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Transient Response of Minimum Variance Control over a Gaussian Communication Channel*

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Abstract— **Recently, a finite horizon minimum variance control problem was proposed using feedback over a Gaussian communication channel. Because only the terminal state is penalized, it was shown that linear communication and control strategies are optimal and achieve the information theoretic minimum cost. However, because the transient state is not penalized, the transient behavior can be poor. In the present paper, we show that if there is at most one open loop unstable plant pole, then the transient response will remain bounded as the control horizon tends to infinity, and will approach a value determined by the solution to a certain algebraic Riccati equation.**

I. INTRODUCTION

Recent years have seen considerable interest in control problems using feedback obtained over a communication channel; a partial list of references is given by [2]–[11]. In the present paper, we consider a finite horizon minimum variance control problem proposed in [4], wherein a performance penalty is imposed only on the terminal output of the system. It is shown in [4] that for this problem the optimum communication and control strategies are linear and time-varying, and achieve an information theoretic lower bound that holds for any causal and nonlinear strategies. Because the transient response is not penalized, it may be poor and even become unbounded in the limit as the control horizon tends to infinity. Of course, imposing a penalty on the transient response will improve its behavior. Except in special cases, however, the resulting terminal response would no longer be equal to its theoretical minimum. As discussed in [4], with a terminal penalty only there is no conflict between the tasks of communication and control, whereas imposing a transient penalty introduces such a conflict. It is thus of theoretical interest to determine conditions under which the optimum terminal cost may be achieved with a well-behaved transient response.

We shall show that, under appropriate hypotheses, the transient response achieved by the optimal control and communication strategies remains bounded, and can be determined from the solution to a constant coefficient Signal-to-Noise Ratio (SNR) constrained Riccati equation [4]. In general, one would expect the transient mean square values of the system output and optimal estimation error to equal the theoretical minimum only for scalar systems. We present a simple example showing that the optimal estimation error variance may equal the theoretical minimum at all transient times even though the system is second order. We also show a simple second order example wherein the mean square value of the system output converges to the variance of the optimal estimation error. However, for the latter example, the theoretical minimum is achieved only at the terminal time.

The remainder of the paper is outlined as follows. In Section II we present preliminary results and review the necessary background from [4]. Section III contains the main results of the paper, namely a set of sufficient conditions for the transient response to remain bounded as the terminal time tends to infinity, and another set of sufficient conditions for the response to become unbounded. We shall show that the response will remain bounded provided that there is at most one unstable open loop pole. Examples illustrating tighness of the bounds for second order systems are given in Section IV. Section V contains conclusions and further research directions.

Notation and Terminology

We use upper case letters to denote random variables, lower case letters to denote realizations of these random variables, subscripts to denote elements of a sequence, and superscripts to denote subsequences, e.g., $x^k \triangleq \{x_0, x_1, \ldots, x_k\}$. Denote the expected value of the random variable X by $\mathcal{E}\{X\}$. We use $\|\cdot\|$ to denote both the Euclidean vector norm and the induced matrix norm. A linear system $x_{k+1} = Ax_k + Bu_k$ is *stable* if all eigenvalues of A have magnitude strictly less than one.

II. PRELIMINARIES

Consider the linear system

$$x_{k+1} = Ax_k + Bu_k + Ed_k, \tag{1}$$

$$y_k = Cx_k, \tag{2}$$

with state $x_k \in \mathbb{R}^n$, control $u_k \in \mathbb{R}$, disturbance $d_k \in \mathbb{R}$, and output $y_k \in \mathbb{R}$. Assume that the initial state x_0 and disturbance d_k are realizations of zero mean Gaussian random variables X_0 and D_k, where X_0 and D_k are independent for all k, X_0 has covariance $\Sigma_{0|-1}$, and D_k is an independent identically distributed (i.i.d.) sequence with variance σ_d^2.

The Gaussian communication channel is modeled as

$$r_k = s_k + n_k. \tag{3}$$

where the channel noise n_k is a realization of an i.i.d. Gaussian random process with zero mean and variance σ_n^2, and is assumed to be independent of the initial state

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and process disturbance. The channel input s_k must satisfy an instantaneous power constraint $\mathcal{E}\{S_k^2\} \leq \mathcal{P}$ for some specified value $\mathcal{P} > 0$.

The channel input is assumed to depend causally on the plant output sequence

$$s_k = f_k(y^k), \tag{4}$$

and the control input to depend causally on the sequence of channel outputs

$$u_k = g_k(r^k), \tag{5}$$

where the encoder (4) and the decoder (5) are potentially nonlinear and time varying.

We shall often need to invoke the following lower bound on channel signal to noise ratio:

$$\mathcal{P}/\sigma_n^2 > -1 + \prod_{i=1}^{m} |\phi_i^2|, \tag{6}$$

where $\{\phi_i,\ i = 1, \ldots, m\}$ is the set of unstable eigenvalues of A. The authors of [5] show that (6) is a *necessary* condition for the existence of a causal, but potentially nonlinear and time-varying, encoder and decoder such that the resulting feedback system is mean square stable.

