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TRANSIENT RESPONSE OF A CLASS OF NONLINEAR SYSTEMS

An algorithm is given to be used in conjunction with the parameter-plane method and the describing-function method for rapid calculation of transient oscillations in the design of a class of nonlinear systems.

This letter gives a straightforward algorithm for plotting the zero-input response of a class of nonlinear systems with arbitrary initial conditions.

The classes of nonlinear systems considered are those for which the stability of self-excited oscillations are determined by the nonlinear differential equation

$$C(s)x + B(s)F(x) = 0 \quad (1)$$

where $s = d/dt$, $C(s)$ and $B(s)$ are polynomials in s with the degree of $C(s)$ being higher than the degree of $B(s)$, and the function $F(x)$ represents the nonlinearity. It will be assumed that the relative stability of nonlinear control systems described by eqn. 1 may be studied in the parameter plane using the approach of Krylov and Bogoliubov and the describing-function method.[1–3] On the basis of this approach, the nonlinear function $F(x)$ is 'linearised' as

$$F(x) = N_1(A)x \quad (2)$$

where

$$N_1(A) = \frac{1}{\pi A}\int_0^{2\pi} F(A\sin\phi)\sin\phi\, d\phi \quad (3)$$

The linearised differential equation corresponding to eqn. 1 is thus

$$\{C(s) + B(s)N_1(A)\}x = 0 \quad (4)$$

The advantage of using the parameter-plane approach in conjunction with the describing-function method is that insight is given as to the effects of the various parameters on transient behaviour in the preliminary stages of design.[3] Also, any system adjustable parameters may be selected, when possible, so that the system equation (eqn. 4) may be approximated to by an equivalent second-order nonlinear system

$$[s^2 - 2\bar{\sigma}(t)s + \{\bar{\sigma}(t)\}^2 + \{\bar{\omega}(t)\}^2]x = 0 \quad (5)$$

which has the desired transient characteristic. The values of $\bar{\sigma}(t)$ and $\bar{\omega}(t)$ are determined from $A(t)$ and the functions $\bar{\sigma}(A)$ and $\bar{\omega}(A)$. The functions $\bar{\sigma}(A)$ and $\bar{\omega}(A)$ are determined using the nonlinearity characteristics $N_1(A)$, together with the parameter-plane constant σ and ω contours or the constant ζ and ω_n contours, where $\sigma = \omega_n\zeta$ and $\omega = \omega_n\sqrt{(1-\zeta^2)}$.[2]

It is the purpose of this letter to develop a straightforward algorithm for calculating $A(t)$, and thus the solution $x(t)$ of eqn. 5, for any specified initial conditions given $\bar{\sigma}(A)$ and $\bar{\omega}(A)$. The solution $x(t)$ may then be used as an approximation to the solution of the corresponding equation (eqn. 4). The developments are extensions of the methods of Grensted.[4]

Solutions of eqn. 5 are assumed to have the form

$$x(t) = A(t)\sin\phi(t) \quad (6)$$

where

$$\frac{dA}{dt} = \sigma_1 A \text{ and } \frac{d\phi}{dt} = \omega_1 \quad (7)$$

i.e. solutions of eqn. 5 are given by

$$x(t) = A_0 \exp\left(\int_{t_0}^{t}\sigma_1 dt\right)\sin\left(\int_0^t \omega_1 dt + \phi_0\right) \quad (8)$$

Substituting eqn. 8 into eqn. 5 results in two equations valid for all A, i.e.

$$\dot{\sigma}_1 + \sigma_1{}^2 - 2\bar{\sigma}\sigma_1 - \omega_1^2 + \bar{\sigma}^2 + \bar{\omega}^2 = 0 \quad (9a)$$

$$2\sigma_1\omega_1 + \dot{\omega}_1 - 2\bar{\sigma}\omega_1 = 0 \quad (9b)$$

If the assumption is made that $\dot{\bar{\sigma}} = \dot{\bar{\omega}} = 0$, eqns. 9*a* and *b* yield $\sigma_1 = \bar{\sigma}$, $\omega_1 = \bar{\omega}$, as in the Krylov–Bogoliubov analysis. By not neglecting the $\dot{\bar{\sigma}}$ and $\dot{\bar{\omega}}$ terms of eqns. 9*a* and *b*, improved results are obtained. The equations may be rewritten neglecting second-order terms, giving σ_1 and ω_1 in terms of $\bar{\sigma}$, $\bar{\omega}$, $\dot{\bar{\sigma}}$ and $\dot{\bar{\omega}}$, i.e.

