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Transient Solutions for One-Dimensional Problems With Strain Softening

Closed-form solutions are presented for the transient response of rods in which strain softening occurs and the stress-strain laws exhibit nonvanishing stresses after the strain-softening regime. It is found that the appearance of any strain softening results in an infinite strain rate if the material is inviscid. For a stress-strain law with a monotonically decreasing stress the strains are infinite also. If the stress increases after the strain-softening portion, the strains remain finite and the strain-softening point moves through the rod.

Introduction

A negative slope is found in the constitutive equations for phenomena such as erosion in penetration, shear banding and other damage mechanisms. Yet, the understanding of the behavior of continua which are governed by such constitutive equations is very limited. In fact, Hadamard (1903) discarded the possibility of such continua by stating that the wavespeed is imaginary, so that the continuum cannot exist. Numerical solutions for such materials are also quite strange. For example, Belytschko et al. (1984, 1985) have recently shown that in spherical geometries, strain-softening models can lead to numerical solutions characterized by many large peaks in strain, and that the locations of these peaks depend very much on the mesh size. However, constitutive models with strain softening are so prevalent and important in practice that their behavior must be understood.

The only closed-form solutions for problems in which the stress tends monotonically to zero are those of Bažant and Belytschko (1985), who presented a transient solution for a one-dimensional rod problem. These solutions exhibited a localization of the strain softening to a domain of measure zero, a discontinuity in the displacement and a singularity in the strains at the point of strain softening. The argument of Hadamard was shown to be irrelevant since the strain softening does not occur in a finite domain. However, the energy dissipation in the strain softening domain was shown to vanish, which raised questions as the applicability of this constitutive model to damage.

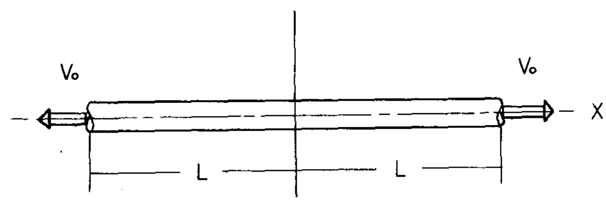

Fig. 1 Problem description: one-dimensional rod of length 2L with velocities prescribed at both ends

We consider here two different stress-strain laws in which the stress does not become a zero in the strain-softening branch of the stress-strain law. In the first material law, the slope of the stress-strain law after the onset of strain softening remains nonpositive; we call this law strain softening/perfectly plastic. In the second material law, the slope of the stress-strain law is negative for an interval and then reverses; we call this law strain softening-rehardening. It is found that if the stress decreases monotonically to any nonzero positive value after strain softening is initiated, a singularity appears in the strain, and the displacement is discontinuous. However, with the rehardening law, the strain remains finite and the strain softening point traverses through the material.

Very few closed-form or numerical-transient solutions with strain softening in which the slope of the stress-strain curve remains nonpositive have appeared in the literature. Some works relevant to this one are Bažant (1976), Aifantis and Serrin (1983), Wu and Freund (1984), Sandler and Wright (1984), Belytschko et al. (1984, 1985), Willam et al. (1984), and Schreyer and Chen (1984). For materials with rehardening, excellent theoretical studies have been reported by James (1980).

Problem Formulation

Consider a bar of length $2L$ with a unit cross section and mass ρ per unit length as shown in Fig. 1. The axis of the bar coincides with the coordinate x; the origin of the coordinate system is the midpoint of the rod so the interval of x is $[-L, +L]$. The equation of motion is

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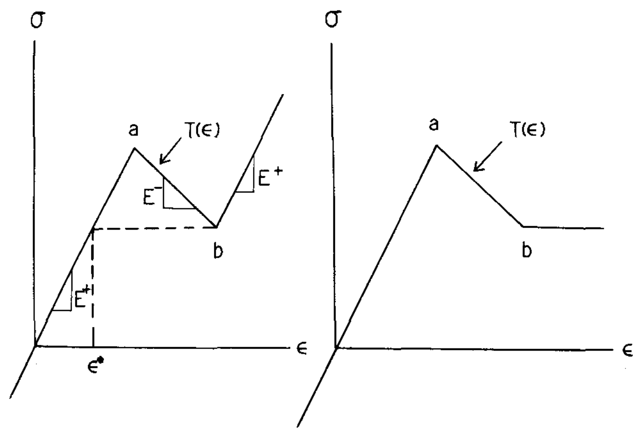

