TRANSITIONS AND STABILITY IN THE NONLINEAR BUCKLING OF ELASTIC PLATES*

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Part I. Introduction. In 1910, von Kármán (1) developed a nonlinear theory for elastic plates by including the squares of the slopes of the deflected middle surface which were neglected in the linear theory. A time-dependent version of the equations, which is obtained by including inertia and damping terms, is studied here. Specifically, we consider the buckling problem, with forces applied normally at the edges. For this problem, the plane state, $w \equiv 0$, is a solution of the static (time-independent) problem for all values of the horizontal thrust λ. It is the unique solution if λ is less than some critical value λ_c . For thrusts beyond λ_c the static problem admits additional solutions (buckled states) which bifurcate from $w \equiv 0$. Indeed, for thrusts beyond λ_c the plane state $w \equiv 0$ becomes unstable and the plate buckles. We consider the dynamic (time-dependent) problem to determine the stability of the static buckled states, the transitions between these and their dependence on initial data. Our method is due to Matkowsky [2, 3] and our approach to this problem will follow closely the work of Reiss and Matkowsky [4] who considered the nonlinear dynamic problem for buckled columns subjected to axial compression.

We consider the problem of a simply supported rectangular plate subjected to a constant edge thrust λ. We obtain a formal asymptotic representation of the solution by a two time method [2, 3]. One time scale, t, describes the initial behavior and the second, of "slow" time $\theta = \epsilon t$ ($\epsilon \ll 1$), is required to describe the long-time large-amplitude response.

The response of the plate is approximated by the leading term in the asymptotic expansion. This consists of a fast-time high-frequency secondary motion superposed on a primary motion that depends only on θ. The primary motion depends strongly on the initial data. For the undamped plate, the primary motion is periodic and may be either a polarized oscillation about one of the two static buckled states or a swaying oscillation between the two static buckled states. When damping is present, the plate always approaches one of the static buckled states as $t \to \infty$. It may sway back and

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forth a finite number of times before the oscillations finally polarize about and damp to one of the static buckled states.

We also consider the effect of larger damping. We use the same technique as described in the preceding paragraph, but the longer-time large-amplitude time scale is now given by $\theta = \epsilon^2 t$. We now find that the primary motion monotonically approaches one of the static buckled states, in contrast to the oscillations that occur for small damping.

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2. Formulation. We consider a simply-supported rectangular plate whose middle surface coincides with the X, Y plane. The undeformed plate occupies the region $-h/2 \le Z \le h/2$, $0 \le X \le a$, $0 \le Y \le b$. It is assumed that the plate is thin, i.e. $h \ll a$ or b. The deformation is caused by applying a constant compressive force H normally at the edges $X = 0, a$.

The deformations of the plate are assumed to be described by a time-dependent form of the von Karman equations. These are two coupled nonlinear fourth-order partial differential equations. Using subscripts to denote partial derivatives, we give a dimensionless form of the equations as

$$\Delta^2 w + \lambda w_{xx} = [f, w] - w_{tt} - \Gamma w_t \,, 0 < x < l, 0 < y < 1, t > 0, \tag{2.1a}$$

$$\Delta^2 f = -\tfrac{1}{2}[w, w], 0 < x < l, 0 < y < 1, t > 0, \tag{2.1b}$$

$$w = \Delta w = 0, x = 0, l, 0 < y < 1, t > 0, y = 0, 1, 0 < x < l, t > 0, \tag{2.1c}$$

$$\begin{aligned} w(x, y, 0) &= G(x, y, \epsilon), \qquad 0 \le x \le l, \\ w_t(x, y, 0) &= J(x, y, \epsilon), \qquad 0 \le y \le 1, \end{aligned} \tag{2.1d}$$

$$f = \Delta f = 0, x = 0, l, 0 < y < 1, t > 0, y = 0, 1, 0 < x < l, t > 0. \tag{2.1e}$$

The symbol [,] is defined as

$$[F, g] \equiv F_{xx} g_{yy} + F_{yy} g_{xx} - 2F_{xy} g_{xy} \,, \tag{2.2}$$

$$l = a/b. \tag{2.3}$$

The parameter λ is called the load and Γ is a coefficient of damping. The initial data are assumed to satisfy the boundary conditions (2.1c), and the parameter ϵ is a measure of the size of the initial data.