Denote the conditional expectation of the plant state X_{k+1} given the channel output history $R^{k-1} = r^{k-1}$ by $\hat{x}_{k|k-1} = \mathcal{E}_{r^{k-1}}\{X_k\}$, the associated state estimation error by $\tilde{x}_{k|k-1} = x_k - \hat{x}_{k|k-1}$, and the error covariance matrix by $\Sigma_{k|k-1} = \mathcal{E}\{\tilde{X}_{k|k-1}\tilde{X}_{k|k-1}^T\}$. Similarly, denote the conditional estimate of the system output by $\hat{y}_{k|k-1}$, and the conditional output estimation error by $\tilde{y}_{k|k-1}$.

Our problem is to choose encoding and decoding sequences $f_k(y^k)$ and $g_k(r^k)$, $k = 0, \ldots, N$, to minimize the mean square value of the system output at terminal time $k = N + 1$, subject to the channel input power constraint $\mathcal{E}\{S_k^2\} \leq \mathcal{P}$. The cost function is thus given by

$$J_{N+1}^* = \inf_{\substack{f_k, g_k \\ k=0,\ldots,N}} \mathcal{E}\{Y_{N+1}^2\}. \tag{7}$$

In the present paper we will also be concerned with the transient regulation cost at times $k < N+1$. Toward that end, it is a standard result that the variance of the plant output is bounded below by that of the conditional estimation error:

$$\mathcal{E}\{Y_{k+1}^2\} \geq \mathcal{E}\{\tilde{Y}_{k+1|k}^2\}, \qquad k = 0, \ldots, N. \tag{8}$$

It is shown in [4] that the lower bound (8) can be achieved with equality at the terminal time $N + 1$ by setting

$$u_N = -(CB)^{-1}CA\hat{x}_{N|N}. \tag{9}$$

Using arguments from information theory, it is also shown in [4] that a theoretical lower bound on the estimation error variance, which must be satisfied by any encoder and decoder of the form (4) and (5), is given by

$$\mathcal{E}\{\tilde{Y}_{k+1|k}^2\} \geq N(Y_{k+1}|R^k), \tag{10}$$

where $N(Y_{k+1}|R^k)$ is the *average conditional entropy power* of Y_{k+1} averaged over the channel output sequence $R^k = r^k$:

$$N(Y_{k+1}|R^k) = \left(\frac{\sigma_n^2}{\mathcal{P} + \sigma_n^2}\right)^{k+1} CA^{k+1}\Sigma_{0|-1}A^{(k+1)T}C^T + \sigma_d^2 \sum_{j=0}^{k} (CA^{k-j}E)^2 \left(\frac{\sigma_n^2}{\mathcal{P} + \sigma_n^2}\right)^{k-j}. \tag{11}$$

Finally, it is shown in [4] that the lower bound (10) can be achieved with equality using linear control and communication strategies. This is done in several steps. First it is assumed that the encoder has access to the plant state and the control input, as well as to feedback from the channel output, and linear control and communication strategies are proposed that yield the minimal estimation error. Second, the assumption that the encoder has feedback from the channel output and access to the control input is removed. Third, the assumption of access to the plant state is replaced, under appropriate hypotheses, by an observer that does not require knowledge of the plant input. We now summarize the first two of these steps; for the third, the reader is referred to [4].

A. Step 1

If the control signal u_N satisfies (9), then the problem of minimizing the terminal cost reduces to one of estimation, and setting $u_k = 0,\ k = 0, \ldots, N-1$, results in the problem of estimating the state of an uncontrolled plant over a channel with feedback. Our approach to this problem is depicted in Figure 1, wherein we define a time-varying linear combination of states $z_k \triangleq H_k x_k$, and consider the channel input

$$s_k = \lambda_k \tilde{z}_{k|k-1}, \tag{12}$$

where $\hat{z}_{k|k-1} = \mathcal{E}\{Z_k|r^{k-1}\}$ and $\tilde{z}_{k|k-1} = z_k - \hat{z}_{k|k-1}$. For given sequences H_k and λ_k, the conditional state estimate satisfies the recursion $\hat{x}_{k+1|k} = A\hat{x}_{k|k-1} + AL_k r_k$, with initial condition $\hat{x}_{0|-1} = 0$. The sequences of estimator gains L_k and error covariance matrices $\Sigma_{k+1|k}$ are given by

$$L_k = \lambda_k \Sigma_{k|k-1} H_k^T / (\lambda_k^2 H_k \Sigma_{k|k-1} H_k^T + \sigma_n^2), \tag{13}$$

$$\Sigma_{k+1|k} = A\Sigma_{k|k-1}A^T - \frac{\lambda_k^2 A\Sigma_{k|k-1}H_k^T H_k \Sigma_{k|k-1}A^T}{\lambda_k^2 H_k \Sigma_{k|k-1} H_k^T + \sigma_n^2} + \sigma_d^2 EE^T, \tag{14}$$

with initial condition $\Sigma_{0|-1}$.

