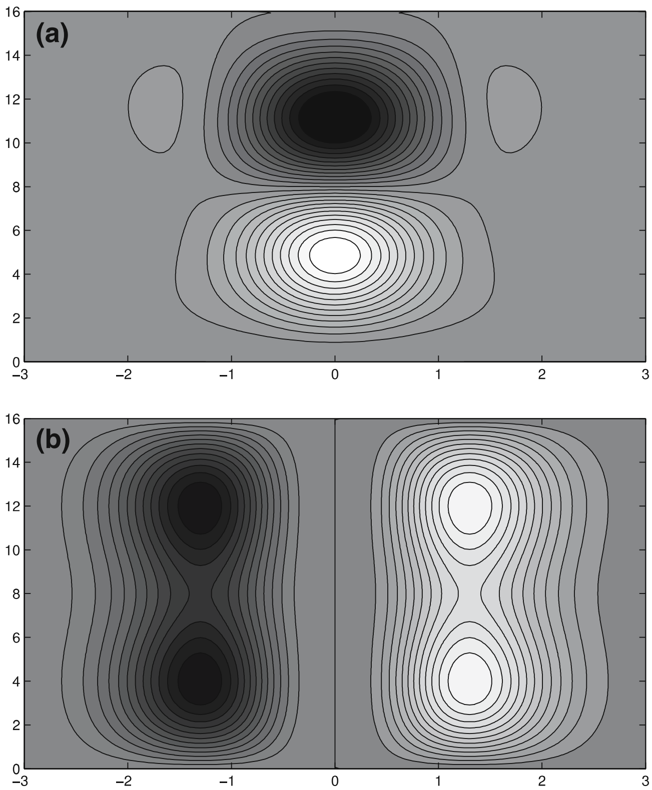

Fig. 2 **a** The modified momentum flux convergence in the equatorially symmetric synoptic-scale heating in the supercluster portion of the MJO envelope. **b** The stream function for the induced circulation; the flow circulates downward at the equator

i.e., G vanishes at either end of the convective envelope, which is true in the congestus/supercluster models of the MJO [10,5]. In models where the zonal mean of the upscale fluxes does not vanish, the vertical flux can be written as a component which satisfies equation (14) plus a remainder; this additional detail is discussed at the end of Sect. 3.1.

In the case of zonal mean zero fluxes, we define a zonal/vertical circulation with an associated pressure and temperature to compensate for the meridional geostrophic constraint

$$\begin{aligned} \overline{U} &= U^* + U_W, \qquad W^{**} = W^* + W_W, \\ \overline{P} &= P^* + P_W, \qquad \overline{\Theta} = \Theta^* + \Theta_W. \end{aligned} \tag{16}$$

where the meridional velocity remains unchanged from before, $V^* = V^{**}$. The equations for the secondary circulation are

$$\begin{aligned} yU_W + P_{W\,y} &= 0 \\ \Theta_{W\,t} + W_W &= -G_{X\,z} \\ P_{W\,z} &= \Theta_W \\ U_{W\,X} + W_{W\,z} &= 0 \end{aligned} \tag{17}$$

and their solution removes the vertical temperature flux, $-G_{X\,z}$ from the long wave equations at the expense of introducing a new term to the zonal momentum flux; in Sect. 3.1 below we discuss the solvability conditions for the equations in (17). The long wave equations then become

$$\begin{aligned} U^*_t - yV^* + P^*_X &= F^{U\,*} - d_0\, U^* \\ yU^* + P^*_y &= 0 \\ \Theta^*_t + W^* &= -d_\theta\, \Theta^* + \overline{S}_\theta \\ P^*_z &= \Theta^* \\ U^*_X + V^*_y + W^*_z &= 0\,, \end{aligned} \tag{18}$$