The solution of this problem is unique, as was proved in [5] by the method of energy integrals.

3. Static theory. The linearized static problem yields the critical loads for which bifurcation may occur. We confine ourselves to a statement of the results. For all values of λ,

$$w \equiv 0, \quad f \equiv 0 \tag{3.1}$$

is a solution to the linear static problem and is the only solution if

$$\lambda \ne \lambda_{mn} = \left(\frac{\Pi}{l}\right)^2 \left(m + \frac{n^2 l^2}{m}\right)^2, \qquad m, n = 1, 2, \cdots . \tag{3.2}$$

If $\lambda = \lambda_{mn}$, the problem also has the solutions

$$w(x, y) = w_{mn}(x, y) = A_{mn}\phi_{m,n}(x, y) \tag{3.3}$$

where

$$\phi_{m,n}(x, y) = \frac{2}{l^{1/2}} \sin \frac{m\Pi x}{l} \sin n\Pi y, \qquad m, n = 1, 2, \cdots \tag{3.4}$$

and the A_{mn} are arbitrary constants. The quantities λ_{mn} are called the critical thrusts. We define

$$\begin{aligned} \lambda_c(l) &\equiv \min_{m,n} \lambda_{mn} \\ &= \frac{\Pi^2}{l^2}(1 + l^2)^2 \quad \text{if} \quad l \leq 1, \\ &= \frac{\Pi^2}{l^2}\left(M + \frac{l^2}{M}\right)^2 \quad \text{if} \quad M \leq l < M + 1 \quad \text{and} \quad l^2 < M(M+1), \\ &= \frac{\Pi^2}{l^2}\left((M+1) + \frac{l^2}{M+1}\right)^2 \quad \text{if} \quad M < l \leq M + 1 \quad \text{and} \quad l^2 > M(M+1), \\ &= \frac{\Pi^2}{l^2}\left(M + \frac{l^2}{M}\right)^2 = \frac{\Pi^2}{l^2}\left((M+1) + \frac{l^2}{(M+1)}\right)^2 \quad \text{if} \quad l^2 = M(M+1), \end{aligned} \tag{3.5}$$

where M is a positive integer.

Thus, we observe that the smallest eigenvalue is simple unless $l^2 = M(M + 1)$; in that case, it has multiplicity two. In this paper, we assume that $\lambda_c(l)$ is a simple eigenvalue. The case of the multiple eigenvalue is treated in [6].

The nonlinear static problem is treated because it yields the possible shapes of the buckled states for $\lambda > \lambda_c$. Employing perturbation theory, which is essentially the method of Lindstedt and Poincaré, we find that

$$w = \epsilon\phi_{i,1}(x, y) + O(\epsilon^3), \tag{3.6}$$

$$f = \epsilon^2 \sum_{m,n=1}^{\infty} \alpha_{mn}A_{mni}\phi_{m,n}(x, y) + O(\epsilon^3), \tag{3.7}$$

$$\lambda = \lambda_c(l) + \epsilon^2 \frac{l^2}{i^2\Pi^2} \sum_{m,n=1}^{\infty} 2\alpha_{mn}A_{mni}^2 + O(\epsilon^3), \tag{3.8}$$

where

$$\alpha_{mn} \equiv [(\Pi^2/l^2)^2(m^2 + n^2l^2)^2]^{-1}, \tag{3.9}$$

$$\begin{aligned} A_{mni} &= (-\tfrac{1}{2}[\phi_{i,1}, \phi_{i,1}], \phi_{m,n}) \\ &= 0 \quad \text{if} \quad m \quad \text{or} \quad n \quad \text{is even}, \\ &= \frac{32i^2\Pi^2}{l^{5/2}}\left[\frac{m^2n^2 - 2n^2 - 2m^2i^2}{mn(m^2 - 4i^2)(n^2 - 4)}\right] \quad \text{if} \quad m \quad \text{and} \quad n \quad \text{are both odd}, \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} i &= 1 && \text{if} \quad l \le 1, \\ &= M && \text{if} \quad M \le l < M + 1 \quad \text{and} \quad l^2 < M(M+1), \\ &= M + 1 && \text{if} \quad M < l \le M + 1 \quad \text{and} \quad l^2 > M(M+1), \end{aligned} \tag{3.11}$$