Fig. 2 Stress-strain laws for problems 1 and 2

$$\frac{\partial \sigma}{\partial x} = \rho \ddot{u} \tag{1}$$

where σ is the stress, $u(x,t)$ the displacement and superposed dots are time derivatives. The stress-strain law is taken to be

$$\dot{\sigma} = E(\sigma,\epsilon,\dot{\epsilon})\frac{\partial v}{\partial x} \tag{2}$$

$$v = \dot{u} \tag{3}$$

In the elastic part of the response, $E>0$ and equations (1)–(3) can be combined to yield

$$c^2 \frac{\partial^2 u}{\partial x^2} = \ddot{u} \qquad c^2 = \frac{E}{\rho} \tag{4}$$

Initially, the bar is undeformed and at rest so

$$u(x,0) = v(x,0) = 0 \quad -L \le x \le L \tag{5a}$$

The boundary conditions are

$$v(L,t) = v_o H(t) \tag{5b}$$

$$v(-L,t) = -v_o H(t) \tag{5c}$$

where $H(\ )$ is the Heaviside step function, v_o is a prescribed constant velocity, and t is the time.

Solutions

The solution to the above system is elastic until the stress associated with the onset of strain softening is reached (we will not be concerned with any purely elastic solutions). Strain softening always occurs first at the midpoint, where the stresses of the two elastic waves are superimposed and would reach a stress of twice the intensity of the initial waves if the material remained elastic.

The procedure of constructing a solution once strain softening is attained depends on the following hypothesis: strain softening is limited to a single point x_s (a set of measure zero) and at that point the strain instantaneously increases at least to where the stress attains a minimum value along the stress-strain curve, so after strain-softening $\epsilon \ge \epsilon_b$ (see Fig. 2).

Remark 1. This hypothesis was demonstrated in Bazant and Belytschko (1985). While this step may need more rigorous proof, it enables all of the governing equations to be satisfied; furthermore, it is borne out by numerical solutions.

Problem 1: Strain Softening-Rehardening. In the first problem, the stress-strain law is shown in Fig. 2(*a*). The stress-strain law can be characterized as follows:

initial conditions: $\sigma = \epsilon = 0;\quad S = \sigma_a$

algorithm: if $\epsilon > \epsilon_a$ and $S > \sigma_b$ and $\dot{\epsilon} > 0$

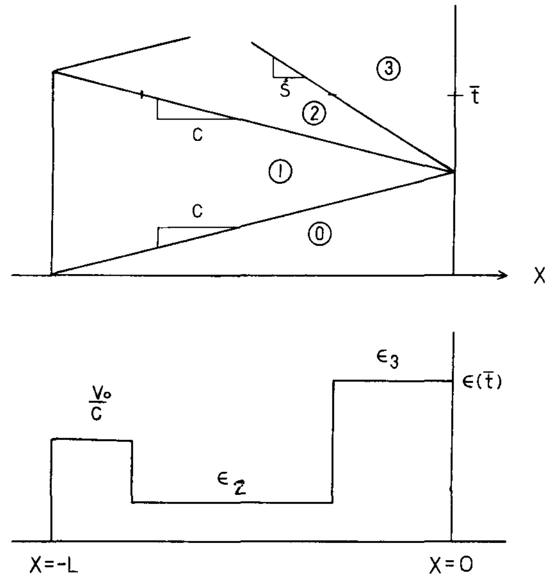

Fig. 3 Wavefronts in problem 1 and the strain distribution $\epsilon(\bar{t})$

$$\text{then } \dot{\sigma} = E^- \dot{\epsilon},\ \dot{S} = \dot{\sigma} \tag{6a}$$

$$\text{otherwise } \dot{\sigma} = E^+ \dot{\epsilon},\ \dot{S} = 0 \tag{6b}$$

In the above, S is a state variable for the material. An alternative algorithm can be written in difference form:

difference algorithm:

