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TRANSIENT BEAM LOADING IN ELECTRON-POSITRON STORAGE RINGS

P.B. Wilson

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1. Introduction

In this note the fundamentals of transient beam loading in electron-positron storage rings will be reviewed. The notation, and some of the material, has been introduced previously in Ref. 1. The present note is, however, more tutorial in nature, and in addition the analysis is extended to include the transient behaviour of the cavity fields and reflected power between bunch passages. Since we are not bound here by the rigid space limitations of a paper for publication, an attempt is made to give a reasonably coherent and complete discussion of transient beam loading that can hopefully be followed even by the uninitiated.

The discussion begins with a consideration of the beam-cavity interaction in the "single-pass" limit. In this limit it is assumed that the fields induced in the cavity by the passage of a bunch have decayed essentially to zero by the time the next bunch has arrived. The problem of the maximum energy that can be extracted from a cavity by a bunch is given particular attention, since this subject seems to be the source of some confusion. The analysis is then extended to the "multiple-pass" case, where the beam-induced fields do not decay to zero between bunches, and to a detailed consideration of the transient variation of cavity fields and reflected power. The note concludes with a brief discussion of the effect of transient beam loading on quantum lifetime.

It is assumed throughout that all rf generators external to the cavity operate in the continuous (cw) mode. The case of pulsed rf will be dealt with in a future note.

2. Some Fundamentals

Cavity voltages are taken to be complex (or phasor) quantities, written with a tilde (e.g., $\tilde{V}$). The absolute value of the voltage, written without a tilde, is the maximum energy in electron volts that can be gained by a small non-perturbing test charge which traverses the cavity at the velocity of light. Assuming $\tilde{V} = V e^{j\omega t}$, a reference plane is defined in the cavity by the position of this maximally accelerated test charge at t = 0. If, for example, the cavity is symmetrical, the plane of symmetry (midplane) is the reference plane. Thus the projection of V on the real axis gives the energy gain for a test charge which crosses the reference plane at an arbitrary phase. It is almost self-evident that the phase of the beam-induced voltage must lie along the negative real axis. We will return to this point again in Section 4.

The analysis following will be carried out assuming a point bunch. Effects due to non-zero bunch length are then readily computed by considering the bunch to be composed of an infinite number of vanishingly small slices and performing an appropriate integration. In carrying out the integration, the principle of superposition must be invoked. Suppose, for example, that a point charge q induces a voltage V_0 in a particular cavity mode. A charge dq then induces a voltage $dV = (V_0/q)dq$. Each element $d\tilde{V}$ of induced voltage is assumed to ring as

$$d\tilde{V} = -dV\,e^{j\omega_0 t'}$$

where ω_0 is the resonant angular frequency of the mode and t' is time of passage of dq across the reference plane as defined above. Each element $d\tilde{V}$ then rings with a different phase, since the passage time t' is different for each dq. By superposition we obtain the total induced voltage by summing over all the elements $d\tilde{V}$, assuming that each element of induced voltage rings independently of any effect due to the presence of all the other charge elements. As an example, it is readily shown in this manner that a gaussian bunch with standard deviation σ_t will induce a total voltage

$$V = V_0 e^{-\omega_0^2 \sigma_t^2/2}$$

where V_0 is the voltage induced by a point bunch of the same charge. Expressions for computing the bunch-shape form factor for bunches of arbitrary shape are given in Ref. 2. More commonly, superposition will be used in this note by assuming that the total voltage seen by a charge crossing a cavity is the vector sum of an externally applied generator voltage and a component due to the voltage induced by the charge itself. The magnitude and phase of each component is assumed to be unaffected by the presence of the other component.

A second powerful law that will be called upon is conservation of energy. In particular, if a charge passes through a cavity, then conservation of energy requires that the energy lost (or gained) by the charge be equal to the increase (or decrease) of energy stored in the cavity fields. As we shall see in the next two sections, these two laws -- superposition and conservation of energy -- are sufficient to prove several interesting and useful theorems about beam loading. This application of superposition and conservation of energy to the computation of the charge-cavity interaction is not new (see, for example, Ref. 3). The present effort extends these previous results and attempts to make the physics of the problem easier to visualize by the liberal use of vector diagrams.

It should be noted that the analysis presented here is carried out in the time domain. The transient behavior of the cavity fields and the reflected power is worked out by applying superposition using simple vector diagrams A parallel analysis can be carried through in the frequency domain.[4] The advantage of the time domain, for the author at any rate, is that it provides a more direct physical feeling for the problem. It is important to recognize however, that the time domain analysis breaks down when the charge is actually inside the cavity. It would be an extremely difficult task indeed to work out in detail the development of the beam-induced fields as a function of time for a cavity of arbitrary shape.* It is simpler to treat the cavity as a black box, with a certain energy transfer between the charge and each of the normal modes in the cavity. For our purposes here, it can be assumed that the beam-induced fields appear instantaneously when the charge crosses the reference plane in the cavity, as defined at the beginning of this section.

3. Efficiency for Energy Extraction from a Cavity

Consider first the energy lost by a charge to all modes in a cavity when the cavity is initially empty of any stored energy. Let ΔU_{0e} be the energy lost to the fundamental mode, where the subscript e emphasizes the fact that the mode is initially empty. The total energy lost to all modes can be written

$$\Delta U_{te} = B\,\Delta U_{0e} \quad ,$$

*Chao and Morton[5] have solved this problem for two infinite conducting plates perpendicular to the beam axis, but without a beam hole. It is difficult enough to solve the problem of the development in time of the beam-induced fields even for this simple geometry.

where B is called the beam loading enhancement factor. The energy lost to higher-order modes only is

$$\Delta U_{hm} = \Delta U_{te} - \Delta U_{oe} = (B - 1)\Delta U_{oe} \quad . \tag{1}$$

After the charge has left the cavity, a beam-induced voltage $\tilde{V}_{bo}$ and a corresponding stored energy W_o remain in the fundamental mode, where

$$W_o = \alpha V_{bo}^2 \quad . \tag{2}$$

By conservation of energy, $\Delta U_{oe} = W_o$ and Eq. (1) becomes

$$\Delta U_{hm} = \alpha(B - 1) V_{bo}^2 \quad . \tag{3}$$

By superposition, $\tilde{V}_{bo}$ will be the same even if there is energy stored in the fundamental mode before the arrival of the bunch. Equation (3) is therefore valid also when the fundamental mode is driven by an external generator.

Assume now that this is the case and consider the superposition of voltages for the fundamental mode. The generator voltage component is assumed to vary as $\tilde{V}_g = V_g e^{j\omega t}$. The phase of this voltage is taken to be ϕ^+ with respect to the real axis just before the arrival of the bunch at the reference plane in the cavity. The bunch is assumed to induce a voltage $\tilde{V}_{bo}$, which appears instantaneously when the bunch crosses the reference plane. As mentioned previously, it is <u>almost</u> obvious that the phase of the induced voltage is exactly opposite to the phase of the voltage which would produce maximum energy gain (see Sec. 4). After the bunch has crossed the reference plane, the net cavity field in the fundamental mode is, by superposition,

$$\tilde{V}_0^- = \tilde{V}_0^+ + \tilde{V}_{bo} \quad ,$$

where $\tilde{V}_{bo}$ is real and negative and $\tilde{V}_0^+$ is the voltage produced by the external generator, $\tilde{V}_0^+ = \tilde{V}_g$. This vector sum is illustrated in Fig. 1.

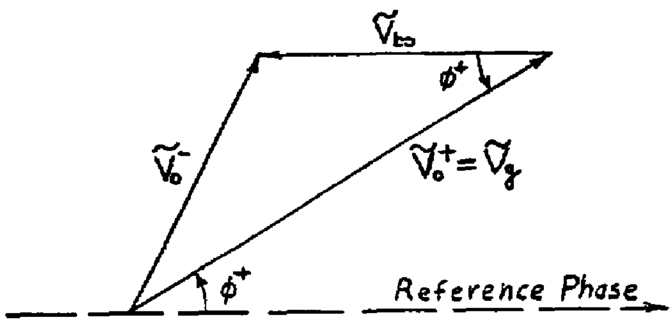

Fig. 1 -- Vector addition of generator and beam-induced voltages in the fundamental mode.