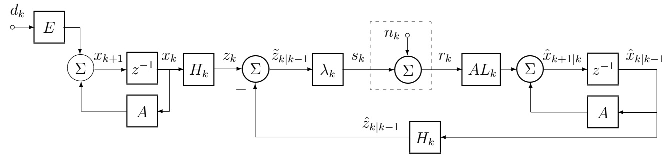

Fig. 1. Estimation over a channel with feedback.

Proposition 1 *Consider the communication channel with feedback depicted in Figure 1. Choose the channel input $s_k,\ k = 0, \ldots, N$, to satisfy* (12)*, where*

$$H_k \triangleq CA^{N+1-k}. \tag{15}$$

Assume that[1] $\Sigma_{0|-1} > 0$*, and choose* λ_k *such that*

$$\lambda_k^2 = \mathcal{P}/H_k\Sigma_{k|k-1}H_k^T. \tag{16}$$

Then the variance of the estimation error at time $k = N+1$ satisfies (10) *with equality.* ■

The fact that $\mathcal{E}\{\tilde{Y}^2_{N+1|N}\}$ satisfies (10) with equality implies that the linear estimation scheme depicted in Figure 1 achieves the information theoretic optimum, and cannot be outperformed by any causal nonlinear estimation scheme.

B. Step 2

We now show that, through use of an appropriately defined control input, the estimation error variance (10) obtained in Proposition 1 can also be achieved without the feedback path around the channel in Figure 1.

Proposition 2 *Consider the plant* (1)-(2) *and the communication channel* (3)*. Define the sequence H_k as in* (15)*, and assume that both $\Sigma_{0|-1} > 0$ and $H_{k+1}B \neq 0$, for $k = 0, \ldots, N$. Let the channel input sequence be given by*

$$s_k = \lambda_k z_k, \qquad z_k = H_k x_k, \qquad k = 0, \ldots, N, \tag{17}$$

where λ_k is chosen as in (16)*. Choose the control sequence*

$$u_k = -F_k\hat{x}_{k|k}, \tag{18}$$

where the control gain is defined by

$$F_k \triangleq (H_{k+1}B)^{-1}H_{k+1}A, \tag{19}$$

and the state estimate $\hat{x}_{k|k}$ satisfies

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + L_k(r_k - \lambda_k H_k \hat{x}_{k|k-1}), \tag{20}$$

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + Bu_k, \tag{21}$$

with initial condition $\hat{x}_{0|-1} = 0$ and estimator gain and covariance matrix

$$L_k = \frac{1}{\lambda_k}\frac{\Sigma_{k|k-1}H_k^T}{H_k\Sigma_{k|k-1}H_k^T}\frac{\mathcal{P}}{\mathcal{P}+\sigma_n^2}, \tag{22}$$

$$\Sigma_{k+1|k} = A\Sigma_{k|k-1}A^T - \frac{\mathcal{P}}{\mathcal{P}+\sigma_n^2}\frac{A\Sigma_{k|k-1}H_k^T H_k \Sigma_{k|k-1}A^T}{H_k\Sigma_{k|k-1}H_k^T} + \sigma_d^2 EE^T. \tag{23}$$

Then at the terminal time $\mathcal{E}\{Y^2_{N+1}\} = \mathcal{E}\{\tilde{Y}^2_{N+1|N}\}$, where $\mathcal{E}\{\tilde{Y}^2_{N+1|N}\}$ is given by (10) *with $k = N$.*

The mean square value of the system output is given by $\mathcal{E}\{Y^2_{k+1}\} = J_{k+1}$, where

$$J_{k+1} = C\Sigma_{k+1|k}C^T + C(A-BF_k)\Gamma_{k|k}(A-BF_k)^T C^T, \tag{24}$$

F_k is defined by (19)*, and $\Gamma_{k|k} \triangleq \mathcal{E}\{\hat{X}_{k|k}\hat{X}^T_{k|k}\}$ satisfies the recursion*

$$\Gamma_{k+1|k+1} = (A-BF_k)\Gamma_{k|k}(A-BF_k)^T + \frac{\mathcal{P}}{\mathcal{P}+\sigma_n^2}\frac{\Sigma_{k|k-1}H_k^T H_k\Sigma_{k|k-1}}{H_k\Sigma_{k|k-1}H_k^T}, \tag{25}$$

with initial condition $\Gamma_{0|0} = \lambda_0^2\Sigma_{0|-1}H_0^T H_0\Sigma_{0|-1}/(\mathcal{P}+\sigma_n^2)$. ■

[1] As noted in [4, Remark 5], the assumption that $\Sigma_{0|-1} > 0$ implies that $H_k\Sigma_{k|k-1}H_k^T > 0,\ k = 0, \ldots, N$.