$$\sigma^{\text{new}} = \sigma^{\text{old}} + E^+ \Delta\epsilon \tag{6c}$$

if $\sigma^{\text{new}} > T(\epsilon^{\text{new}})$ then, replace above by $\sigma^{\text{new}} = T(\epsilon^{\text{new}})$ (6*d*)

$$T(\epsilon) = \begin{cases} E^+ \epsilon & \text{for } \epsilon \le \epsilon_a \\ \sigma_a + E^-(\epsilon - \epsilon_a) & \text{for } \epsilon_a < \epsilon < \epsilon_b \\ \sigma_b + E^+(\epsilon - \epsilon_b) & \text{for } \epsilon_b < \epsilon \end{cases} \tag{6e}$$

It is assumed that $Ev_o/c > \sigma_a/2$, so that when the two waves meet at $x=0$, strain softening is initiated, so that equation (6*a*) applies. Since it is hypothesized that at the strain softening point the strain jumps instantaneously, the stress then instantaneously takes on a value which we will call σ_s, $\sigma_s \ge \sigma_b$.

We then have 2 boundary-value problems (BVP):

BVP(A) $-L \le x \le 0$:

$$\text{Equations (4) and (5a), } \sigma(0,t) = \sigma_s \qquad \frac{L}{c} \le t \le t_3 \tag{7a}$$

BVP(B) $0x \le L$:

$$\text{Equations (4) and (5b), } \sigma(0,t) = \sigma_s \qquad \frac{L}{c} \le t \le t_3 \tag{7b}$$

where t_3 is a time to be determined as part of the solution. Since these two BVPs are symmetric with respect to the origin, we consider only BVP(A).

It will be shown in the following that a solution to this problem can be found if σ_s assumes any value in the range

$$\sigma_b \le l_s \le \sigma_a \tag{8}$$

see Fig. 2. We will parametrize this family of solutions by the strain in the initial reflected wave, the strain in domain 2 in Fig. 3, which is denoted by ϵ_2.

The structure of the solution is shown in Fig. 3. In domain 1, behind the initial elastic wavefront, the velocity, stress, and strain are given by

$$\dot{u}_1 = -v_o \quad \epsilon_1 = \frac{v_o}{c} \quad \sigma_1 = \frac{Ev_o}{c} \tag{9}$$

(subscripts on the left-hand variables designate the domain to which the variables pertain).

It will be shown that the reflected elastic wave moves faster than the wave associated with strain softening, so that we have elastic behavior in domain 2; hence

$$\epsilon_2 = \frac{\sigma_2}{E} \tag{10}$$

Also, as can be seen from Fig. 2(*a*), since $\sigma_s \geq \sigma_b$, the stress-strain law gives

$$\sigma_s = \sigma_b + E(\epsilon_2 - \epsilon^*) \tag{11}$$

and since $\sigma_s \geq \sigma_b$,

$$\epsilon_2 \geq \epsilon^* \tag{12}$$

where $\epsilon^* = \sigma_b / E$. The remainder of the solutions will be constructed by using the jump conditions

$$[\dot{u}] = \dot{s}[\epsilon] \quad [\sigma] = \rho \dot{s}^2 [\epsilon] \tag{13}$$

where [ ] designates a jump and $\dot{s}$ is the velocity of the discontinuity.

From the velocity jump condition between domains 1 and 2, we obtain

$$\dot{u}_2 - \dot{u}_1 = c(\epsilon_2 - \epsilon_1) \tag{14a}$$

which, upon the use of equations (9), gives

$$\dot{u}_2 = c\left(\epsilon_2 - \frac{2v_o}{c}\right) \tag{14b}$$

The displacement field in domains 1 and 2 is then given by

$$u(x,t) = -v_o < \xi - \frac{2x}{c} > + (c\epsilon_2 - v_o) < \xi > \tag{15}$$

where $\xi = t - (L - x)/c$ and $<f> = fH(\ )$. Hence $u(0,t) = (c\epsilon_2 - 2v_o) < t - L/c >$. Therefore, if the displacement field is to remain continuous at $x = 0$, another wave must emanate from that point; the only exception is the unusual situation where $c\epsilon_2 = 2v_o$, which will be examined later. The speed of this wave will be denoted by $\dot{s}$ and it represents the interface between domains 2 and 3 in Fig. 3.