Applying the law of cosine to the vector triangle in Fig. I,

$$V_0^{-2} = V_0^{+2} + V_{bo}^2 - 2V_0^+V_{bo} \cos \phi^+ \quad .$$

From conservation of energy, the energy gained by the bunch from the fundamental mode must be equal to the difference in the energy stored in the cavity before and after the passage of the bunch. Since the constant α in Eq. (2) relates any stored energy to the square of the corresponding voltage, we have

$$\Delta U_0 = W_0^+ - W_0^- = \alpha(V_0^{+2} - V_0^{-2})$$

$$= \alpha(2V_0^+V_{bo} \cos \phi^+ - V_{bo}^2) \quad .$$

To obtain the net energy extracted from the cavity, we must subtract off the energy radiated back into higher-order modes, as given by Eq. (3). Thus

$$\begin{aligned}\Delta U_{net} &= \Delta U_0 - \Delta U_{hm} \\ &= \alpha(2V_0^+ V_{b0} \cos \phi^+ - BV_{b0}^2) \quad .\end{aligned}$$

This result has been derived previously by Morton and Neil.[3] The efficiency for net energy extraction is now

$$\eta = \frac{\Delta U_{net}}{W_0^+} = 2\left(\frac{V_{b0}}{V_0^+}\right)\cos \phi^+ - B\left(\frac{V_{b0}}{V_0^+}\right)^2 \quad . \tag{4}$$

The maximum efficiency as a function of V_{b0} for a given V_0^+ is readily obtained as

$$\eta_{max} = \frac{\cos^2 \phi^+}{B} \tag{5}$$

at a beam-induced voltage

$$V_{b0} = \frac{V_0^+ \cos \phi^+}{B} \quad .$$

Equation (5) is seen to be the condition for maximum energy extraction derived by Keil, Schnell and Zotter.[6] For brevity we will call it the KSZ criterion. The variation in efficiency as a function of V_{b0} (which is proportional to the charge q) is shown in Fig. 2 below. The constant of proportionality between V_{b0} and q will be derived in Section 4.

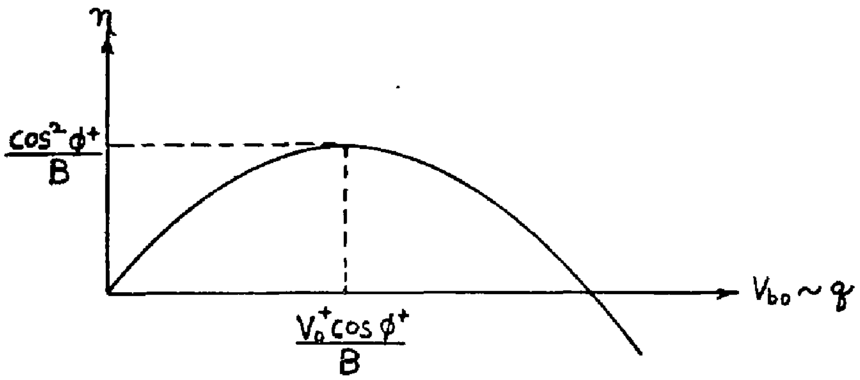

Fig. 2 -- Efficiency for net energy extraction as a function of beam-induced voltage.

Two comments about Eq. (5) are in order. We note first of all that ϕ^+ is not the synchronous phase angle. It is the phase of the cavity voltage before the arrival of the bunch. The synchronous phase angle ϕ is the phase of the effective cavity field acting in the bunch and will lie at some angle $\phi > \phi^+$, since the bunch will "see" some fraction of its own beam-induced voltage. The exact relation between ϕ and ϕ^+ will be derived in the following section.

The second comment concerns the maximum efficiency for energy extraction when there are two counter-rotating beams in a storage ring. We assume that the cavity is located so that the fields induced by the q^+ and q^- charges are coherent; that is, the cavities are located an integral multiple of half-wavelengths from an interaction point. On the other hand, it is reasonable to assume that the fields induced in the higher cavity modes are, on the average, incoherent (see discussion in Ref. 7). It can be shown that the

KSZ criterion becomes for this case

$$\eta_{max} = \frac{\cos^2 \phi^+}{(B + 1)/2} \quad . \tag{6}$$

The proof is left to the interested reader.

4. The Fundamental Theorem of Beam Loading

Consider a point charge crossing an initially empty cavity. After the charge has passed out of the cavity, a beam-induced voltage V_{bn} remains in each mode. What fraction of V_{bn} does the charge itself see as it crosses the cavity? Since the induced voltage for mode n starts at zero and ends up at V_{bn}, the most naïve assumption is to take the average, or $\frac{1}{2}V_{bn}$, as the effective fraction of the self-induced voltage acting on the charge. In this section we will prove that this factor of one-half is exact for any cavity. The fact that a charge sees exactly one-half of its own beam-induced voltage we will call the fundamental theorem of beam loading. This theorem is basic to a correct computation of the effective cavity voltage acting on a bunch when both a generator voltage and a beam-induced voltage are present. The theorem also provides the key which relates the energy loss by a charge crossing a cavity to the cavity parameters computed in the absence of charge. Following is one of several possible proofs of the theorem.

Assume that a charge q_1 crossing a cavity sees a fraction f of its own induced voltage V_{b1} for a particular mode. The energy lost by the charge is then

$$\Delta U_1 = q f V_{b1} \quad . \tag{7}$$

Now let a second equal charge follow the first charge by exactly one-half of an rf wavelength of the mode in question. The sequence of events is illustrated in the diagram of Fig. 3. When the second charge arrives at the cavity reference plane, the field induced by the first charges has shifted phase by 180° and is now maximally accelerating for the second charge. Moreover, the induced field $\tilde{V}_{b2}$ of the second equal charge will exactly cancel the field from the first charge, which is now $-\tilde{V}_{b1}$. Thus no stored energy will remain in the cavity after the second charge has passed through. By superposition, the energy gained by the second charge will be

$$\Delta U_{1} = q(V_{b1} - fV_{b2}) = qV_{b1}(1 - f) \quad , \tag{8}$$

since the charges are equal and $V_{b1} = V_{b2}$. Because the stored energy in the in the cavity is zero before and after the passage of the two charges, we must have by conservation of energy that the energy lost by the first charge is equal to the energy gained by the second charge. Equating $\Delta U_1 = \Delta U_2$ in Eqs. (7) and (8),

$$q\,f\,V_{b1} = q\,V_{b1}(1 - f)$$

$$f = \frac{1}{2} \quad .$$

By an extension of the proof, it can be shown that conservation of energy will also be violated if the phase of the beam-induced voltage is not exactly -180° with respect to the phase of the field which would maximally accelerate a test charge following the same space-time trajectory.

Substituting $f = \frac{1}{2}$ in Eq. (7), we have for the energy lost to an initially empty cavity

