Transition Probabilities for Symmetric Jump Processes

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Abstract. We consider symmetric Markov chains on the integer lattice in d dimensions, where $\alpha \in (0,2)$ and the conductance between x and y is comparable to $|x-y|^{-(d+\alpha)}$. We establish upper and lower bounds for the transition probabilities that are sharp up to constants.

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1. Introduction.

There is a huge literature on the subject of transition probabilities of random walks on graphs. For a recent and comprehensive account, see the book [Wo]. The vast majority of the work, however, has been for nearest neighbor Markov chains. The purpose of this paper is to obtain good transition probability estimates for Markov chains on the integer lattice $\mathbb{Z}^d$ in d dimensions in the case when the probability of a jump from a point x to a point y is comparable to that of a symmetric stable process of index $\alpha \in (0,2)$.

To be more precise, for $x, y \in \mathbb{Z}^d$ with $x \neq y$, let C_{xy} be positive finite numbers such that $\sum_z C_{xz} < \infty$ for all x. Set $C_{xx} = 0$ for all x. We call C_{xy} the *conductance* between x and y. Define a symmetric Markov chain by

$$\mathbb{P}(X_1 = y \mid X_0 = x) = \frac{C_{xy}}{\sum_z C_{xz}}, \qquad x, y \in \mathbb{Z}^d. \tag{1.1}$$

In this paper we will assume that $\alpha \in (0,2)$ and there exists $\kappa > 1$ such that for all $x \neq y$

$$\frac{\kappa^{-1}}{|x-y|^{d+\alpha}} \leq C_{xy} \leq \frac{\kappa}{|x-y|^{d+\alpha}}. \tag{1.2}$$

Write $p(n, x, y)$ for $\mathbb{P}^x(X_n = y)$. The main result of this paper is

Theorem 1.1. *There exist positive finite constants c_1 and c_2 such that*

$$p(n, x, y) \leq c_1 \Big(n^{-d/\alpha} \wedge \frac{n}{|x-y|^{d+\alpha}} \Big), \tag{1.3}$$

and for $n \geq 2$

$$p(n, x, y) \geq c_2 \Big(n^{-d/\alpha} \wedge \frac{n}{|x-y|^{d+\alpha}} \Big). \tag{1.4}$$

If $n = 1$ and $x \neq y$, (1.4) also holds.

The Markov chain X_n is discrete in time and in space. Closely related to X_n is the continuous time process Y_t, which is the process that waits at a point in $\mathbb{Z}^d$ a length of time that is exponential with parameter 1, jumps according to the jump probabilities of X, then waits at the new point a length of time that is exponential with parameter 1 and independent of what has gone before, and so on. A continuous-time continuous state space process related to both X_t and Y_t is the process U_t on $\mathbb{R}^d$ whose Dirichlet form is

$$\mathcal{E}(f,f) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (f(y) - f(x))^2 C(x,y)\, dx\, dy,$$

where $C(x,y)$ is a measurable function with

$$\frac{\kappa^{-1}}{|x-y|^{d+\alpha}} \leq C(x,y) \leq \frac{\kappa}{|x-y|^{d+\alpha}}.$$

The process U_t stands in the same relationship to X_n as the diffusion process corresponding to a uniformly elliptic operator in divergence form does to a nearest neighbor Markov chain.

The methods of this paper allow one to obtain bounds for the transition probabilities of Y_t and the transition densities of U_t. In fact, these are considerably easier than the bounds for X_n, so we concentrate in this paper only on the estimates for X_n. Some results for Y_t are needed, however, along the way.

Our methods are quite different from those used for diffusions or nearest neighbor chains. Recall that for a nearest neighbor Markov chain on $\mathbb{Z}^d$, the transition probabilities are bounded above and below by expressions of the form

$$c_1 n^{-d/2} \exp(-c_2 |x-y|^2/n)$$

as long as $|x-y|$ is not larger than n; see [SZ]. One way of obtaining these results is to use a method of Davies as developed in [CKS]. The lack of a suitably fast decay in the conductances in (1.2) makes the powerful theorem of [CKS] only partially successful. We use that theorem to handle the small jumps and use a perturbation argument to handle the large jumps. Another difficulty that shows up is that, unlike the diffusion case, $\mathbb{P}^x(|X_n - y| < 1)$ is not comparable to $\mathbb{P}^x(\max_{k\le n} |X_k - x| > |x-y|)$ when $|x-y|$ is relatively large. We circumvent this by proving a parabolic Harnack inequality and using another perturbation argument.

Previous work related to this paper includes [Kl] and [Km]. In both these works partial results were obtained for estimates for the process U_t mentioned above. [SY] studies nearest neighbor chains on $\mathbb{Z}^d$. In [HS-C] upper bounds of Gaussian type were obtained for Markov chains whose jumps had bounded range or where the conductances decayed at a Gaussian rate.

After some preliminaries, we obtain in Section 2 a tightness (or large deviations) estimate for our Markov chain X_n. This is followed in Section 3 by a parabolic Harnack inequality. In Section 4 we obtain the upper bound in Theorem 1.1, and in Section 5 we prove the lower bound.

2. Tightness.

We denote the ball of radius r centered at x by $B(x,r)$; throughout we use the Euclidean metric. T_A will denote the first hit of a set A by whichever process is under consideration, while τ_A will denote the first exit. The letter c with subscripts will denote positive finite constants whose exact value is unimportant and may change from occurrence to occurrence.

We assume we are given reals C_{xy} satisfying (1.2) and we define the transition probabilities for the Markov chain X_n by

$$p(1,x,y) = \mathbb{P}^x(X_1 = y) = \frac{C_{xy}}{C_x}, \qquad x \neq y, \tag{2.1}$$

where $C_x = \sum_z C_{xz}$, and $p(1,x,x) = 0$ for every x. The process X_n is symmetric (or reversible): C_x is an invariant measure for which the kernel $C_x p(1,x,y)$ is symmetric in x, y. Note that $c_1^{-1} \le C_x/C_y \le c_1$ for some positive and finite constant c_1.

Our main goal in this section is to get a tightness, or large deviations, estimate for X_n. See Theorem 2.8 for the exact statement.

We will need Y_t, the continuous time version of X_n, which we construct as follows: Let $U_1, U_2, \ldots$ be an i.i.d. sequence of exponential random variables with parameter 1 that is independent of the chain X_n. Let $T_0 = 0$ and $T_k = \sum_{i=1}^k U_i$. Define $Y_t = X_n$ if $T_n \le t < T_{n+1}$. If we define $A(x,y) = |x-y|^{d+\alpha} C_{xy}/C_x$, then by (1.2), $\kappa^{-1} \le A(x,y) \le \kappa$, and the infinitesimal generator of Y_t is

$$\sum_{y \ne x} [f(y) - f(x)] \frac{A(x,y)}{|x-y|^{d+\alpha}}.$$

We introduce now several processes related to Y_t, needed in what follows. The rescaled process $V_t = D^{-1} Y_{D^\alpha t}$ takes values in $\mathcal{S} = D^{-1}\mathbb{Z}^d$ and has infinitesimal generator

$$\sum_{y \in \mathcal{S}, y \ne x} [f(y) - f(x)] \frac{A^D(x,y)}{D^d |x-y|^{d+\alpha}} ,$$

where $A^D(x,y) = A(Dx, Dy)$ for $x, y \in \mathcal{S}$. If the large jumps of V_t are removed, we obtain the process W_t with infinitesimal generator

$$\mathcal{A} f(x) = \sum_{\substack{y \in \mathcal{S}, y \ne x \\ |x-y| \le 1}} [f(y) - f(x)] \frac{A^D(x,y)}{|x-y|^{d+\alpha}} .$$

To analyze W_t, we compare it to a Lévy process with a comparable transition kernel: Let Z_t be the Lévy process which has no drift and no Gaussian component and whose Lévy measure is

$$n_Z(dh) = \sum_{\substack{y \ne 0, |y| \le 1 \\ y \in \mathcal{S}}} \frac{1}{D^d |y|^{d+\alpha}} \delta_y(dh).$$

Write $q_Z(t,x,y)$ for the transition density for Z_t.

Proposition 2.1. *There exist c_1, c_2 such that the transition density $q_Z(t,x,y)$ satisfies*

$$q_Z(t,x,y) \le \begin{cases} c_1 D^{-d} t^{-d/\alpha}, & t \le 1, \\ c_2 D^{-d} t^{-d/2}, & t > 1. \end{cases}$$

Proof. The characteristic function $\varphi_t(u)$ of Z_t is periodic with period $2\pi D$ since Z_t is supported on $\mathcal{S} = D^{-1}\mathbb{Z}^d$. By the Lévy-Khintchine formula and the symmetry of n_Z,

$$\varphi_t(u) = \exp\Big(-2t \sum_{x \in \mathcal{S}, |x| \le 1} [1 - \cos u \cdot x] \frac{1}{D^d |x|^{d+\alpha}} \Big). \tag{2.2}$$

Let

$$Q(a) = \{(u_1, \dots, u_d) : -a < u_i \le a, i = 1, \dots, d\}. \tag{2.3}$$

We estimate φ_t as follows.

