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TRANSIENT DISTURBANCES IN A HALF-SPACE DURING THE FIRST STAGE OF FRICTIONLESS INDENTATION OF A SMOOTH RIGID DIE OF ARBITRARY SHAPE*

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Abstract. A solution is obtained by the method of self-similar potentials for the time-spatial distribution of the contact stress between a homogeneous, isotropic, linearly elastic half-space and a smooth rigid die having an arbitrary indenting velocity and shape. The solution holds as long as the outward speed of the contact zone does not fall below the speed of the dilatational wave in the elastic medium. A proof is given that the instantaneous value of the force required to indent the die during this stage of contact is directly proportional to the product of the area of contact and the velocity of indentation at that instant.

Introduction. In this paper the method of self-similar solutions [1–7] is used to solve problems in which rigid dies of arbitrary shape are pressed into a linearly elastic, homogeneous isotropic half-space at a rate sufficient to cause the contact to be superseismic. For these problems, no disturbance propagates along the surface of the half-space more rapidly than the boundary of the region of contact. Consequently, there will be no deformation of the surface at points beyond the region of contact. Moreover, the deformation of the surface within the contact zone will be completely defined by the portion of the rigid die which has crossed the original position of the surface of the half-space.

In [5-7] it was shown that the rate of penetration must exceed a certain value for the contact between a half-space and a die with a cusp at the point of initial contact to be superseismic. However, the situation is quite different if the die is smooth in the region of contact. For problems of this type, the contact must always be superseismic for a finite interval of time. Moreover, the length of this interval of time can easily be computed from the shape and indentation velocity of the die and the velocity of the dilatational wave in the half-space.

The problem in the title belongs to that class of elastodynamic problems the boundary conditions of which may be expressed in terms of functions which are homogeneous functions of space and time. For such problems, solutions may readily be obtained by either the self-similar potential approach or by the more familiar transform methods [8]. The virtual equivalence at these methods when applied to self-similar problems has most recently been demonstrated by Norwood [9].

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Two-dimensional problems. For plane strain contact problems, it is convenient to denote the vertical displacement of points on the surface of the $y = 0$ half space by $U(x, t)$. (We use the notation of [5-7] in this paper.) If the contact is frictionless, the boundary conditions defined below are sufficient completely to determine the stress and displacement fields at every point in the half-space:

$$\begin{aligned} u_y(x, 0, t) &= U(x, t), \\ \sigma_{xy}(x, 0, t) &= 0. \end{aligned} \tag{1}$$

If $U(x, t)$ is a homogeneous function of x and t, the problem can be solved readily with the aid of the method of self-similar potentials described in [5]. However, if $U(x, t)$ is not a homogeneous function of x and t, the vertical displacement can always be expressed as the superposition of homogeneous functions in the following manner:

$$U(x, t) = \int_0^t \int_{-\infty}^{\infty} U(\xi, t_0) \cdot u_y^0(x - \xi, t - t_0)\, d\xi\, dt_0 \tag{2}$$

where $u_y^0(x - \xi, t - t_0)$ is a vertical displacement which satisfies the conditions

$$u_y^0(x - \xi, t - t_0) = 0 \text{ for } \xi \neq x \text{ or } t \neq t_0\,, \quad \int_0^t \int_{-\infty}^{\infty} u_y^0(x - \xi, t - t_0)\, d\xi\, dt_0 = 1. \tag{3}$$

That is, $u_y^0(x - \xi, t - t_0) = \delta(x - \xi) \cdot \delta(t - t_0)$ where δ is the Dirac delta function. It is apparent that u_y^0 may be interpreted as the displacement equivalent of a vertical impulse. That is, the relationship between u_y^0 and a "distributed displacement" $U(x, t)$ is completely analogous to the relationship between an impulse of unit magnitude and a distributed load.

From Eqs. (3), it is apparent that u_y^0 is a homogeneous function of $x - \xi$ and $t - t_0$. Thus, the solution of this problem can be expressed in terms of self-similar potentials by the method described in [5]. The principal features of this method as they pertain to the problem of an impulsive displacement acting at $x = y = t = 0$ are repeated here for completeness.