$$\Delta U = \frac{1}{2}\,q\,V_b \quad . \tag{9}$$

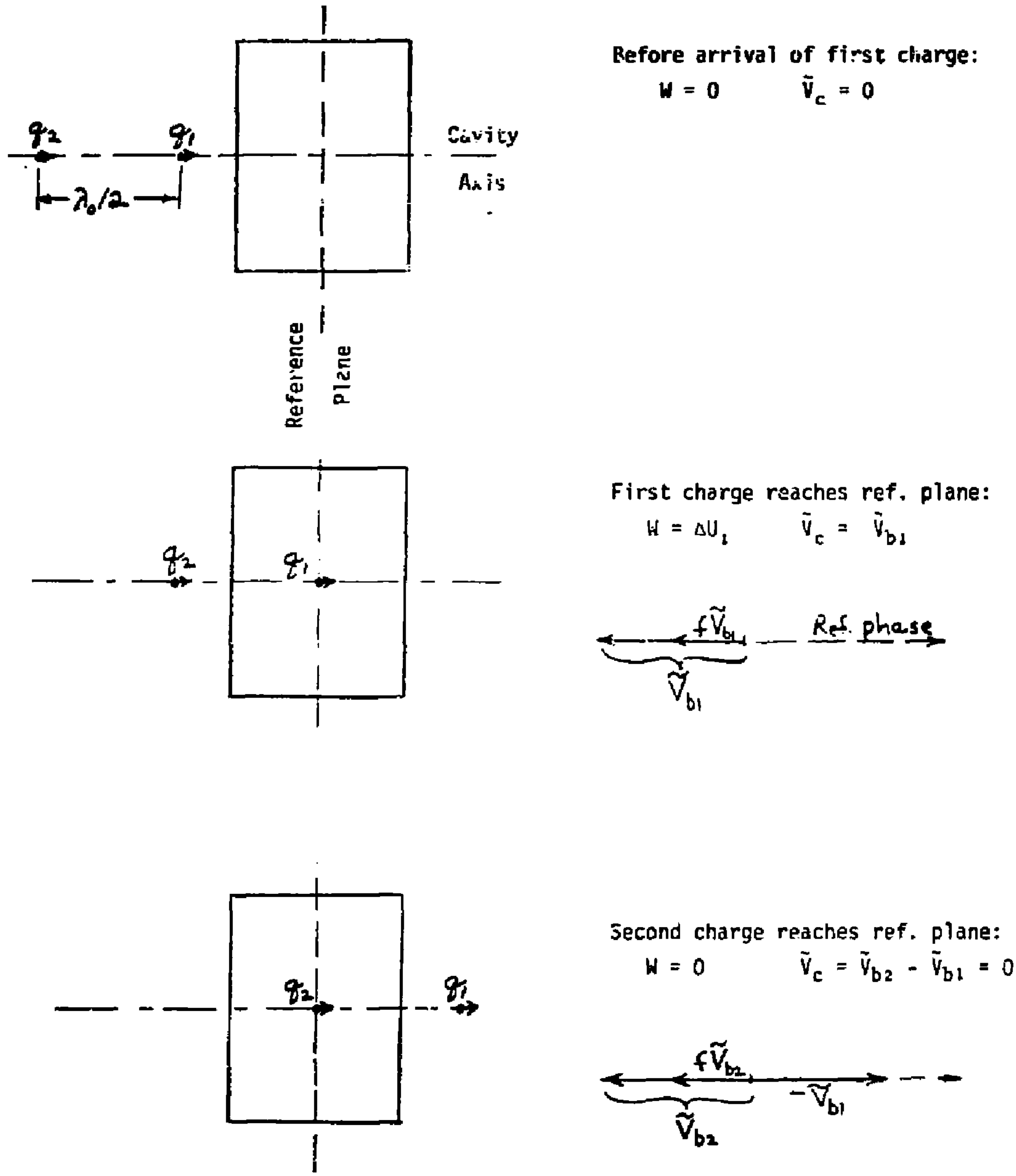

Fig. 3 -- Thought experiment proving the fundamental theorem of beam loading.

The loss parameter k for radiation into an empty cavity is defined by

$$\Delta U = kq^2 \quad . \tag{10}$$

From Eqs. (2), (9) and (10) we obtain

$$V_b = 2kq = \frac{1}{2\alpha} q \tag{11a}$$

$$k = \frac{V_b^2}{4\Delta U} = \frac{V_b^2}{4W_b} = \frac{V^2}{4W} \quad , \tag{11b}$$

where $W_b = \Delta U$ is the stored energy in the cavity for a beam-induced voltage V_b. The expression for k in Eq. (11b) is valid whatever the source of cavity voltage, so $k = V^2/4W$ can be computed from the electromagnetic properties of a charge-free cavity. In this case V is the maximum voltage seen by a non-perturbing test charge traveling across the cavity at the speed of light when the stored energy is W. It is obtained from cavity programs (e.g., LALA and SUPERFISH) by finding the absolute value of the integral

$$V = \left| \int_0^L E_z(z)\; e^{(j\omega_0 z/c)} dz \right|$$

along the cavity axis, where ω_0 is the mode frequency and L the cavity length.

The parameter R/Q for a cavity is defined by

$$\frac{R}{Q} = \frac{V^2}{\omega W} \quad .$$

From Eq. (11b) we then have

$$k = \frac{\omega}{4} \left(\frac{R}{Q} \right) \quad . \tag{12}$$

The effective voltage "seen" by a charge crossing an initially empty cavity can be defined as

$$\Delta U = qV_{eff}$$

where ΔU is the energy lost to a particular mode. From Eqs. (9) and (10) we then have

$$V_{eff} = kq = \frac{1}{2} V_b \quad . \tag{13}$$

Since V_{eff} is a voltage, superposition can be applied to construct a vector diagram (Fig. 4) which shows graphically how the single-pass net energy gain for a charge q can be computed, taking into account both the voltage provided by an external generator and the voltage induced by the charge itself.

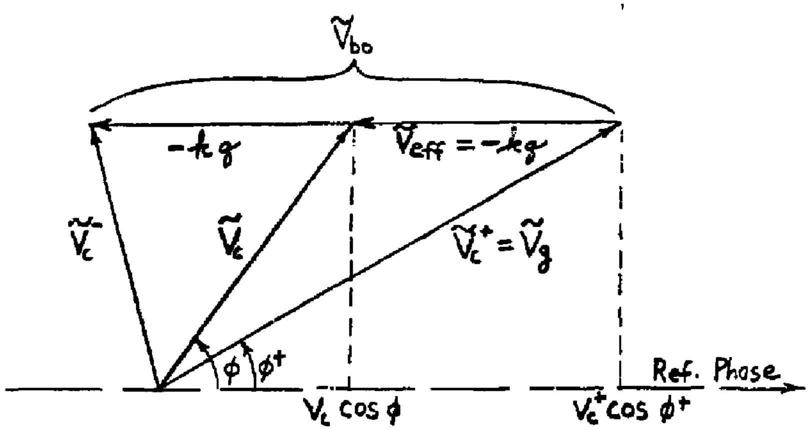

Fig. 4 -- Vector diagram showing construction of the net single-pass energy gain V_c cos ϕ for a driven cavity with beam loading.

Let us return now to the question of the maximum energy extraction according to the KSZ criterion given in Eq. (5). Recall that KSZ was derived by invoking superposition and conservation of energy. The formalism of Fig. 4 for computing the net accelerating voltage is based on the fundamental theorem of beam loading, which was also derived assuming superposition and conservation of energy. It seems reasonable to expect, therefore, that the net beam-loaded energy gain $V_c \cos\phi$ as computed using the construction of Fig. 4 will be consistent with KSZ. That this is indeed so is shown explicitly in Appendix A.

5. Multiple-pass Build-up of the Beam-induced Voltage

We next calculate the turn-to-turn build-up of the field in a cavity when a charge (bunch) passes repetitively through the cavity, as is the case in a storage ring. The situation is illustrated graphically in Fi . 5.

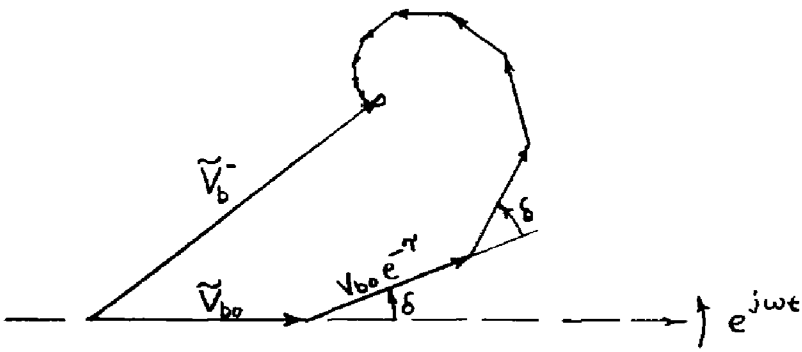

Fig. 5 -- Diagram illustrating the multiple-pass build-up of a beam-induced voltage in a cavity.

Here $\tilde{V}_{b0}$ is the single-pass beam-induced voltage (for simplicity, shown here as positive and real), $e^{-\tau}$ is the decay of the cavity field during one turn, δ is the net phase shift per turn (subtracting off multiples of 2π) and $\tilde{V}_b^-$ is the final cavity voltage for $t \to \infty$. The minus superscript emphasizes that fact that $\tilde{V}_b^-$ is the equilibrium voltage just <u>after</u> the passage of a bunch. The decay parameter τ can be written

$$\tau = \frac{T_b}{T_f} \quad , \tag{14a}$$

where T_b is the passage time between bunches (assuming equal bunches in a multiple-bunch machine) and $T_f = 2Q_L/\omega_0$ is the cavity filling time. The phase shift δ is

$$\delta = T_b\omega_0 - 2\pi h_b = T_b(\omega_0 - \omega) \quad , \tag{14b}$$

where ω_0 is the resonant frequency of the cavity and h_b, an integer, is the harmonic number for a single-bunch machine, or the number of rf wavelengths between bunches for a ring with more than one bunch. In constructing Fig. 5 we can therefore consider the reference phase to be rotating at the angular frequency ω of the external rf generator. It is natural to use the external generator as the basic clock for describing field variations in the cavity, since the spacing of bunches in a storage ring is determined by the generator frequency and not by the cavity resonant frequency.

