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# TRANSIENT RESPONSE OF AN IMPULSIVELY LOADED PLASTIC STRING ON A PLASTIC FOUNDATION

By

M. MIHĂILESCU-SULICIU (*Institute of Mathematics, P.O. Box 1-764, RO-70700, Bucharest, Romania*),

I. SULICIU (*Institute of Mathematics, P.O. Box 1-764, RO-70700, Bucharest, Romania*),

T. WIERZBICKI (*Department of Ocean Engineering, Massachusetts Institute of Technology, Cambridge*),

AND

M. S. HOO FATT (*Department of Ocean Engineering, Massachusetts Institute of Technology, Cambridge*)

**Abstract.** The problem of an impulsive loading of a long rigid-plastic string resting on a rigid-plastic foundation is studied. A closed form solution is obtained by disregarding the longitudinal motion and considering an arbitrarily large transversal motion. Expressions for the final shape of the string are derived in terms of the magnitude of the applied impulse. It is found that the stress and the foundation reaction force are not uniquely determined, while the shape of the string is.

**Notation.**

$a = c_0^2/(q_0 X_0)$ = constant in Eq. (15),

$c_0 = (\overline{N}/\varrho_0)^{1/2}$ = transversal wave speed,

$\vec{i}, \vec{j}$ = unit vectors along $ox, oy$ axes, respectively,

$\overline{I}_0 = p_0 t_0$ = applied impulse per unit length,

$\overline{N}$ = yield stress of the string,

$p, p_0$ = pressure load, pressure amplitude,

$q$ = foundation reaction force,

$q_0$ = yield limit of the foundation reaction force,

$Q$ = dimensionless foundation reaction force,

$\vec{r} = \vec{r}(X, t)$ = position vector,

$\mathfrak{R}$ = real numbers,

$S$ = dimensionless stress in the string,

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$t, t_0$ = time, maximum time the pressure $p$ acts on the string,

$\widetilde{T}, \overline{T}$ = different stress measures for the string,

$w$ = dimensionless vertical displacement of the string,

$X$ = initial coordinate along the string,

$2X_0$ = initial distance pressure $p(X, t)$ acts on,

$x = x(X, t)$, $y = y(X, t)$, actual plane coordinates of the string,

$z$ = dimensionless horizontal coordinate of the string,

$\varepsilon, \tilde{\varepsilon}, \overline{\varepsilon}$ = different strain measures,

$\tau$ = dimensionless time measure,

$\zeta$ = dimensionless spatial coordinate along the undeformed string,

$\varrho_0$ = initial mass density per unit length of the string.

**1. Introduction.** The problem of plane motion of a long rigid plastic string resting on a rigid plastic foundation is formulated and solved in this paper. We extend here the earlier results presented in the references [2, 3] and [7 9, 11], where the reader may also find detailed justification of the relation between a plastic string on plastic foundations and cylindrical shells, submarine pipelines, etc., as well as references to earlier work on related topics.

The stress and strain measure used here for the string are different from those commonly used in the earlier literature (before the seventies; see [1] and also Sec. 3.1 of this work). However, when we apply our results to relatively small strains, which is the case in the problem under investigation, the quantitative differences are insignificant.

We consider the string initially at rest along a straight line and glued to a plastic foundation (which can move in one direction only, i.e., it can only be compressed). At $t = 0+$ we apply an impulsive load resulting, for example, from a contact explosion, [3], [8]. We assume that both the string and the foundation behavior are rigid-perfect plastic as described in Fig. 1 and the accompanying text.

Our aim is to find a closed-form solution of the above problem. First, we show that the impulsive loading problem can be transformed into a simpler but discontinuous initial-value problem. The shock wave solution can then be constructed without any simplifying assumption. This solution is written down in Fig. 3. Due to the assumption that the string has a very large (infinite) Young's modulus, the solution implies a horizontal shock wave across which the stress jumps to the yield stress.

Behind the shock wave the unloading follows and the problem becomes much more complicated. In order to construct a closed-form solution we disregard the longitudinal motion of the string. Then the vertical displacement of the string is found as a function of time and distance. The solution is piecewise smooth and depends on a single dimensionless parameter, which is a function of the applied impulse and other material parameters (see Fig. 6). As far as the distribution of stress and foundation reaction force is concerned, there are many possible solutions, but all of them lead to the same shape of the string. However, if a unique solution in stress and foundation reaction force is required, a viscosity selection criterion similar to the one discussed in [10] may be added.

**2. Formulation of the problem.** We consider the plane motion of a long (infinite) string. The position vector $\vec{r} = \vec{r}(X,t)$ of each string particle $X$ at any time $t \geq 0$ is given by

$$\vec{r} = x(X,t)\vec{i} + y(X,t)\vec{j}, \quad X \in \Re,\ t \geq 0. \tag{1}$$

We assume that the string is initially the straight line

$$\vec{r}(X,0) = X\vec{i}, \qquad X \in \Re. \tag{2}$$

We use the strain measure

$$2\tilde{\varepsilon}(X,t) = \left(\frac{\partial x}{\partial X}\right)^2 + \left(\frac{\partial y}{\partial X}\right)^2 - 1. \tag{3}$$

Since the stress vector at any string cross section $X$ is tangent to the string, we choose a stress magnitude $\widetilde{T}(X,t)$ which is conjugated (with respect to the mechanical power) to the strain measure (3). This means that the mechanical power is given by $\widetilde{T}\dot{\tilde{\varepsilon}}/\varrho_0$, where $\dot{\tilde{\varepsilon}}$ is the strain rate and $\varrho_0$ is the initial mass density. There are other strain and stress measures used in the literature on the subject; for instance, $\bar{\varepsilon}, \overline{T}$ related to $\tilde{\varepsilon}, \widetilde{T}$ by

$$\bar{\varepsilon} = \sqrt{1+2\tilde{\varepsilon}} - 1, \qquad \overline{T} = \widetilde{T}\sqrt{1+2\tilde{\varepsilon}} \tag{4}$$

are used in Cristescu [1], and of course we have $\widetilde{T}\dot{\tilde{\varepsilon}}/\varrho_0 = \overline{T}\dot{\bar{\varepsilon}}/\varrho_0$.

The string is considered to have a rigid perfect-plastic behavior, i.e., we assume that there is a yield stress $\overline{N}$ (the ordinate of the line $AB$ in Fig. 1) such that the strain can increase only if the path $(\tilde{\varepsilon}(X,t), \widetilde{T}(X,t))$ is along $AB$. Otherwise, the strain stays constant, and the stress can decrease or increase along a segment $BC$. Initially $C$ coincides with $O$.

The plastic string is assumed to be "glued" to a plastic foundation (in the sense that they cannot separate from each other and there is no friction between them) which acts on the plastic string as a body force $q(X,t)$. The reaction force $q(X,t)$ is related to the

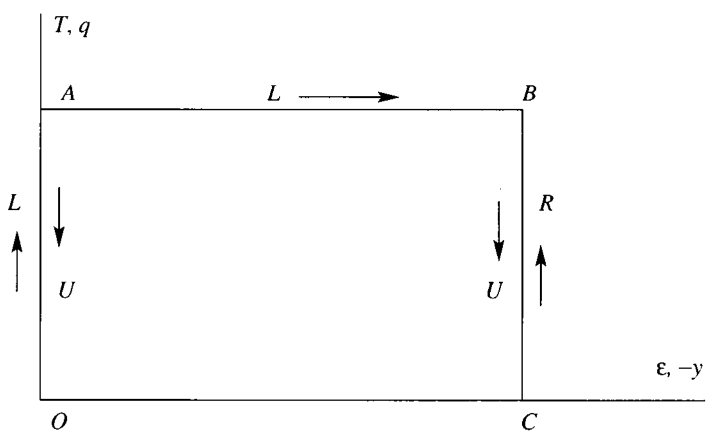


FIG. 1. Rigid perfect plastic behavior of the string and of the foundation. $L$—Load, $U$—Unload, $R$—Reload.