Case 1: $|u| \le \frac{1}{2}$.

Since $|x| \le 1$, we have $1 - \cos u \cdot x \ge c_3 (u \cdot x)^2 = c_3 |u|^2 |x|^2 h_u(x)$, where $h_u(x) = (u \cdot x)^2 / |u|^2 |x|^2$. Thus

$$\begin{aligned}
\sum_{|x| \le 1} [1 - \cos u \cdot x] \frac{1}{D^d |x|^{d+\alpha}} &\ge c_3 |u|^2 \sum_{|x| \le 1} h_u(x) |x|^{2-d-\alpha} D^{-d} \\
&\ge c_4 D^{\alpha - 2} |u|^2 \int_{B(0,D)} |x|^{2-d-\alpha} h_u(x) dx \\
&= c_4 D^{\alpha-2} |u|^2 \int_0^D r^{1-\alpha} \left[\int_{S(r)} h_u(s) \sigma_r(ds) \right] dr,
\end{aligned}$$

where $S(r)$ is the $(d-1)$-dimensional sphere of radius r centered at 0, and $\sigma_r(ds)$ is normalized surface measure on $S(r)$. Since $h_u(x)$ depends on x only through $x/|x|$, the inner integral does not depends on r. Furthermore, by rotational invariance, it does not depend on u. Thus,

$$\sum_{|x| \le 1} [1 - \cos u \cdot x] \frac{1}{D^d |x|^{d+\alpha}} \le c_5 |u|^2 .$$

Case 2: $\frac{1}{2} \le |u| \le D/32$.

Let $A = \{x \in \mathcal{S} : \frac{1}{4|u|} \le |x| \le \frac{4}{|u|} \wedge 1, 1 \ge u \cdot x \ge \frac{1}{16}\}$. If $x \in A$, then $[1 - \cos u \cdot x] \ge c_6$, the minimum value of $|x|^{-d-\alpha}$ is $c_7 |u|^{d+\alpha}$, and a bit of geometry shows that there are at least $c_8 |u|^{-d} D^d$ points in A. (Notice that $|u| < D/32$ is required to prevent A from being empty.) We then have

$$\sum_{|x| \le 1} [1 - \cos u \cdot x] \frac{1}{D^d |x|^{d+\alpha}} \ge \sum_A [1 - \cos u \cdot x] \frac{1}{D^d |x|^{d+\alpha}} \ge c_6 c_7 |u|^{d+\alpha} c_8 |u|^{-d} = c_9 |u|^\alpha.$$

Case 3: $D/32 < |u|$, $u \in Q(\pi D)$.

At least one component of u must be larger than $c_{10} D$ where $c_{10} = 1/(32\sqrt{d})$; without loss of generality we may assume it is the first component. Let $y_0 = (D^{-1}, 0, \dots, 0)$. Since $|u_1| \le \pi D$ and $u \cdot y_0 \ge c_{10}$, then $1 - \cos u \cdot y_0 \ge c_{11}$. Hence

$$\sum_{|x| \le 1} [1 - \cos u \cdot x] \frac{1}{D^d |x|^{d+\alpha}} \ge c_{11} D^{-d} |y_0|^{-d-\alpha} \ge c_{12} D^\alpha \ge c_{13} |u|^\alpha,$$

since $u \in Q(\pi D)$.

For $u \in Q(\pi D)$, we then have that $\varphi_t(u)$ is real and

$$0 < \varphi_t(u) \le e^{-c_{14}t|u|^2} + e^{-c_{15}t|u|^\alpha}.$$

Since Z_t is supported on $\mathcal{S}$,

$$\begin{aligned} q_Z(t,x,y) &= \frac{1}{|Q(\pi D)|}\int_{Q(\pi D)} e^{iu\cdot(x-y)}\varphi_t(u)du \\ &\le \frac{1}{|Q(\pi D)|}\int_{Q(\pi D)} \varphi_t(u)du \\ &\le \frac{c_{16}}{D^d}\int_{\mathbb{R}^d} (e^{-c_{14}t|u|^2} + e^{-c_{15}t|u|^\alpha})du, \end{aligned}$$

where $|Q(\pi D)|$ denotes the Lebesgue measure of $Q(\pi D)$. Our result follows from applying a change of variables to each of the integrals on the right hand side. □

We now obtain bounds for the transition probabilities of W_t:

Proposition 2.2. *If $q_W(t,x,y)$ is the transition density for W, then*

$$q_W(t,x,y) \le \begin{cases} c_1 D^{-d} t^{-d/\alpha}, & t \le 1, \\ c_2 D^{-d} t^{-d/2}, & t > 1. \end{cases}$$

The proof of Proposition 2.2 is almost identical with that of Theorem 1.2 in [BBG], and is omitted here.

To obtain off-diagonal bounds for q_W we again proceed as in [BBG]. Let

$$\begin{aligned} \Gamma(f,f)(x) &= \sum_{\substack{y\in\mathcal{S} \\ 0<|x-y|\le 1}} (f(x)-f(y))^2 \frac{A^D(x,y)}{D^d|x-y|^{d+\alpha}}, \\ \Lambda(\psi)^2 &= \|e^{-2\psi}\Gamma(e^\psi,e^\psi)\|_\infty \vee \|e^{2\psi}\Gamma(e^{-\psi},e^{-\psi})\|_\infty, \\ E(t,x,y) &= \sup\{|\psi(x)-\psi(y)| - t\Lambda(\psi)^2 : \Lambda(\psi) < \infty\}. \end{aligned}$$

Proposition 2.3. *For $t \le 1$ and $x, y \in \mathcal{S}$,*

$$q_W(t,x,y) \le c_1 D^{-d} t^{-d/\alpha} e^{-E(2t,x,y)}.$$

Proof. Allowing for slight differences in notation, the proof is very similar to the proof of Lemma 1.4 in [BBG]. The principal difference is the following. Let K be an integer larger than $\frac{1}{2}+\frac{1}{\alpha}$. Let M be a sufficiently regular manifold with volume growth given by $V(x,r) \approx r^{2Kd}$, $r > 1$ and $V(x,r) \approx r^d$, $r < 1$, where $V(x,r)$ is the volume of the ball in M of radius r centered at x. We can then find a symmetric Markov process $\widetilde{V}_t$ on M independent of W whose transition density with respect to a measure m on M satisfies

$$\begin{aligned} q_{\widetilde{V}}(t,x,y) &\le c_2 t^{-d/2}, && 0<t\le 1,\\ q_{\widetilde{V}}(t,x,y) &\le c_2 t^{-dK}, && 1<t<\infty,\\ q_{\widetilde{V}}(t,x,x) &\ge c_3 t^{-d/2}, && 0<t\le 1,\\ q_{\widetilde{V}}(t,x,x) &\ge c_3 t^{-dK}, && 1<t<\infty. \end{aligned}$$

Then $q_W(t,x,y)q_{\widetilde{V}}(t,x',y') \le c_4 D^{-d} t^{-d(\frac{1}{2}+\frac{1}{\alpha})}$ for all t while $q_W(t,x,y)q_{\widetilde{V}}(t,0,0) \ge c_5 D^{-d} t^{-d(\frac{1}{2}+\frac{1}{\alpha})}$ for $t\le 1$. With these changes, the proof is now as in [BBG]. □

The next step is to estimate $E(t,x,y)$ and use this in Proposition 2.3.

Proposition 2.4. *Suppose $t \le 1$. Then*

$$q_W(t,x,y) \le c_1 D^{-d} t^{-d/\alpha} e^{-|x-y|}.$$

In particular, for $\frac{1}{4} \le t \le 1$,

$$q_W(t,x,y) \le c_1 D^{-d} e^{-|x-y|}.$$

Proof. Let $\psi(\xi) = B\cdot\xi$, where $B = (y-x)/|y-x|$. Note that if $|\xi-\zeta| \le 1$, then $(e^{\psi(\zeta)-\psi(\xi)}-1)^2 = (e^{B\cdot(\zeta-\xi)}-1)^2$ is bounded by $c_2|B|^2|\zeta-\xi|^2 = c_2|\zeta-\xi|^2$. Hence

$$e^{-2\psi(\xi)}\Gamma(e^\psi,e^\psi)(\xi) = \sum_{\substack{\zeta\in\mathcal{S}\\ 0<|\xi-\zeta|\le 1}} (e^{\psi(\zeta)-\psi(\xi)}-1)^2 \frac{A^D(\xi,\zeta)}{D^d|\xi-\zeta|^{d+\alpha}}$$

is bounded by

$$c_3 \sum_{\substack{\zeta\in\mathcal{S}\\ 0<|\xi-\zeta|\le 1}} D^{-d}|\xi-\zeta|^{2-d-\alpha}.$$

Since the sum is over $\zeta\in\mathcal{S}$ that are within a distance 1 from ξ, this in turn is bounded by c_4. We have the same bound when ψ is replaced by $-\psi$, so $\Lambda(\psi)^2 \le c_4^2$. Moreover the bound does not depend on x or y. On the other hand,

$$\psi(y)-\psi(x) = (y-x)\cdot(y-x)/|y-x| = |y-x|.$$

Using this in Proposition 2.3 and recalling $t\le 1$, we have our result. □

From the above estimate we can obtain a tightness estimate for W_t.