The final field is now readily obtained as the sum of a geometric series

$$\begin{aligned} \tilde{V}_b^- &= \tilde{V}_{b0}(1 + e^{-\tau} e^{j\delta} + e^{-2\tau} e^{j2\delta} + \ . \ . \ . \ ) \\ \frac{\tilde{V}_b^-}{\tilde{V}_{b0}} &= \frac{1}{1 - e^{-\tau} e^{j\delta}} \end{aligned} \tag{15}$$

After many turns, when the induced field approaches $\tilde{V}_b^-$, what is the effective field seen by a bunch passing through the cavity? By superposition, the bunch will see the field present in the cavity from all previous bunch passages, plus exactly one-half of the total induced voltage arising from its current passage. Let $\tilde{V}_b^+$ be the voltage in the cavity just before the bunch arrives. Then the net voltage $\tilde{V}_b$ acting on the bunch is

$$\tilde{V}_b = \tilde{V}_b^+ + \frac{1}{2}\tilde{V}_{bo} \quad .$$

Since

$$\tilde{V}_b^- = \tilde{V}_b^+ + \tilde{V}_{bo} \quad ,$$

we have also

$$\tilde{V}_b = \tilde{V}_b^- - \frac{1}{2}\tilde{V}_{bo} \quad . \tag{16}$$

The diagram in Fig. 5 shows the build-up of the induced voltage just after the passage of a bunch: $\tilde{V}_{bo}$ is the induced field for the bunch that has just passed, and the other vectors are the diminished and rotated voltages remaining from previous bunch passages. The relationship between $\tilde{V}_b$, $\tilde{V}_{bo}$, $\tilde{V}_b^+$ is shown in Fig. 6, where we now assume that $\tilde{V}_{bo}$ lies along the negative real axis, representing an energy loss; that is, $\tilde{V}_{bo} = - V_{bo}$. From Eqs. (15) and (16),

$$\frac{\tilde{V}_b}{-V_{bo}} = \frac{1}{1 - e^{-\tau} e^{j\delta}} - \frac{1}{2} = F_1(\tau,\delta) + jF_2(\tau,\delta) \tag{17a}$$

$$F_1(\tau,\delta) = \frac{1 - e^{-2\tau}}{2(1 - 2e^{-\tau}\cos\delta + e^{-2\tau})} \tag{17b}$$

$$F_2(\tau,\delta) = \frac{e^{-\tau}\sin\delta}{(1 - 2e^{-\tau}\cos\delta + e^{-2\tau})} \quad . \tag{17c}$$

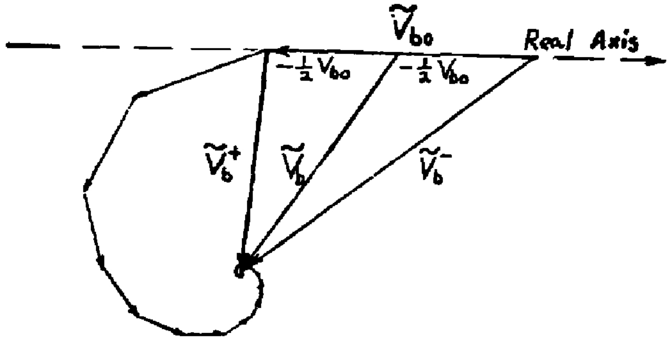

Fig. 6 -- Diagram showing relation between the effective beam-induced voltage $\tilde{V}_b$ and the induced voltages just before and after the passage of a bunch.

The quantities δ and V_{b_0} in Eqs. (17) can be expressed in terms of more usual cavity parameters. The tuning angle ψ is defined by

$$\tan\psi = 2Q_L \frac{(\omega_0 - \omega)}{\omega_0} = T_f(\omega_0 - \omega) \quad .$$

From Eqs. (14) we then have

$$\tan\psi = \delta\left(\frac{T_f}{T_b}\right) = \frac{\delta}{\tau} \quad . \tag{18}$$

From Eqs. (11a) and (12) we can also write

$$V_{bo} = 2kq = \frac{\omega_0}{2}\left(\frac{R}{Q}\right)q = \frac{i_0 R}{1+\beta}\,\tau \quad , \tag{19}$$

where β is the cavity coupling coefficient such that $Q_0 = (1 + \beta)Q_L$, and $i_0 = q/T_b$ is the circulating current (a single beam is assumed for the moment). It is useful to introduce the parameter τ_0, where

$$\tau_0 = \frac{\tau}{1 + \beta} = \frac{T_b}{T_{f0}} \quad . \tag{20}$$

Here, T_{f0} is the unloaded filling time of the cavity, $T_{f0} = 2Q_0/\omega_0$. Equation (19) then becomes

$$V_{b0} = i_0 R \tau_0 \tag{21}$$

and Eqs. (17) become

$$\tilde{V}_b = - i_0 R\tau_0 \left[F_1(\beta,\psi) + jF_2(\beta,\psi)\right] \quad , \tag{22a}$$

where

$$F_1(\beta,\psi) = \frac{1 - e^{-2\tau_0(1+\beta)}}{2D} \tag{22b}$$

$$F_2(\beta,\psi) = \frac{e^{-\tau_0(1+\beta)} \sin\left[\tau_0(1 + \beta) \tan\psi\right]}{D} \tag{22c}$$

$$D = 1 - 2e^{-\tau_0(1+\beta)} \cos\left[\tau_0 (1 + \beta) \tan\psi\right] + e^{-2\tau_0(1+\beta)} \quad .$$

6. Computation of the Generator Power

In a storage ring the desired cavity voltage is usually given. That is, a certain accelerating voltage $V_c \cos\phi$ and synchronous phase angle ϕ are required. The beam-induced voltage is given in terms of the beam current and cavity parameters by Eqs. (22). The required generator voltage is then obtained from

$$\tilde{V}_g = \tilde{V}_c - \tilde{V}_b \quad , \tag{23}$$

since the net cavity voltage is the vector sum of the generator and beam-induced voltages. This sum is illustrated in Fig. 7, in which a constant generator voltage $\tilde{V}_g$ has been added to the beam-induced voltages shown in Fig. 6. For a resonant cavity, the generator voltage and generator power

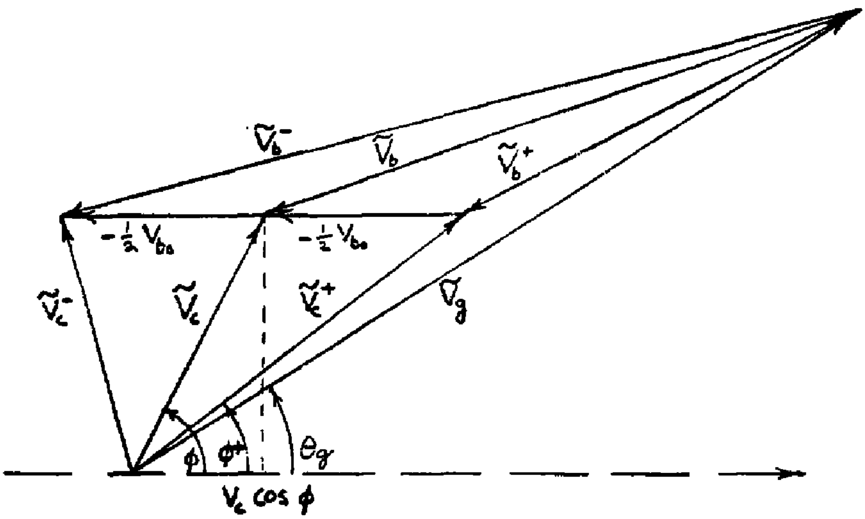

Fig. 7 -- Vector sum of voltages in a beam-loaded cavity driven by an external generator.

are related (see, for example, Ref. 1) by

$$V_g = (P_g R)^{\frac{1}{2}} \frac{2\beta^{\frac{1}{2}}}{1+\beta} \cos\psi \quad . \tag{24}$$

Here, P_g is the incident power, and it is assumed that the generator is matched by (for example) the use of a circulator. We now take the real and imaginary components of Eq. (23). With the aid of Fig. 7 and Eq. (22a) we can write the following expressions for the two components:

$$V_g \cos\theta_g = V_c \cos\phi + V_{b0} F_1(\beta,\psi) \tag{25a}$$

$$V_g \sin\theta_g = V_c \sin\phi + V_{b0} F_2(\beta,\psi) \quad , \tag{25b}$$

where V_{b0} is given by Eq. (21) and θ_g is the phase angle of the generator voltage. Squaring and adding the two expressions, and then eliminating V_g with the aid of Eq. (24), we have finally

$$P_g = \frac{V_c^2}{R\cos^2\psi} \cdot \frac{(1+\beta)^2}{4\beta} \cdot \left\{\left[\cos\phi + \frac{i_0 R \tau_0}{V_c} F_1(\beta,\psi)\right]^2 + \left[\sin\phi + \frac{i_0 R \tau_0}{V_c} F_2(\beta,\psi)\right]^2\right\} . \tag{26}$$

It will prove useful also to find the phase angle of the generator voltage. Dividing Eq. (25b) by Eq. (25a),

$$\tan\theta_g = \frac{V_c \sin\phi + V_{b0} F_2(\beta,\psi)}{V_c \cos\phi + V_{b0} F_1(\beta,\psi)} \quad . \tag{27}$$

For two equal counter-rotating beams of electrons and positrons, Eqs. (26) and (27) can be applied if i_0 is replaced by $2i_0$.