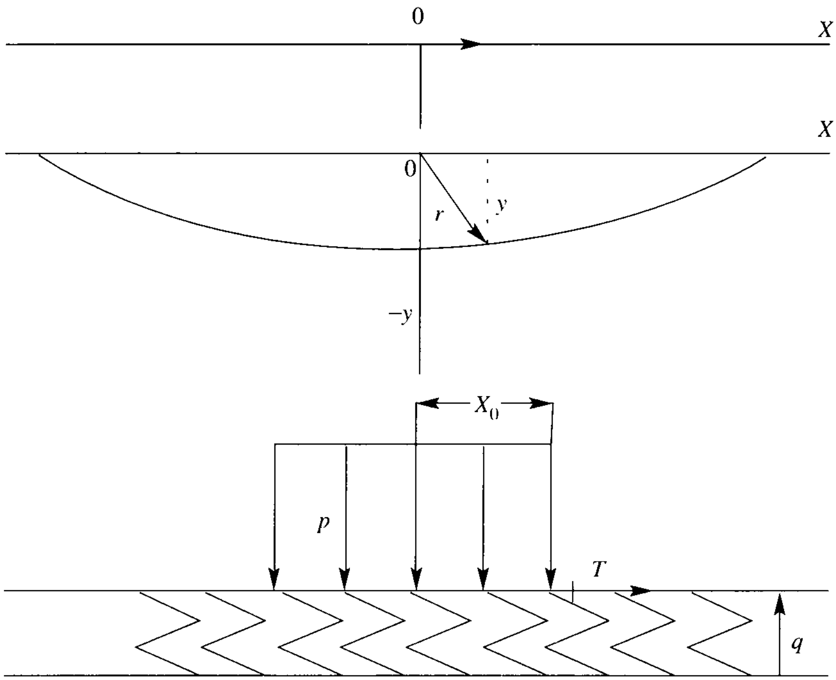


FIG. 2. String on a plastic foundation

vertical displacement $y(X,t)$ of the string as in Fig. 1, i.e., there is a constant $q_0$ (the ordinate of the line $AB$ in Fig. 1) such that the string ordinate $-y(X,t)$ can increase only if the path $(-y(X,t), q(X,t))$ is along $AB$; otherwise, $-y(X,t)$ stays constant and the foundation reaction force $q$ can decrease or increase along a segment $BC$. The string may therefore move only downwards, its upward motion being prevented by the permanent (plastic) deformation of the foundation. For detailed discussion of the meaning and engineering interpretation of the force $q(X,t)$ see [2, 3, 9] and also Fig. 2.

We assume that at $t = 0$ on a string interval $(-X_0, X_0)$, a uniform pressure $p$ is suddenly applied, held constant for a time $t_0 > 0$, and then suddenly removed. In other words, we assume that a downwards pressure

$$p(X,t) = \begin{cases} p_0 & \text{for } X \in (-X_0, X_0),\ t \in (0, t_0), \\ 0 & \text{otherwise}, \end{cases} \tag{5}$$

is acting on the string.

The balance law of momentum (see [1], Chap. 4, Sec. 2) gives

$$\begin{aligned} \varrho_0 \frac{\partial^2 x}{\partial t^2} - \frac{\partial}{\partial X}\left(\widetilde{T}\frac{\partial x}{\partial X}\right) &= 0, \\ \varrho_0 \frac{\partial^2 y}{\partial t^2} - \frac{\partial}{\partial X}\left(\widetilde{T}\frac{\partial y}{\partial X}\right) &= \varrho_0 (q - p), \end{aligned} \tag{6}$$

with $p = p(X,t)$ given by (5).

The system (6) together with the constitutive relations

$$\widetilde{T} \sim \tilde{\varepsilon}, \qquad q \sim -y \tag{7}$$

described above and Eq. (3) furnish a complete system of equations for the unknowns $x(X,t), y(X,t), \tilde{\varepsilon}(X,t), \widetilde{T}(X,t)$, and $q(X,t)$. The string is assumed to be initially undeformed and at rest along the $OX$-axis and subjected to a uniform stress $T_0$, $0 \le T_0 \le \overline{N}$, so that the initial conditions of the problem are

$$\begin{aligned} &x(X,0) = X, \quad y(X,0) = 0, \quad \frac{\partial x(X,0)}{\partial t} = 0, \quad \frac{\partial y(X,0)}{\partial t} = 0, \\ &\widetilde{T}(X,0) = T_0 \in [0, \overline{N}], \quad q(X,0) = 0, \ x \in \mathfrak{R}. \end{aligned} \tag{8}$$

Introduce now the total impulse $I(X)$ defined by

$$I(X) = \int_{0-}^{\infty} p(X,t)\, dt = \int_{0-}^{t_0} p(X,t)\, dt = \begin{cases} p_0 t_0 & \text{if } X \in (-X_0, X_0), \\ 0 & \text{otherwise,} \end{cases} \tag{9}$$

and require $p_0$ to be such that

$$\lim_{t_0 \to 0} p_0 t_0 = \overline{I}_0 = \text{Const.} \tag{10}$$

The initial-value problem (3), (5)–(8), where the function $p(X,t)$ satisfies (10), is called an impulsive loading problem.

The impulsive loading problem can be converted into an initial-value problem by writing equation $(6)_2$ in the integral form

$$\begin{aligned} &\int_{X_1}^{X_2} \varrho_0 \left( \frac{\partial y(X,t_2)}{\partial t} - \frac{\partial y(X,t_1)}{\partial t} \right) dX \\ &\qquad - \int_{t_1}^{t_2} \left( \left( \widetilde{T} \frac{\partial y}{\partial X} \right)(X_2,t) - \left( \widetilde{T} \frac{\partial y}{\partial X} \right)(X_1,t) \right) dt \\ &\qquad\qquad = \int_{X_1}^{X_2} \int_{t_1}^{t_2} \varrho_0 q(X,t)\, dX\, dt - \int_{X_1}^{X_2} \int_{t_1}^{t_2} \varrho_0 p(X,t)\, dX\, dt, \end{aligned} \tag{11}$$

which must hold for any $X_1, X_2 \in \mathfrak{R}$, $X_1 < X_2$, and any $t_1, t_2 \in \mathfrak{R}$, $t_1 < t_2$. Let us take $t_1 = 0-, t_2 = t_0$; then since $q, \widetilde{T}$, and $\partial y / \partial X$ are bounded, we find from (11) for $t_0 \to 0$, by taking (9) and (10) into account,

$$\int_{X_1}^{X_2} \left( \frac{\partial y(X,0+)}{\partial t} - \frac{\partial y(X,0-)}{\partial t} \right) dX = - \int_{X_2}^{X_2} I(X) dX.$$

But $X_1, X_2$ are arbitrary, and by using (8) we get

$$\frac{\partial y(X,0+)}{\partial t} = -I(X) = \begin{cases} -\overline{I}_0 & \text{if } X \in (-X_0, X_0), \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

For $0 < t_1 < t_2$ in (11) and $t_0 \to 0$ we get Eq. $(6)_2$ with the term $-\varrho_0 p(X,t)$ dropped from the right-hand side.