where M is a positive integer. The inner product (ϕ, ψ) is defined by

$$(\phi, \psi) = \int_0^1 \int_0^l \phi\psi \, dx \, dy \tag{3.12}$$

and the symbol [,] is defined by (2.2). The calculations which led to the above results may be found in [5].

4. Nonlinear dynamic theory. We consider the nonlinear dynamic problem with small but finite-amplitude data and with λ slightly greater than $\lambda_c(l)$. Thus we define a small parameter ϵ by

$$\lambda - \lambda_c(l) \equiv \epsilon^2. \tag{4.1a}$$

The parameter ϵ is thus a measure of the nearness of λ to λ_c. We assume that

$$w(x, y, 0) = G(x, y, \epsilon) \sim \sum_{j=1}^{\infty} G_j(x, y)\epsilon^j, \tag{4.1b}$$

$$w_t(x, y, 0) = J(x, y, \epsilon) \sim \sum_{j=1}^{\infty} J_j(x, y)\epsilon^j. \tag{4.1c}$$

We also assume that $G_j(x, y)$ and $J_j(x, y)$ can be expanded in a double Fourier sine series with a finite number of terms. That is,

$$G_j(x, y) = \sum_{m,n=1}^{N_j} G_{jmn}\phi_{m,n}(x, y) \qquad j = 1, 2, \cdots, \tag{4.1d}$$

$$J_j(x, y) = \sum_{m,n=1}^{N_j} J_{jmn}\phi_{m,n}(x, y) \qquad j = 1, 2, \cdots. \tag{4.1e}$$

Such data is termed quiescent. We assume that the damping is small, i.e.

$$\Gamma = \epsilon\gamma, \tag{4.1f}$$

where γ is independent of ϵ. The case $\Gamma = O(1)$ is discussed later.

We hypothesize that the solution depends essentially on two time scales. One scale, which we designate by t, describes the small-time behavior. The second scale, designated by θ describes the longer-time large-amplitude response. For this reason, we define the auxiliary time θ by

$$\theta \equiv \epsilon t. \tag{4.2}$$

We seek a formal asymptotic solution of the nonlinear dynamic problem, valid as ϵ approaches zero, in the form:

$$w(x, y, t, \epsilon) \sim \sum_{j=1}^{\infty} w^{(j)}(x, y, t, \theta)\epsilon^j, \tag{4.3a}$$

$$f(x, y, t, \epsilon) \sim \sum_{j=1}^{\infty} f^{(j)}(x, y, t, \theta)\epsilon^j. \tag{4.3b}$$

The expansions are formal since we do not prove the asymptotic nature of the solutions. The expansion coefficients $w^{(j)}, f^{(j)}, j = 1, 2, 3, \cdots$, are assumed to be bounded functions of all their arguments. They are determined by substituting (4.1)–(4.3) into (2.1) and equating coefficients of the same power of ϵ. Thus, we obtain for $j = 1, 2 \cdots$

$$Lw^{(j)} = -w_{xx}{}^{(j-2)} + \sum_{n=1}^{j-1} [f^{(n)}, w^{(j-n)}] - 2w_{t\theta}{}^{(j-1)} \qquad (4.4a)$$
$$- w_{\theta\theta}{}^{(j-2)} - \gamma w_t{}^{(j-1)} - \gamma w_\theta{}^{(j-2)} \equiv s_j\,, \quad 0 < x < l, \quad 0 < x < l, \quad t > 0,$$

$$\Delta^2 f^{(j)} = -\frac{1}{2} \sum_{n=1}^{j-1} [w^{(n)}, w^{(j-n)}] \equiv q_j\,, \quad 0 < x < l, \quad 0 < y < 1, \quad t > 0, \qquad (4.4b)$$