The velocity-strain jump condition gives

$$\dot{s}(\epsilon_3 - \epsilon_2) = \dot{u}_3 - \dot{u}_2 = 2v_o - c\epsilon_2 \tag{16a}$$

where the last equality is obtained by noting $\dot{u}_3 = 0$ because of symmetry and using equation (14*b*). The stress jump condition gives

$$\sigma_3 - \sigma_2 = \rho \dot{s}^2 (\epsilon_3 - \epsilon_2) \tag{16b}$$

and the stress-strain law in the strain softening domain gives

$$\sigma_3 - \sigma_b = E(\epsilon_3 - \epsilon_b) \tag{16c}$$

Equations (16) are solved as follows: we can put equation (16*c*) in the form

$$\sigma_3 - \sigma_2 = \rho c^2 (\epsilon_3 - \epsilon_b - \epsilon_2 + \epsilon^*) \tag{17}$$

and using equations (16*b*) and (17) yields

$$\dot{s}^2 (\epsilon_3 - \epsilon_2) = c^2 [\epsilon_3 - \epsilon_b - \epsilon_2 + \epsilon^*] \tag{18}$$

Using equation (16*a*) to eliminate $\dot{s}$ from equation (18) yields a quadratic equation for ϵ_3

$$\epsilon_3^2 - \epsilon_3(2\epsilon_2 + \epsilon_b - \epsilon^*) + \epsilon_2(\epsilon_2 + \epsilon_b - \epsilon^*) - \left(\frac{2v_o}{c} - \epsilon_2\right)^2 = 0 \tag{19}$$

which gives a one-parameter of solutions for ϵ_3

$$\epsilon_3 = \frac{2\epsilon_2 + \epsilon_b - \epsilon^*}{2} + \left[\frac{1}{4}(\epsilon_b - \epsilon^*)^2 + \left(\frac{2v_o}{c} - \epsilon_2\right)^2\right]^{1/2} \tag{20}$$

in terms of the parameter ϵ_2. Only the solution with the positive sign on the radical has been selected in the above since it is necessary that $\epsilon_3 > \epsilon_b$; this inequality is violated with the negative sign.

Combining equations (16*b*) and (16*c*) to eliminate the stresses and using equation (16*a*) to then eliminate ϵ_3, we obtain the following equation for $\dot{s}$

$$\left(\frac{\dot{s}}{c}\right)^2 + 2A\left(\frac{\dot{s}}{c}\right) - 1 = 0 \tag{21a}$$

where

$$A = \frac{c(\epsilon_b - \epsilon^*)}{2(2v_o - c\epsilon_2)} \tag{21b}$$

Hence

$$\dot{s} = c(-A + \sqrt{1 + A^2}) \tag{22}$$

and it follows immediately that if $A >$ [illegible] en by the triangle inequality $\dot{s} < c$. The condition that A [illegible] satisfied if

$$2v_o - c\epsilon_2 > 0 \tag{23}$$

which must be satisfied if strain softening [illegible] itiated.

Thus we have a one-parameter family of so[illegible] this problem in which the parameter ϵ_2 is restricted by

$$\epsilon^* \leq \epsilon_2 \leq \frac{\sigma_a}{E}$$

An interesting case, which we will see is usually obtained in numerical solutions of these equations, corresponds to $\epsilon_2 = \epsilon^*$. Equations (20) then becomes

$$\epsilon_3 = \frac{\epsilon_b + \epsilon^*}{2} + \left[\frac{(\epsilon_b - \epsilon^*)^2}{4} + \left(\frac{2v_o}{c} - \epsilon^*\right)^2\right]^{1/2} \tag{25}$$

The strain ϵ_3 can then be shown to be bounded by

$$\frac{1}{2}(2\epsilon_2 + \epsilon_b + \epsilon^*) \leq \epsilon_3 \leq \frac{2v_o}{c} + (\epsilon_b - \epsilon^*) \tag{26}$$

Note that if $2v_o \rightarrow c\epsilon^*$, equation (21*b*) shows that $A \rightarrow \infty$ and from equation (22), $\dot{s} \rightarrow 0$.