Thus, the solution of the impulsive loading problem can be obtained by solving the initial-value problem (3), (5)–(8) and then taking the limit when $t_0 \to 0$ under the condition (10) or by solving directly the initial-value problem

$$\begin{gathered}
\varrho_0 \frac{\partial^2 x}{\partial t_2} - \frac{\partial}{\partial X}\left(\widetilde{T}\frac{\partial x}{\partial X}\right) = 0, \\
\varrho_0 \frac{\partial^2 y}{\partial t^2} - \frac{\partial}{\partial X}\left(\widetilde{T}\frac{\partial y}{\partial X}\right) = \varrho_0 q, \\
2\tilde{\varepsilon} = \left(\frac{\partial x}{\partial X}\right)^2 + \left(\frac{\partial y}{\partial X}\right)^2 - 1, \\
\tilde{\varepsilon} \sim \widetilde{T}, \qquad -y \sim q, \\
x(X,0) = X, \qquad y(X,0) = 0, \\
\frac{\partial x(X,0)}{\partial t} = 0, \qquad \frac{\partial y(X,0)}{\partial t} = -I(X) = \begin{cases} -\overline{I}_0 & \text{for } X \in (-X_0, X_0), \\ 0 & \text{otherwise}, \end{cases} \\
\widetilde{T}(X,0) = T_0 \in [0, \overline{N}], \qquad q(X,0) = 0.
\end{gathered} \tag{13}$$

Let us introduce the following dimensionless variables:

$$\begin{gathered}
\zeta = \frac{X}{X_0}, \qquad \tau = \frac{c_0 t}{X_0}, \qquad c_0 = \left(\frac{\overline{N}}{\varrho_0}\right)^{1/2}, \\
z = \frac{c_0^2}{q_0 X_0^2} x, \qquad w = \frac{c_0^2}{q_0 X_0^2} y, \qquad S = \frac{1}{\varrho_0 c_0^2}\widetilde{T}, \\
\varepsilon = \frac{c_0^4}{q_0^2 X_0^2}\tilde{\varepsilon}, \qquad Q = \frac{q}{q_0}, \qquad I_0 = \frac{c_0}{q_0 X_0}\overline{I}_0.
\end{gathered} \tag{14}$$

The dimensionless form of Eqs. (13) is

$$\begin{gathered}
\frac{\partial^2 z}{\partial \tau^2} - \frac{\partial}{\partial \zeta}\left(S\frac{\partial z}{\partial \zeta}\right) = 0, \\
\frac{\partial^2 w}{\partial \tau^2} - \frac{\partial}{\partial \zeta}\left(S\frac{\partial w}{\partial \zeta}\right) = Q, \\
2\varepsilon = \left(\frac{\partial z}{\partial \zeta}\right)^2 + \left(\frac{\partial w}{\partial \zeta}\right)^2 - a^2, \\
\varepsilon \sim S, \qquad -w \sim Q, \\
z(\zeta,0) = a\zeta, \qquad w(\zeta,0) = 0, \\
\frac{\partial z(\zeta,0)}{\partial \tau} = 0, \qquad \frac{\partial w(\zeta,0)}{\partial \tau} = -I(\tau) = \begin{cases} -I_0 & \text{for } -1 < \zeta < 1, \\ 0 & \text{otherwise}, \end{cases} \\
S(\zeta,0) = S_0 \in [0,1], \qquad Q(\zeta,0) = 0.
\end{gathered} \tag{15}$$

**3. Solution of the problem.** The initial data of the problem (15) have a jump discontinuity at the points $\zeta = \pm 1$; therefore, shock waves will be formed. They must satisfy the following compatibility conditions (see for instance [1], Chap. 4, Sec. 13, and also [4, 5]), obtained in a standard manner when we look for weak solutions of (15):

$$\begin{aligned} c[\vec{v}] + [\widetilde{T}\vec{u}] &= 0, \\ [\vec{v}] + c[\vec{u}] &= 0, \end{aligned} \tag{16}$$

where

$$\begin{aligned} \vec{v} &= \frac{\partial \vec{r}}{\partial \tau} = (z_\tau, w_\tau), \\ \vec{u} &= \frac{\partial \vec{r}}{\partial \zeta} = (z_\zeta, w_\zeta), \end{aligned} \tag{17}$$

and $c = d\zeta/d\tau$ is the shock-wave speed.

Since the problem (15) is symmetric with respect to $\zeta$, it is sufficient to construct the solution for $\zeta \geq 0$, $\tau \geq 0$ only.

3.1. *Solution of the Riemann problem with step data at $\zeta = 1, \tau = 0$.* The step-data problem (15) at the point ($\zeta = 1, \tau = 0$) is called a Riemann problem. The discontinuous structure of the solution at that point can be obtained directly from (16). It may also be obtained in the same way as discussed in [4] or [5]. In order to show some properties of the solution, let us assume for a moment that instead of the rigid perfectly plastic model of Fig. 1 we have an elastic perfectly plastic model. This model has for loading processes the following stress-strain relation:

$$\widetilde{T} = \begin{cases} E\tilde{\varepsilon} & \text{if } 0 \leq \tilde{\varepsilon} \leq \overline{N}/E, \\ \overline{N} & \text{if } \tilde{\varepsilon} > \overline{N}/E, \end{cases} \tag{18}$$

where $E = \text{const.} > 0$ is Young's modulus. In terms of the stress and strain measures $\bar{\varepsilon}$ and $\overline{T}$ defined in [4] the above relation (18) leads to

$$\overline{T} = \begin{cases} E(\bar{\varepsilon} + \bar{\varepsilon}^2/2), & 0 \leq \bar{\varepsilon} \leq \sqrt{(1 + \overline{N}/E)} - 1 = \varepsilon_Y, \\ \overline{N}(1 + \bar{\varepsilon}), & \varepsilon_Y < \bar{\varepsilon}. \end{cases} \tag{19}$$

It is known (see [6] and also [1]) that whenever we have a stress-strain relation of the form $\overline{T} = \overline{N}(1 + \bar{\varepsilon}), \overline{N} = \text{const.}$, on some strain interval, problem (15) becomes linear on that interval. However, when unloading is involved the linearity is lost.

Taking the above remark into account, the solution of the Riemann problem at the point $(1, 0)$ can be directly determined from [4]. Then we take $E \to \infty$ to get the rigid perfect plastic model, and the solution of the Riemann problem in various regions of the $(\zeta, \tau)$-plane is given in Fig. 3 (see p. 334).

We shall describe now some properties of the solution. In Fig. 3 the $O\zeta$-axis is a shock wave across which stress jumps from $S_0 \in [0, 1]$ to $S = 1$. $AB$ and $BC$ are shock waves in two opposite directions (with speeds $c = -1$ and $c = 1$, respectively). In the triangle $OAB$ and below $BC$ for $\zeta > 1$, the solution can be calculated and is presented in Fig. 3. The values $\vec{V}(1, 0+)$, etc., are the one-sided limit values at $\zeta = 1$, $\tau = 0$ between the shocks $AB$ and $BC$. We note that both shock waves are loading waves since across them the strain jumps from zero to

$$2\tilde{\varepsilon}(1, 0+) = \vec{u} \cdot \vec{u} - a^2 = \frac{I_0^2}{2} > 0. \tag{20}$$