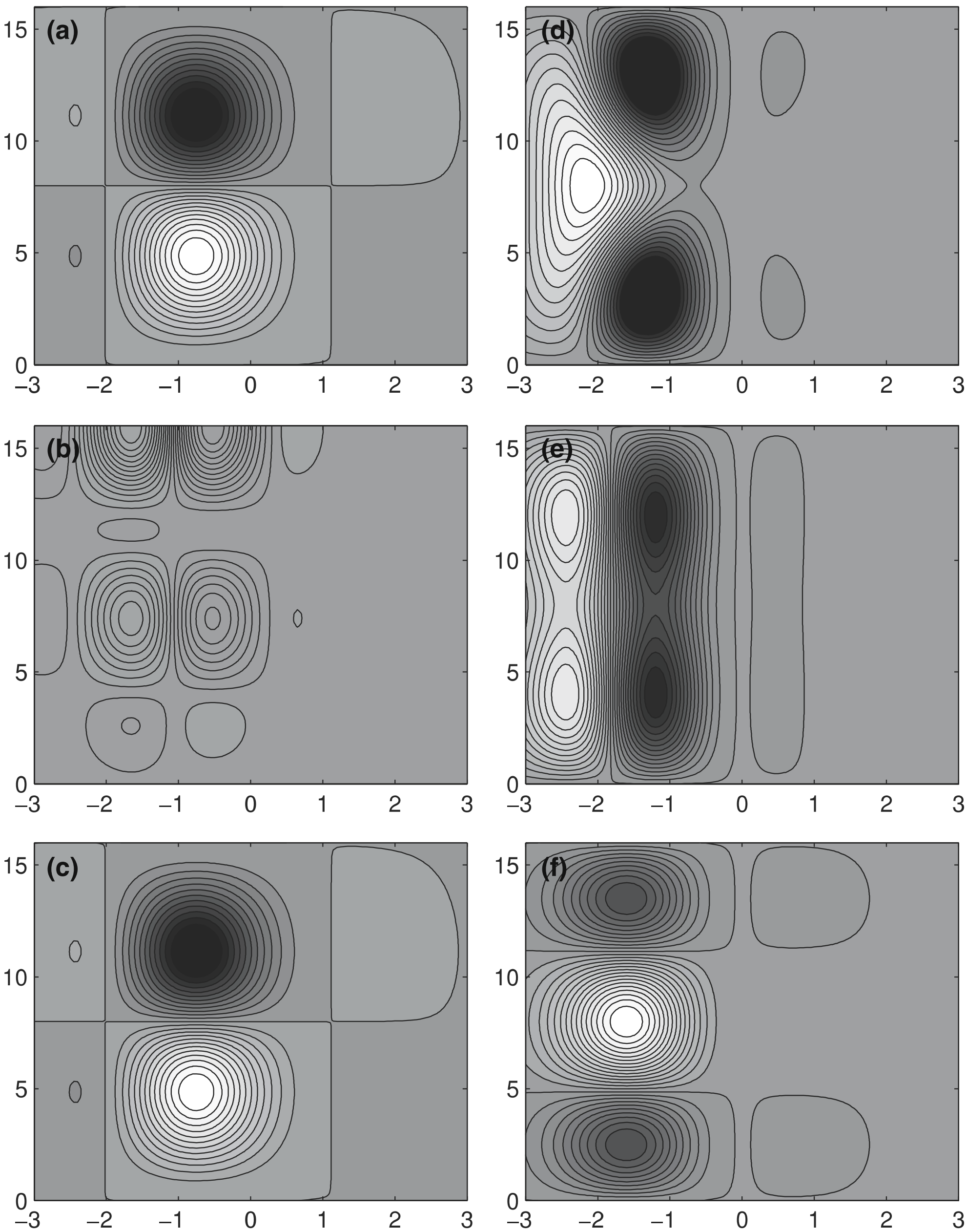

Fig. 3 The eddy flux convergences as a function of latitude (in 1,000 km) and height (in km) in the troposphere for off-equatorial synoptic-scale heating in the supercluster portion of the convective envelope for the MJO models, as in Fig. 1

which are exactly the original long wave equations but describing the transformed velocities, temperature and pressure, without any temperature flux convergence and using the transformed momentum forcing

$$F^{U*} = -\left(v'\,u'\right)_y - \left(w'\,u'\right)_z + y\;\left(v'\,\theta'\right)_z - U_{W\,t} - P_{W\,X}. \tag{19}$$

3.1 General solution of the Walker-like circulation

Notice that the equations for the zonal circulation (17) are similar to those of a forced Kelvin wave but without an equation for the zonal momentum. We would therefore expect their homogeneous solutions to be less