For $\tau_D \ll 1$, Eq. (26) reduces to

$$P_g = \frac{V_c^2}{R\cos^2\psi} \cdot \frac{(1+\beta)^2}{4\beta} \cdot \left\{ \left[\cos\phi + \frac{i_0 R}{V_c(1+\beta)} \cos^2\psi \right]^2 + \left[\sin\phi + \frac{i_0 R}{V_c(1+\beta)} \cos\psi \sin\psi \right]^2 \right\} ,$$

which is the expression for the required generator power in the absence of transient loading effects (see, for example, Ref. 1). The minimum generator power as a function of ψ is, for this case,

$$P_g(\psi_m) = \frac{V_c^2}{R} \cdot \frac{(1+\beta)^2}{4\beta} \left[1 + \frac{i_0 R \cos\phi}{V_c(1+\beta)} \right] \qquad (28a)$$

where

$$\tan\psi_m = -\frac{i_0 R}{V_c(1+\beta)} \sin\phi \quad . \qquad (28b)$$

The coupling coefficient can in turn be adjusted to minimize the generator power given by Eq. (28a). The result is

$$P_g(\beta_m) = \frac{V_c^2\,\beta_m}{R} \qquad (29a)$$

at

$$\beta_m = 1 + \frac{i_0 R \cos\phi}{V_c} \qquad (29b)$$

The tuning angle for β_m becomes, combining Eqs. (28b) and (29b),

$$\tan\psi(\beta_m) = -\left(\frac{\beta_m - 1}{\beta_m + 1}\right)\tan\phi \quad . \qquad (29c)$$

In the general case where $\tau_0 > 0$, the minimum generator power as a function of β and ψ can be obtained from Eq. (26) for a given cavity voltage V_c, synchronous phase angle ϕ, beam current i_0 and cavity parameters R and τ_0. It is not possible to write simple analytic expressions equivalent to Eqs. (28) and (29), so the optimizations of β and ψ must be done numerically. It is found that the increase in minimum generator power due to transient effects is not very large (less than a few percent) for typical storage ring designs at τ_0 up to about 0.5. For $\tau_0 > 1$, the generator power increases rapidly compared to the power required in the continuous beam limit. For large τ_0, where the time between bunches is large compared to the cavity filling time, it is clear that some sort of pulsed rf system would be desirable. In such a system power is applied to the cavities for about a filling time preceding the arrival of the bunch. For most of the period between bunches, there is no stored energy in the cavities and hence no power dissipation.

From Fig. 7, the net accelerating voltage per turn acting on charge q is $V_a = V_c \cos \phi$. From Eq. (25a) the energy gained by the charge from the fundamental cavity mode is, using $V_{b0} = 2kq$,

$$\Delta U = qV_c \cos \phi = qV_g \cos \theta_g - kq^2\left[2F_1(\tau)\right] \quad .$$

Since kq^2 is the energy loss for a single passage of the bunch through an empty cavity, the factor $2F_1(\tau)$ takes into account the cumulative effect of the charge passing through the cavity on successive revolutions. From Eq. (17b) the factor $2F_1$ is seen to approach unity for large τ, as it should. For small τ, it is convenient to rewrite the net accelerating voltage in a form which is more natural for a nearly continuous beam,

$$V_a = V_g \cos\theta_g - \frac{i_0 R}{1+\beta}\left[\tau F_1(\tau)\right] \quad .$$

As τ approaches zero, τF_1 reduces to $\cos^2\psi$.

7. Time Variation of the Cavity Voltage between Bunches

In Appendix B it is shown that if $\tilde{V}(\infty)$ is the steady-state voltage a resonant cavity would eventually reach for $t \to \infty$, the transient approach of the cavity voltage to $\tilde{V}(\infty)$ is given by

$$\tilde{V}(t) = \tilde{V}(\infty) + \left[\tilde{V}(0) - \tilde{V}(\infty)\right] e^{-(t/T_f)(1-j\tan\psi)} \quad , \qquad (30)$$

where $\tilde{V}(0)$ is the value of $\tilde{V}(t)$ at $t = 0$. To see how to apply this expression to the case of a beam-loaded cavity, we must first draw the appropriate vector diagram. This is shown in Fig. 8. When the bunch crosses the cavity reference plane, the cavity voltage changes instantaneously (in our model) from $\tilde{V}_c^+$ to $\tilde{V}_c^-$. The magnitude of the change is $- V_{b0}$. The voltage then begins to charge toward $\tilde{V}_g$ along the spiral path shown. At the precise moment the voltage once again reaches $\tilde{V}_c^+$, another bunch comes by to repeat the cycle. We can now make the following correspondences between the voltages in Eq. (30) and those in Fig. 8:

$$\begin{aligned} \tilde{V}(t) &\sim \tilde{V}_c(t) \\ \tilde{V}(0) &\sim \tilde{V}_c^- \\ \tilde{V}(\infty) &\sim \tilde{V}_g \quad . \end{aligned}$$

We have therefore

$$\tilde{V}_c(t) = \tilde{V}_g + (\tilde{V}_c^- - \tilde{V}_g)\, e^{-(t/T_f)(1-j\tan\psi)} \quad .$$

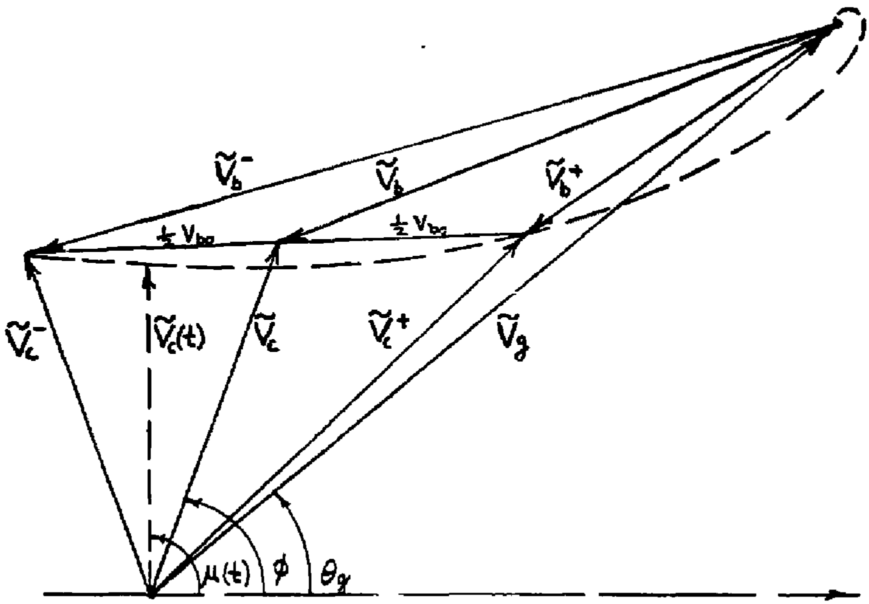

Fig. 8 -- Vector diagram illustrating the transient variation of voltages in a beam-loaded cavity.

But from the diagram

$$\tilde{V}_c^- - \tilde{V}_g = \tilde{V}_b^-$$

$$\tilde{V}_g + \tilde{V}_b^- + \frac{1}{2} V_{bo} = \tilde{V}_c \quad .$$

Therefore

$$\tilde{V}_c(t) = \tilde{V}_c + \tilde{V}_b^- \left[e^{-(t/T_f)(1-j\tan\psi)} - 1 \right] - \frac{1}{2} V_{bo} \quad .$$

To simplify the notation, we introduce a normalized time $x = t/T_b$, such that

$x = 1$ when t is equal to the arrival time of the next bunch. Recall also that $\tan\psi = \delta/\tau$. Substituting for $\tilde{V}_b^-$ from Eq. (15), and again taking into account that $\tilde{V}_{b0} = -V_{b0}$, we find

$$\tilde{V}_c(x) = \tilde{V}_c - \frac{V_{b0}[e^{-x\tau}e^{jx\delta} - 1]}{1 - e^{-\tau}e^{j\delta}} - \frac{V_{b0}}{2} \quad .$$