The class of plane strain problems which have displacement boundary conditions that are homogeneous functions of degree -2 are uniquely solved by displacement fields whose components satisfy the equations

$$u_x = \operatorname{Re}\left\{\frac{\partial}{\partial t}\left(\frac{\partial \Phi}{\partial x} + \frac{\partial \psi}{\partial y}\right)\right\}, \tag{4a}$$

$$u_y = \operatorname{Re}\left\{\frac{\partial}{\partial t}\left(\frac{\partial \Phi}{\partial y} - \frac{\partial \psi}{\partial x}\right)\right\} \tag{4b}$$

if $\Phi(\theta_1)$ and $\psi(\theta_2)$ denote potentials which are functions of the θ_i defined by the equations

$$t - \theta_1 x - y(a^{-2} - \theta_1^2)^{1/2} = 0, \qquad t - \theta_2 x - y(b^{-2} - \theta_2^2)^{1/2} = 0. \tag{5}$$

The constants a and b, which respectively denote the speeds of propagation of the dilatational (P) and rotational (S) waves, are related to the Lamé constants (λ, μ) and the density (ρ) of the half-space by the equations

$$a^2 = (\lambda + 2\mu)/\rho, \qquad b^2 = \mu/\rho. \tag{6}$$

To simplify the determination of the unknown potentials, it is convenient to integrate

Eqs. (3) and (4b) twice with respect to time. This leads to the following equations for points on the $y = 0$ surface (where $\theta_1 = \theta_2 = \theta = t/x$):

$$u_F(x, 0, t) = -\text{Re}\left\{\int_0^\theta [(a^{-2} - \theta^2)^{1/2}\Phi' - \theta\psi'(\theta)]\, d\theta\right\} \tag{7}$$

where $\Phi' = \partial\Phi(\theta)/\partial\theta$, $\psi' = \partial\psi(\theta)/\partial\theta$, and

$$u_F(x, 0, t) = \int_0^t \int_0^\tau u_y^0(x, \tau_0)\, d\tau_0\, d\tau = t\cdot\delta(x). \tag{8}$$

For reasons discussed in [5], the term within the braces in Eq. (6) is identical to the complex-valued function $u_F^*(\theta)$ constructed from u_F by means of the Cauchy integral formula for the half-plane:

$$u_F^*(\theta) = \frac{1}{i\pi}\int_{-\infty}^{\infty} \frac{u_F(\eta, 0, t)}{\eta - x}\, d\eta = t/i\pi x = -\theta/i\pi. \tag{9}$$

Consequently,

$$-(a^{-2} - \theta^2)^{1/2}\Phi' + \theta\psi' = -\frac{1}{i\pi} \tag{10}$$

at every point on the $y = 0$ surface.

A second equation involving the unknown potentials follows from the absence of tangential traction (friction) on the $y = 0$ surface. Combining Eq. (4) and the strain-displacement relation

$$\sigma_{xy} = \mu\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) \tag{11}$$

results, with the aid of Eq. (5), in the expression

$$\sigma_{xy}(x, y, t) = \mu\frac{\partial^2}{\partial t^2}\left\{\frac{2\theta_1(a^{-2} - \theta_1^2)^{1/2}\Phi'}{\delta_1'} + \frac{(b^{-2} - 2\theta_2^2)\psi'}{\delta_2'}\right\} \tag{12}$$

where $\delta_1' = -x + y\theta_1(a^{-2} - \theta_1^2)^{-1/2}$, $\delta_1' = -x + y\theta_2(b^{-2} - \theta_2^2)^{-1/2}$. For the tangential traction to be zero, the integrand of the equation defining σ_{xy} must be zero at every point on the $y = 0$ surface. Thus

$$2\theta(a^{-2} - \theta^2)^{1/2}\Phi' + (b^{-2} - 2\theta^2)\psi' = 0 \tag{13}$$

The potentials which solve the impulsive displacement problem can now be found by solving Eqs. (10) and (13) and replacing θ by the appropriate θ_i . The result is

$$\Phi'(\theta_1) = -\frac{b^2(b^{-2} - 2\theta_1^2)}{i\pi(a^{-2} - \theta_1^2)^{1/2}}, \qquad \psi'(\theta_2) = \frac{2b^2\theta_2}{i\pi}. \tag{14}$$