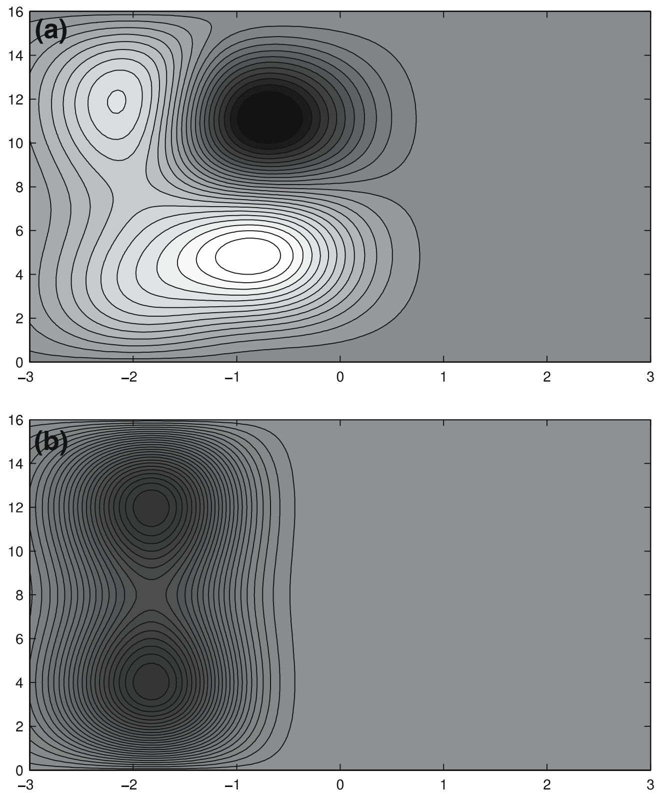

Fig. 4 **a** The modified momentum flux convergence in the supercluster portion of the MJO envelope for synoptic-scale heating centered at 900 km south. **b** The stream function for the induced circulation; the flow circulates downward above the heating region at 900 km south

restricted than the unforced Kelvin waves. Eliminating the other variables in favor of U_W the zonal velocity is described by

$$y\, U_{zz\,t} + U_{X\,y} = G_{X\,y\,zz} \tag{20}$$

where the W subscript has been omitted for ease of notation. The most general assumption for the structure of the forcing and the zonal velocity response is a Fourier series in the vertical direction and Fourier integrals in the horizontal direction and time

$$\begin{bmatrix} G \\ U \end{bmatrix} = \sum_{n=1} \int\int \begin{bmatrix} \tilde{\phi}^n(\omega,\ k,\ y) \\ \tilde{\Phi}^n(\omega,\ k,\ y) \end{bmatrix} e^{i\,(k\,X-\omega\,t)} \cos(nz)\ \mathrm{d}k\ \mathrm{d}\omega. \tag{21}$$

Substituting Eq. (21) into (20) and taking the zonal, vertical and temporal Fourier transforms yields

$$s\,n^2\,y\,\tilde{\Phi}^n + \tilde{\Phi}^n_y = -\,n^2\,\tilde{\phi}^n_y \tag{22}$$

where s is the phase speed of each mode, $s \equiv \omega/k$. Eq. (22) can be solved by using the integrating factor, $e^{s\,n^2\,y^2/2}$, yielding

$$\tilde{\Phi}^n(\omega,\ k,\ y) = \tilde{\Phi}^n_0\, e^{-sn^2y^2/2} - n^2 \int_0^y e^{sn^2\left(y'^2-y^2\right)/2}\ \tilde{\phi}^n_y(\omega,\ k,\ y')\ \mathrm{d}y'. \tag{23}$$

The solvability condition for this equation is that no component of the forcing corresponds to a westward-traveling wave; in order for solutions to exist, the function $\tilde{\phi}^n$ must be nonzero only in regions of the (ω, k) plane where $s \geq 0$. The prominent observed planetary-scale waves envelope in the troposphere, such as the MJO, move eastward and satisfy this condition [5]. The components of the heat flux that do not satisfy this criterion cannot be removed with this transformation.

Notice that there is no restriction on the homogeneous solution, $\tilde{\Phi}_0^n(\omega,\, k)$, since we have not specified any boundary conditions (except that the solutions do not blow up far from the equator). Since we are making a transformation of the dependent variables, we are free to specify the boundary conditions as we choose, and we shall choose the simplest. Therefore we require the homogeneous part of the solution to vanish, $\tilde{\Phi}_0^n(\omega,\, k) = 0\ \forall\ s \neq 0$, meaning that the Walker-cell solution vanishes at the equator. For $s = 0$ we simply let $\tilde{\Phi}^n = -n^2\, \tilde{\phi}^n$.

It is clear that the $k = 0$ components of the forcing, i.e. the zonal mean, cannot be removed with this transformation and they will have to be separated out of the forcing function, $F^{\theta **}$ from Eq. (14), before this solution is carried out. There is no unique way to isolate the zonal mean temperature flux convergence from the nonzero mean piece, so we can invoke physical arguments to guide us to a choice. The most obvious choice is to subtract from the vertical temperature flux convergence a function with the same zonal mean but which is constant as a function of longitude. This has the benefits that it is a simple splitting and produces what amounts to a mean climate, but its drawback is that the induced momentum forcing is no longer zonally localized (as is the case for the MJO models of [5]).