Separating this expression into real and imaginary components,

$$V_c(x)\cos\mu = V_c\cos\phi + V_{b0}F_A(x) \tag{31a}$$

$$V_c(x)\sin\mu = V_c\sin\phi + V_{b0}F_B(x) \tag{31b}$$

where

$$F_A(x) = \left[1 - e^{-2\tau} - 2e^{-x\tau}\cos x\delta + 2e^{-(1+x)\tau}\cos\delta(1-x)\right]/2D \tag{32a}$$

$$F_B(x) = \left[e^{-\tau}\sin\delta - e^{-x\tau}\sin x\delta - e^{-(1+x)\tau}\sin\delta(1-x)\right]/D \tag{32b}$$

$$D = 1 - 2e^{-\tau}\cos\delta + e^{-2\tau} \quad .$$

Squaring and adding Eqs. (31), and using $V_{b0} = i_0 R\tau_0$,

$$\frac{V_c^2(x)}{V_c^2} = \left[\cos\phi + \frac{i_0 R\tau_0}{V_c}F_A(x)\right]^2 + \left[\sin\phi + \frac{i_0 R\tau_0}{V_c}F_B(x)\right]^2 , \tag{33}$$

where $\tau = \tau_0(1 + \beta)$ and $\delta = \tau\tan\psi$. For a given τ_0, β and ψ are obtained by minimizing the generator power as given by Eq. (26). Equation (33), together with the definitions of F_A and F_B given by Eqs. (32), determine the transient variation in the amplitude of the cavity voltage. The transient variation in the phase of the cavity voltage is obtained by taking the ratio of Eqs. (31),

$$\tan \mu(x) = \frac{V_c \sin \phi + V_{bo} F_B(x)}{V_c \cos \phi + V_{bo} F_A(x)} \quad . \tag{34}$$

8. Reflected Power

The reflected power P_r can be computed using conservation of energy,

$$P_r = P_g - P_c - dW/dt \quad , \tag{35}$$

where P_g is the incident generator power, $P_c = V_c^2(t)/R$ is the instantaneous cavity dissipated power, and W is the stored energy given by

$$W(t) = \frac{V^2(t)}{\omega(R/Q)} = \frac{1}{2} T_{f0} P_c(t) \quad ,$$

where T_{f0} is the unloaded filling time. Equation (35) now becomes

$$P_r(t) = P_g - P_c(t) - \frac{1}{2} T_{f0} \frac{d}{dt} \left[P_c(t)\right] \quad . \tag{36}$$

If a normalized cavity voltage $v(t) = V_c(t)/V_c$ is introduced, the above expression can be written in normalized form, using $x = t/T_b$, as

$$P_r(x) = P_g - \frac{V_c^2}{R} \left\{ v^2(x) + \frac{1}{2\tau_0} \frac{d}{dt} \left[v^2(x)\right] \right\} \quad .$$

The function $v^2(x)$ is just that given by Eq. (33).

The above derivation does not give the phase of the reflected power, which may sometimes be of interest. An alternative derivation of the reflected power is given in Appendix C. Although considerably longer, it gives both the magnitude and phase of the reflected power.

9. Effect of Transient Beam Loading on Quantum Lifetime

The quantum lifetime is given by[8]

$$\tau_q = \frac{\tau_\varepsilon}{2} \frac{e^\xi}{\xi} \tag{37}$$

where τ_ε is the damping time for energy oscillations and

$$\xi = \frac{1}{T_0 H_0} \int_{t_u}^{t_s} \left[V(t) - V_a\right] dt \tag{38}$$

$$= \frac{V_a}{2\pi h H_0} \int_{\phi_u}^{\phi_s} \left[\frac{V(\omega t)}{V_a} - 1\right] d(\omega t)$$

Here $V(t)$ is the total voltage, including the beam-induced voltage, seen by an electron oscillating through the bunch. V_a is the total energy loss in volts per turn, including losses due to synchrotron radiation, parasitic modes in the vacuum chamber, and higher-order modes in the rf cavities. The subscripts s and u refer to the stable and unstable synchronous phase angles. T_0 is the revolution time, h is the harmonic number and H_0 is a lattice parameter related to the momentum compaction factor α and energy spread σ_ε by

$$H_0 = \alpha E_0 (\sigma_\varepsilon / E_0)^2 \quad .$$

The integral in Eq. (38) can be represented geometrically as the area lying below the total cavity voltage (including the effect of the beam-induced voltage) as a function of time, and above a horizontal line giving the total energy loss per turn, V_a. This area, which is proportional to the depth of the potential well trapping a single particle oscillating incoherently through the bunch, can be computed with the aid of the diagram in Fig. 9.

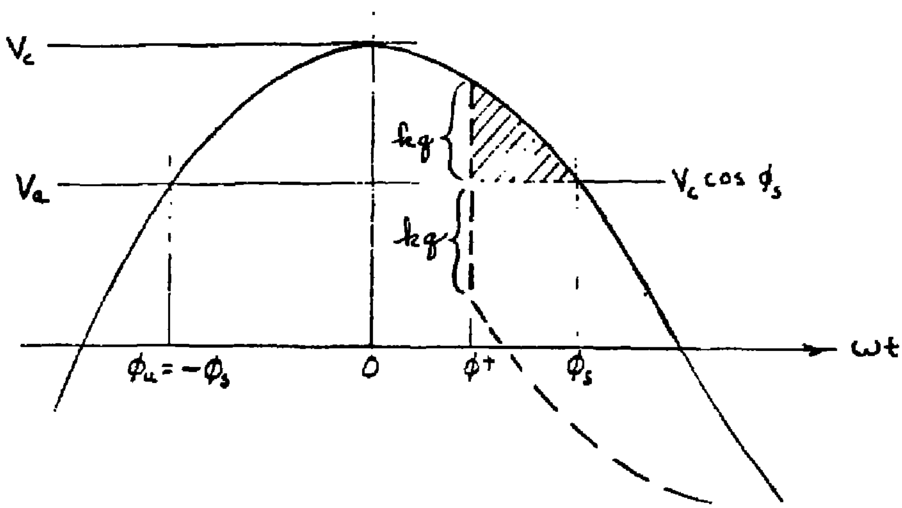

Fig. 9 -- Diagram showing the reduction due to beam loading in the potential well trapping incoherent oscillations.

The solid curve in Fig. 9 gives the rf voltage for the case of negligible beam loading, while the dashed curve shows the voltage for the case of a short bunch which produces a discontinuity of magnitude $- V_{bo} = - 2kq$ in the cavity voltage. We see that the parameter ξ will be reduced by beam loading in proportion to the shaded area.

Call the integral in Eq. (38) A_0 for the case of no beam loading and A with beam loading. Then

$$A_0 = \int_{-\phi_s}^{\phi_s} \left(\frac{\cos \omega t}{\cos \phi_s} - 1 \right) d(\omega t) = 2(\tan \phi_s - \phi_s) \tag{39}$$

$$A = A_0 - \int_{\phi^+}^{\phi_s} \left(\frac{\cos \omega t}{\cos \phi_s} - 1 \right) d(\omega t)$$

$$= (\tan \phi_s - \phi_s) + \frac{\sin \phi^+}{\cos \phi_s} - \phi^+ \quad ;$$

thus

$$\frac{\xi}{\xi_0} = \frac{(\tan \phi_s + \sin \phi^+/\cos \phi_s) - (\phi_s + \phi^+)}{2(\tan \phi_s - \phi_s)} \tag{40}$$

The relation between ϕ^+ and ϕ_s is given by the basic beam-loading vector triangle. This is repeated for convenience in Fig. 10.

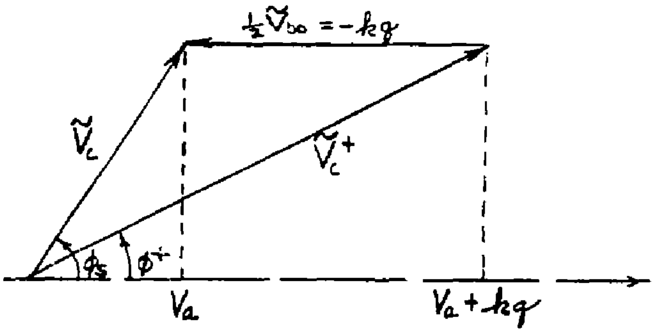

Fig. 10 -- The beam-loading vector triangle.

From the triangle in Fig. 10,

$$V_c^+ \cos \phi^+ = V_a + kq$$

$$V_c^+ \sin \phi^+ = V_a \tan \phi_s$$

$$\tan \phi^+ = \frac{\tan \phi_s}{1 + kq/V_a} \qquad (41)$$

As a numerical example, consider the parameters for the LEP-70 ring:[9]

$$\tau_q = 24 \text{ hours}$$

$$\tau_\varepsilon = 5.8 \text{ ms at } E_0 = 15 \text{ GeV}$$

$$\xi_0 = 20.2$$

$$V_a = 1079 \text{ MV}$$

$$R = 32.3 \text{ G}\Omega$$

$$i_0 = 10.5 \text{ mA per beam}$$

$$\tau_0 = T_b/T_{f0} = 0.52$$

$$kq = i_0 R \tau_0 = 176 \text{ MV (two beams)}$$

$$\phi_s = 29.8^\circ$$

Using values for ϕ_s, V_a and kq in Eq. (41), we compute that $\phi^+ = 26.2^\circ$. from Eq. (40), we then have $\xi/\xi_0 = 0.990$. Equation (37) implies

$$\frac{d\tau_q}{\tau_q} = (\xi - 1) \frac{d\xi}{\xi} \quad .$$

The reduction in quantum lifetime due to beam loading is therefore about 20%. However, from Eq. (39) we can work out that, using $\cos \phi_s = V_a/V_c$,

$$\frac{d\xi}{\xi_0} = \frac{dA}{A_0} = \frac{2 \tan \phi_s}{A_0} \frac{dV_c}{V_c} \quad .$$

From Eq. (39), A_0 is about 0.1 for $\phi_s = 29.8^{\circ}$. Thus only a trivial increase in V_c (about 0.1%) is required to restore the quantum lifetime to 24 hours.