$$\bar{\sigma}_1 = \bar{\sigma} - \frac{\bar{\omega}\dot{\bar{\omega}}}{2(\dot{\bar{\sigma}} + \bar{\omega}^2)} \quad \omega_1^2 = \sigma + \omega^2 \quad (10)$$

In order to calculate the constants A_0 and ϕ_0 of eqn. 8, consider its time derivative; i.e.

$$\dot{x}(t) = A_0 \exp\left(\int_0^t \sigma_1 dt\right)\left\{\sigma_1 \sin\left(\int_{t_0}^t \omega_1 dt + \phi_0\right) + \omega_1 \cos\left(\int_{t_0}^t \omega_1 dt + \phi_0\right)\right\} \quad (11)$$

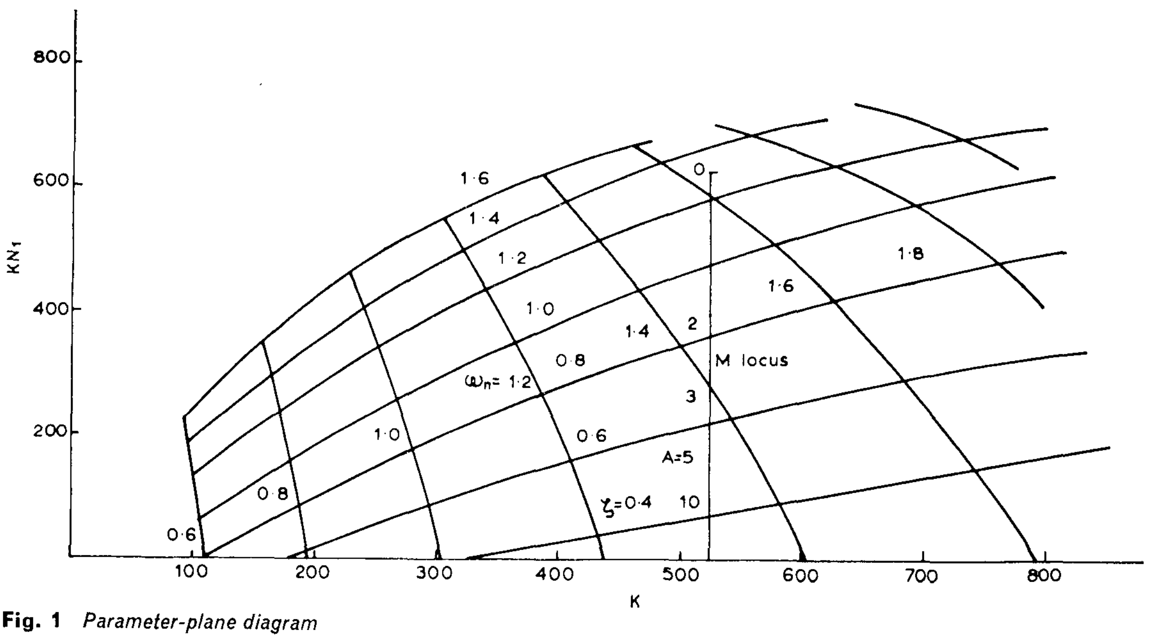

Fig. 1 *Parameter-plane diagram*

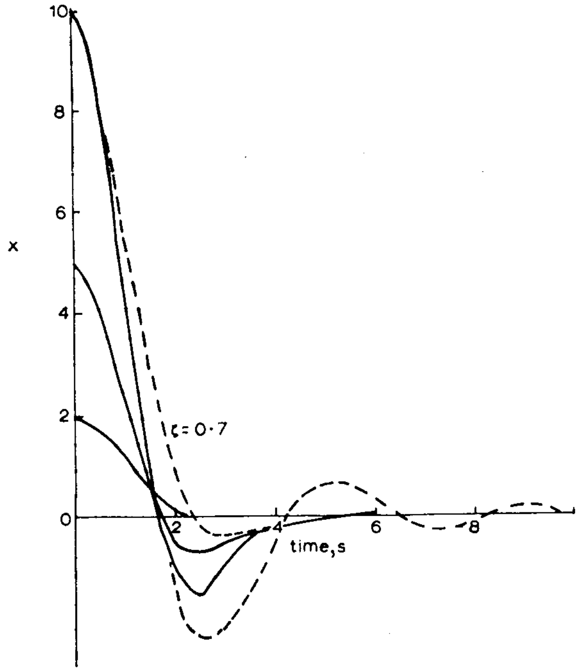

Fig. 2 *System response*

Assuming that the boundary conditions of eqn. 5 are given as $x(t_0) = x_0$ and $\dot{x}(t_0) = 0$, eqns. 8 and 11 give ϕ_0 and A_0 as

$$\phi_0 = \tan^{-1}\frac{\omega_1(t_0)}{\sigma_1(t_0)} \quad A_0 = \frac{x_0}{\sin \phi_0} \quad . \quad . \quad . \quad . \quad (12)$$

Thus ϕ_0 and A_0 may be calculated from eqns. 12 and 10 using estimations for $\dot{\bar{\omega}}$ and $\dot{\bar{\sigma}}$ (perhaps just taken as zero), and the transient oscillations may be determined from eqn. 8 using the following algorithm:

(*a*) Read in $\bar{\sigma}(A)$, $\bar{\omega}(A)$ curves and initial values of ω_1 and σ_1, and the values of x_0 and a suitable Δt.

(*b*) Calculate ϕ_0 and A_0 using eqn 12. Let $t = \Delta t$, $n = 0$.

(*c*) Calculate $A_{n+1} = A_n \exp\{\sigma_1(A_n)\}\Delta t$; $\phi_{n+1} = \phi_n + \omega_1(A_n)\Delta t$ and $x_{n+1} = A_{n+1} \sin(\phi_{n+1})$. Plot x_{n+1}, A_{n+1}, t. Let $t = t + \Delta t$ and let $n = n + 1$. Calculate $\sigma_1(A_n)$, $\omega_1(A_n)$ from the input data, A_n and eqn. 10, where $\dot{\bar{\omega}} = \{\bar{\omega}(A_n) - \bar{\omega}(A_{n-1})\}/\Delta t$ and $\dot{\bar{\sigma}} = \{\bar{\sigma}(A_n) - \bar{\sigma}(A_{n-1})\}/\Delta t$.

(*d*) Repeat step (*c*) until a sufficient portion of the transient oscillations are calculated.