A second, more convenient method is to exploit the structure of the planetary scale mean heating; for example, in a Matsuno–Gill model, the first baroclinic heating is zonally localized over the first positive range of a cosine of width 10,000 km. The MJO models of the authors consider just such a mean forcing profile,

$$\overline{S^\theta} = F(X)\, H(y)\, \left[\sin(z) - \overline{\alpha}(X)\sin(2z)\right], \tag{24}$$

where $F(X) > 0$ within the region of synoptic-scale forcing. The remaining temperature flux convergence from Eq. (14) is no longer assumed to have zero zonal mean and so we write

$$F^{\theta **} = \sum_{n=1} F_n(X, y, t)\, \sin(nz). \tag{25}$$

We define the new function

$$F^{\theta\,\dagger} = \sum_{n=1} \left[F_n(X, y, t) - F(X)\, \Lambda^n(y, t)\right]\, \sin(nz), \tag{26}$$

and require each mode to have zero zonal mean, therefore

$$\Lambda^n(y, t) = \frac{\int F_n(X, y, t)\, \mathrm{d}X}{\int F(X)\, \mathrm{d}X}. \tag{27}$$

$F^{\theta\,\dagger}$ can now be used to construct the Walker-like circulation whereas the remainder term can be added to the mean forcing

$$\begin{aligned}\overline{S}^\dagger &= \overline{S} + F(X) \sum_{n=1} \Lambda^n(y, t)\, \sin(nz) \\ &= F(X) \left\{ H(y)\left[\sin(z) - \overline{\alpha}(X)\sin(2z)\right] + \sum_{n=1} \Lambda^n(y, t)\, \sin(nz) \right\}.\end{aligned} \tag{28}$$

3.2 Example: removing variations in the mean heating

The most straightforward example of removing a zonal mean-zero temperature forcing in favor of a Walker-like cell and a modified momentum forcing comes from the second baroclinic mode of the planetary-scale mean heating discussed in the MJO models of [10,5]. As was discussed above, in those models the planetary-scale mean heating consists of a first and second baroclinic mode as in Eq. (24) where the envelope, $F(X)$, is positive over a zonal extent of 10,000 km, akin to the Matsuno–Gill models,

$$F(X) = \begin{cases} \cos\left(\frac{\pi X}{2L_*}\right) & |X| < L_* \\ 0 & |X| \geq L_* \end{cases} \tag{29}$$

and the meridional structure is equatorially symmetric

$$H(y) = e^{-y^2/2}. \tag{30}$$

The second baroclinic mode is multiplied by a function of longitude, $\overline{\alpha}(X)$, which determines the height in the troposphere of the heating maximum: the upper half of the troposphere for positive $\overline{\alpha}$, the lower half of the troposphere for negative $\overline{\alpha}$. The specific form of the function was chosen in [5] to model the mean heating from deep convective superclusters in the western half of a convective envelope and lower troposphere congestus heating in the eastern half of the envelope.

In order to construct the zonal/vertical circulation, the second baroclinic mean heating mode need only have zero zonal mean. Since in $|X| \leq L_*,\ X\,F(X)$ is qualitatively similar to $\sin(X)$, and for ease of explanation, we use this simpler function to describe its zonal structure,

$$\overline{\alpha}\,F(X) = \begin{cases} -\sin\left(\frac{\pi X}{L_*}\right) & |X| \leq L_* \\ 0 & |X| > L_* \end{cases}. \tag{31}$$

Considering the case where this heating is steady the equations for the zonal circulation (for $|X| < L_*$) become

$$\begin{aligned} yU + P_y &= 0 \\ W &= \sin\left(\frac{\pi X}{L_*}\right) e^{-y^2/2} \sin(2z) \\ P_z &= \Theta \\ U_X + W_z &= 0 \end{aligned} \tag{32}$$

for which the vertical velocity is manifest, the zonal velocity is

$$U = \frac{2L_*}{\pi}\left(1 + \cos\left(\frac{\pi X}{L_*}\right)\right) e^{-y^2/2} \cos(2z), \tag{33}$$

the stream function for the Walker circulation is

$$\Psi = \frac{L_*}{\pi}\left(1 + \cos\left(\frac{\pi X}{L_*}\right)\right) e^{-y^2/2} \sin(2z) \tag{34}$$

where $(U, W) = (\Psi_z, -\Psi_X)$ and the pressure is

$$P = \frac{2L_*}{\pi}\left(1 + \cos\left(\frac{\pi X}{L_*}\right)\right) e^{-y^2/2} \cos(2z). \tag{35}$$