The displacements can now be determined from Eqs. (4) and (14). The stresses can be computed from Eqs. (12) and (14) and the following equations derived by a procedure analogous to that used to obtain Eq. (12):

$$\begin{aligned}
\sigma_{xx} &= \mu\frac{\partial^2}{\partial t^2}\left\{(2\theta_1 + b^{-2} - 2a^{-2})\frac{\Phi'}{\delta_1'} + 2\theta_2(b^{-2} - \theta_2^2)^{1/2}\frac{\psi'}{\delta_2},\right\} \\
\sigma_{yy} &= \mu\frac{\partial^2}{\partial t^2}\left\{(b^{-2} - 2\theta_1^2)\frac{\Phi'}{\delta_1'} - 2\theta_2(b^{-2} - \theta_2^2)^{1/2}\frac{\psi'}{\delta_2'}\right\}, \\
\sigma_{zz} &= \nu(\sigma_{xx} + \sigma_{yy}), \qquad \sigma_{xz} = \sigma_{yz} = 0
\end{aligned} \tag{15}$$

(where ν is Poisson's ratio). Hence, the normal traction which induces an impulsive displacement at the origin is defined by the equation

$$\sigma_{yy}{}^0(x, t) = -\text{Re}\left(\frac{\mu b^2}{i\pi x}\frac{\partial^2}{\partial t^2}\frac{R(\theta^2)}{(a^{-2} - \theta^2)^{1/2}}\right) \tag{16}$$

where $\theta = t/x$. It follows that the normal traction required to cause any deformation $u(x, t)$ of the $y = 0$ surface is determined by the equation

$$\begin{aligned}\sigma_{yy}{}^0(x, t) &= \text{Re}\int_{-\infty}^{\infty}\int_0^t u(\xi, t - \tau)\sigma_y{}^0(x - \xi, \tau)\, d\tau\, d\xi \\ &= -\text{Re}\left(\frac{\mu b^2}{i\pi}\int_{-\infty}^{\infty}\int_0^t \frac{u(\xi, t - \tau)}{x - \xi}\frac{\partial^2}{\partial\tau^2}\frac{R(\theta)^2}{a^{-2} - \theta^2}\, d\tau\, d\xi\right.\end{aligned} \tag{17}$$

where $\theta = \tau(x - \xi)^{-1}$, $\tau = t - t_0$ and $R(\theta)^2$ denotes the Rayleigh function [6].

Eq. (17) determines the contact stress beneath a rigid die only as long as the contact remains superseismic. Once the contact becomes subseismic, the displacement is no longer completely known at every point on the surface. That is, the quantity $u(x, t)$ can no longer be determined solely from the known shape and velocity of the rigid die.

Three-dimensional problems. It follows by analogy from the work of the previous section that the contact stress for any frictionless three-dimensional superseismic contact problem is defined by the equation

$$\sigma_y(r, \omega, t) = \int_0^{\infty}\int_0^t\int_0^{2\pi} U(r_0, \omega_0, t - \tau)\cdot\sigma_{yy}{}^0(r - r_0, \omega - \omega_0, \tau)r_0\, d\omega_0\, d\tau\, dr_0 \tag{18}$$

1) $U(r_0, \omega_0, t - \tau)$ defines the vertical displacement at every point on the $y = 0$ surface during the interval of time in which the contact is superseismic, and

2) $\sigma_{yy}{}^0(r - r_0, \omega - \omega_0, \tau)$ defines the normal traction required to force a vertical impulsive displacement of unit magnitude at $r = r_0$, $\omega = \omega_0$ and $\tau = 0$. This impulsive displacement satisfies the equation

$$u_y{}^0(r - r_0, \omega - \omega_0, \tau) = \delta(r - r_0)\cdot\delta(\omega - \omega_0)\cdot\delta(\tau). \tag{19}$$