Example: For the system equation

$$[s^4 + 36s^3 + 335s^2 + \{300 + 15KN_1(A)\}s + 300K]\,x = 0 \quad . \quad . \quad . \quad . \quad (13)$$

the $\alpha\beta$ parameter-plane diagram is in Fig. 1, where $\alpha = K$ and $\beta = KN_1(A)$. The design problem is to select K and $N_1(A)$ so that the transient oscillations have less than 15% overshoot for initial values of x from $x = 0$ to $x = 10$, and the 'fall' time and settling time are to be as short as possible. This quasioptimisation problem has been solved in the parameter plane using a value of 525 for K and choosing $N_1(A)$ corresponding to a saturating element (see the M locus of Fig. 1). The solution $x(t)$ of eqn. 13 may be calculated approximately using the parameter-plane diagram information and the algorithm above. This has been done for three sets of initial conditions and the results are plotted in Fig. 2. Various responses of a second-order linear system are also given in Fig. 2 for comparison. The exact response of eqn. 13 for the case of Fig. 1 is given, to within a few percent, by the approximate responses of Fig. 2. Other cases have been calculated with a similar accuracy, including one in which two nonlinearities were involved. Extensions to the case when the M locus enters the part of the parameter plane for which the dominant roots are real are readily made.

We comment that, for the second-order nonlinear system (eqn. 5), it is possible to determine the response from the location of the roots and the time derivative of the location of the roots to a high degree of approximation, if the calculations of ϕ_0 and A_0 (eqn. 12) are sufficiently good. This result, together with the parameter-plane method, forms the basis of useful approximate design procedures for higher-order systems.

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