The induced zonal momentum forcing from Eq. (19) is just the negative of the zonal pressure gradient of the Walker circulation,

$$F^{U*} = -P_X = 2\sin\left(\frac{\pi X}{L_*}\right) e^{-y^2/2} \cos(2z). \tag{36}$$

Figure 5 shows the results of applying this zonal momentum transformation to the mean heating model, which interpolates between upper troposphere heating in the west and lower troposphere heating in the east of a longitudinally localized forcing region; frame (a) shows this heating profile above the equator. The stream function induced by the transformation and evaluated at the equator is shown in frame (b); it consists of an easterly mid-troposphere jet and westerly return jets at the bottom and top of the troposphere. The induced momentum forcing, frame (c), reinforces the Walker circulation in the eastern portion of the jet (mid-troposphere westerlies, upper and lower troposphere easterlies) and counteracts it in the western portion of the jet (mid-troposphere easterlies, upper and lower troposphere westerlies). Frame (d) shows the remaining, first baroclinic mean heating which cannot be removed by the transformation.

The four frames in Fig. 6 are the zonal flows above the equator corresponding to each of the frames in Fig. 5. It is clear that the induced momentum forcing [frame (c)] has a small effect on the total. Clearly, the induced Walker circulation [frame (b)], which is confined to the forcing region, accounts for most of the structure in the total flow [frame (a)] over and above the circulation due to first baroclinic heating [frame (d)].

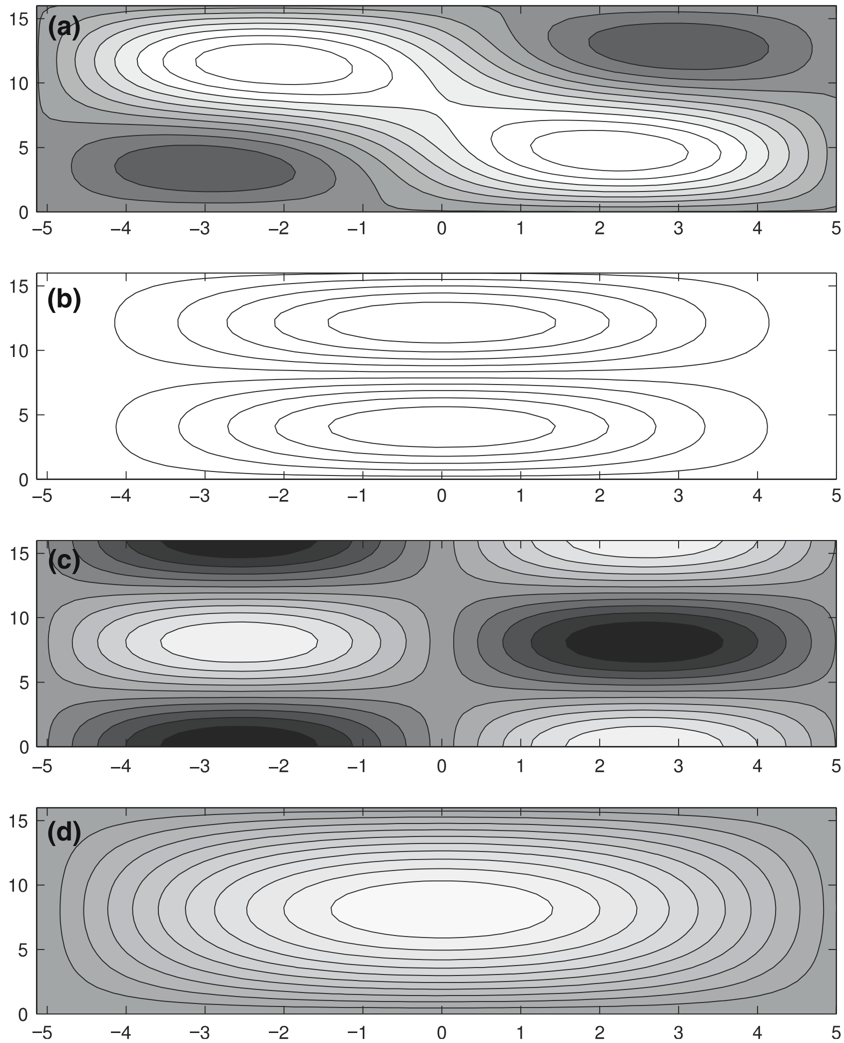

Fig. 5 **a** Zonal mean heating above the equator for an envelope that interpolates between upper troposphere heating in the west and lower troposphere heating in the east: (*white*) positive, (*black*) negative. The figures are only shown above the forcing region which extends from $\pm 5,000$ km and to 16 km above the base of the free troposphere. **b** The Walker stream function above equator showing an easterly mid-troposphere jet. **c** Induced zonal momentum forcing above the equator: (*white*) westerly, (*black*) easterly. **d** Resultant mean heating above the equator after removing the second baroclinic heating