Proposition 2.5. *There exists c_1 such that if $t \le 1$ and $\lambda > 0$, then*

$$\mathbb{P}^x(\sup_{s\le t} |W_s - x| > \lambda) \le c_1 e^{-\lambda/8}.$$

Proof. From Proposition 2.4 and summing, if $t \in [\frac14, 1]$ and $\lambda > 0$,

$$\mathbb{P}^x(|W_t - x| \ge \lambda) \le \sum_{\substack{y\in\mathcal{S}\\ |y-x|\ge\lambda}} c_2 t^{-d/\alpha} D^{-d} e^{-|y-x|} \le c_3 e^{-\lambda/2}. \tag{2.4}$$

Let $S_\lambda = \inf\{t : |W_t - W_0| \ge \lambda\}$. Then using (2.4),

$$\begin{aligned}\mathbb{P}^x(\sup_{s\le 1/2} |W_s - x| \ge \lambda) = \mathbb{P}^x(S_\lambda \le 1/2)&\\ &= \mathbb{P}^x(|W_1 - x| > \lambda/2) + \mathbb{P}^x(S_\lambda \le 1/2, |W_1 - x| \le \lambda/2)\\ &\le c_3 e^{-\lambda/4} + \int_0^{1/2} \mathbb{P}^x(|W_1 - W_s| > \lambda/2, S_\lambda \in ds).\end{aligned}$$

By the Markov property, the last term on the right is bounded by

$$\int_0^{1/2} \mathbb{E}^x[\mathbb{P}^{W_s}(|W_{1-s} - W_0| > \lambda/2); S_\lambda \in ds] \le c_3 e^{-\lambda/4} \int_0^{1/2} \mathbb{P}^x(S_\lambda \in ds) \le c_3 e^{-\lambda/4},$$

using (2.4) again.

Adding gives

$$\mathbb{P}^x(\sup_{s\le t} |W_s - x| > \lambda) \le c_4 e^{-\lambda/4} \tag{2.5}$$

as long as $t \le \frac12$. For $t \in (\frac12, 1]$, note that if $\sup_{s\le t} |W_s - x| > \lambda$, then $\sup_{s\le \frac12} |W_s - x| > \lambda/2$ or $\sup_{\frac12 < s \le 1} |W_s - W_{1/2}| > \lambda/2)$. The probability of the first event is bounded using (2.5), while the probability of the second event is bounded using the Markov property at time $\frac12$ and (2.5). $\square$

Define $\mathcal{B}$ to be the infinitesimal generator of V_t without small jumps:

$$\mathcal{B}f(x) = \sum_{\substack{y\in\mathcal{S}\\ |y-x|>1}} [f(y) - f(x)] \frac{A^D(x,y)}{D^d |x-y|^{d+\alpha}}.$$

Our next goal is to obtain tightness estimates for the process $V_t = D^{-1} Y_{D^\alpha t}$, whose generator is $\mathcal{A} + \mathcal{B}$.

Proposition 2.6. *Let V_t be the process whose generator is $\mathcal{A}+\mathcal{B}$. There exist c_1, c_2 and δ_0 such that if $\delta \le \delta_0$ and $\lambda \ge 1$, then*

$$\mathbb{P}^x(\sup_{s\le\delta} |V_s - x| > \lambda) \le c_1 e^{-c_2\lambda} + c_1\delta.$$

Proof. Since summing $D^d|y-x|^{-d-\alpha}$ over $|y-x|\ge 1$ is a constant,

$$|\mathcal{B}f(x)| \le c_3 \|f\|_\infty, \tag{2.6}$$

and hence $\mathcal{B}$ is a bounded operator on L^∞.

Define $Q_t^W f(x) = \sum q_W(t,x,y) f(y)$. Let Q_t^V be the corresponding transition semigroup for V_t. Let $S_0(t) = Q_t^W$ and for $n \ge 1$, let $S_n(t) = \int_0^t S_{n-1}\mathcal{B}Q_{t-s}^W ds$. Then

$$Q_t^V = \sum_{n=0}^\infty S_n(t);$$

see [Le], Theorem 2.2, for example. Obviously Q_t^W is a bounded operator on L^∞ of norm 1, so for $t < \delta_0 = 1/(2c_3)$ the sum converges by (2.6). In particular, for $t \le \delta \le \delta_0$, we have

$$|Q_t^W f(x) - Q_t^V f(x)| \le c_4 \delta \|f\|_\infty.$$

Fix x and apply this to $f(y) = 1_{B(x,\lambda)}(y)$. We obtain

$$\mathbb{P}^x(|V_t - x| > \lambda) = Q_t^V f(x) \le Q_t^W f(x) + c_4\delta = \mathbb{P}^x(|W_t - x| > \lambda) + c_4\delta \le c_5 e^{-\lambda/8} + c_4\delta.$$

We now obtain our result by applying the method of proof of Proposition 2.5. □

Now notice that $Y_t = DV_{t/D^\alpha}$. Translating Proposition 2.6 in terms of Y_t, we have

Corollary 2.7. *If $\lambda \ge 1$ and $\delta \le \delta_0$,*

$$\mathbb{P}^x(\sup_{s\le\delta D^\alpha} |Y_s - x| > \lambda D) \le c_1 e^{-c_2\lambda} + c_1\delta. \tag{2.7}$$

for every D.

We can now obtain the tightness result for X_n.

Theorem 2.8. *Given $C > 1$ and $\beta \in (0,1)$, there exists γ such that*

$$\mathbb{P}^x(\max_{k\le[\gamma S^\alpha]} |X_k - x| > CS) \le \beta \tag{2.8}$$

for all $S > 0$.

Proof. Let $\beta \in (0,1)$. By Corollary 2.7 we may choose λ and $\delta \le \delta_0/2$ so that

$$\mathbb{P}^x(\sup_{s \le 2\delta D^\alpha} |Y_s - x| > \lambda D) \le \beta/2$$

for every D. Define $D = CS/\lambda$. We may suppose Y is constructed as in Section 2. Then

$$\begin{aligned}\mathbb{P}^x(\max_{k \le [\delta D^\alpha]} |X_k - x| > CS)& \\ &\le \mathbb{P}^x(\sup_{s \le 2\delta D^\alpha} |Y_s - x| > CS) \\ &\qquad + \mathbb{P}^x(|T_{[\delta D^\alpha]} - [\delta D^\alpha]| > [\delta D^\alpha]) \\ &\le \frac{\beta}{2} + \frac{c_3}{\delta D^\alpha}.\end{aligned}$$

We used Chebyshev's inequality and the fact that $T_{[\delta D^\alpha]}$ is the sum of i.i.d. exponentials to bound the second probability on the right hand side. Choose S_0 large so that $c_3/\delta D^\alpha < \beta/2$ if $S \ge S_0$. We thus have the desired result of $S \ge S_0$.

Finally choose γ smaller if necessary so that $\gamma S_0^\alpha < 1$. If $S < S_0$, then $\gamma S^\alpha < 1$. But X_k needs at least one unit of time to make a step; hence the left hand side of (2.8) is 0 if $S < S_0$. □

Remark 2.9. Given the above tightness estimate, one could formulate a central limit theorem. Under a suitable normalization a sequence of Markov chains whose jump structure is similar to that of a symmetric stable process should converge weakly to a process such as the U_t described in Section 1.

Remark 2.10. We expect that our techniques could also give tightness for Markov chains where the conductances decay more rapidly than the rates given in this paper. In this case one might have a central limit theorem where the limiting distributions are those of processes corresponding to elliptic operators in divergence form. It would be quite interesting to formulate a central limit theorem for Markov chains where the limit processes are diffusions but the Markov chains do not have bounded range.

3. Harnack inequality.

It is fairly straightforward at this point to follow the argument of [BL] and obtain a Harnack inequality of Moser type for functions that are harmonic with respect to X_n. In this paper, however, we are primarily interested in transition probability estimates. As a tool for obtaining these, we turn to a parabolic Harnack inequality.

Let $\mathcal{T} = \{0, 1, 2, \ldots\} \times \mathbb{Z}^d$. We will study here the $\mathcal{T}$-valued Markov chain (V_k, X_k), where $V_k = V_0 + k$. We write $\mathbb{P}^{(j,x)}$ for the law of (V_k, X_k) started at (j, x). Let $\mathcal{F}_j = \sigma((V_k, X_k) : k \le j)$. A bounded function $q(k, x)$ on $\mathcal{T}$ will be said to be *parabolic* on $D \subset \mathcal{T}$ if $q(V_{k\wedge\tau_D}, X_{k\wedge\tau_D})$ is a martingale.