The solution for $-L \leq x < -s$ is then

$$u(x,t) = -v_o < \xi - \frac{2x}{c} > + (c\epsilon^* - v_o) < \xi > \tag{27a}$$

$$\epsilon = \frac{v_o}{c} H\left(\xi - \frac{2x}{c}\right) + \left(\epsilon^* - \frac{v_o}{c}\right) H(\xi) \tag{27b}$$

For $-s \leq x \leq 0$

$$u = \epsilon_3 x H < \xi > \tag{27c}$$

$$\epsilon = \epsilon_3 H < \xi > \tag{27d}$$

The character of the solution is shown in Fig. 3. A noteworthy feature which distinguishes it from an elastic-plastic solution is the unloading wave emanating from the center.

Remark 2. Although the point of strain softening moves in the solution, this does not contradict the statement in Bazant and Belytschko (1985) (for the case in which $\sigma_b = 0$ and $E \leq 0$ in the softening domain) that the strain softening/elastic interface must be stationary. In the case considered here with $\sigma_b \neq 0$ and E becoming positive again after softening, the strain softening occurs instantaneously and the point subsequently becomes elastic. Thus, the interface $s(t)$ can be considered to be between two elastic domains.

Remark 3. Note that if equation (16*a*) is satisfied, $\dot{s} > 0$ as required, since $\epsilon_3 > \epsilon_2$.

Remark 4. The solution poses some peculiar mathematical difficulties, for at the points $x = \pm s$ the stress takes on the values in the range $\sigma_a \leq \sigma \leq \sigma_s$ twice in one point in time; thus whether it is differentiable, and whether $\sigma_{,x}$ in the governing equation (1) is defined, is not clear.

Remark 5. The propagation of the jump discontinuity and

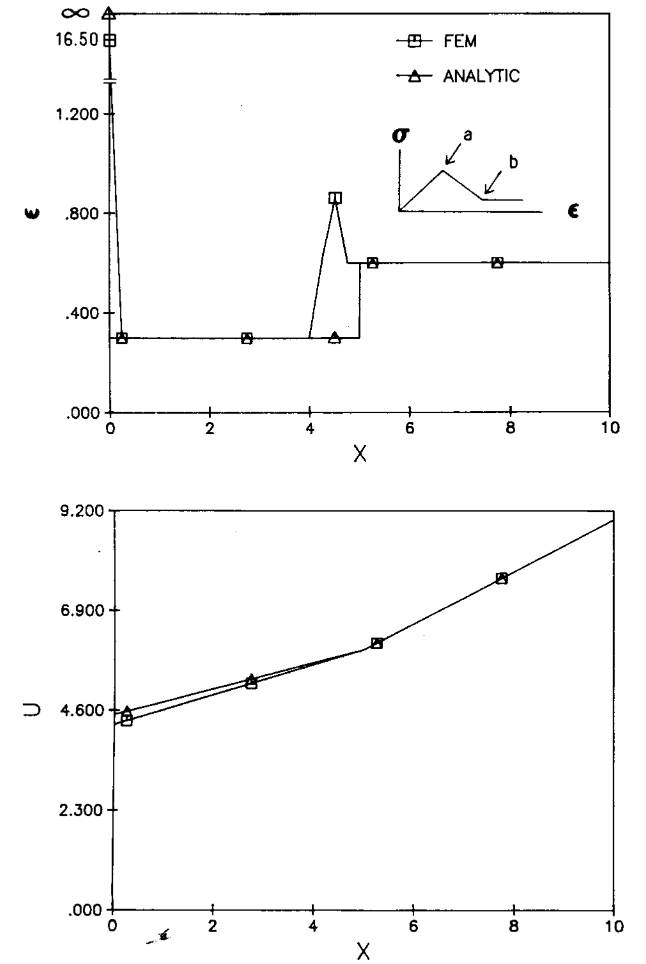

Fig. 4 Strain and displacement distributions at time $t=15$ for problem 2 for the case when $\sigma_b=0.3$