The preceding results are illustrated in [4, Example 5.7]. In that example, which considers a second order system with two unstable eigenvalues, it is seen that the transient response becomes very large. Furthermore, simulations reveal that the transient response increases without bound as the terminal time tends to infinity.

III. PROPERTIES OF THE TRANSIENT RESPONSE

We now analyze the transient response of the estimation error and regulation cost, and state conditions under which this response remains bounded as the horizon tends to infinity. To do so, we must study properties of the SNR constrained Riccati equation (23). Although SNR constrained Riccati equations are studied extensively in [4], the analysis therein is performed for constant coefficient difference equations and the corresponding algebraic Riccati equations (cf. [4, eqns. (74)-(75)]). By way of contrast, we must now work with the SNR constrained Riccati equation (23) that has a *time-varying* coefficient H_k given by (15). Under appropriate hypotheses, we shall show that for any finite value of k, and in the limit as $N \to \infty$, the Riccati equation (23) converges to one having *constant coefficients*. We then show that under additional hypotheses, including the assumption that A has at most one unstable eigenvalue, the solution to the constant coefficient Riccati equation converges to a steady state value that determines the transient behavior of the solution to the original Riccati equation (23).

Denote the eigenvalues of A by $\{\phi_i : i = 1, \ldots, n\}$. *Assume throughout the paper that, for simplicity, these eigenvalues are distinct.* Hence A has a complete set of right eigenvectors v_i and left eigenvectors w_i; the latter are row vectors with the property that $w_iA = \phi_i w_i$. The modal decomposition of A thus has the form $A = \sum_{i=1}^n \phi_i v_i w_i$. Recall that left and right eigenvectors have the properties that $w_iv_j = 1, i = j$ and $w_iv_j = 0, i \neq j$.

Our next two results, taken together, describe the behavior of the covariance matrix (23) in the limit as both $k \to \infty$ and $N - k \to \infty$. Proposition 3 provides the latter limit, and the former is given in Proposition 4. In order to demonstrate convergence as $N \to \infty$, it is useful to note that the various sequences defined by (15)-(16), (19), and (22)-(25) all depend implicitly upon the value of N used to define the terminal cost (7). When convenient, we shall write, for example, $\Sigma^\infty_{k+1|k} = \lim_{N\to\infty}\Sigma_{k+1|k}$.

Proposition 3 *In addition to the hypotheses of Proposition 2, assume that (A, B) is reachable and that A has* *a unique real eigenvalue ϕ_1 with largest magnitude. Then for any finite value of k, and in the limit as $N \to \infty$, the Riccati equation* (23) *with H_k^N given by* (15) *converges to the constant coefficient Riccati equation*

$$\Sigma_{k+1|k}^{\infty} = A\Sigma_{k|k-1}^{\infty}A^T - \frac{\mathcal{P}}{\mathcal{P}+\sigma_n^2}\frac{A\Sigma_{k|k-1}^{\infty}w_1^Tw_1\Sigma_{k|k-1}^{\infty}A^T}{w_1\Sigma_{k|k-1}^{\infty}w_1^T} + \sigma_d^2EE^T, \quad (26)$$

with initial condition $\Sigma_{0|-1}^{\infty} = \Sigma_{0|-1}$. The associated control gain F_k^N (19) *and estimator gain L_k^N* (22) *converge to*

$$F^{\infty} = \phi_1(w_1B)^{-1}w_1, \quad (27)$$

$$L_k^{\infty} = \frac{1}{\lambda_k}\frac{\Sigma_{k|k-1}^{\infty}w_1^T}{w_1\Sigma_{k|k-1}^{\infty}w_1^T}\frac{\mathcal{P}}{\mathcal{P}+\sigma_n^2} \quad (28)$$

Furthermore, $\mathcal{E}\{\tilde{Y}_k^2\} \to C\Sigma_{k+1|k}^{\infty}C^T$, and $\mathcal{E}\{Y_{k+1}^2\} \to J_{k+1}^{\infty}$, where

$$J_{k+1}^{\infty} = C\Sigma_{k+1|k}^{\infty}C^T + C(A-BF^{\infty})\Gamma_{k|k}^{\infty}(A-BF^{\infty})^TC^T, \quad (29)$$

and $\Gamma_{k|k}^{\infty}$ satisfies

$$\Gamma_{k+1|k+1}^{\infty} = (A-BF^{\infty})\Gamma_{k|k}^{\infty}(A-BF^{\infty})^T + \frac{\mathcal{P}}{\mathcal{P}+\sigma_n^2}\frac{\Sigma_{k|k-1}^{\infty}w_1^Tw_1\Sigma_{k|k-1}^{\infty}}{w_1\Sigma_{k|k-1}^{\infty}w_1^T}, \quad (30)$$

with $\Gamma_{0|0}^{\infty} = \Gamma_{0|0}$.