Define

$$Q(k, x, r) = \{k, k+1, \ldots, k + [\gamma r^\alpha]\} \times B(x, r). \tag{3.0}$$

Our goal in this section is the following result:

Theorem 3.1. *There exists c_1 such that if q is bounded and nonnegative on $\mathcal{T}$ and parabolic on $Q(0, z, R)$, then*

$$\max_{(k,y)\in Q([\gamma R^\alpha],z,R/3)} q(k, y) \le c_1 \min_{y\in B(z,R/3)} q(0, y).$$

We prove this after first establishing a few intermediate results.

From Theorem 2.8 there exists γ such that for all $r > 0$

$$\mathbb{P}^x(\max_{k\le[\gamma r^\alpha]} |X_k - x| > r/2) \le \frac{1}{4}. \tag{3.1}$$

Without loss of generality we may assume $\gamma \in (0, \frac{1}{3})$.

We will often write τ_r for $\tau_{Q(0,x,r)}$. For $A \subset Q(0, x, r)$ set $A(k) = \{y : (k, y) \in A\}$. Define $N(k, x)$ to be $\mathbb{P}^{(k,x)}(X_1 \in A(k + 1))$ if $(k, x) \notin A$ and 0 otherwise.

Lemma 3.2. *Let*

$$J_n = 1_A(V_n, X_n) - 1_A(V_0, X_0) - \sum_{k=0}^{n-1} N(V_k, X_k).$$

Then $J_{n\wedge T_A}$ is a martingale.

Proof. We have

$$\begin{aligned}\mathbb{E}\,[J_{(k+1)\wedge T_A} - J_{k\wedge T_A} \mid \mathcal{F}_k] = \mathbb{E}\,[&1_A(V_{(k+1)\wedge T_A}, X_{(k+1)\wedge T_A}) - 1_A(V_{k\wedge T_A}, X_{k\wedge T_A}) \\ &- N(V_{k\wedge T_A}, X_{k\wedge T_A}) \mid \mathcal{F}_k].\end{aligned}$$

On the event $\{T_A \le k\}$, this is 0. If $T_A > k$, this is equal to

$$\mathbb{P}^{(V_k,X_k)}((V_1, X_1) \in A) - N(V_k, X_k) = \mathbb{P}^{X_k}(X_1 \in A(V_k + 1)) - N(V_k, X_k) = 0.$$

$\square$

Given a set $A \subset \mathcal{T}$, we let $|A|$ denote the cardinality of A.

Proposition 3.3. *There exists θ_1 such that if $A \subset Q(0,x,r/2)$ and $A(0)=\emptyset$, then*

$$\mathbb{P}^{(0,x)}(T_A < \tau_r) \geq \theta_1 \frac{|A|}{r^{d+\alpha}}.$$

Proof. Observe that T_A cannot equal τ_r. If $\mathbb{P}^{(0,x)}(T_A \leq \tau_r) \geq \frac{1}{4}$ we are done, so assume without loss of generality that $\mathbb{P}^{(0,x)}(T_A \leq \tau_r) < \frac{1}{4}$. Let $S = T_A \wedge \tau_r$. From Lemma 3.2 and optional stopping we have

$$\mathbb{E}^{(0,x)} 1_A(S, X_S) \geq \mathbb{E}^{(0,x)} \sum_{k=0}^{S-1} N(k, X_k).$$

Note that if $(k,x) \in Q(0,x,r)$

$$N(k,x) = \mathbb{P}^{(k,x)}(X_1 \in A(k+1)) \geq \sum_{y \in A(k+1)} \frac{c_1}{|x-y|^{d+\alpha}} \geq \frac{c_2}{r^{d+\alpha}} |A(k+1)|.$$

So on the set $(S \geq [\gamma r^\alpha])$ we have $\sum_{k=0}^{S-1} N(k, X_k) \geq c_3 |A|/r^{d+\alpha}$. Therefore, since $\tau_r \leq [\gamma r^\alpha]$,

$$\begin{aligned}\mathbb{E}^{(0,x)} 1_A(S, X_S) &\geq c_4 \frac{|A|}{r^{d+\alpha}} \mathbb{P}^x(S \geq [\gamma r^\alpha]) \\ &\geq c_4 \frac{|A|}{r^{d+\alpha}} [1 - \mathbb{P}^x(T_A \leq \tau_r) - \mathbb{P}^x(\tau_r < [\gamma r^\alpha])].\end{aligned}$$

Now $\mathbb{P}^x(\tau_r < [\gamma r^\alpha]) \leq \frac{1}{4}$ by (3.1). Therefore $\mathbb{E}^{(0,x)} 1_A(S, X_S) \geq c_5 |A|/r^{d+\alpha}$. Since $A \subset Q(0,x,r/2)$, the proposition follows. □

With $Q(k,x,r)$ defined as in (3.0), let $U(k,x,r) = \{k\} \times B(x,r)$.

Lemma 3.4. *There exists θ_2 such that if $(k,x) \in Q(0,z,R/2)$, $r \leq R/4$, and $k \geq [\gamma r^\alpha]+2$, then*

$$\mathbb{P}^{(0,z)}(T_{U(k,x,r)} < \tau_{Q(0,z,R)}) \geq \theta_2 r^{d+\alpha}/R^{d+\alpha}.$$

Proof. Let $Q' = \{k, k-1, \ldots, k-[\gamma r^\alpha]\} \times B(x, r/2)$. By Proposition 3.3,

$$\mathbb{P}^{(0,z)}(T_{Q'} < \tau_{Q(0,z,R)}) \geq c_1 r^{d+\alpha}/R^{d+\alpha}.$$

Starting at a point in Q', by (3.1) there is probability at least $\frac{3}{4}$ that the chain stays in $B(x,r)$ for at least time γr^α. So by the strong Markov property, there is probability at least $\frac{3}{4} c_1 r^{d+\alpha}/R^{d+\alpha}$ that the chain hits Q' before exiting $Q(0,z,R)$ and stays within $B(x,r)$ for an additional time $c_2 r^\alpha$, hence hits $U(k,x,r)$ before exiting $Q(0,z,R)$. □

Lemma 3.5. *Suppose $H(k,w)$ is nonnegative and 0 if $w \in B(x,2r)$. There exists θ_3 (not depending on x, r, or H) such that*

$$\mathbb{E}^{(0,x)}[H(V_{\tau_r}, X_{\tau_r})] \le \theta_3 \mathbb{E}^{(0,y)}[H(V_{\tau_r}, X_{\tau_r})], \qquad y \in B(x, r/3).$$

Proof. Fix x and r and suppose $k \le [\gamma r^\alpha]$ and $w \notin B(x,2r)$. Assume for now that $[\gamma r^\alpha] \ge 4$. We claim there exists c_1 such that

$$M_j = 1_{(k,w)}(V_{j\wedge\tau_r}, X_{j\wedge\tau_r}) - \sum_{i=0}^{j-1} \frac{c_1}{|w-x|^{d+\alpha}} 1_{(i<\tau_r)} 1_{k-1}(V_i)$$

is a submartingale. To see this we observe

$$\mathbb{E}\,[1_{(k,w)}(V_{(i+1)\wedge\tau_r}, X_{(i+1)\wedge\tau_r}) - 1_{(k,w)}(V_{i\wedge\tau_r}, X_{i\wedge\tau_r}) \mid \mathcal{F}_i]$$

is 0 if $i \ge \tau_r$ and otherwise it equals

$$\mathbb{E}^{(V_i,X_i)} 1_{(k,w)}(V_{1\wedge\tau_r}, X_{1\wedge\tau_r}).$$

This is 0 unless $k = V_i + 1$. When $k = V_i + 1$ and $i < \tau_r$ this quantity is equal to

$$\mathbb{P}^{X_i}(X_1 = w) \ge \frac{c_2}{|X_i - w|^{d+\alpha}} \ge \frac{c_3}{|x-w|^{d+\alpha}}.$$

Thus $\mathbb{E}\,[M_{i+1} - M_i \mid \mathcal{F}_i]$ is 0 if $i \ge \tau_r$ or $k \ne V_i + 1$ and greater than or equal to 0 otherwise if c_1 is less than c_3, which proves the claim.