It is convenient to deduce $\sigma_y{}^0$ from the solution of a problem in which the stress and velocity boundary condition are self-similar, since problems of this class can frequently be solved, virtually by inspection, with the aid of the method of rotational superposition described in [2, 3, 4, 5]. Consequently, we shall initially solve the problem having the boundary conditions

$$v_y(r - r_0, \omega - \omega_0, t) = \frac{t^2}{2}\cdot\delta(r - r_0)\cdot\delta(\omega - \omega_0), \qquad \sigma_{ry}(r - r_0, \omega - \omega_0, t) = 0 \tag{20}$$

for a traction $\sigma_y{}^4$ whose fourth derivative with respect to time may be interpreted as the required $\sigma_y{}^0$. It is obviously necessary to determine $\sigma_y{}^4$ in such a way that no terms are included which correspond to self-equilibrating stress singularities at $r = r_0$ and $\omega = \omega_0$. For this reason, $\sigma_y{}^4$ will be determined by a simple limiting process from the solution of the problem in which there is no tangential traction and the vertical velocity of surface points is zero except within the region $r - r_0 \leq \alpha t$ where it is equal to Δ. For this problem

$$
\begin{aligned}
v_y(r, 0, t) &= \Delta \quad \text{for} \quad r \le \alpha t \\
&= 0 \quad \text{for} \quad r > \alpha t \\
\sigma_{ry}(r, 0, t) &= 0 \quad \text{for} \quad 0 \le r \le \infty,
\end{aligned} \tag{21}
$$

where, for convenience, the axis of symmetry is assumed to pass through the point (r_0, ω_0).

With the aid of the technique of Kostrov [4], it is not difficult to find and solve the single symmetric plane strain problem whose rotational superposition yields the solution of the problem with the boundary conditions defined by Eq. (21). A complete development of this method of solution is given in [5] where it is proved that self-similarity of the stress and velocity boundary conditions for any axially-symmetric problem requires that the analogous conditions for the corresponding plane problem also be self-similar. Moreover, it is established that the boundary conditions for corresponding plane and axially symmetric problems must satisfy the relations

$$
\sigma_{ry}(r, t) = \int_0^\pi \sigma_{xy}(x, t) \cos \omega \, d\omega = \operatorname{Re} \int_c \frac{\theta_0^2}{2} \frac{\sigma_{xy}^*(\theta^2)}{\theta^{3/2}} \frac{d\theta^2}{(\theta^2 - \theta_0^2)^{1/2}}, \tag{22a}
$$

$$
\sigma_{yy}(r, t) = \int_0^\pi \sigma_{yy}(x, t) \, d\omega = \operatorname{Re} \int_c \frac{\theta_0}{2} \frac{\sigma_{yy}^*(\theta^2)}{\theta^2} \frac{d\theta^2}{(\theta^2 - \theta_0^2)^{1/2}}, \tag{22b}
$$

$$
v_y(r, t) = \int_0^\pi v_y(x, t) \, d\omega = \operatorname{Re} \int_c \frac{\theta_0}{2} \frac{v_y^*(\theta^2)}{\theta^2} \frac{d\theta^2}{(\theta^2 - \theta_0^2)^{1/2}}, \tag{22c}
$$

where $x = r \cos \omega$, $\theta_0 = t/r$, and θ is the value on the $y = 0$ surface of the θ_i $(i = 1, 2)$ defined implicitly by Eq. (5). In Eqs. (22), σ_{xy}^*, σ_{yy}^* and v_y^* denote complex-valued functions of θ^2 whose real parts on the $y = 0$ surface are equal, respectively, to the tractions and the vertical velocity of surface points for the plane problem. That is,

$$
\operatorname{Re} \sigma_{xy}^*(\theta^2) = \sigma_{xy}(x, t), \quad \operatorname{Re} \sigma_{yy}^*(\theta^2) = \sigma_{yy}(x, t), \quad \operatorname{Re} v_y^*(\theta^2) = v_y(x, t). \tag{23}
$$

These functions of θ^2 must be analytic at every point off the positive real θ^2 axis (see [5]). Consequently, the contour of integration C_1, obtained by the change in the variable of integration from ω to θ^2, can be deformed into any closed contour which begins just above $\theta_0^2 = t^2/(+r)^2$, crosses the real θ^2 axis to the left of any non-analyticities in the integrands of Eqs. (22), and ends just below $\theta_0^2 = t/-r^2$. Three possible contours of this type are shown in Fig. 1. The wide line between θ_0^2 and $+\infty$ in this figure indicates the only non-analyticities of $(\theta^2 - \theta_0^2)^{1/2}$ in the entire complex θ^2 plane. The value of this radical at the origin is specified to be positive imaginary for all $\theta^2 < \theta_0^2$.