Proof: Define

$$\bar{H}_k^N = \frac{1}{\phi_1^{N+1-k}}H_k^N, \quad (31)$$

and note that the Riccati equation (23) may be rewritten as

$$\Sigma_{k+1|k} = A\Sigma_{k|k-1}A^T - \frac{\mathcal{P}}{\mathcal{P}+\sigma_n^2}\frac{A\Sigma_{k|k-1}\bar{H}_k^{NT}\bar{H}_k^N\Sigma_{k|k-1}A^T}{\bar{H}_k^N\Sigma_{k|k-1}\bar{H}_k^{NT}} + \sigma_d^2EE^T. \quad (32)$$

It follows from the modal decomposition of A that

$$A^k = \phi_1^k(v_1w_1 + \beta^k\Delta_k), \quad (33)$$

where $\beta = \phi_2/\phi_1$ satisfies $|\beta| < 1$, and Δ_k is a bounded sequence. Together (15), (31), and (33) yields $\bar{H}_k^N = Cv_1w_1 + \beta^{N+1-k}C\Delta_k$, and we see that

$$\lim_{N\to\infty}\bar{H}_k^N = Cv_1w_1. \quad (34)$$

Substituting (34) into (32) yields (26). It is straightforward to show from (26) and the assumption $\Sigma_{0|-1} > 0$ that $w_1\Sigma_{k|k-1}^{\infty}w_1^T > 0$, $k = 0,\ldots,N$. The assumption that (A,B) is reachable implies that $w_1B \neq 0$, and thus that (27) is well defined. ∎

Proposition 4 *In addition to the hypotheses of Proposition 3, assume that (A,E) is reachable, that A has* at most one *unstable eigenvalue, $m \leq 1$, and that the channel SNR satisfies the bound* (6). *Then, in the limit as $k \to \infty$, the sequence $\Sigma_{k+1|k}^{\infty}$ defined by* (26) *converges to the unique positive semidefinite solution to the algebraic Riccati equation*

$$\Sigma^{\infty} = A\Sigma^{\infty}A^T - \frac{\mathcal{P}}{\mathcal{P}+\sigma_n^2}\frac{A\Sigma^{\infty}w_1^Tw_1\Sigma^{\infty}A^T}{w_1\Sigma^{\infty}w_1^T} + \sigma_d^2EE^T. \quad (35)$$

Furthermore, $w_1\Sigma^{\infty}w_1^T > 0$, and the eigenvalues of $A - \lambda^{\infty}AL^{\infty}w_1$ lie inside the open unit disk, where

$$L^{\infty} = \frac{1}{\lambda^{\infty}}\frac{\Sigma^{\infty}w_1^T}{w_1\Sigma^{\infty}w_1^T}\frac{\mathcal{P}}{\mathcal{P}+\sigma_n^2}, \quad (36)$$

and $(\lambda^{\infty})^2 = \mathcal{P}/w_1\Sigma^{\infty}w_1^T$. The sequences J_{k+1}^{∞} and $\Gamma_{k+1|k+1}^{\infty}$ defined by (29)-(30) *converge to*

$$J^{\infty} = C\Sigma^{\infty}C^T + C(A-BF^{\infty})\Gamma^{\infty}(A-BF^{\infty})^TC^T, \quad (37)$$

$$\Gamma^{\infty} = (A-BF^{\infty})\Gamma^{\infty}(A-BF^{\infty})^T + \frac{\mathcal{P}}{\mathcal{P}+\sigma_n^2}\frac{\Sigma^{\infty}w_1^Tw_1\Sigma^{\infty}}{w_1\Sigma^{\infty}w_1^T}. \quad (38)$$

Proof: These results follow from [4, Propositions 7.1 and 7.6], provided we can show that (A, w_1) is detectable. If A has only stable eigenvalues, this is trivial. Otherwise, by the orthogonality property of left and right eigenvectors, (A, w_1) is detectable if and only if A has exactly one unstable eigenvalue. ∎

We now illustrate the results of Proposition 4.

Example 5 Consider

$$A = \begin{bmatrix} 1.1 & 1 \\ 0 & 0.8 \end{bmatrix}, \quad E = B = \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad (39)$$

where $\sigma_d^2 = 1, \mathcal{P} = 10, \sigma_n^2 = 5$. Figure 2 contains plots of $\mathcal{E}\{Y_{k+1}^2\}$ and $\mathcal{E}\{\tilde{Y}_{k+1|k}^2\}$, together with the information theoretic lower bound (10). Note that both the transient estimation error variance and regulation cost remain bounded even though the plant is open loop unstable. Also plotted are the steady state values of the estimation error covariance $C\Sigma^{\infty}C^T$ and regulation cost J^{∞} obtained from (35) and (37). ∎

The next result shows that, if A has two real and distinct unstable eigenvalues, then the sequence $\Sigma_{k+1|k}^{\infty}$ diverges.