Since $\mathbb{P}^y(\max_{i\le[\gamma r^\alpha]} |X_i - X_0| > r/2) \le \frac{1}{4}$, then

$$\mathbb{E}^{(0,y)}\tau_r \ge [\gamma r^\alpha] \mathbb{P}^{(0,x)}(\tau_r \ge [\gamma r^\alpha]) \ge [\gamma r^\alpha]/2. \tag{3.3}$$

The random variable τ_r is obviously bounded by $[\gamma r^\alpha]$, so by optional stopping,

$$\mathbb{P}^{(0,y)}((V_{\tau_r}, X_{\tau_r}) = (k,w)) \ge \Big(\mathbb{E}^{(0,y)}\tau_r - 1\Big) \frac{c_4}{|x-w|^{d+\alpha}} \ge \frac{c_4 r^\alpha}{|x-w|^{d+\alpha}}.$$

Similarly, there exists c_5 such that

$$1_{(k,w)}(V_{j\wedge\tau_r}, X_{j\wedge\tau_r}) - \sum_{i=1}^{j-1} \frac{c_5}{|w-x|^{d+\alpha}} 1_{(i<\tau_r)} 1_{k-1}(V_i)$$

is a supermartingale and so

$$\mathbb{P}^{(0,x)}((V_{\tau_r}, X_{\tau_r}) = (k,w)) \le \Big(\mathbb{E}^{(0,x)}\tau_r\Big) \frac{c_6}{|x-w|^{d+\alpha}} \le \frac{c_6 r^\alpha}{|x-w|^{d+\alpha}}.$$

Letting $\theta_3 = c_6/c_4$, we have

$$\mathbb{E}^{(0,x)}[1_{(k,w)}(V_{\tau_r}, X_{\tau_r})] \le \theta_3 \mathbb{E}^{(0,y)}[1_{(k,w)}(V_{\tau_r}, X_{\tau_r})].$$

It is easy to check that θ_3 can be chosen so that this inequality also holds when $[\gamma r^\alpha] < 4$. Multiplying by $H(k,w)$ and summing over k and w proves our lemma. □

Proposition 3.6. *For each n_0 and x_0, the function $q(k,x) = p(n_0 - k, x, x_0)$ is parabolic on $\{0, 1, \dots, n_0\} \times \mathbb{Z}^d$.*

Proof. We have

$$\begin{aligned}\mathbb{E}\,[q(V_{k+1}, X_{k+1}) \mid \mathcal{F}_k] &= \mathbb{E}\,[p(n_0 - V_{k+1}, X_{k+1}, x_0) \mid \mathcal{F}_k] \\ &= \mathbb{E}^{(V_k, X_k)}[p(n_0 - V_1, X_1, x_0)] \\ &= \sum_z p(1, X_k, z) p(n_0 - V_k - 1, z, x_0).\end{aligned}$$

By the semigroup property this is

$$p(n_0 - V_k, X_k, x_0) = q(V_k, X_k).$$

□

Proof of Theorem 3.1. By multiplying by a constant, we may suppose

$$\min_{y \in B(z, R/3)} q(0, y) = 1\,.$$

Let v be a point in $B(z, R/3)$ where $q(0,v)$ takes the value one. Suppose $(k,x) \in Q([\gamma R^\alpha], z, R/3)$ with $q(k,x) = K$. By Proposition 3.3 there exists $c_2 \le 1$ such that if $r < R/3$, $C \subset Q(k+1, x, r/3)$, and $|C|/|Q(k+1,x,r/3)| \ge \frac{1}{3}$, then

$$\mathbb{P}^{(k,x)}(T_C < \tau_r) \ge c_2. \tag{3.4}$$

Set

$$\eta = \frac{c_2}{3}, \qquad \zeta = \frac{1}{3} \wedge (\theta_3 \eta). \tag{3.5}$$

Define r to be the smallest number such that

$$\frac{|Q(0, x, r/3)|}{R^{d+\alpha}} \ge \frac{3}{\theta_1 \zeta K} \tag{3.6}$$

and

$$\frac{r^{d+\alpha}}{R^{d+\alpha}} \ge \frac{2}{\zeta K \theta_2}. \tag{3.7}$$

This implies

$$r/R = c_3 K^{-1/(d+\alpha)}. \tag{3.8}$$

Let

$$A = \{(i,y) \in Q(k+1,x,r/3) : q(i,y) \geq \zeta K\}.$$

Let $U = \{k\} \times B(x, r/3)$. If $q \geq \zeta K$ on U, we would then have by Lemma 3.4 that

$$\begin{aligned} 1 = q(0,v) &= \mathbb{E}^{(0,v)} q(V_{T_U \wedge \tau_{Q(0,z,R)}}, X_{T_U \wedge \tau_{Q(0,z,R)}}) \\ &\geq \zeta K \mathbb{P}^{(0,v)}(T_U < \tau_{Q(0,z,R)}) \geq \frac{\theta_2 r^{d+\alpha} \zeta K}{R^{d+\alpha}}, \end{aligned}$$

a contradiction to our choice of r. So there must exist at least one point in U for which q takes a value less than ζK.

If $\mathbb{E}^{(k,x)}[q(V_{\tau_r}, X_{\tau_r}); X_{\tau_r} \notin B(x,2r)] \geq \eta K$, then by Lemma 3.5 we would have

$$\begin{aligned} q(k,y) &\geq \mathbb{E}^{(k,y)}[q(V_{\tau_r}, X_{\tau_r}); X_{\tau_r} \notin B(x,2r)] \\ &\geq \theta_3 \mathbb{E}^{(k,x)}[q(V_{\tau_r}, X_{\tau_r}); X_{\tau_r} \notin B(x,2r)] \geq \theta_3 \eta K \geq \zeta K \end{aligned}$$

for $y \in B(x, r/3)$, a contradiction to the preceding paragraph. Therefore

$$\mathbb{E}^{(k,x)}[q(V_{\tau_r}, X_{\tau_r}); X_{\tau_r} \notin B(x,2r)] \leq \eta K. \tag{3.9}$$

By Proposition 3.3,

$$\begin{aligned} 1 = q(0,v) &\geq \mathbb{E}^{(0,v)}[q(V_{T_A}, X_{T_A}); T_A < \tau_{Q(0,z,R)}] \\ &\geq \zeta K \mathbb{P}^{(0,v)}(T_A < \tau_{Q(0,z,R)}) \geq \frac{\theta_1 |A| \zeta K}{R^{d+\alpha}}, \end{aligned}$$

hence

$$\frac{|A|}{|Q(k+1,x,r/3)|} \leq \frac{R^{d+\alpha}}{\theta_1 |Q(k+1,x,r/3)| \zeta K} \leq \frac{1}{3}.$$

Let $C = Q(k+1,x,r/3) - A$. Let $M = \max_{Q(k+1,x,2r)} q$. We write

$$\begin{aligned} q(k,x) = \mathbb{E}^{(k,x)}&[q(V_{T_C}, X_{T_C}); T_C < \tau_r] \\ &+ \mathbb{E}^{(k,x)}[q(V_{\tau_r}, X_{\tau_r}); \tau_r < T_C, X_{\tau_r} \notin B(x,2r)] \\ &+ \mathbb{E}^{(k,x)}[q(V_{\tau_r}, X_{\tau_r}); \tau_r < T_C, X_{\tau_r} \in B(x,2r)]. \end{aligned}$$

The first term on the right is bounded by $\zeta K \mathbb{P}^{(k,x)}(T_C < \tau_r)$. The second term on the right is bounded by ηK. The third term is bounded by $M \mathbb{P}^{(k,x)}(\tau_r < T_C)$. Therefore

$$K \leq \zeta K \mathbb{P}^{(k,x)}(T_C < \tau_r) + \eta K + M(1 - \mathbb{P}^{(k,x)}(T_C < \tau_r)).$$

It follows that

$$M/K \geq 1+\beta$$

for some β not depending on x or r, and so there exists a point $(k',x') \in Q(k+1,x,2r)$ such that $q(k',x') \geq (1+\beta)K$.

We use this to construct a sequence of points: suppose there exists a point (k_1,x_1) in $Q([\gamma R^\alpha],z,R/6)$ such that $q(k_1,x_1)=K$. We let $x=x_1, k=k_1$ in the above and construct $r_1=r, x_2=x'$, and $k_2=k'$. We define r_2 by the analogues of (3.6) and (3.7). We then use the above (with (k,x) replaced by (k_2,x_2) and (k',x') replaced by (k_3,x_3)) to construct k_3,x_3, and so on. We thus have a sequence of points (k_i,x_i) for which $k_{i+1}-k_i \leq (2r_i)^\alpha$, $|x_{i+1}-x_i| \leq 2r_i$, and $q(k_i,x_i) \geq (1+\beta)^{i-1}K$. By (3.8) there exists K' such that if $K \geq K'$, then $(k_i,x_i) \in Q([\gamma R^\alpha],z,R/3)$ for all i. We show this leads to a contradiction. One possibility is that for large i we have $r_i<1$, which means that $B(x_i,r_i)$ is a single point and that contradicts the fact that there is at least one point in $B(x_i,r_i)$ for which $q(k_i,\cdot)$ is less than $\eta(1+\beta)^{i-1}K$. The other possibility is that $q(k_i,x_i) \geq (1+\beta)^{i-1}K' > \|q\|_\infty$ for large i, again a contradiction. We conclude q is bounded by K' in $Q([\gamma R^\alpha],z,R/3)$. □

4. Upper bounds.

In this section our goal is to obtain upper bounds on the transition probabilities for our chain X_n. We start with a uniform upper bound.