Since each contour of integration is closed, the non-analyticities of the integrands and their character at infinity can be determined using Morera's theorem and the Cauchy integral formula. It is then a simple matter to deduce the functions defining σ_{xy}^*, σ_{yy}^* and v_y^* since these functions are completely determined by their non-analyticities (which occur only on the real $\theta^2 > 0$ axis) and their character at infinity.

For example, in order that σ_{ry} be zero for all $r > 0$, $\sigma_{xy}^*/\theta^{3/2}$ must be analytic for all $0 < \theta^2 < \theta_0^2$ when Eq. (22a) is integrated along C_2. Thus σ_{xy}^* must be a function of the form

$$
\sigma_{xy}^*(\theta^2) = E(\theta^2) \cdot \theta^{3/2} \tag{24}
$$

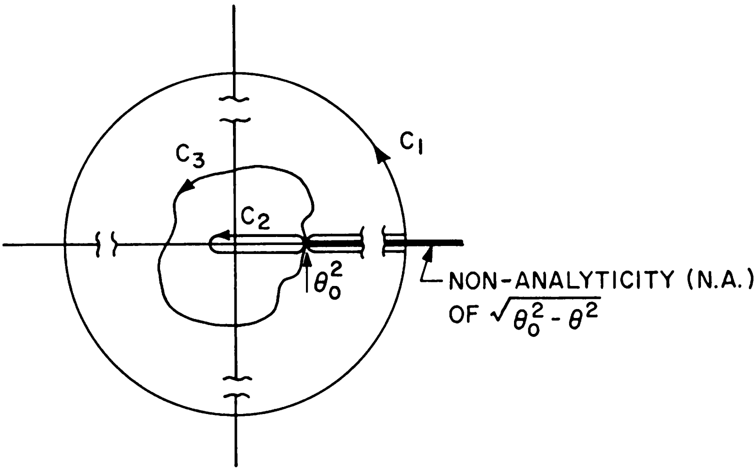

FIG. 1. Integration contours, non-analyticity of $\theta_0^2 - \theta^2$ in the complex θ^2 plane.

where $E(\theta^2)$ is an integral function, i.e. a function having no singularities or poles in the finite portion of the θ^2 plane. However, each function of the type defined by Eq. (23) gives rise to a singular stress field at $r = 0$. Hence, $E(\theta^2)$ must necessarily be zero and, as a result,

$$\sigma_{xy}^*(\theta^2) = \sigma_{xy}(x, 0, t) = 0. \tag{25}$$

The condition that $v_y = 0$ for $r > \alpha t$ requires only that $v_y^*(\theta^2)/\theta^2$ be analytic for all $0 \leq \theta^2 \leq \alpha^{-2}$. However, for v_y to have a constant nonzero value when $r \leq \alpha t$, $v_y^*(\theta^2)$ must be a function of the form

$$v_y^*(\theta^2) = \sum_{n=0}^{\infty} F_n(\theta^2)\cdot(\alpha^{-2} - \theta^2)^{-n-1/2} + G(\theta^2) \tag{26}$$

where n is an integer and the $F_n(\theta^2)$ and $G(\theta^2)$ are integral functions.

For the problem being considered, there should be no singularities in the stress field at $r = 0$. Hence σ_{yy}^* must tend to infinity less rapidly than θ^2, since $\sigma_{yy}^* = F\theta^2 a(2\pi^2)$ solves the axially symmetric problem in which a point load of magnitude Ft^2 acts vertically downward at $r = 0$ (see [5]). Because

$$\partial\sigma_{yy}^*/\partial\theta^2 = -[\mu b^2 R(\theta^2)/(\alpha^{-2} - \theta^2)^{1/2}][\partial v_y^*/\partial\theta^2] \tag{27}$$

for all problems in which $\sigma_{xy}(x, 0, t) = 0$, $v_y^*(\theta^2)$ must tend to infinity less rapidly than θ. Thus, $G(\theta^2)$ and F_0 must be constants, F_1 can only be a linear function of θ^2, F_2 can only be a quadratic function of θ^2, etc. As a result, the most general form of $v_y^*(\theta^2)$ is the infinite series

$$v_y^*(\theta^2) = \sum_{n=0}^{\infty} [A_n(\alpha^{-2} - \theta^2)^{-n-1/2} + G_n] \tag{28}$$