Acknowledgement

It is a pleasure to acknowledge the hospitality of the ISR Division at CERN during the period when this report was conceived and much of the analysis carried out. Numerous discussions with colleagues at CERN and SLAC have contributed to the development and clarification of the beam-loading concepts presented here. I have profited particularly from discussions with H. Henke, A. Hutton, E. Keil, G. Loew, P. Morton and B. Zotter. Special thanks are also due to Cathy Nissen for working nights and weekends to type the manuscript.

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APPENDIX A

Consistency of the Beam-loaded Energy Gain with the KSZ Criterion

The beam-loaded energy gain is given by $V_c \cos \phi$ in the diagram of Fig. 4. We will prove that this energy gain is always consistent with the KSZ criterion as expressed by Eq. (5). Let V_s be the sum of the synchrotron radiation loss and all parasitic mode losses external to the rf cavities. Then the fundamental mode must provide a voltage gain

$$V_c \cos \phi = V_s + V_{hm} \quad ,$$

where $qV_{hm} = \Delta U_{hm}$ is the energy loss to higher modes. From Eqs. (1) and (10),

$$V_{hm} = (B - 1)kq \quad .$$

From Fig. 4, the voltage V_c^+ before the arrival of the bunch is related to $V_c \cos \phi$ by

$$V_c^+ \cos \phi^+ = V_c \cos \phi + kq \quad .$$

Combining the preceding three expressions,

$$V_c^+ \cos \phi^+ = V_s + Bkq \quad . \qquad \text{(A-1)}$$

The net energy extracted from the rf cavities is simply qV_s. The energy stored in the cavities before the arrival of the bunch is $W^+ = \alpha V_c^{+2}$. Therefore the efficiency for net energy extraction is

$$\eta = \frac{qV_s}{\alpha V_c^{+2}} \quad .$$

Using $1/\alpha = 4k$ from Eq. (11a), and eliminating V_c^{+2} with the aid of Eq. (A-1), the efficiency becomes

$$\eta = \frac{\cos^2 \phi^+}{F} \tag{A-2}$$

$$F = \frac{(V_s + Bkq)^2}{4\,kq\,V_s} \quad .$$

To be consistent with KSZ, the factor F must never be less than B. By differentiation, the minimum value of F as a function of kq is found to be F(min) = B at $kq = V_s/B$. Thus the actual efficiency for energy extraction as expressed by Eq. (A-2) can never exceed $\cos^2 \phi^+/B$ as long as the proper allowance for higher-mode losses, given by

$$V_{hm} = (B - 1)kq = i_0 Z_{hm}$$

$$Z_{hm} = \frac{\tau_0}{2}(B - 1)R \quad ; \quad \tau_0 = \frac{\omega_0 T_b}{2Q_0}$$

has been included in computing the total required voltage gain, $V_c \cos \phi$. It should be noted that the preceding expression for the higher-mode-loss impedance for the rf cavities is valid in the single-pass limit. If the induced higher-mode fields do not decay away between bunch passages, the actual loss can be more or less, depending on the phase length of the ring circumference for the modes in question.

APPENDIX B

Transient Response of a Resonant Cavity to a Step Change in Driving Voltage

Consider first an undriven cavity with resonant frequency ω_0 and damping time T_f. Suppose the cavity is initially charged to voltage $\tilde{V}_d(0)$, and that this voltage is allowed to decay as e^{-t/T_f} while viewed in a reference frame rotating at angular frequency ω (the rf driving frequency). The time variation of the cavity voltage is

$$\tilde{V}_d(t) = \tilde{V}_d(0)e^{-t/T_f}\, e^{jt\Delta\omega} \quad , \tag{B-1}$$

where $\Delta\omega = \omega_0 - \omega$. The time variation of $V_d(t)$ [the reason for the subscript will become clear shortly] is illustrated in Fig. B-1.

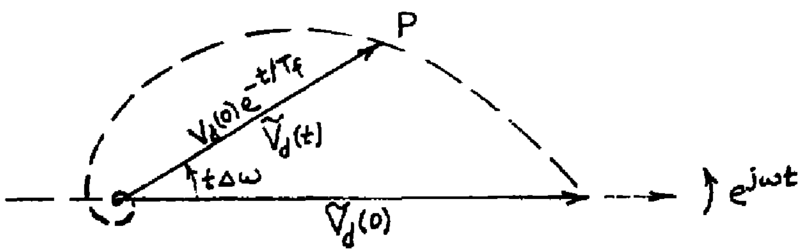

Fig. B-1 -- Discharge of a resonant cavity viewed in a rotating coordinate frame.

The relevance of this seemingly simple physical picture may not be obvious at first glance. In a storage ring we are dealing with driven rf cavities, and the bunch repetition frequency is also a sub-harmonic of the driving frequency ω. Thus all steady-state driven voltages are phasors viewed in a coordinate system rotating at the driving frequency ω. Transient variations can, however, be viewed as the superposition of a final steady-state voltage level plus an undriven discharge toward this voltage, which occurs at the natural cavity resonant frequency ω_0. Thus, by adding a final steady-state vector $\tilde{V}(\infty)$ to the diagram in Fig. B-1, we obtain the general transient variation of the cavity voltage $\tilde{V}(t)$, where $\tilde{V}(t) = \tilde{V}(0)$ at $t = 0$. $\tilde{V}_d(t)$ in Eq. (B-1) now gives the time variation of the "difference vector",

$$\tilde{V}_d(t) = \tilde{V}(t) - \tilde{V}(\infty) \tag{B-2a}$$

where

$$\tilde{V}_d(0) = \tilde{V}(0) - \tilde{V}(\infty) \quad . \tag{B-2b}$$

The relationship of these vectors is shown in Fig. B-2.

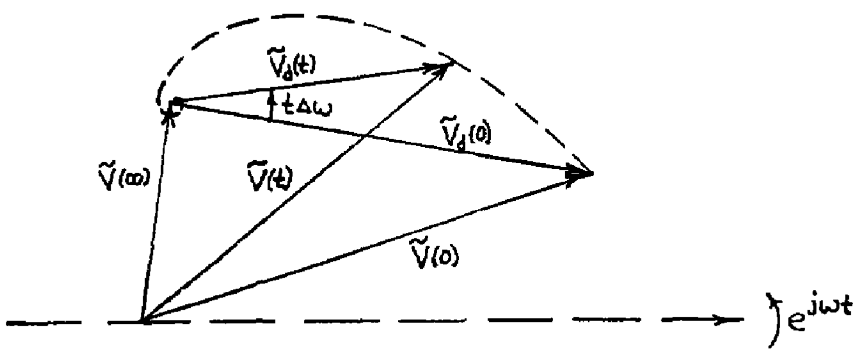

Fig. B-2 -- Transient change of the cavity voltage between $\tilde{V}(0)$ and $\tilde{V}(\infty)$.

Using the definition of the tuning angle, $\tan\psi = T_f\Delta\omega$, Eq. (B-1) becomes

$$\tilde{V}_d(t) = \tilde{V}_d(0)\, e^{-(t/T_f)(1-j\tan\psi)} \tag{B-3}$$

Substituting for $\tilde{V}_d(t)$ and $\tilde{V}_d(0)$ in this expression using Eqs. (B-2), we obtain

$$\tilde{V}(t) = \tilde{V}(\infty) + \left[\tilde{V}(0) - \tilde{V}(\infty)\right] e^{-(t/T_f)(1-j\tan\psi)} \quad . \tag{B-4}$$

This expression can also be considered as giving the transient reponse of a resonant cavity to a step change in driving voltage from $\tilde{V}(0)$ to $\tilde{V}(\infty)$ at time $t = 0$.