Proposition 6 *Assume that (A,E) is reachable, that the bound* (6) *is satisfied, and that A has at least two real unstable eigenvalues with $|\phi_1| > |\phi_2| > 1$. Assume also that $\Sigma_{0|-1} > 0$. Then the sequence $\Sigma_{k+1|k}^{\infty}$ defined by* (26) *diverges.*

Proof: It follows from [1, p. 81] that the constant coefficient Riccati equation (26) may be rearranged as

$$\Sigma_{k+1|k}^{\infty} = (A - AL_k^{\infty}w_1^T)\Sigma_{k|k-1}^{\infty}(A - AL_k^{\infty}w_1^T)^T + AL_k^{\infty}\sigma_n^2(AL_k^{\infty})^T + \sigma_d^2EE^T, \quad (40)$$

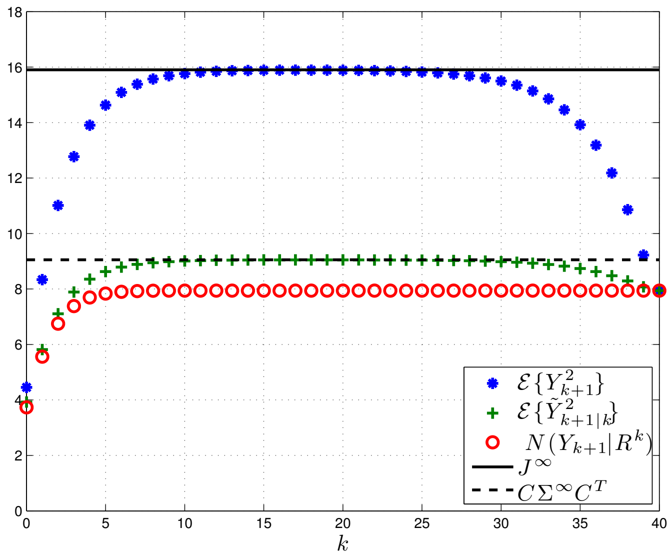

Fig. 2. The transient responses of $\mathcal{E}\{Y_{k+1}^2\}$ and $\mathcal{E}\{\tilde{Y}_{k+1|k}^2\}$ converge to J^∞ and $C\Sigma^\infty C^T$, where J^∞ is given by (37) and Σ^∞ is given by (35).

from which it follows that

$$\Sigma_{k+1|k}^\infty \geq (A - AL_k^\infty w_1^T)\Sigma_{k|k-1}^\infty (A - AL_k^\infty w_1)^T. \quad (41)$$

Note that $(A - AL_k^\infty w_1^T)$ has an eigenvalue ϕ_2, and let w_{2k} denote its associated left eigenvector. Assume with no loss of generality that $\|w_{2k}\| = 1,\ \forall k$, and define a sequence $\eta_k \triangleq w_{2k}\Sigma_{k|k-1}^\infty w_{2k}^T$. Then $\|\Sigma_{k+1|k}\| \geq \eta_{k+1} \geq \phi_2^2\eta_k$, and thus $\|\Sigma_{k+1|k}\| \geq \phi_2^{2(k+1)}\eta_0 \to \infty$ as $k \to \infty$. ■

We next consider what happens when there exist eigenvalues $|\phi_1| > |\phi_2| = 1$, and show that the response to the disturbance input grows without bound.

Lemma 7 *Suppose that*

$$A = \begin{bmatrix}\phi_1 & 0\\ 0 & \phi_2\end{bmatrix}, \quad E = \begin{bmatrix}e_1\\ e_2\end{bmatrix}, \quad (42)$$

with $|\phi_1| > |\phi_2| = 1$. Assume that e_1 and e_2 are both nonzero, and that the bound (6) is satisfied. Then the sequence $\Sigma_{k+1|k}^\infty$ defined by (26) diverges.

Proof: Denote the individual elements of $\Sigma_{k+1|k}^\infty$ by $\sigma_{k+1|k}^\infty(i,j),\ i,j = 1,2$. It is straightforward to show from (26) that

$$\sigma_{k+1|k}^\infty(1,1) = \frac{\phi_1^2\sigma_n^2}{\mathcal{P}+\sigma_n^2}\sigma_{k|k-1}^\infty(1,1) + \sigma_d^2 e_1^2, \quad (43)$$

$$\sigma_{k+1|k}^\infty(1,2) = \frac{\phi_1\phi_2\sigma_n^2}{\mathcal{P}+\sigma_n^2}\sigma_{k|k-1}^\infty(1,2) + \sigma_d^2 e_1 e_2, \quad (44)$$

$$\sigma_{k+1|k}^\infty(2,2) = \phi_2^2\sigma_{k|k-1}^\infty(2,2) - \frac{\mathcal{P}}{\mathcal{P}+\sigma_n^2}\frac{\phi_2^2\sigma_{k|k-1}^\infty(1,2)^2}{\sigma_{k|k-1}^\infty(1,1)} + \sigma_d^2 e_2^2. \quad (45)$$