Let is begin by considering the Lévy process Z_t whose Lévy measure is

$$n(dx)=\sum_{y\in\mathbb{Z}^d, y\neq 0} |y|^{-(d+\alpha)}\delta_y(dx).$$

Proposition 4.1. *The transition density for Z_t satisfies $q_Z(t,x,y) \leq c_1 t^{-d/\alpha}$.*

Proof. The proof is similar to Proposition 2.1 (with $D=1$). The characteristic function $\varphi_t(u)$ is given by

$$\varphi_t(u)=\exp\Big(-2t\sum_{x\in\mathbb{Z}^d}[1-\cos(u\cdot x)]\frac{1}{|x|^{d+\alpha}}\Big).$$

For $|u| \leq 1/32$, we proceed similarly to Case 2 of the proof of Proposition 2.1: we set $D=1$, set $A=\{x\in\mathbb{Z}^d : \frac{1}{4|u|} \leq |x| \leq \frac{4}{|u|}, 1 \geq u\cdot x \geq \frac{1}{16}\}$, and obtain

$$\sum[1-\cos u\cdot x]\frac{1}{|x|^{d+\alpha}} \geq c_2|u|^\alpha.$$

Let $Q(a)$ be defined by (2.3). For $|u|>1/32$ with $u\in Q(\pi)$, we proceed as in Case 3 of the proof of Proposition 2.1 and obtain the same estimate. We then proceed as in the remainder of the proof of Proposition 2.1 to obtain our desired result. □

Proposition 4.2. *The transition densities for Y_t satisfy*

$$q_Y(t,x,y) \le c_1 t^{-d/\alpha}.$$

Proof. This is similar to the proof of Proposition 2.2, but considerably simpler, as we do not have to distinguish between $t \le 1$ and $t > 1$. □

Now we can obtain global bounds for the transition probabilities for X_n.

Theorem 4.3. *There exists c_1 such that the transition probabilities for X_n satisfy*

$$p(n,x,y) \le c_1 n^{-d/\alpha}, \qquad x,y \in \mathbb{Z}^d.$$

Proof. Recall the construction of Y_t in Section 1. First, by the law of large numbers $T_n/n \to 1$ a.s. Thus there exists c_2 such that $\mathbb{P}(T_{[n/2]} \le \frac{3}{4}n < T_n) \ge c_2$ for all n.

Let $C_x = \sum_z C_{xz}$, and set $r(n,x,y) = C_x p(2n,x,y)$. Since $C_x p(1,x,y)$ is symmetric, it can be seen by induction that $C_x p(n,x,y)$ is symmetric. The kernel $r(n,x,y)$ is nonnegative definite because

$$\begin{aligned}
\sum_x \sum_y f(x) r(n,x,y) f(y) &= \sum_x \sum_y \sum_z f(x) C_x p(n,x,z) p(n,z,y) f(y) \\
&= \sum_x \sum_y \sum_z f(x) f(y) C_z p(n,z,x) p(n,z,y) \\
&= \sum_z C_z \Big(\sum_x f(x) p(n,z,x) \Big)^2 \ge 0.
\end{aligned}$$

If we set $r_M(n,x,y) = r(n,x,y)$ if $|x|, |y| \le M$ and 0 otherwise, we have an eigenfunction expansion for r_M:

$$r_M(n,x,y) = \sum_i \lambda_i^n \varphi_i(x) \varphi_i(y), \tag{4.1}$$

where each $\lambda_i \in [0,1]$. By Cauchy-Schwarz,

$$\begin{aligned}
r_M(n,x,y) &\le \Big(\sum_i \lambda_i^n \varphi_i(x)^2 \Big)^{1/2} \Big(\sum_i \lambda_i^n \varphi_i(y)^2 \Big)^{1/2} \\
&= r_M(n,x,x)^{1/2} r_M(n,y,y)^{1/2}.
\end{aligned}$$

Also, by (4.1) $r_M(n,x,x)$ is decreasing in n. Letting $M \to \infty$ we see that $p(2n,x,x)$ is decreasing in n and

$$p(2n,x,y) \le p(2n,x,x)^{1/2} p(2n,y,y)^{1/2}.$$

Suppose now that n is even and $n \geq 8$. It is clear from (1.2) and (2.1) that there exists c_3 such that $p(3,z,z) \geq c_3$ for all $z \in \mathbb{Z}^d$. If k is even and $k \leq n$, then $\mathbb{P}^x(X_k = x) \geq \mathbb{P}^x(X_n = x)$. If k is odd and $k \leq n$, then

$$\mathbb{P}^x(X_k = x) = p(k,x,x) \geq p(k-3,x,x)p(3,x,x) \geq c_3\mathbb{P}^x(X_{k-3} = x) \geq c_3\mathbb{P}^x(X_n = x).$$

Setting $t = \frac{3}{4}n$, using Proposition 4.2, and the independence of the T_i from the X_k, we have

$$\begin{aligned}
c_4 t^{-d/\alpha} \geq \mathbb{P}^x(Y_t = x) &= \sum_{k=0}^{\infty} \mathbb{P}^x(X_k = x, T_k \leq t < T_{k+1}) \\
&\geq \sum_{[n/2] \leq k \leq n} \mathbb{P}^x(X_k = x)\mathbb{P}(T_k \leq t < T_{k+1}) \\
&\geq c_3\mathbb{P}^x(X_n = x)\mathbb{P}^x(T_{[n/2]} \leq t < T_n) \geq c_2c_3\mathbb{P}^x(X_n = x).
\end{aligned}$$

We thus have an upper bound for $p(n,x,x)$ when n is even, and by the paragraph above, for $p(n,x,y)$ when $n \geq 8$ is even.

Now suppose n is odd and $n \geq 5$. Then

$$c_5(n+3)^{-d/\alpha} \geq p(n+3,x,y) \geq p(n,x,y)p(3,y,y) \geq c_3 p(n,x,y),$$

which implies the desired bound when n is odd and $n \geq 5$.

Finally, since $p(n,x,y) = \mathbb{P}^x(X_n = y) \leq 1$, we have our bound for $n \leq 8$ by taking c_1 larger if necessary. $\square$

We now turn to the off-diagonal bounds, that is, when $|x-y|/n^{1/\alpha}$ is large. We begin by bounding $\mathbb{P}^x(Y_{t_0} \in B(y, rt_0^{1/\alpha}))$. To do this, it is more convenient to look at $W_t = t_0^{-1/\alpha} Y_{t_0 t}$ and to obtain a bound on $\mathbb{P}^x(W_1 \in B(y,r))$ for $x,y \in \mathcal{S} = t_0^{-1/\alpha}\mathbb{Z}^d$. The infinitesimal generator for W_t is

$$\sum_{y \in \mathcal{S}} [f(y) - f(x)] \frac{A^{t_0^{1/\alpha}}(x,y)}{t_0^{d/\alpha}|x-y|^{d+\alpha}}.$$

Fix D and let $E = D^{1/2}$. Let Q_t be the transition operator for the process V_t corresponding to the generator

$$\mathcal{A}f(x) = \sum_{\substack{y \in \mathcal{S} \\ |y-x| \leq E}} [f(y) - f(x)] \frac{A^{t_0^{1/\alpha}}(x,y)}{t_0^{d/\alpha}|x-y|^{d+\alpha}}.$$

Define

$$\mathcal{B}f(x) = \sum_{\substack{y \in \mathcal{S} \\ |y-x| > E}} [f(y) - f(x)] \frac{A^{t_0^{1/\alpha}}(x,y)}{t_0^{d/\alpha}|x-y|^{d+\alpha}}$$

and $\|f\|_1 = \sum_{\mathcal{S}} |f(y)|$.

Proposition 4.4. *There exists c_1 such that*

$$\|Q_t f\|_1 \le c_1 \|f\|_1, \qquad \|Q_t f\|_\infty \le \|f\|_\infty. \tag{4.2}$$

Also

$$\|\mathcal{B} f\|_1 \le \frac{c_1}{E^\alpha}\|f\|_1, \qquad \|\mathcal{B} f\|_\infty \le \frac{c_1}{E^\alpha}\|f\|_\infty. \tag{4.3}$$

Proof. The second inequality in (4.2) follows because Q_t is a Markovian semigroup. Notice that $C_x Q_t(x,y)$ is symmetric in x, y. Then

$$\|Q_t f\|_1 \le \sum_x \sum_y Q_t(x,y)|f(y)| = \sum_y |f(y)| \sum_x Q_t(x,y) \le c_2 \sum_y |f(y)|$$

because $\sum_x Q_t(x,y) = \sum_x \frac{C_y}{C_x} Q_t(y,x) \le c_2 \sum_x Q_t(y,x) = c_2$. This establishes the first inequality.