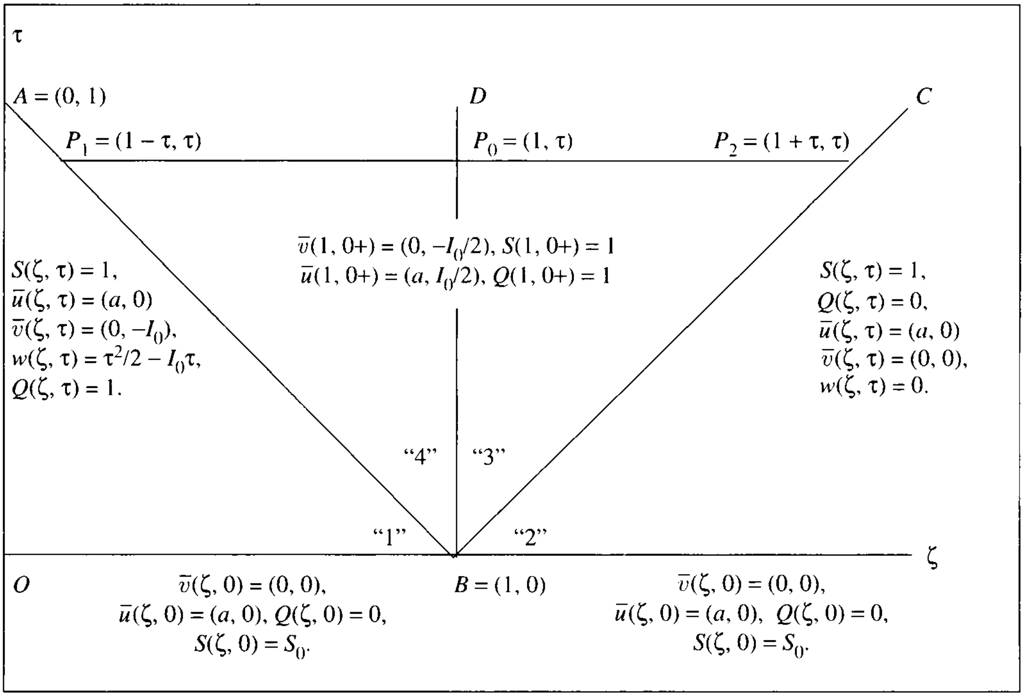


FIG. 3. Solution before and just after shocks

3.2. *Solution in the open region above ABC of Figure* 3. The second wave structure for the system (15) implies there exists also a second-order wave at point $B$, namely the vertical line $BD$. A simple calculation using the jump relations across $AB$, $BC$, and $BD$ at the point $B$ (see Appendices B and C) leads to the one-sided limit values of the second-order partials of $\vec{r}$ at $B(1,0)$.

Thus, the common value of

$$\dot{\varepsilon}(1,0+) = (\frac{\partial \vec{r}}{\partial \zeta} \cdot \frac{\partial^2 \vec{r}}{\partial \zeta \partial \tau})(1,0+)$$

in regions "3" and "4" (of Fig. 3) is

$$(\dot{\varepsilon}(1,0+))^{3,4} = -\frac{1}{8}\left(I_0 + (I_0^2 + 4a^2)\left(\frac{\partial S}{\partial \zeta}(1,0+)\right)^4\right) \tag{21}$$

and, if it is strictly positive, then $S = 1$ in regions "3" and "4", at least in a neighborhood of $(1,0)$, and $(\frac{\partial S}{\partial \zeta}(1,0+))^4 = 0$, which is impossible according to (21). Therefore, $(\dot{\varepsilon}(1,0+))^{3,4} = 0$ and

$$\left(\frac{\partial S}{\partial \zeta}(1,0+)\right)^4 = -\frac{I_0}{I_0^2 + 4a^2} < 0, \qquad \left(\frac{\partial S}{\partial \tau}(1,0+)\right)^{3,4} = -\frac{I_0}{I_0^2 + 4a^2} < 0, \tag{22}$$

i.e., regions "3" and "4" are unloading regions (with $S < 1$ and $\dot{\varepsilon} = 0$) at least in a neighborhood of $(1,0)$. In what follows we shall prove that the unloading extends in the whole region above $ABC$.

Due to the complexity of the unloading process, in order to get a closed form solution we shall neglect the horizontal motion of the string by making the following simplifying hypothesis:

$$z(\zeta,\tau) = a\zeta \quad \text{for all } \zeta \in \mathfrak{R},\ \tau \geq 0 \tag{23}$$

(which can only be satisfied approximately since otherwise by $(15)_1$ we get that the stress is constant with respect to $\zeta$). Then problem (15) takes the form

$$\begin{gathered}
\frac{\partial w_\tau}{\partial \tau} - \frac{\partial}{\partial \zeta}(S w_\zeta) = Q, \\
w(\zeta,0) = 0, \qquad w_\tau(\zeta,0) = \begin{cases} -I_0 & \text{for } |\zeta| \leq 1, \\ 0 & \text{for } |\zeta| > 1, \end{cases} \\
S \sim \varepsilon = \frac{1}{2}\left(\frac{\partial w}{\partial \zeta}\right)^2, \\
Q \sim -w.
\end{gathered} \tag{24}$$

In order to obtain the solution in regions "3" and "4" we need to find $w_\tau$ and $w_\zeta$ on $AB$ in region "4" and on $BC$ in region "3", i.e., we need to determine $(w_\tau(1-\tau,\tau),\ w_\zeta(1-\tau,\tau))^4$ and $(w_\tau(1+\tau,\tau), w_\zeta(1+\tau,\tau))^3, \tau > 0$. For that let us consider the points $P_0(1,\tau)$ on $BD$, $P_1(1-\tau,\tau)$ on $AB$, and $P_2(1+\tau,\tau)$ on $BC$ (see Fig. 3); we integrate Eq. $(24)_1$ on the triangles $BP_0P_1$ and $BP_2P_0$, respectively. We obtain, by using the jump relations across $AB$ and $BC$,

$$\begin{aligned}
\int_0^\tau S(1,\alpha)\,d\alpha &= \frac{2}{I_0}\left(\frac{\tau^2}{2} - \tau w_\tau^3(1+\tau,\tau)\right), \\
\int_0^\tau S(1,\alpha)\,d\alpha &= \frac{2}{I_0}(I_0\tau - \tau^2 + \tau w_\tau^4(1-\tau,\tau)),
\end{aligned} \tag{25}$$

since $\partial^2 w/\partial\zeta\partial\tau = 0$ in the regions "3" and "4". From (25) we get

$$\begin{gathered}
w_\tau^3(1+\tau,\tau) = w_\tau^4(1-\tau,\tau) = \frac{3}{4}\tau - \frac{I_0}{2}, \\
w_\zeta^3(1+\tau,\tau) = \frac{I_0}{2} - \frac{3}{4}\tau, \qquad w_\zeta^4(1-\tau,\tau) = \frac{I_0}{2} - \frac{1}{4}\tau.
\end{gathered} \tag{26}$$

Now, by integration of $(24)_1$ and taking into account (26) we obtain

$$\begin{gathered}
w^3(\zeta,\tau) = \tfrac{1}{4}(3+2I_0)(\zeta-\tau-1) + \tfrac{3}{8}((1+\tau)^2 - \zeta^2), \\
S^3(\zeta,\tau) = \frac{2I_0 + 1 - 2\tau - \zeta}{2I_0 + 3(1-\zeta)}, \quad Q^3(\zeta,\tau) = 1, \qquad \zeta \in (1, 1+\tau),\ \tau \geq 0, \\
w^4(\zeta,\tau) = \frac{\tau}{8}(3\tau - 4I_0) + \frac{\zeta-1}{8}(\zeta - 1 + 4I_0), \\
S^4(\zeta,\tau) = \frac{2I_0 + 1 - 2\tau - \zeta}{2I_0 - 1 + \zeta}, \quad Q^4(\zeta,\tau) = 1, \qquad \zeta \in (1-\tau, 1),\ \tau \geq 0,
\end{gathered} \tag{27}$$

and this solution is valid as long as $w_\tau < 0$, i.e., for $\tau \in [0, 2I_0/3]$ in both regions. For $\tau = 2I_0/3$ we have $w_\tau^{3,4}(\zeta, 2I_0/3) = 0$ and, since $w_{\tau\tau}^{3,4}(\zeta, 2I_0/3) = 3/4 > 0$, the velocity $w_\tau$