The assumption that the bound (6) is satisfied implies that (43)-(44) have steady state solutions given by

$$\sigma^\infty(1,1) = \frac{\sigma_d^2 e_1^2}{1 - \frac{\phi_1^2\sigma_n^2}{\mathcal{P}+\sigma_n^2}}, \quad \sigma^\infty(1,2) = \frac{\sigma_d^2 e_1 e_2}{1 - \frac{\phi_1\phi_2\sigma_n^2}{\mathcal{P}+\sigma_n^2}}. \quad (46)$$

Substituting (46) into (45) and rearranging yields

$$\sigma_{k+1|k}^\infty(2,2) = \phi_2^2\sigma_{k|k-1}^\infty(2,2) + \left(1 - \frac{\phi_2^2(\mathcal{P}/\sigma_n^2 + 1 - \phi_1^2)\mathcal{P}/\sigma_n^2}{(\mathcal{P}/\sigma_n^2 + 1 - \phi_1\phi_2)^2}\right)\sigma_d^2 e_2^2. \quad (47)$$

Under our assumptions, $\phi_2^2 = 1$. Then it is easy to verify that the second term on the right hand side of (47) is equal to zero if and only if $\mathcal{P}/\sigma_n^2 = -1$, which is not possible. Hence $\sigma_{k+1|k}^\infty(2,2) \to \infty$ as $k \to \infty$. ■

IV. Tightness of the Bounds

Combining the bounds (8) and (10) yields

$$\mathcal{E}\{Y_k^2\} \geq \mathcal{E}\{\tilde{Y}_{k|k-1}^2\} \geq N(Y_k|R^{k-1}). \quad (48)$$

The results of [4] described in Propositions 1-2 show that both these bounds may be achieved with equality at the terminal time $k = N+1$. This fact is unusual since the control and communication schemes used to achieve these bounds are linear whereas the bounds (48) hold for any nonlinear encoder and decoder of the form (4) and (5). In general, one would not expect either bound to be tight at transient times. As discussed in [5], an exception to this general rule is obtained in the case of first order systems, where linear time invariant schemes satisfy both bounds in (48) with equality at each time step.

We now prove analytically that the *rightmost* bound in (48) may be tight at each time step for a second order system with one eigenvalue equal to zero.

Proposition 8 *Consider the system*

$$A = \begin{bmatrix}\phi_1 & 0\\ 0 & 0\end{bmatrix}, \quad E = \begin{bmatrix}e_1\\ e_2\end{bmatrix}, \quad C = \begin{bmatrix}c_1 & c_2\end{bmatrix}. \quad (49)$$

At each time step k, the estimation error $C\Sigma_{k+1|k}C^T$, where $\Sigma_{k+1|k}$ is given by (23), satisfies the lower bound (10) with equality.

Proof: Denote the elements of $\Sigma_{k+1|k}$ by $\sigma_{k+1|k}(i,j),\ i,j = 1,2$. Substituting (49) into (23), it is not hard to show that

$$\sigma_{k+1|k}(1,1) = \left(\frac{\phi_1^2\sigma_n^2}{\mathcal{P}+\sigma_n^2}\right)\sigma_{k|k-1}(1,1) + \sigma_d^2 e_1^2, \quad (50)$$

$$\sigma_{k+1|k}(1,2) = \sigma_d^2 e_1 e_2, \quad \sigma_{k+1|k}(2,2) = \sigma_d^2 e_2^2. \quad (51)$$

Iterating (50) and using (51) yields

$$C\Sigma_{k+1|k}C^T = \left(\frac{\sigma_n^2}{\mathcal{P}+\sigma_n^2}\right)^{k+1} c_1^2\phi_1^{2(k+1)}\sigma_{0|-1}(1,1) + \sigma_d^2 c_1^2 e_1^2 \sum_{j=0}^{k}\phi_1^{2(k-j)}\left(\frac{\sigma_n^2}{\mathcal{P}+\sigma_n^2}\right)^{k-j} + \sigma_d^2(2c_1c_2e_1e_2 + c_2^2e_2^2). \quad (52)$$

On the other hand, substituting (49) into (11) also yields (52), and it follows that the variance of the linear estimation error is identical to the theoretical lower bound. ∎

To see why the linear estimation scheme achieves the information theoretic optimum for a second order plant, consider the linear estimation scheme over a channel with feedback depicted in Figure 1. In this figure, the channel input is given by $s_k = \lambda_k H_k \tilde{x}_{k|k-1},\ k = 0, \ldots, N$, where H_k is given by (15). Note from the structure of H_k that the channel input depends only on the first plant state, and not the delay state. An inspection of Figure 1 reveals that the estimate of the delay state is simply kept equal to zero, and does not depend on the channel output. Hence, the problem of optimal transmission over the Gaussian channel reduces to that of transmitting the state of a scalar plant.