Note

$$\sum_{\substack{y\in\mathcal{S}\\ |y-x|>E}} \frac{A^{t_0^{1/\alpha}}(x,y)}{t_0^{d/\alpha}|x-y|^{d+\alpha}} \le c_3 E^{-\alpha}. \tag{4.4}$$

Then

$$|\mathcal{B} f(x)| \le 2\|f\|_\infty \sum_{\substack{y\in\mathcal{S}\\ |y-x|>E}} \frac{A^{t_0^{1/\alpha}}(x,y)}{t_0^{d/\alpha}|x-y|^{d+\alpha}} \le 2c_3 E^{-\alpha}\|f\|_\infty.$$

To get the first inequality in (4.3),

$$\begin{aligned}\sum_x |\mathcal{B} f(x)| &\le \sum_x \sum_{|y-x|>E} |f(y)| \frac{A^{t_0^{1/\alpha}}(x,y)}{t_0^{d/\alpha}|x-y|^{d+\alpha}} + \sum_x |f(x)| \sum_{|y-x|>E} \frac{A^{t_0^{1/\alpha}}(x,y)}{t_0^{d/\alpha}|x-y|^{d+\alpha}}\\ &\le \sum_y |f(y)| \sum_{|x-y|>E} \frac{A^{t_0^{1/\alpha}}(x,y)}{t_0^{d/\alpha}|x-y|^{d+\alpha}} + c_3 E^{-\alpha} \sum_x |f(x)|.\end{aligned}$$

Applying (4.4) completes the proof. □

Let K be the smallest integer larger than $2(d+\alpha)/\alpha$ and let

$$A_n = D^{(1/2)+(n/4K)}.$$

Let us say that a function g is in $\mathcal{L}(n,\eta)$ if

$$|g(z)| \le \eta\Big[\frac{1}{D^{d+\alpha}} + \frac{1}{|z-y|^{d+\alpha}} 1_{B(y,A_n)^c}(z) + H(z)\Big]$$

for all z, where H is a nonnegative function supported in $B(y, A_n)$ with $\|H\|_1 + \|H\|_\infty \le 1$.

Lemma 4.5. *Suppose $D^{1/(4K)} \geq 4$ and $n \leq K$. There exists c_1 such that if $g \in \mathcal{L}(n,\eta)$, then*

(a) $\mathcal{B}g \in \mathcal{L}(n+1, c_1\eta)$;

(b) for each $s \leq 1$, $Q_s g \in \mathcal{L}(n+1, c_1\eta)$.

Proof. In view of (4.2) and (4.3), $\|\mathcal{B}(D^{-d+\alpha})\|_\infty \leq c_2 D^{-d+\alpha}$ and the same bound holds when $\mathcal{B}$ is replaced by Q_t.

Next, set

$$v(z) = \frac{1}{|z-y|^{d+\alpha}} 1_{B(y,A_n)^c}(z).$$

Note $\|v\|_1 + \|v\|_\infty \leq c_3$, where c_3 does not depend on n or D. Let

$$J_0(z) = |\mathcal{B}(v+H)(z)| 1_{B(y,A_{n+1})}(z)$$

and $J(z) = J_0(z)/(\|J_0\|_1 + \|J_0\|_\infty)$. Because of (4.2), we see that J_0 has L^1 and L^∞ norms bounded by a constant, so J is a nonnegative function supported on $B(y, A_{n+1})$ with $\|J\|_1 + \|J\|_\infty \leq 1$. The same argument serves for Q_t in place of $\mathcal{B}$.

It remains to get suitable bounds on $|\mathcal{B}v|$ and $Q_t v$ when $|z-y| \geq A_{n+1}$. We have

$$|\mathcal{B}v(z)| \leq \sum_{|w-z|>E} v(w) \frac{c_4}{|w-z|^{d+\alpha}} + \sum_{|w-z|>E} v(z) \frac{c_4}{|w-z|^{d+\alpha}}. \tag{4.5}$$

Clearly the second sum is bounded by $c_5 v(z)$ as required. We now consider the first sum. Let $C = \{w : |w-z| \geq |w-y|\}$. If $w \in C$, then $|w-z| \geq |y-z|/2$. Hence

$$\begin{aligned}\sum_{w\in C, |z-w|>E} & \frac{1}{|w-z|^{d+\alpha}} \frac{1}{|w-y|^{d+\alpha}} \\ & \leq c_6 \frac{1}{|y-z|^{d+\alpha}} \sum_{|w-y|>1} \frac{1}{|w-y|^{d+\alpha}} \\ & \leq \frac{c_7}{|y-z|^{d+\alpha}}.\end{aligned}$$

If $w \in C^c$, then $|w-y| \geq |y-z|/2$, and we get a similar bound. Combining gives the desired bound for (4.5).

Finally, we examine $Q_t v(z)$ when $z \in B(y, A_{n+1})^c$. We write

$$Q_t v(z) = \sum_{|z-w| \leq A_{n+1}/2} Q_t(z,w) v(w) + \sum_{|z-w| > A_{n+1}/2} Q_t(z,w) v(w). \tag{4.6}$$

If $|z-y| \geq A_{n+1}$ and $|z-w| \leq A_{n+1}/2$, then $|w-y| \geq |z-y|/2$. For such w, $v(w) \leq c_8/|z-y|^{d+\alpha}$, and hence the first sum in (4.6) is bounded by

$$\frac{c_8}{|z-y|^{d+\alpha}} \sum_w Q_t(z,w) = \frac{c_8}{|z-y|^{d+\alpha}}.$$

For $|z-w| > A_{n+1}/2$, v is bounded, and the second sum in (4.6) is less than or equal to

$$\sum_{|z-w|>A_{n+1}/2} Q_t(z,w) \le \mathbb{P}^z(|V_t - z| \ge A_{n+1}/2) \le c_9 e^{-c_{10}(A_{n+1}/2A_n)}$$

using Proposition 2.5. This is less than

$$c_{11}\Big(\frac{A_n}{A_{n+1}}\Big)^{8K^2} \le c_{12} D^{-d-\alpha}.$$

Combining the estimates proves the lemma. □

Proposition 4.6. *There exists c_1 such that $\mathbb{P}^x(Y_t \in B(y,1)) \le c_1/|x-y|^{d+\alpha}$.*

Proof. Let $D = |x-y|$. Assume first that $D \ge D_0$, where $D_0 = 4^{4K}$. Let $f = 1_{B(y,1)}$. Clearly there exists η such that $f \in \mathcal{L}(1,\eta)$. Then $Q_t f \in \mathcal{L}(2, c_2\eta)$ for all $t \le 1$ by Lemma 4.5. Set $S_0(t) = Q_t$ and $S_1(t) = \int_0^t Q_s \mathcal{B} Q_{t-s} ds$. Since $Q_1 f \in \mathcal{L}(2, c_2\eta)$ and $|x-y| = D > A_2$ we have

$$|S_0(1) f(x)| \le c_3 |x-y|^{-d-\alpha}.$$

By Lemma 4.5, for each $s \le t \le 1$, $Q_s \mathcal{B} Q_{t-s} f \in \mathcal{L}(4, c_2^3\eta)$. Hence $|Q_s \mathcal{B} Q_{t-s} f(x)| \le c_4 D^{-d-\alpha}$. Integrating over $s \le t$, we have

$$|S_1(t) f(x)| \le c_4 D^{-d-\alpha}.$$

Set $S_2(t) = \int_0^t S_1(s) \mathcal{B} Q_{t-s} ds = \int_0^t \int_0^s Q_r \mathcal{B} Q_{s-r} \mathcal{B} Q_{t-s} dr\, ds$. By Lemma 4.5 we see that $Q_r \mathcal{B} Q_{s-r} \mathcal{B} Q_{t-s} f \in \mathcal{L}(6, c_2^5\eta)$ and therefore $|Q_r \mathcal{B} Q_{s-r} \mathcal{B} Q_{t-s} f(x)| \le c_6 D^{-d-\alpha}$. Integrating over r and s, we have

$$|S_2(t) f(x)| \le c_6 D^{-d-\alpha}.$$

We continue in this fashion and find that for all $n \le K$ we have

$$|S_n(1) f(x)| \le c_7(n) D^{-d-\alpha}.$$

On the other hand, by Proposition 4.4

$$\|\mathcal{B}\|_\infty \le c_8/E^\alpha.$$

Take D_0 larger if necessary so that $c_8 D_0^{-\alpha/2} < \frac{1}{2}$. If $D \ge D_0$, we have by the argument of Proposition 2.6 that

$$\|S_n(1) f\|_\infty \le (c_8/E^\alpha)^n.$$

Consequently,

$$\sum_{n=K}^{\infty} |S_n f(x)| \le c_9/E^{\alpha K} \le c_{10} D^{-d-\alpha}.$$

If we set $P_t = \sum_{n=0}^{\infty} S_n(t)$, we then have

$$|P_1 f(x)| \le \Big(\sum_{n=0}^{K} c_7(n) + c_{10}\Big) D^{-d-\alpha} = c_{11} D^{-d-\alpha}.$$

This is precisely what we wanted to show because by [Le], P_t is the semigroup corresponding to W_t.

This proves the result for $D \ge D_0$. For $D < D_0$ we have our result by taking c_1 larger if necessary. □

From the probabilities of being in a set for Y_t we can obtain hitting probabilities.