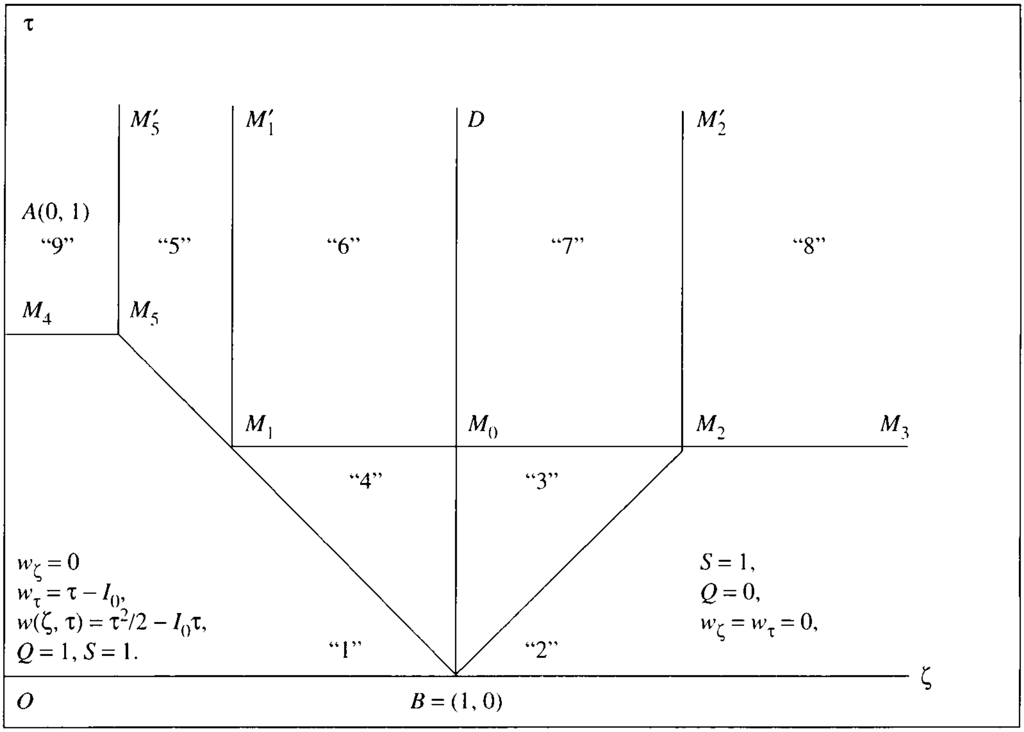


FIG. 4. The solution structure for $I_0$ in $(0, 1]$

remains equal to zero for $\tau > 2I_0/3$ in both regions (i.e., the string motion stops there), since the string is not allowed to move upwards. Now, depending on the magnitude of the applied impulse $I_0$, the value $\tau = 2I_0/3$ may be smaller or larger than the value $\tau = 1$ when the shock wave $AB$ meets its symmetric shock wave $AB'$ with $B' = (-1, 0)$ (see also Fig. 5), and the two cases lead to different solutions. On the other hand, for $2I_0/3 < 1$ (i.e., $I_0 < 3/2$) the shock wave $AB$ disappears at $\tau = I_0$ (i.e., all the jumps across it vanish) and the solution will be different depending on whether the shock $AB$ disappears before meeting the shock $AB'$ at $\tau = 1$ (i.e., for $I_0 < 1$) or not (i.e., for $I_0 > 1$). We have therefore to consider three cases, namely $I_0 \in (0, 1]$, $I_0 \in (1, 3/2]$, and $I_0 > 3/2$.

*The case* $I_0 \in (0, 1]$. The solution (27) is valid only for $\tau \in [0, 2I_0/3]$. To extend the solution for larger times we consider the points $M_1 = (1 - 2I_0/3, 2I_0/3)$, $M_0 = (1, 2I_0/3)$, $M_2 = (1 + 2I_0/3, 2I_0/3)$, $M_3 = (\infty, 2I_0/3)$ in Fig. 4. According to $(27)_1$ we have $w_\tau^3(M_2) = w_\zeta^3(M_2) = 0$, while $S^3(\zeta, \tau)$ has different limit values at $M_2$ on each line $\zeta - (1 + 2I_0/3) = m(\tau - 2I_0/3)$, with $m \in [1, \infty)$, so that these values $S^3(M_2; m)$ vary from $S^3(M_2; 1) = 1$ to $S^3(M_2; \infty) = 1/3$.

Now, if we study in some detail the structure of the rarefaction and shock waves under the simplifying hypothesis (23), i.e., $z(\zeta, \tau) = a\zeta$ for all $\zeta \in \mathfrak{R}$, $\tau > 0$, we find that there are infinitely many solutions for the Riemann problem at the point $M_2$, but they only differ in the stress distribution, while for all of them we get the same $w$ in regions "7" and "8", i.e.,

$$\begin{aligned} w^7_\zeta(\zeta,\tau) &= w^3_\zeta(\zeta, \tfrac{2}{3}I_0), \\ w^7_\tau(\zeta,\tau) = 0, &\quad \text{for } \zeta \in (1, 1+\tfrac{2}{3}I_0),\ \tau > \tfrac{2}{3}I_0, \end{aligned} \tag{28}$$

and

$$w^8(\zeta,\tau) = 0 \quad \text{for } \zeta > 1+\tfrac{2}{3}I_0,\ \tau > \tfrac{2}{3}I_0. \tag{29}$$

A unique solution for the Riemann problem at $M_2$ may be obtained if we construct the solution according to the shock and rarefaction wave structure without the additional hypothesis (23). This solution consists of a shock wave $M_2M_3$ and two second-order waves, namely the lines $M_2M_0$, $M_2M_2'$, and leads to the same solution (28)–(29) for $w$ in regions "7" and "8".

At the point $M_1$ (see Fig. 4) we have again a Riemann problem. A similar argument to that used to solve the Riemann problem at the point $(1,0)$ shows there is only one shock wave at $M_1$, namely $M_1M_5$. The horizontal line $M_1M_0$ and the vertical line $M_1M_1'$ are both second-order waves. The solution in regions "5" and "6" is

$$\begin{aligned} w^5_\zeta(\zeta,\tau) &= w^5_\zeta(\zeta, 1-\zeta) = I_0 - 1 + \zeta, \\ w^5_\tau(\zeta,\tau) = 0, &\quad \text{for } \zeta \in (1-\tau, 1-2I_0/3),\ \tau > 2I_0/3, \\ w^6_\zeta(\zeta,\tau) &= w^4_\zeta(\zeta, 2I_0/3), \\ w^6_\tau(\zeta,\tau) = 0, &\quad \text{for } \zeta \in (1-2I_0/3, 1),\ \tau > 2I_0/3. \end{aligned} \tag{30}$$

Let us consider now the point $M_5 = (1-I_0, I_0) \in BM_1$ (see Fig. 4), where $w^1_\tau(M_5) = w^5_\zeta(M_5) = 0$ (i.e., where the string motion stops completely; see also Fig. 3 and formula $(30)_1$). We also have $w^1_\zeta(M_5) = w^5_\zeta(M_5) = 0$ and $w^1_\tau(M_5) = w^5_\tau(M_5) = 0$, as it happened at point $M_2$, and $S$ has infinitely many limit values at $M_5$ in region "5". Thus we get at $M_5$ a similar Riemann problem to that at $M_2$, and the same argument we used at $M_2$ leads to the following solution: the horizontal line $M_4M_5$ is a stress shock wave; the vertical line $M_5M_5'$ is a second-order wave, and in region "9" and beyond we have

$$\begin{aligned} w_\tau(\zeta,\tau) = 0 &\quad \text{for all } \zeta \ge 0,\ \tau \ge I_0, \\ w^9_\zeta(\zeta,\tau) = 0 &\quad \text{for } \zeta \in (0, 1-I_0),\ \tau > I_0. \end{aligned} \tag{31}$$

The string (vertical) motion stops at the value $\tau = \tau^*$ (in this case $\tau^* = I_0$) when the velocity $w_\tau(\zeta,\tau^*) = 0$, and $w_\tau(\zeta,\tau)$ remains equal to zero for $\tau > \tau^*$ in the whole string; the final shape of the string is therefore given by $w(\zeta,\tau^*)$, as the plastic behavior of the foundation prevents any other modification of $w$.