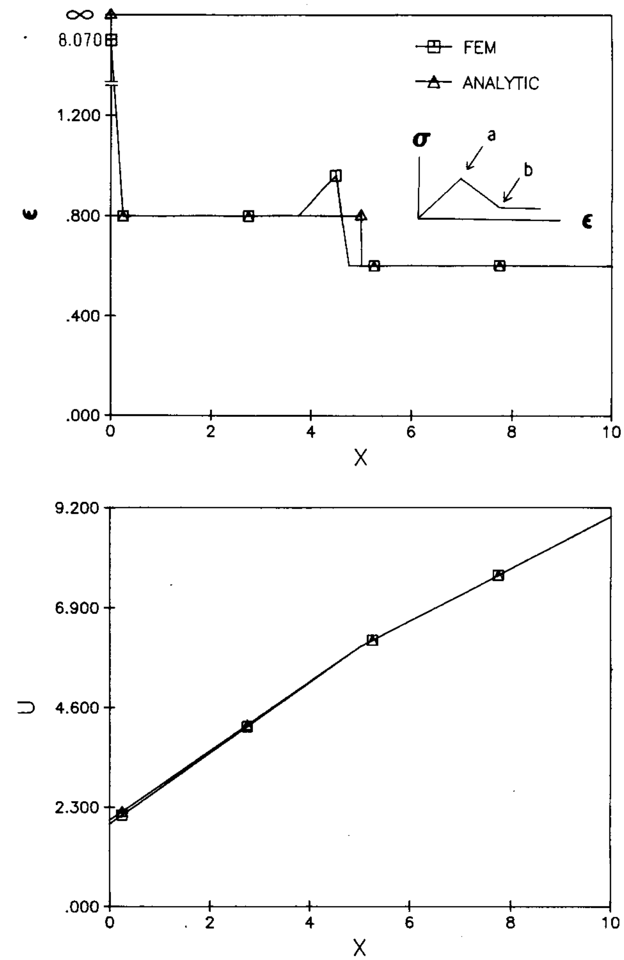

Fig. 5 Strain and strain displacement distributions at time $t=15$ for problem 2 for the case when $\sigma_b=0.8$

subsequent loading may require that $\sigma_3=T(\epsilon_3)>\sigma_a$. Under certain conditions this requirement is not met.

Solution 2. The same BVP is solved with the stress-strain law shown in Fig. 2(*b*). The constitutive algorithm is as follows:

$$\text{initial conditions:}\quad \sigma=\epsilon=0 \quad S=\sigma_a, \tag{28a}$$

$$\text{if } \sigma=S=\sigma_b \text{ and } \dot{\epsilon}>0, \quad \dot{\sigma}=0 \tag{28b}$$

$$\text{if } \sigma_b\leq S\leq\sigma_a,\ \sigma=\dot{S} \text{ and } \dot{\epsilon}>0, \quad \dot{\sigma}=E^-\dot{\epsilon} \tag{29a}$$

$$\text{otherwise} \quad \dot{\sigma}=E^+\dot{\epsilon} \tag{29b}$$

The difference algorithm of equation (6*d*–*e*) with $T(\epsilon)$ redefined as in Fig. 2(*b*) may also be used.

In constructing the solution for this material law, we note that the stress at the strain-softening point becomes σ_b after the jump in the strain, so the elastic solution in domains 1 and 2 becomes

$$\epsilon_2=\epsilon^*, \quad \sigma_2=\sigma_b \tag{30a}$$

$$u(x,t)=-v_o<\xi-\frac{2x}{c}>+(c\epsilon^*-v_o)<\xi> \tag{30b}$$

The size of the strain-softening domain is characterized by s with $s(t=0)=0$, and from (30*b*), the elastic solution at $s=0$ is given by

$$u(0,t)=(c\epsilon^*-2v_o)<t-\frac{L}{c}> \tag{31}$$

If strain softening has occurred, $c\epsilon^*-2v_o<0$, so since the stresses in both the elastic and strain-softening domain are σ_b, there is no mechanism for developing a wave to eliminate the displacement discontinuity. The only way to satisfy the boundary value problems (7) is to allow a discontinuity in the displacement at $x=0$ and an associated infinite strain. Hence equation (30*b*) holds in the left-hand plane with $s=0$ and the magnitude of the displacement discontinuity is $2c\epsilon^*-4v_o$ and the strain field is given by

$$\epsilon(x,t)=\frac{v_o}{c}H\left(t-\frac{L+x}{c}\right)-\left(\frac{v_o}{c}-\epsilon_o\right)H\left(t-\frac{L-x}{c}\right)$$
$$+(2c\epsilon^*-4v_o)\delta(x)H\left(t-\frac{L}{c}\right) \quad \text{for } x\leq 0. \tag{32}$$

where $\delta(\ )$ is the Dirac delta function.