4 Transforming the boundary conditions associated with the Ekman dissipation

The Ekman pump from the boundary layer induces a meridional/vertical circulation within the free troposphere that can be removed in a similar manner as the meridional temperature flux convergence. In particular, the induced meridional/vertical flow is of lower order than the synoptic-scale fluctuating component. This is an interesting exercise as it sheds light on the nature of the boundary-layer dissipation without making reference to baroclinic/barotropic mode splitting (see [5] for details of this decomposition).

Using the boundary conditions for the vertical velocity in the free troposphere, Eqs. (7) and (8), we can construct the induced vertical component of the Ekman flow in the free troposphere. This flow is compensated by a meridional flow, and thus a stream function can be constructed. The flow is given by

$$\begin{bmatrix} V_E \\ W_E \end{bmatrix} = \begin{bmatrix} -\Delta v^B \\ -(\pi - z)\,\Delta v^B_y \end{bmatrix} = \begin{bmatrix} \psi_z \\ -\psi_y \end{bmatrix} \quad \text{where} \quad \psi = (\pi - z)\,\Delta v^B. \tag{37}$$

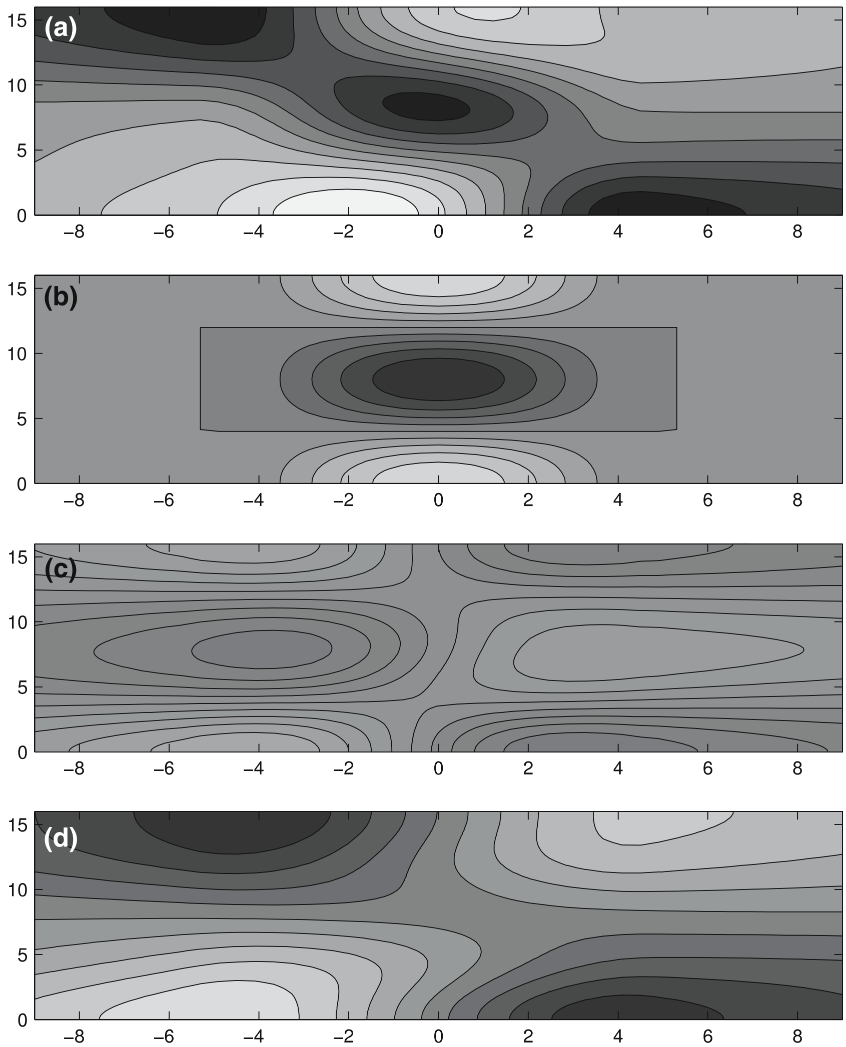

Fig. 6 Zonal flow above the equator for zonal extent of $\pm 10,000$ km, westerly (*white*), easterly (*black*). **a** Total flow associated with forcing in frame (a) of Fig. 5. **b** Zonal flow associated with induced stream function from frame (b) of Fig. 5. **c** Zonal flow associated with induced momentum forcing, frame (c) of Fig. 5. **d** Zonal flow associated with first baroclinic heating, frame (d) of Fig. 5