The final shape of the deformed string in this case is given by

$$w(\zeta, I_0) = \begin{cases} -\frac{I_0^2}{2}, & \zeta \in [0, 1-I_0), \\ \frac{1}{2}(\zeta^2 - 1) + (I_0 - 1)(\zeta - 1), & \zeta \in [1-I_0, 1-2I_0/3), \\ -\frac{I_0^2}{6} + \frac{\zeta-1}{8}(\zeta - 1 + 4I_0), & \zeta \in [1-2I_0/3, 1), \\ \frac{3}{8}(1-\zeta^2) + \frac{3+2I_0}{4}(\zeta - 1) - \frac{I_0^2}{6}, & \zeta \in [1, 1+2I_0/3), \\ 0, & \zeta \ge 1 + 2I_0/3. \end{cases} \tag{32}$$

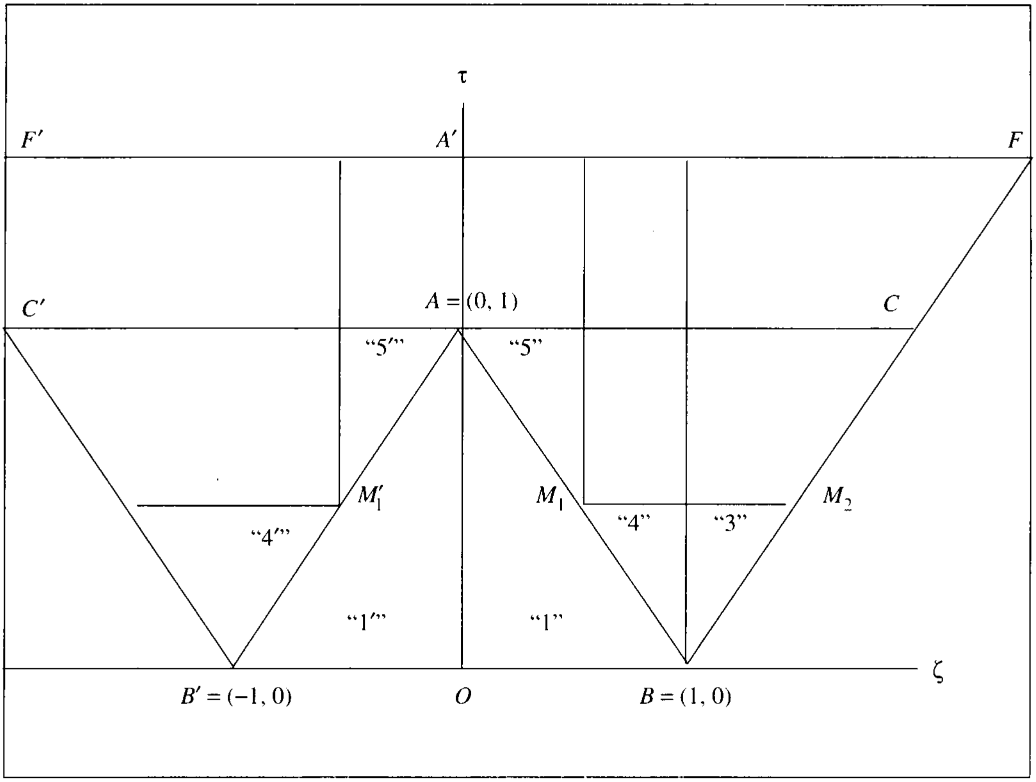


FIG. 5. The solution structure for $I_0 > 1$

*The case* $I_0 \in (1, 3/2]$. In this case the construction of the solution follows the same steps as in the case of $I_0 \leq 1$. The only difference is that while the point $M_1$ (where $w_\tau^1(M_1) = 0$) still lies on $AB$ between $B$ and $A$ (as $\tau(M_1) = 2I_0/3 < 1$), the point $M_5$ (where $w_\tau^1(M_5) = w_\zeta^5(M_5) = 0$) is pushed on $AB$ beyond point $A$ (as $\tau(M_5) = I_0 > 1$). Thus regions "1" and "5" in this case extend along $AB$ up to $\tau = 1$, and region "9" disappears. Therefore, we have now the same expressions for the solution in all the regions "1" through "8", in the whole domain $\{\zeta \in \mathfrak{R},\ \tau \in (0, 1)\}$ as in the case $I_0 \leq 1$ (see Fig. 5).

Let us remember that problem (24) is symmetric in $\zeta$ and, therefore, at the point $A = (0, 1)$ the shock wave $AB$ meets its symmetric shock wave $AB'$. We have, according to (30),

$$w_\tau^5(A) = w_\tau^{5'}(A) = 0, \qquad w_\zeta^5(A) = -w_\zeta^{5'}(A) = I_0 - 1 > 0. \tag{33}$$

The shock wave structure resulting from (16) does not allow the two shocks $AB$ and $AB'$ to continue for $\tau > 1$, since beyond them at point $A$ we get a strictly positive particle velocity $w_\tau$; then $AA'$ has to be a shock wave at $A$ (according to (D.4) or (D.3) of Appendix D and (33)), and the two one-sided limits of stress with respect to the shock $AA'$ must be equal to zero. The solution of the Riemann problem at the point $A$ consists then of two horizontal shock waves $AC$ and $AC'$ across which only $S$ jumps from $S = 1$ to $S = 0$ and the vertical shock wave $AA'$ across which only $w_\zeta$ jumps from $1 - I_0$ (for

$\zeta = 0-$) to $I_0 - 1$ (for $\zeta = 0+$). Therefore, we have

$$\begin{aligned} w_\zeta(\zeta,\tau) &= w_\zeta(\zeta,1) \quad \text{for } \zeta \in \mathfrak{R} - \{0\},\ \tau \geq 1, \\ w_\tau(\zeta,\tau) &= 0 \quad \text{for } \zeta \in \mathfrak{R},\ \tau \geq 1, \end{aligned} \tag{34}$$

and the final shape of the string is given by

$$w(\zeta, I_0) = \begin{cases} \frac{1}{2}(\zeta-1)^2 + I_0(\zeta-1), & \zeta \in [0, 1-2I_0/3), \\ \frac{1}{8}(\zeta-1)^2 + \frac{I_0}{2}(\zeta-1) - \frac{I_0^2}{6}, & \zeta \in [1-2I_0/3, 1), \\ -\frac{3}{8}(\zeta-1)^2 + \frac{I_0}{2}(\zeta-1) - \frac{I_0^2}{6}, & \zeta \in [1, 1+2I_0/3), \\ 0, & \zeta \geq 1 + 2I_0/3. \end{cases} \tag{35}$$

*The case* $I_0 > 3/2$. The construction of the solution being the same in this case as for the previous ones, we remark that the point $M_1$ is now pushed on $AB$ beyond $A$ (as $\tau(M_1) = 2I_0/3 > 1$); thus, the regions "3" and "4" extend along $BM_2$ and $BA$, respectively, up to $\tau = 1$, and regions "5" to "8" disappear (see Fig. 5). Now, at point $A$, the shock wave $AB$ separates regions "1" and "4" and the shock wave $AB'$ separates regions "1′" and "4′"; we have

$$\begin{aligned} w_\zeta^4(A) &= \frac{2I_0 - 1}{4} = -w_\zeta^{4'}(A), \\ w_\tau^4(A) &= w_\tau^{4'}(A) = \frac{3 - 2I_0}{4}. \end{aligned}$$

The solution of the Riemann problem at $A$ consists again of two horizontal shock waves $AC$ and $AC'$ across which the stress jumps from the value $S = 1$ to the value $S = 0$ and a vertical shock wave $AA'$ across which only $w_\zeta$ has a jump; the stress remains equal to zero for $\zeta = 0, \tau > 1$. We have another Riemann problem at point $C$; its solution consists of the shock wave $CF$ and possibly of the horizontal shock wave $AC$. We still have $\dot{\varepsilon} = 0$ for $\tau > 1$, and thus $w_\tau$ depends on $\tau$ only, and $Q = 1$ for $\tau > 1$ as long as $w_\tau < 0$.