The energy dissipation due to nonlinear material behavior in the region $-s\leq x\leq s$, where s tends to zero, results strictly from the Dirac delta term in equation (32) and is given by

$$W=2\sigma_b(c\epsilon^*-2v_o) \tag{33}$$

This agrees with the result of Bazant and Belytschko (1985) when $\sigma_b\rightarrow 0$, i.e., the dissipation vanishes when the stress goes to zero in the strain-softening domain. When $\sigma_b\neq 0$, a finite dissipation of energy can be achieved, but it is solely due to the plastic response and equivalent to that of an ideal plastic material with yield stress σ_b.

Figures 4 and 5 show the strain and displacement fields for problem 2 in the left-hand plane. Both the analytic solution and the finite element solution are shown. In these examples, $c=1$, $Ev_o/c\sigma_a=0.6$, and $\sigma_b=(0.3,\ 0.8)$ in Figs. 4 and 5, respectively. For the finite element solution, 40 elements were

Table 1 Parameters for problem 1 – Figs. 6-8

$\epsilon_a = 1.0$	$\sigma_a = 1.0$
$\epsilon_b = 1.2$	$\sigma_b = 0.2$
$v_o = 0.8$	$L = 50$

no. of elements for $(0 \le x \le L) = 80$
$c = 1.0$
Courant number ~ 0.7
$\dot{s}$ (for closed form solution) $= 0.704$