Again, the flow in the free troposphere is split into a component due to the Ekman pump and one which satisfies homogeneous boundary conditions for the vertical velocity at the top and bottom boundaries

$$\begin{bmatrix} \overline{V} \\ \overline{W} \end{bmatrix} = \begin{bmatrix} V^* \\ W^* \end{bmatrix} + \begin{bmatrix} -\Delta v^B \\ -(\pi - z)\,\Delta v^B_y . \end{bmatrix} \tag{38}$$

Therefore, in these new variables, the equations governing unforced planetary scale flows (5) only in the presence of boundary layer dissipation become

$$\begin{aligned} \overline{U}_t - yV^* + \overline{P}_X &= -\Delta\, d\, \frac{y^2}{d^2 + y^2}\, \overline{U}_0 \\ y\overline{U} + \overline{P}_y &= 0 \\ \overline{\Theta}_t + W^* &= (\pi - z)\ \Delta \left[\frac{d\, y}{y^2 + d^2}\, \overline{U}_0 \right]_y \\ \overline{P}_z &= \overline{\Theta} \\ \overline{U}_X + V^*_y + W^*_z &= 0 \end{aligned} \tag{39}$$

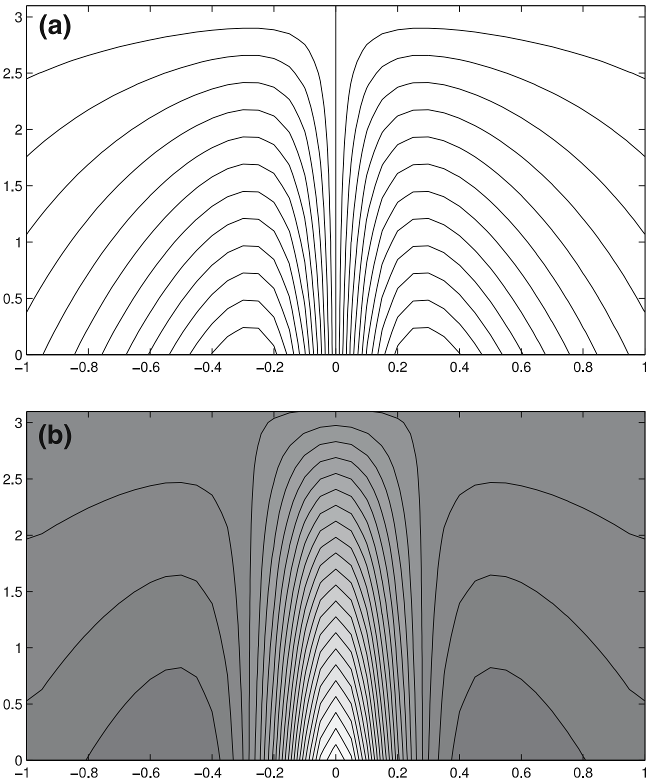

Fig. 7 **a** Stream function induced by Ekman pump of a lower troposphere, equatorially confined, westerly jet. **b** Induced heating associated with Ekman pump: (*white*) positive, (*gray*) negative

where Eq. (9) has been used to eliminate the boundary-layer velocity in favor of the zonal velocity at the base of the troposphere, $\overline{U}_0$. Since $\overline{U}_0$ is clearly independent of height in the troposphere this form of the equations lends itself to a straightforward interpretation: the Ekman boundary-layer dissipation has three consequences for the free troposphere.

The first consequence is a vertical flow emanating from the base of the troposphere which compensates for the meridional convergence of flow in the barotropic boundary layer. Wherever the zonal velocity at the base of the troposphere is positive (negative) near the equator the flow in the boundary layer diverges away from (converges toward) the equator driving a downward (upward) flow in the free troposphere to compensate. Since this circulation is divergence free, downward (upward) pumping in the free troposphere induces a meridional flow which is toward (away from) the equator. Notice that the tropospheric meridional flow is in the opposite direction of the boundary-layer meridional flow.

The second consequence is a momentum forcing which only dissipates the barotropic mode since it is independent of height; this term appears on the right-hand side of the momentum equation in (39). The strength of this forcing is proportional to the total zonal velocity at the base of the troposphere and of the opposite sign, thereby coupling the barotropic mode to all of the baroclinic modes.