Now, let $\tau > 1$, $A' = (0,\tau)$ with $w_\tau(\zeta,\tau) < 0$; we integrate Eq. $(24)_1$ on the domain $ACFA'$, and we get

$$2w_\tau(\zeta,1) - (1+\tau)w_\tau(\zeta,\tau) = -\tfrac{1}{2}(\tau^2 + 2\tau - 3),$$

i.e.,

$$w_\tau(\zeta,\tau) = \begin{cases} \frac{\tau^2+2\tau-2I_0}{2(1+\tau)}, & \zeta \in [0, 1+\tau),\ \tau \in [1, \sqrt{1+2I_0} - 1), \\ 0, & \zeta > 1+\tau,\ \tau \in [1, \sqrt{1+2I_0} - 1), \\ 0, & \zeta \in \mathfrak{R},\ \tau \geq \sqrt{1+2I_0} - 1; \end{cases} \tag{36}$$

$$w_\zeta(\zeta,\tau) = \begin{cases} w_\zeta(\zeta,1), & \zeta \in [0,2),\ \tau \geq 1, \\ -w_\tau(\zeta,\zeta-1), & \zeta \in (2, \sqrt{1+2I_0}),\ \tau \geq \zeta - 1, \\ 0, & \zeta \in (2, \sqrt{1+2I_0}),\ \tau \in (1, \zeta-1), \\ 0, & \zeta > \sqrt{1+2I_0},\ \tau > 1. \end{cases} \tag{37}$$

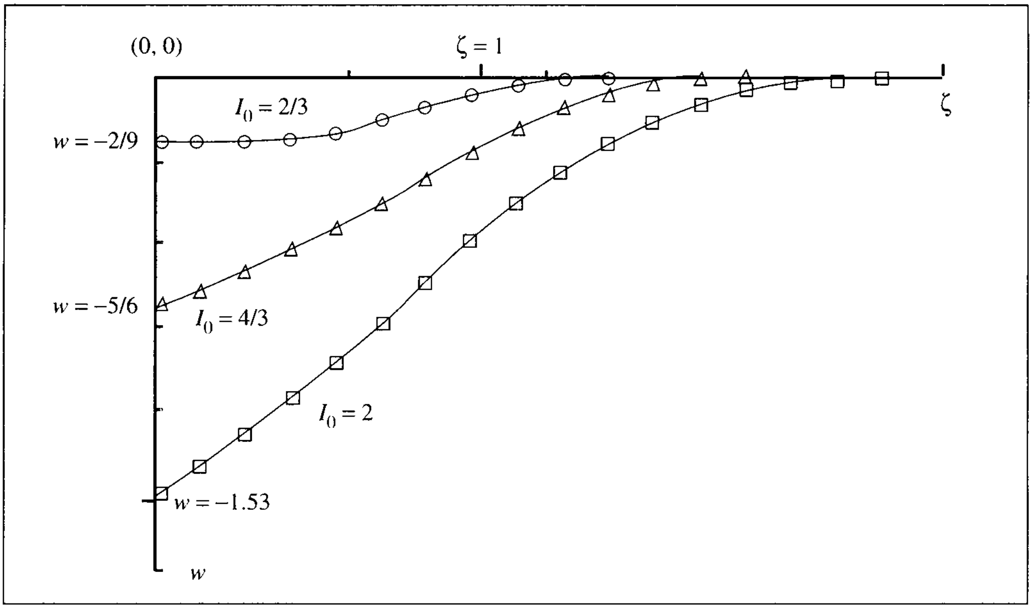


FIG. 6. Final string shape for different $I_0$

The final shape of the string will then become

$$w(\zeta, I_0) = \begin{cases} \frac{1}{8}(\zeta-1)^2 + \frac{I_0}{2}(\zeta-1) - \frac{3}{8} - \frac{1+2I_0}{2}\ln\left(\frac{\sqrt{1+2I_0}}{2}\right), & \zeta \in [0,1), \\ -\frac{3}{8}(\zeta-1)^2 + \frac{I_0}{2}(\zeta-1) - \frac{3}{8} - \frac{1+2I_0}{2}\ln\left(\frac{\sqrt{1+2I_0}}{2}\right), & \zeta \in [1,2), \\ -\frac{\zeta^2}{4} + \frac{1+2I_0}{4} - \frac{1+2I_0}{2}\ln\left(\frac{\sqrt{1+2I_0}}{\zeta}\right), & \zeta \in [2, \sqrt{1+2I_0}), \\ 0, & \zeta \geq \sqrt{1+2I_0}. \end{cases} \tag{38}$$

In Fig. 6 the final shape of the string is plotted for three different values of $I_0$, i.e., $I_0 = 2/3, 4/3$, and 2. These values fall into the three different intervals on which the final form of the string has different expressions. By $(14)_9$, $I_0 = \frac{c_0}{q_0 X_0}\overline{I}_0$, and for $\overline{I}_0$ we can substitute $p_0 t_0$. Clearly, for the string to deform we need an applied pressure larger than the yield limit of the foundation reaction force, i.e., $q_0 < p_0$, and we can see the meaning of the three cases ($I_0 \in (0,1], I_0 \in (1,3/2], I_0 > 3/2$). We remark that, for $I_0 < 1$, there is a flat portion in the middle of the string. For larger values of $I_0$, i.e., $I_0 > 1$, a slope discontinuity appears in the middle of the string, and this angle becomes sharper when $I_0$ increases. This behavior is mainly due to the constitutive relation for $q$ and the fact that the string is "glued" on the foundation.

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**Appendix A. The system (15) has no rarefaction waves.** Indeed, let $P_0 = (\zeta_0, \tau_0)$ be a point and let us denote $\lambda = (\zeta - \zeta_0)/(\tau - \tau_0)$. Then a rarefaction wave at $P_0$ will be described by the functions $\vec{u}(\lambda), \vec{v}(\lambda), S(\lambda)$ verifying $(15)_{1,2}$ and (17), i.e.,

$$\lambda \frac{d\vec{v}}{d\lambda} + \frac{d}{d\lambda}(S\vec{u}) = (0,0), \qquad \frac{d\vec{v}}{d\lambda} + \lambda \frac{d\vec{u}}{d\lambda} = (0,0), \tag{A.1}$$

which leads to

$$(S - \lambda^2)\frac{d\vec{u}}{d\lambda} + \vec{u}\frac{dS}{d\lambda} = (0,0). \tag{A.2}$$

The inner product of (A.2) by $\vec{u}$ (as $|\vec{u}| \geq 0$) gives

$$\frac{1}{2}(S - \lambda^2)\frac{d}{d\lambda}(|\vec{u}|^2) + |\vec{u}|^2 \frac{dS}{d\lambda} = 0.$$

Now, the constitutive relation $S \sim \varepsilon = \frac{1}{2}(|\vec{u}|^2 - a^2)$ does not allow $S$ and $|\vec{u}|^2$ to vary in the same time; therefore, we always have $\frac{dS}{d\lambda} = 0$ and $\frac{d}{d\lambda}(|\vec{u}|^2) = 0$, and there is no rarefaction wave.

For the system (24), rarefaction waves are possible only at points $P_0 = (\zeta_0, \tau_0)$ where $w_\zeta(P_0) = 0$. Indeed, in this case (A.2) becomes

$$\frac{1}{2}(S - \lambda^2)\frac{dw_\zeta}{d\lambda} + w_\zeta \frac{dS}{d\lambda} = 0$$

and, for $w_\zeta(P_0) \neq 0$, the conclusion is the same as above; however, when $w_\zeta(P_0) = 0$, rarefaction waves may appear, but $\frac{dw_\zeta}{d\lambda} = \frac{dw_\tau}{d\lambda} = 0$ and only $\frac{dS}{d\lambda} \neq 0$.