where each A_n and G_n are constants. The requirement that $v_y^*(\theta^2)/\theta^2$ be analytic at the origin will be satisfied only if $G_n = A_n\alpha^{2n+1}$. It follows directly from Eqs. (21), (22c) and (28) that $A_0 = -\Delta(\pi\alpha)^{-1}$ regardless of the values of each $A_1, A_2, \cdots$. Furthermore, it can be shown with the aid of techniques described in [5] that only the first term in this series gives rise to a stress field with no singularity in σ_{yy} at points on the circle $r = \alpha t$. Thus, it may be concluded that the problem whose boundary conditions are defined by Eq. (21) is essentially solved by the functions

$$v_y^*(\theta^2) = -\frac{\Delta}{\pi}\left[\frac{\alpha^{-1}}{(\alpha^{-2} - \theta^2)^{1/2}} - 1\right], \tag{29a}$$

$$\frac{\partial \sigma_{yy}^*(\theta^2)}{\partial \theta^2} = \frac{\mu b^2 \Delta}{\pi} \frac{R(\theta^2)}{(a^{-2} - \theta^2)^{1/2}} \frac{\alpha^{-1}}{(\alpha^{-2} - \theta^2)^{3/2}}. \tag{29b}$$

The functions which solve the problem whose boundary conditions are defined by Eq. (20) can now be obtained from Eqs. (29) by setting $\Delta = (2\alpha^2\pi)^{-1}$ and then taking the limit as α tends to zero. This yields the functions

$$v_y^*(\theta^2) = -\theta^2/4\pi^2, \tag{30a}$$

$$\frac{\partial \sigma_{yy}^*(\theta^2)}{\partial \theta^2} = \frac{\mu b^2 R(\theta^2)}{4\pi^2(a^{-2} - \theta^2)^{1/2}}. \tag{30b}$$

The equation defining $\sigma_{yy}(r, 0, t)$ for this problem is most conveniently determined from Eq. (30b) and the relationship

$$r \frac{\partial \sigma_{yy}}{\partial t}(r, 0, t) = \mathrm{Re} \int_c \frac{\partial \sigma_{yy}^*}{\partial \theta^2} \frac{\partial \theta^2}{(\theta^2 - \theta_0^2)^{1/2}}. \tag{31}$$

The final result is the simple expression

$$\begin{aligned} \sigma_{yy}(r, 0, t) &= 0 && \text{for} \quad r > at \\ &= \frac{\mu b^2}{4\pi}(K_1\theta_0^5 + K_2\theta_0^3 + K_3\theta_0 + C_1) && \text{for} \quad bt < r < at \\ &= \frac{\mu b^2}{4\pi}(K_4\theta_0^3 + K_5\theta_0 + C_1 + C_2) && \text{for} \quad 0 \le r < bt, \end{aligned} \tag{32}$$

where $\theta_0 = t/r$,

$$\begin{aligned} K_1 &= -3/5, \qquad C_1 = -2b^{-4}a^{-1} + 16b^{-2}a^{-3}/3 + 64a^{-5}/15 \\ K_2 &= (4b^{-2} - 2a^{-2})/3, \qquad C_2 = -16b^{-5}/15 \\ K_3 &= -2b^{-4} + 4a^{-2}b^{-2} - 3a^{-4} \\ K_4 &= 2(b^{-2} - a^{-2})/3 \\ K_5 &= -3b^{-4} + 4a^{-2}b^{-2} - 3a^{-4} \end{aligned} \tag{33}$$

Since this normal traction causes the displacement

$$\begin{aligned} u_y(r, 0, t) &= t^3/6 && \text{for} \quad r = 0 \\ &= 0 && \text{for} \quad 0 < r \le \infty, \end{aligned} \tag{34}$$

the fourth derivative with respect to time of this traction may be interpreted as the normal traction which causes an impulsive vertical displacement at the origin. That is to say, the traction