The third consequence is that the Ekman pump provides active heating to the baroclinic modes. According to the right-hand side of the potential temperature equation, this heating decreases linearly with height in the troposphere and is positive (negative) wherever the zonal velocity at the base of the troposphere near the equator is negative (positive). In particular a downward (upward) Ekman pump in the free troposphere induces heating (cooling) near the equator. Though this may seem counterintuitive, downward flow induces heating because it is advecting the background *potential* temperature, which increases with height in the upper troposphere.

Figure 7 shows the stream function [frame (a)] and the induced heating [frame (b)] associated with a westerly jet at the base of the troposphere. The induced flow in the troposphere is downward at the equator and equator-ward elsewhere. The induced heating is positive above the equator and negative elsewhere.

5 Discussion

We describe a pair of velocity transformations that separately remove the meridional and vertical components of the upscale temperature flux in favor of induced circulation and modified momentum forcing in the multiscale IPESD framework derived by Majda and Klein in [11]. We have used examples from the MJO models of Biello and Majda [10,5,7] to demonstrate that convergence is possible for the transformation of the meridional temperature flux without any restriction. We also demonstrate that, if the zonal mean of the vertical component of the temperature flux convergence vanishes, an analogous transformation can be performed to remove this component in favor of a forcing in the momentum equations. The meridional transformation also allows a reinterpretation of lower troposphere Ekman dissipation as active heating due to the vertical advection of the background potential temperature gradient by the induced circulation, plus zonal momentum drag. The meridional temperature flux transformation and induced meridional circulation is a new, tropical wave example of the transformed Eulerian mean theory in the case of strong vertical stratification of potential temperature [2–4]. Constructing the expression for conservation of pseudo-momentum for the linear long wave equations (5) we find

$$(P_r + P_G) + \nabla \cdot \begin{bmatrix} \frac{1}{2}\left(\overline{U}^2 + \overline{\Theta}^2\right) \\ \overline{U}\,\overline{V} \\ \overline{U}\,\overline{W} - y\overline{V}\,\overline{\Theta} \end{bmatrix} = \left(\overline{U}_y - y\overline{\Theta}_z\right)\left(yF_z^\theta - F_y^U\right) + U_z\,F^\theta + \theta\,F_z^U \tag{40}$$

where

$$P_r = -\frac{1}{2}\left(\overline{U}_y - y\,\overline{\Theta}_z\right)^2 \quad \text{and} \quad P_G = \overline{U}_z\,\overline{\Theta} \tag{41}$$

and the equations have not been zonally averaged as in the standard pseudo-momentum expressions. We note that the meridional transformation is a symmetry of the pseudo-momentum equations (40) however the zonal/vertical transformation does not arise from this expression in any neat way. This reflects the fact that the meridional transformation is valid without restriction, whereas the zonal transformation only applies for certain types of momentum forcing.

The asymptotic ordering of the flows, that is to say, a much weaker planetary-scale component of the meridional and vertical circulation, means that the removal of the meridional temperature flux convergence has implications for how planetary-scale heating rates are inferred from velocity convergence measurements [20]. An interesting question for both observations and theory is how much of an upscale eddy temperature flux convergence is actually mean heating on the planetary scales; the method for deriving heating rates from dynamic measurements was outlined in [20]. The multiscale MJO models of Biello and Majda [10,5,7] assume a separation of scales between a planetary-scale mean and synoptic-scale fluctuating component of the mean heating. In turn, planetary-scale flows are linearly proportional to the strength of the planetary-scale mean heating rate and proportional to the square of the synoptic-scale fluctuating heating, through the upscale fluxes of momentum and temperature. Though the separation of the heating rate into multiple scales is clearly defined in theory, it is not necessarily so clearly achieved in practice, especially when these heating rates depend on the unresolved dynamics. The transformation of the meridional circulation discussed in Sect. 2 removes the meridional temperature flux convergence in favor of a modified momentum forcing and a weak meridional/vertical (Hadley-like) circulation, which is unlikely to be measurable above the signal of the synoptic scale fluctuations. If one were to invert the IPESD model to infer heating rates from the dynamics, then such a prescription would seem tractable for synoptic-scale heating rates where each of the components of the velocity field are the same order of magnitude. On the planetary scales, inverting the planetary scale velocities to infer heating and forcing rates is both difficult because of the small magnitude of the fields and accelerations, but also ambiguous due to the transformation inducing the weak meridional circulation. This fact may have some bearing on how measurements of heating rates are interpreted in the context of multiscale theories. The authors shall pursue this reexamination of measurements in future work.

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