**Appendix B. Second-order waves at the point** $B = (1, 0)$**.** Let $\mathbf{C} : \zeta = \overline{\zeta}(\tau)$ be a second-order wave at $B$, between $AB$ and $BC$, and let $c = \frac{d\overline{\zeta}}{d\tau}(0)$ be its slope at this point; then $c^2 < 1$. Furthermore, since $Q, S, w_\zeta, w_\tau$ do not jump across $\mathbf{C}$ at point $B$, we have the following jump relations for the system (15):

$$\begin{gathered} [\![\vec{v}_\tau]\!] - [\![S_\zeta]\!]\vec{u} - [\![\vec{u}_\zeta]\!] = (0,0), \qquad [\![\vec{v}_\tau]\!] + c[\![\vec{v}_\zeta]\!] = (0,0), \\ [\![\vec{u}_\tau]\!] + c[\![\vec{u}_\zeta]\!] = (0,0), \qquad [\![S_\tau]\!] + c[\![S_\zeta]\!] = 0. \end{gathered} \tag{B.1}$$

Let us assume that $c \neq 0$; then, by means of $(B.1)_{2-4}$ relation $(B.1)_1$ becomes

$$(c^2 - 1)[\![\vec{u}_\tau]\!] - [\![S_\tau]\!]\vec{u} = (0,0) \tag{B.2}$$

and

$$(c^2 - 1)\vec{u}[\![\vec{u}_\tau]\!] - [\![S_\tau]\!]|\vec{u}|^2 = 0; \tag{B.3}$$

besides, let us also remark that, according to the constitutive relation, $S_\tau^+ \neq 0$ (or $S_\tau^- \neq 0$) implies $\vec{u} \cdot \vec{u}_\tau^+ = 0$ (or $\vec{u} \cdot \vec{u}_\tau^- = 0$) and also $\vec{u} \cdot \vec{u}_\tau = \dot{\varepsilon} \geq 0$ everywhere.

We may have the following cases:

$\alpha$) If $[\![S_\tau]\!] = 0$ then by (B.2) we get $[\![\vec{u}_\tau]\!] = (0,0)$, and by (B.1) all jumps vanish and there is no wave.

$\beta$) If $[\![S_\tau]\!] \neq 0$ then:

$\beta_1$) If $S_\tau^+ \neq 0$ and $S_\tau^- \neq 0$, then $\vec{u} \cdot [\![\vec{u}_\tau]\!] = 0$, which is impossible according to (B.3);

$\beta_2$) If $S_\tau^+ \neq 0, S_\tau^- = 0$, we have $\vec{u} \cdot \vec{u}_\tau^+ = 0$, $\vec{u} \cdot \vec{u}_\tau^- > 0$ ($\vec{u} \cdot \vec{u}_\tau^-$ cannot be zero according to (B.3)), and from (B.3) we get $S_\tau^+ > 0$; but $\dot{\varepsilon}^- = \vec{u} \cdot \vec{u}_\tau^- > 0$ implies $S = 1$ on **C** in the neighborhood of point $B$; therefore, $S_\tau^+ > 0$ implies $S$ has to strictly increase in time from the value $S = 1$ on **C**, which is impossible.

$\beta_3$) If $S_\tau^+ = 0$, $S_\tau^- \neq 0$, we have $\dot{\varepsilon}^+ = \vec{u} \cdot \vec{u}_\tau^+ > 0$, $\vec{u} \cdot \vec{u}_\tau^- = 0$, and from (B.3) we obtain $S_\tau^- < 0$; but $\dot{\varepsilon}_+ > 0$ implies $S = 1$ on **C** in a neighborhood of point $B$, and $S_\tau^- < 0$ means $S$ has strictly decreased in time to a value $S = 1$ on **C**, which is again impossible.

In conclusion, there is no possible second-order wave at the point $B$ with slope $c \neq 0$.

The second-order wave analysis at point $B$ for the system (24) follows the same argument as above and starts from $(\text{B.2})_1$, i.e.,

$$(c^2 - 1)[\![w_{\zeta\tau}]\!] - [\![S_\tau]\!] w_\zeta = 0.$$

We reach the same conclusion as before provided $w_\zeta^{3,4} \neq 0$, and, as $(w_\zeta(1,0))^{3,4} = I_0/2$, there is no second-order wave with $c \neq 0$ at point $B$ for the system (24).

**Appendix C. The one-sided limit values at $(1,0)$ of the derivatives of $\vec{u}, \vec{v}, S$.** Let us denote these one-sided limit values at $(1,0)$ by

$$\vec{u}_\tau^i = \vec{u}_\tau(1,0) \quad \text{in region } i, \text{ etc.,}$$
$$S_\tau^i = S_\tau(1,0) \quad \text{in region } i, \text{ etc.}$$

For the system (15), across both shock waves $AB$, $BC$ we have the jump relations

$$[\![\vec{v}]\!] - D[\![\vec{u}]\!] = (0,0) \tag{C.1}$$

with $D = -1$ for $AB$ and $D = 1$ for $BC$ (where $D$ is the shock wave speed). Then differentiation of (C.1) along the shock gives

$$[\![\vec{v}_\tau]\!] + 2D[\![\vec{v}_\zeta]\!] + D^2[\![\vec{u}_\zeta]\!] = (0,0), \tag{C.2}$$

while from $(15)_{1,2}$, we get

$$[\![\vec{v}_\tau]\!] - [\![\vec{u}_\zeta]\!] = [\![S_\zeta \vec{u}]\!] + (0, [\![Q]\!]). \tag{C.3}$$

Across the second-order wave $BD$ we have the jump relations

$$\begin{gathered} [\![\vec{u}_\tau]\!] = [\![\vec{v}_\tau]\!] = (0,0), \qquad [\![S_\tau]\!] = 0, \\ [\![S_\zeta]\!]\vec{u}^{3,4} + [\![\vec{u}_\zeta]\!] = (0,0). \end{gathered} \tag{C.4}$$

Then from (C.2)–(C.4) we obtain

$$\vec{r}_{\zeta\tau}^{\,3,4} = \vec{u}_\tau^{\,3,4} = -\left(\frac{a}{2}S_\zeta^4, \frac{1}{4} + \frac{I_0}{4}S_\zeta^4\right). \tag{C.5}$$

**Appendix D. Shock waves with zero speed (contact discontinuities).** The jump relations (16) for shock waves, for the system (15) when the shock speed $c = 0$, lead to

$$[\![\vec{v}]\!] = 0, \qquad [\![S\vec{u}]\!] = 0. \tag{D.1}$$

Relation $(\mathrm{D.1})_2$ may be written as

$$S^+|\vec{u}|^+ \frac{\vec{u}^+}{|\vec{u}|^+} = S^-|\vec{u}|^- \frac{\vec{u}^-}{|\vec{u}|^-} = \Delta,$$

and there are two possibilities.

1) If $\Delta \neq 0$, then

$$\left[\!\!\left[\frac{\vec{u}}{|\vec{u}|}\right]\!\!\right] = 0 \quad \text{and} \quad S^+|\vec{u}|^+ = S^-|\vec{u}|^-. \tag{D.2}$$

2) If $\Delta = 0$, then

$$S^+ = S^- = 0 \quad \text{and} \quad [\![|\vec{u}|]\!] \neq 0, \ \left[\!\!\left[\frac{\vec{u}}{|\vec{u}|}\right]\!\!\right] \neq 0. \tag{D.3}$$

The same jump relations (D.1) for the system (24) reduce to

$$[\![w_\tau]\!] = 0, \qquad [\![Sw_\zeta]\!] = 0$$

and therefore

$$S^+w_\zeta^+ = S^-w_\zeta^-. \tag{D.4}$$

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