We next observe that the *leftmost* bound in (48) may be satisfied with equality asymptotically, in the sense that

$$J^\infty = C\Sigma^\infty C^T \tag{53}$$

in Proposition 4. For motivation, we note that combining (18), (20), and (21) yields

$$\hat{x}_{k+1|k} = (A - BF_k)\left(\hat{x}_{k|k-1} + L_k(r_k - \gamma_k H_k \hat{x}_{k|k-1})\right).$$

If the hypotheses of Proposition 3 are satisfied, then for sufficiently large values of N, $F_k \approx F^\infty$ defined by (27). It follows that $A - BF^\infty$ will have one eigenvalue at zero and the remainder at $\phi_k,\ k = 2, \ldots, n$. The latter are stable by the hypotheses of Proposition 4, which also implies that $L_k \approx L^\infty$ given by (36). It is straightforward to find examples for which $F^\infty L^\infty = 0$. In this case, since the eigenvalues of $A - BF^\infty$ are stable, $\hat{x}_{k+1|k} \to 0$, and thus the leftmost bound in (48) will be satisfied with equality.

Example 9 Consider

$$A = \begin{bmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{bmatrix}, \quad E = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Then L^∞ is proportional to $\begin{bmatrix} \sigma^\infty(1,1) & \sigma^\infty(2,1) \end{bmatrix}^T$, with values given by (46). One may show that setting

$$e_2 = (e_1 b_2 \phi_1)/(b_1 \phi_2)\left(1 - \frac{\phi_1\phi_2\sigma_n^2}{\mathcal{P} + \sigma_n^2}\right) \Big/ \left(1 - \frac{\phi^2\sigma_n^2}{\mathcal{P} + \sigma_n^2}\right)$$

yields $F^\infty L^\infty = 0$. Figure 3 contains plots of the resulting sequences for $\phi = 1.1, \phi_2 = 0.9, b_1 = b_2 = 1, e_1 = 1, C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \mathcal{P} = 10, \sigma_n^2 = 5$. With this set of parameters, (53) is satisfied, and thus the transient response of $\mathcal{E}\{Y_{k+1}^2\}$ converges to that of $\mathcal{E}\{\tilde{Y}_{k+1|k}^2\}$ at intermediate times. ∎

V. Conclusions and Further Directions

In this paper we have analyzed the transient response of a control and communication scheme first proposed in [4] in the limit as the control horizon becomes very large. We have presented both sufficient conditions for the transient response to converge to a finite value obtained from a constant coefficient Riccati equation, and sufficient conditions for the transient response to diverge. We have also studied cases

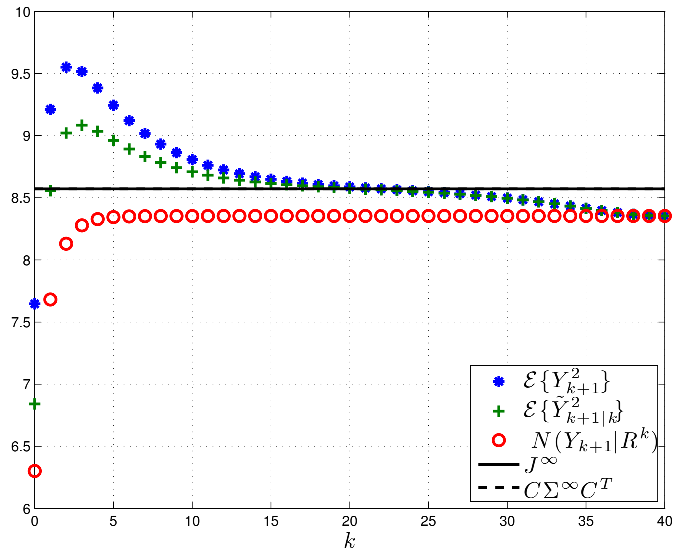

Fig. 3. Plots of $\mathcal{E}\{Y_{k+1}^2\}$, $\mathcal{E}\{\tilde{Y}_{k+1|k}^2\}$, and $N(Y_{k+1}|R^k)$ for terminal time $N+1 = 61$. For the chosen parameters, $J^\infty = C\Sigma^\infty C^T$.

wherein equality is achieved in one of the two fundamental bounds that relate the mean square value of the system output to the mean square estimation error and the information theoretic minimum. It remains to obtain a set of necessary and sufficient conditions for convergence of the transient response to a finite value; to do so would require a more thorough treatment of complex eigenvalues.

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