Proposition 4.7. *There exist c_1 and c_2 such that*

$$\mathbb{P}^x(Y_t \text{ hits } B(y, c_1 t_0^{1/\alpha}) \text{ before time } t_0) \le c_2 \Big(\frac{t_0^{1/\alpha}}{|x-y|}\Big)^{d+\alpha}.$$

Proof. There is nothing to prove unless $|x-y|/t_0^{1/\alpha}$ is large. Let $D = |x-y|$ and let A be the event that Y_t hits $B(y, t_0^{1/\alpha})$ before time t_0. Let C be the event that $\sup_{s \le t_0} |Y_s - Y_0| \le c_3 t_0^{1/\alpha}$. From Theorem 2.8, $\mathbb{P}^z(C) \ge \frac{1}{2}$ if c_3 is large enough. By the strong Markov property,

$$\mathbb{P}^x(Y_{t_0} \in B(y, (1+c_3) t_0^{1/\alpha})) \ge \mathbb{E}^x[\mathbb{P}^{Y_S}(C); A] \ge \tfrac{1}{2}\mathbb{P}^x(A),$$

where $S = \inf\{t : Y_t \in B(y, t_0^{1/\alpha})\}$. We can cover $B(y, (1+c_3) t_0^{1/\alpha})$ by a finite number of balls of the form $B(z, t_0^{1/\alpha})$, where the number M of balls depends only on c_3 and the dimension d. Then by Proposition 4.6, the left hand side is bounded by $c_4 M (t_0^{1/\alpha}/D)^{d+\alpha}$.
□

We now get the corresponding result for X_n. We suppose that Y_t is constructed in terms of X_n and stopping times T_n as in Section 2.

Proposition 4.8. *There exist c_1 and c_2 such that*

$$\mathbb{P}^x(X_n \text{ hits } B(y, c_1 n_0^{1/\alpha}) \text{ before time } n_0) \le c_2 \Big(\frac{n_0^{1/\alpha}}{|x-y|}\Big)^{d+\alpha}.$$

Proof. Let A be the event that X_n hits $B(y, n_0^{1/\alpha})$ before time n_0, C the event that Y_t hits $B(y, n_0^{1/\alpha})$ before time $2n_0$, and D the event that $T_{n_0} \le 2n_0$. By the independence of A and D, we have

$$\mathbb{P}^x(A)\mathbb{P}(D) = \mathbb{P}^x(A \cap D) \le \mathbb{P}^x(C).$$

Using the bound on $\mathbb{P}^x(C)$ from Proposition 4.7 and the fact that $\mathbb{P}(D) > c_2$, where c_2 does not depend on n_0, proves the proposition. $\square$

We now come to the main result of this section.

Theorem 4.9. *There exists c_1 such that*

$$p(n,x,y) \le c_1\Big(n^{-d/\alpha} \wedge \frac{n}{|x-y|^{d+\alpha}}\Big).$$

Proof. Let $D = |x-y|$. Fix c_2 sufficiently large. If $D \le c_2 n^{1/\alpha}$, the result follows from Theorem 4.3. So suppose $D > c_2 n^{1/\alpha}$. Let $m = n + [\gamma n]$. By Proposition 4.8,

$$\mathbb{P}^x(X_m \in B(y, m^{1/\alpha})) \le c_3 \frac{m^{1+d/\alpha}}{D^{d+\alpha}}.$$

On the other hand, the left hand side is $\sum_{z\in B(y,m^{1/\alpha})} p(m,x,z)$. So for at least one $z \in B(y, m^{1/\alpha})$, we have $p(m,x,z) \le c_4 m/D^{d+\alpha} \le c_5 n/D^{d+\alpha}$. Let

$$q(k,w) = p(n+[\gamma n]-k, w, x)\,.$$

By Proposition 3.6, q is parabolic in $\{0,1,\ldots,[\gamma n]\} \times \mathbb{Z}^d$, and we have shown that

$$\min_{w\in B(z,n^{1/\alpha})} q(0,w) \le c_5 n/D^{d+\alpha}.$$

Thus by Theorem 3.1 we have

$$p(n,x,y) = \frac{C_y}{C_x} p(n,y,x) = \frac{C_y}{C_x} q([\gamma n], y) \le c_6 n/D^{d+\alpha}.$$

$\square$

5. Lower bounds.

Lower bounds are considerably easier to prove.

Proposition 5.1. *There exist c_1 ands c_2 such that if $|x-y| \le c_1 n^{1/\alpha}$ and $n \ge 2$, then*

$$p(n,x,y) \ge c_2 n^{-d/\alpha}.$$

Proof. Let $m = n - [\gamma n]$. By Theorem 2.8 there exists c_3 not depending on x or m such that

$$\mathbb{P}^x(\max_{k\le m} |X_k - x| > c_3 m^{1/\alpha}) \le \frac{1}{2}.$$

By Theorem 4.9 provided m is sufficiently large, there exists $c_4 < c_3/2$ not depending on x or m such that

$$\mathbb{P}^x(X_m \in B(x, c_4 m^{1/\alpha})) \le \frac{1}{4}.$$

Let $E = B(x, c_3 m^{1/\alpha}) - B(x, c_4 m^{1/\alpha})$. Therefore

$$\mathbb{P}^x(X_m \in E) \ge \frac{1}{4}.$$

This implies, since $\mathbb{P}^x(X_m \in E) = \sum_{z\in E} p(m, x, z)$, that for some $z \in E$ we have $p(m, x, z) \ge c_5 m^{-d-\alpha} \ge c_6 n^{-d-\alpha}$. If $w \in E$, then by Theorem 3.1 with $q(k, \cdot) = p(n-k, x, \cdot)$, we have

$$p(n, x, w) \ge c_7 n^{-d/\alpha}.$$

This proves our proposition when n is greater than some n_1.

By (1.2) and (2.1) it is easy to see that there exists c_8 such that

$$p(2, x, x) \ge c_8, \qquad p(3, x, x) \ge c_8.$$

If $n \le n_1$ and $n = 2\ell + 1$ is odd,

$$p(n, x, y) \ge p(2, x, x)^\ell p(1, x, y) \ge c_8^\ell n^{-d/\alpha}.$$

The case when $n \le b_1$ and n is even is done similarly. □

Theorem 5.2. *There exists c_1 such that if $n \ge 2$*

$$p(n, x, y) \ge c_1 \Big(n^{-d/\alpha} \wedge \frac{n}{|x-y|^{d+\alpha}} \Big).$$

Proof. Again, our result follows for small n as a consequence of (1.2) and (2.1), so we may suppose n is larger than some n_1. In view of Proposition 5.1 we may suppose $|x - y| \ge c_2 n^{1/\alpha}$. Let $A = B(y, n^{1/\alpha})$. Let $N(z) = \mathbb{P}^z(X_1 \in A)$ if $z \notin A$ and 0 otherwise. For $z \in B(x, n^{1/\alpha})$ note $N(z) \ge c_3 n^{d/\alpha}/D^{d+\alpha}$. As in the proof of Lemma 3.2,

$$1_A(X_{j\wedge T_A}) - 1_A(X_0) - \sum_{i=1}^{j\wedge T_A} N(X_i)$$

is a martingale. By optional stopping at the time $S = n \wedge \tau_{B(x,n^{1/\alpha})}$ we have

$$\mathbb{P}^x(X_S \in A) = \mathbb{E}^x \sum_{i=1}^{S} N(X_i) \ge \frac{c_3 n^{d/\alpha}}{D^{d+\alpha}} \mathbb{E}^x[S - 1].$$

Arguing as in (3.3), $\mathbb{E}^x S \geq c_4 n$. We conclude that

$$\mathbb{P}^x(X_n \text{ hits } B(y, n^{1/\alpha}) \text{ before time } n) \geq c_5 n^{1+d/\alpha}/D^{d+\alpha}.$$

By Theorem 2.8, starting at $z \in B(y, n^{1/\alpha})$, there is positive probability not depending on z or n such that the chain does not move more than $c_6 n^{1/\alpha}$ in time n. Hence by the strong Markov property, there is probability at least $c_7 n^{1+d/\alpha}/D^{d+\alpha}$ that $X_n \in B(y, c_8 n^{1/\alpha})$. Let $m = n - [\gamma n]$ Applying the above with m in place of n,

$$\mathbb{P}^x(X_m \in B(y, c_9 n^{1/\alpha})) \geq c_{10} \frac{m^{1+d/\alpha}}{D^{d+\alpha}} \geq c_{11} \frac{n^{1+d/\alpha}}{D^{d+\alpha}}.$$

However this is also $\sum_{w \in B(y, c_9 n^{1/\alpha})} p(m, x, w)$. So there must exist $w \in B(y, c_9 n^{1/\alpha})$ such that $p(m, x, w) \geq c_{12} n/D^{d+\alpha}$. A use of Theorem 3.1 as in the proof of Theorem 5.1 finishes the current proof. □

Proof of Theorem 1.1. This is a combination of Theorems 4.9 and 5.2. If $n = 1$ and $x \neq y$, the result follows from (1.2) and (2.1). □

Remark 5.3. Similar (but a bit easier) arguments show that the transition probabilities for Y_t satisfy

$$c_1 \Big(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \Big) \leq q_Y(t, x, y) \leq c_2 \Big(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \Big). \tag{5.1}$$

One can also show that the transition densities of the process U_t described in Section 1 also satisfy bounds of the form (5.1). One can either modify the proofs suitably or else approximate U_t by a sequence of processes of the form Y_t but with state space $\varepsilon \mathbb{Z}^d$, and then let $\varepsilon \to 0$.

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