$$\begin{aligned}
\sigma_{yy}{}^0 &= 0 && \text{for} \quad r > at \\
&= \frac{\mu b^2}{4\pi}\left(\frac{120K_1 t}{r^5} + \frac{6K_2}{r^3}\,\delta(t - r/a)\right. \\
&\qquad\qquad \left. + \frac{K_3}{r}\frac{\partial^2\delta(t - r/a)}{\partial t^2} + \frac{C_1\partial^3\delta(t - r/a)}{\partial t^3}\right) && \text{for} \quad bt < r < at \\
&= \frac{\mu b^2}{4\pi}\left(\frac{6K_4}{r^3}\,\delta(t - r/b) + K_5\frac{\partial^2\delta(t - r/b)}{\partial t^2}\right. \\
&\qquad\qquad \left. + (C_1 + C_2)\frac{\partial^3\delta(t - r/b)}{\partial t^3}\right) && \text{for} \quad 0 \le r < bt
\end{aligned} \tag{35}$$

results in the displacement

$$\begin{aligned}
u_y(r, 0, t) &= \delta(t) \quad \text{for} \quad r = 0 \\
&= 0 \qquad \text{for} \quad 0 < r \le \infty
\end{aligned} \tag{36}$$

when $\sigma_{ry}(r, 0, t) = 0$ for $0 \le r \le \infty$.

Eq. (35) may be interpreted as the influence function which is to be used as shown in Eq. (18) to compute the contact stress beneath any rigid three-dimensional die which indents a half-space superseismically. For computational purposes it is convenient to integrate Eq. (18) by parts to reduce the singularity in the $\sigma_{yy}{}^0$ term. The following result is obtained:

$$\sigma_{yy}(r, w, t) = \int_0^\infty \int_0^{2\pi} \left(U\sigma_{yy}{}^1|_0{}^t - V\sigma_{yy}{}^2|_0{}^t + \int_0^t A\sigma_{yy}{}^2\,d\tau\right) dw\,dr_0$$

where $A = \partial V(t - \tau)/d\tau$, $V = \partial U(t - \tau)/\partial\tau$, $U = U(r_0\,, w_0\,, t - \tau)$ and where

$$\begin{aligned}
\sigma_{yy}{}^1(r - r_0\,, w - w_0\,, \tau) &= \int_0^\tau \sigma_y{}^0(r - r_0\,, w - w_0\,, \tau_1)\,d\tau_1\,, \\
\sigma_{xy}{}^2(r - r_0\,, w - w_0\,, \tau) &= \int_0^\tau \sigma_y{}^1(r - r_0\,, w - w_0\,, \tau_1)\,d\tau_1\,.
\end{aligned} \tag{37}$$

It should be emphasized that Eqs. (18) and (36) hold only as long as the contact remains superseismic. Moreover, they hold for dies of any shape, not just those which are axially symmetric.

The force required to indent a rigid die superseismically. In the work above, the contact stress defined by Eq. (35) was used as an influence function in computing the contact stress beneath a rigid die indenting superseismically. A similar procedure is used now to find the force required to press the die into the elastic half-space.

In [5, 7] it was established that the formula

$$P(t) = -2\pi^2 t^2 \left.\frac{\partial\sigma_{yy}{}^*}{\partial\theta^2}\right|_{\theta^2=\theta} \tag{38}$$

gives the total force on the $y = 0$ surface for any axially symmetric problem having self-similar stresses and velocities. Consequently, a force

$$P(t) = -\rho\,\frac{a}{2}\,t^2 \tag{39}$$

is required to cause the point displacement defined by Eq. (34). It follows that a force

$$P(t) = -\rho a \tag{40}$$

is required if

$$\begin{aligned} v_y(r, 0, t) &= 1 \quad \text{for} \quad r = 0 \\ &= 0 \quad \text{for} \quad 0 < r \leq \infty . \end{aligned} \tag{41}$$

Thus a force

$$P(t) = -\rho a V(t) \tag{42}$$

is required if the velocity of the point $r = 0$ has an arbitrary time dependence denoted by $V(t)$.

When a rigid die indents a half-space superseismically, each point within the region of contact moves down with the same velocity at any time. Consequently, the force necessary to indent the die is given by

$$P(t) = -\rho a V(t) \cdot A(t) \tag{43}$$

where $A(t)$ denotes the area of contact at time t. This extremely simple relationship holds for dies of any shape with any time variation of the indentation velocity, as long as the contact remains superseismic. Moreover, Eq. (43) indicates that the force at any time depends only on the area of contact between the die and the half-space and the indentation velocity at that instant of time. That is, the instantaneous force $P(t)$ does not depend on the time histories of either the area of contact or the indentation velocity prior to the instant of time being considered.

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