It is interesting to show that Eq. (B-3) represents an equiangular spiral. That is, the tangent to the curve at any point P in Fig. (B-1) makes a constant angle with respect to the difference vector joining point P to the origin. The derivative $\dot{\tilde{V}} = d\tilde{V}/dt$ is tangent to the curve $\tilde{V}(t)$. From Eq. (B-3),

$$\dot{\tilde{V}}_d(t) = -\tilde{V}_d(t)(1 - j\tan\psi)/T_f \quad .$$

Since

$$e^{-j\psi} = (1 - j\tan\psi)\cos\psi \quad ,$$

we have

$$\dot{\tilde{V}}_d(t) = -\tilde{V}_d(t)\,\frac{e^{-j\psi}}{T_f\cos\psi} \quad .$$

Thus if $\dot{\tilde{V}}_d(t)$ is rotated by angle $+\psi$, it will lie along the direction of $-\tilde{V}_d(t)$ as shown in Fig. B-3.

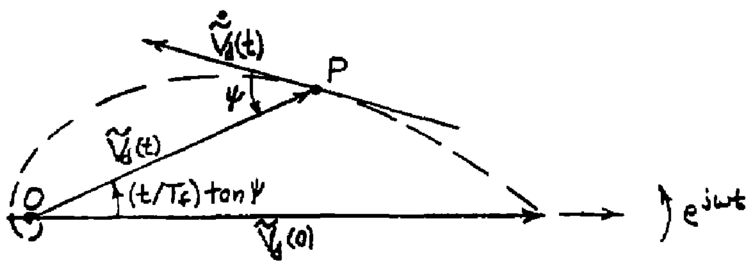

Fig. B-3 -- Diagram showing the equiangular property of Eq. (B-3).

APPENDIX C

Alternative Derivation of the Reflected Power

For reasons which will become apparent later, we must first compute the phase angle between the instantaneous cavity voltage $\tilde{V}_c(t)$ and the generator voltage at resonance, $\tilde{V}_{gr}$. The generator voltage off-resonance is related to $\tilde{V}_{gr}$ by*

$$\tilde{V}_g = \tilde{V}_{gr} \cos \psi \; e^{j\psi} \quad .$$

That is, as the cavity is tuned off resonance, the phase rotates through angle ψ and the magnitude decreases by a factor $\cos \psi$. The relationship of $\tilde{V}_c(t)$ to $\tilde{V}_g$ and $\tilde{V}_{gr}$ is shown in Fig. C-1.

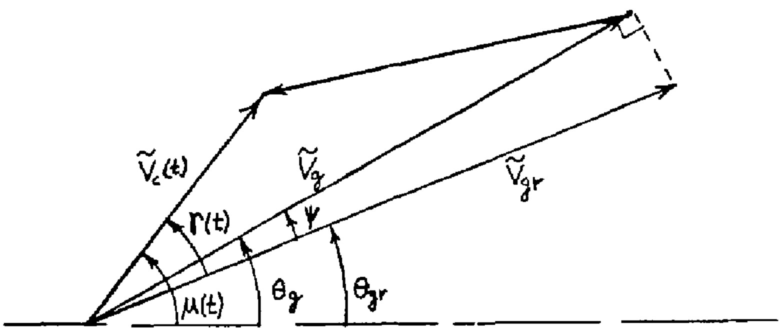

Fig. C-1 -- Diagram showing phase angles used to compute angle $\Upsilon(t)$ between $\tilde{V}_c(t)$ and $\tilde{V}_{gr}$.

It is seen that angle $\Upsilon(t)$ between $\tilde{V}_c(t)$ and $\tilde{V}_{gr}$ is given by

*In tuning-angle notation, the impedance Z of a parallel resonant circuit off-resonance is related to the impedance on resonance Z_r by $Z/Z_r = (1 - j \tan \psi)^{-1}$, where $\tan \psi \equiv 2Q_L(\omega_0 - \omega)/\omega_0$.

$$\gamma(t) = \mu(t) - \theta_g + \psi \quad , \qquad \text{(C-1)}$$

where θ_g is the phase angle of the generator voltage and $\mu(t)$ is the instantaneous phase angle of the time-varying cavity voltage. The two angles θ_g and $\mu(t)$ have been computed previously by Eqs. (27) and (34), and $\gamma(t)$ is therefore determined through Eq. (C-1).

We next switch our attention to the input transmission line to the cavity, where we consider a different kind of superposition of voltages. The total reflected wave traveling away from the cavity coupling aperture (or loop) can be considered to be the vector sum of a reflected wave equal in magnitude to the incident wave from the generator, but reversed in phase after reflection from the plane of the coupling aperture, and an emitted wave radiating from the coupling aperture. The emitted wave is the wave that would be present if the generator were suddenly switched off. The amplitude of the emitted wave is therefore proportional to $V_c(t)$, and the phase is fixed by the phase of $\tilde{V}_c(t)$. The superposition gives

$$\tilde{V}_r = -\tilde{V}_i + \tilde{V}_e \quad .$$

This vector triangle is shown in Fig. C-2.

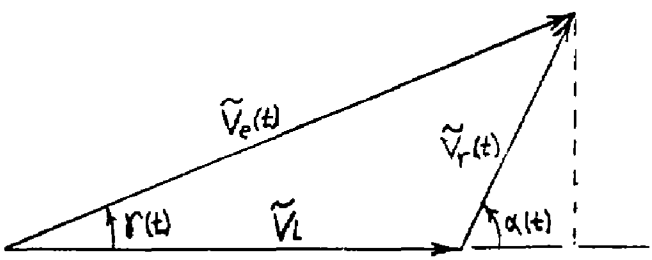

Fig. C-2 -- Vector triangle for waves traveling away from the cavity in the input transmission line.

The diagram represents wave voltages at the plane of the detuned short in the input transmission line. If the cavity is shorted out, $\tilde{V}_e = 0$ and $\tilde{V}_r = -\tilde{V}_i$. Assume now the cavity is exactly at resonance, with no beam. $\tilde{V}_r$ must then also be real at the plane of the detuned short; that is, it is co-linear with $\tilde{V}_i$. Hence $\tilde{V}_e$ must also be co-linear with $\tilde{V}_i$. But inside the cavity we know that $\tilde{V}_c$ is then co-linear with $\tilde{V}_{gr}$. The angle γ in Fig. C-2 is therefore the same as angle γ in Fig. C-1, since the phase and amplitude of $\tilde{V}_e$ are determined by $\tilde{V}_c$.

Applying the law of cosines to the triangle in Fig. C-2,

$$\frac{V_r^2}{V_i^2} = 1 + \frac{V_e^2}{V_i^2} - 2\frac{V_e}{V_i}\cos\gamma \quad . \qquad \text{(C-2)}$$

Now use the fact that, when the generator is switched off, the emitted power is related to the power dissipated in the cavity through the definition of the coupling coefficient,

$$P_e = \beta P_c = \beta V_c^2(t)/R \quad .$$

The amplitude of the emitted voltage wave is then given by

$$\frac{V_e}{V_i} = \left(\frac{P_e}{P_g}\right)^{\frac{1}{2}} = \left[\frac{\beta V_c^2(t)}{RP_g}\right]^{\frac{1}{2}} \quad . \qquad \text{(C-3)}$$

This expression for the voltage of the emitted wave is valid whether or not there is a generator voltage present. Substituting this expression, together with $V_r^2/V_i^2 = P_r/P_g$ in Eq. (C-2),

$$\frac{P_r(t)}{P_g} = 1 + \frac{\beta V_c^2(t)}{RP_g} - 2\left[\frac{\beta V_c^2(t)}{RP_g}\right]^{\frac{1}{2}} \cos\gamma(t) \quad . \qquad \text{(C-4)}$$

The diagram in Fig. C-2 is just the familiar Smith chart (reflection coefficient plot) of transmission line theory. Angle α, measured with respect to $\tilde{V}_i$, is the phase angle of the reflection coefficient. Angle α is readily obtained in terms of the geometry of Fig. C-2 as

$$\tan\alpha = \frac{(V_e/V_i)\sin\gamma}{(V_e/V_i)\cos\gamma - 1} \quad .$$

The computation of the voltage $\tilde{V}_c(t)$ inside the cavity and $\tilde{V}_r(t)/V_i$ in the input transmission line, carried out in the preceding section for the case of equal bunch charges with equal spacing in time, can be extended to any combination of bunch charges and bunch spacings by superposition. In particular, suppose unequal electron and positron bunches pass through a cavity, with the positron bunch delayed in time by Δt. If $\tilde{V}_{ce}(t)$ and $\tilde{V}_{cp}(t)$ are the voltages that would result if these bunches passed separately through the cavity at $t = 0$, then the net cavity voltage which results when both bunches pass through the cavity is

$$\tilde{V}_c(t) = \tilde{V}_{ce}(t) + \tilde{V}_{cp}(t - \Delta t) \quad .$$

When the net cavity voltage has been computed (both magnitude and phase), then the reflected voltage wave can be computed using Eq. (C-4), or, alternatively, by conservation of energy using Eq. (36).