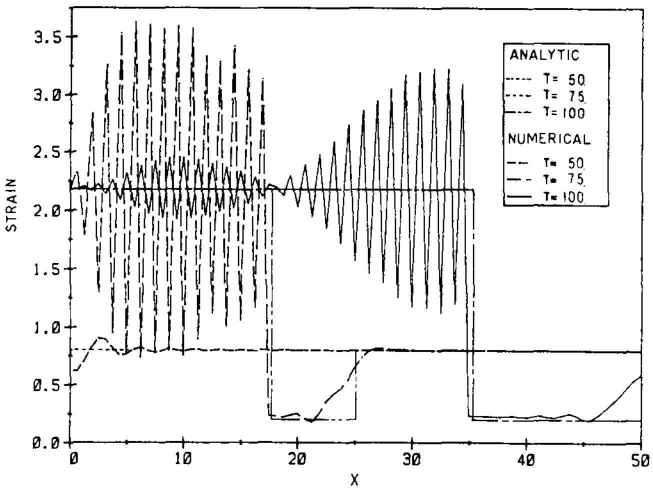

Fig. 6 Strain for example of problem 1

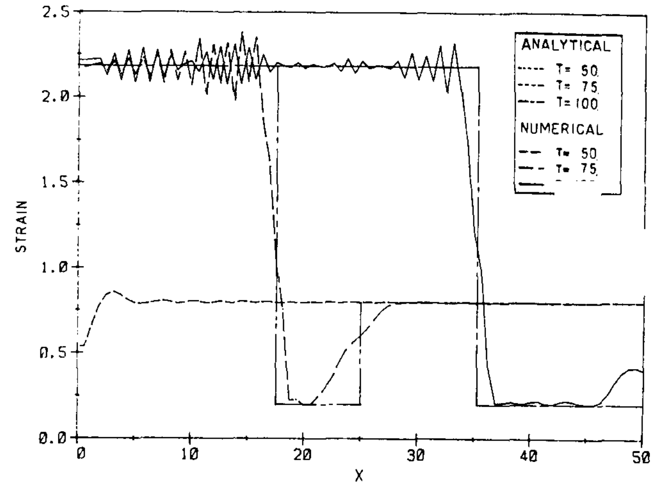

Fig. 7 Strain for example of problem 1 with five-point spatial averaging filter

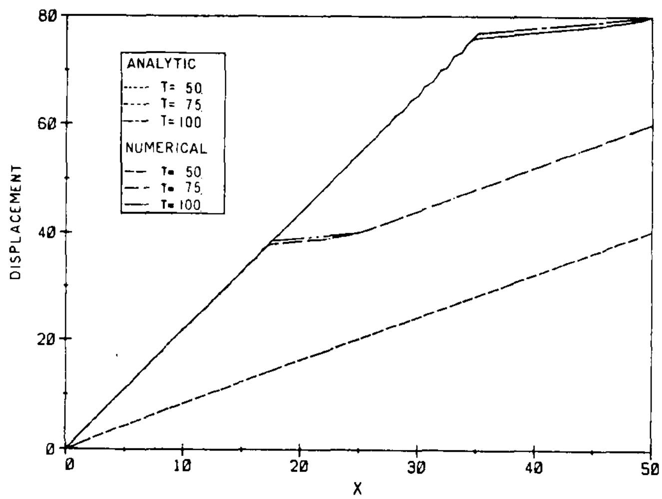

Fig. 8 Displacements for example of problem 1

used on the left-hand plane and the Courant number was 0.6. Whereas the analytic solution predicts an infinite strain at $x = 0$, the strain in the finite element solution is finite but much larger than the surrounding strains. The analytic and finite element solutions otherwise agree quite well except at the wavefront generated by strain-softening (at $x = 5$ in the figures).

Figure 6 shows the strains in the right-hand plane for a finite element solution for problem 1; the corresponding analytic solution with $\epsilon_2 = \epsilon^*$ is also shown. The problem parameters are listed in Table 1. Several features are noteworthy: (*i*) the finite element solution exhibits the unloading wave (at which $\epsilon = 0.2$) which precedes the strain-softening wave $\dot{s}$; (*ii*) the finite element solution correctly captures the wave speed $\dot{s}$; (*iii*) the strains behind the wave $s(t)$ are extremely noisy, which probably reflects the difficulty the numerical solution has in reproducing the complex stress path associated with the wavefront (see Remark 4).

The noise significantly exceeds that found in finite element solutions of elastic wave propagation problems (see Holmes and Belytschko, 1976). Figure 7 shows the same solution with a five-point "averaging" digital filter described in Holmes and Belytschko (1976) applied to the strains and stresses. The filtering technique more clearly brings out the similarities of the finite element and analytic solutions. Figure 8 shows the displacements at 3 times, which again illustrates the presence of the unloading wave and the excellent agreement of the closed form and numerical solutions.

Capturing the unloading wave in a numerical solution does require some care. We used a time-step control so that during a time step no element can pass more than 10 percent beyond the point (ϵ_b, σ_b) in the stress-strain law. Attempts to obtain the same fidelity by reducing the Courant number (time step) to about 0.1 were unsuccessful because at such low Courant numbers the wavefronts are excessively dispersed.

A convergence study was made in the L_2-norm for this solution using meshes of 40 to 320 elements. The rates of convergence were quite sensitive to the time step and amount of artificial viscosity. The rates of convergence varied from $h^{0.6}$ to $h^{1.5}$, which is substantially below the h^2 rate in linear, static problems for this element.

Discussion and Conclusions

Closed-form transient solutions have been developed for rods with strain softening, a negative slope in the stress-strain curve. Two types of stress-strain curves were considered, one where the stress increases again and one where it remains constant after the strain softening. Finite element solutions were also obtained for representative problems. The following conclusions are drawn:

1 If the stress remains constant after the strain softening, a discontinuity appears in the displacement.

2 If the stress increases after strain softening, no discontinuity appears in the displacement, but the strain-softening point moves through the rod with jump discontinuities in the stresses and strains at a speed that is slower than the elastic wavespeed.

3 Finite element solutions reproduce the salient features of these solutions but exhibit excessive noise and slow rates of convergence.

4 When the stress remains constant, the displacement discontinuity is associated with a finite dissipation of energy; when the stress monotonically decreases to zero, the failure of the material associated with the discontinuity in displacements

requires no energy to be dissipated because it occurs on a set of measure zero.

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