$$w^{(j)} = \Delta w^{(j)} = f^{(j)} = \Delta f^{(j)} = 0,$$
$$x = 0, l, \quad 0 < y < 1, \quad t > 0, \quad y = 0, 1, \quad 0 < x < l, \quad t > 0, \qquad (4.4c)$$

$$w^{(j)}(x, y, 0, 0) = \sum_{m,n=1}^{N_j} G_{jmn}\phi_{m,n}(x, y), \qquad j = 1, 2, 3, \cdots, \qquad (4.4d)$$

$$w_t{}^{(1)}(x, y, 0, 0) = \sum_{m,n=1}^{N_j} J_{1mn}\phi_{m,n}(x, y), \qquad (4.4e)$$

$$w_t{}^{(j)}(x, y, 0, 0) + w_\theta{}^{(j-1)}(x, y, 0, 0) = \sum_{n=1}^{N_j} J_{jmn}\phi_{m,n}(x, y), \qquad j = 2, 3, \cdots. \qquad (4.4f)$$

The sums in (4.4a) and (4.4b) are defined to be zero if $j = 1$ and $w^{(j)} = f^{(j)} \equiv 0$ when $j \le 0$. The linear differential operator L is defined by

$$Lw^{(j)} \equiv \Delta^2 w^{(j)} + \lambda_c(l) w_{xx}{}^{(j)} + w_{tt}{}^{(j)}. \qquad (4.5)$$

To determine $w^{(1)}$ and $f^{(1)}$, we set $j = 1$ in (4.4). Then

$$w^{(1)}(x, y, t, \theta) = \sum_{m,n=1}^{\infty} A_{1mn}(t, \theta)\phi_{m,n}(x, y), \qquad (4.6a)$$

where for each $m, n = 1, 2 \cdots$, the coefficients A_{1mn} satisfy

$$(A_{1mn})_{tt} + \frac{m^2\Pi^2}{l^2}(\lambda_{mn} - \lambda_c(l))A_{1mn} = 0, \qquad (4.6b)$$

$$A_{1mn}(0, 0) = G_{1mn}\,, \qquad (A_{1mn})_t(0, 0) = J_{1mn}\,. \qquad (4.6c)$$

Thus, we find that

$$A_{1mn} = a_{1i1}(\theta)t + b_{1i1}(\theta)$$
$$= a_{1mn}(\theta) \sin \omega_{mn} t + b_{1mn}(\theta) \cos \omega_{mn} t \qquad (4.7a)$$

if $m \ne i$ and $n \ne 1$ simultaneously, where

$$\omega_{mn}{}^2 = \frac{m^2\Pi^2}{l^2}(\lambda_{mn} - \lambda_c(l)), \qquad m \ne i, \quad n \ne 1, \qquad (4.7b)$$

and i is defined in (3.11). The functions $a_{1mn}(\theta)$, $b_{1mn}(\theta)$, $m, n = 1, 2 \cdots$ are to be determined. Since $w^{(1)}$ is to be a bounded function of all its arguments, we conclude that $a_{1i1}(\theta) = 0$. This latter condition requires us to make a restriction on our initial data, namely

$$g_{1i1} = 0. \tag{4.7c}$$

We define

$$b(\theta) \equiv b_{1i1}(\theta) \tag{4.8a}$$

and we have

$$b(0) = G_{1i1} . \tag{4.8b}$$

The other initial conditions are

$$\begin{aligned} a_{1mn}(0) &= J_{1mn}/\omega_{mn} && \text{if} \quad 1 \le m \le N_1 , \quad 1 \le n \le N_1 , \\ &= 0 && \text{if} \quad m > N_1 , \quad n > N_1 , \\ b_{1mn}(0) &= G_{1mn} && \text{if} \quad 1 \le m \le N_1 , \quad 1 \le n \le N_1 , \\ &= 0 && \text{if} \quad m > N_1 , \quad n > N_1 . \end{aligned} \tag{4.9}$$

The solution $f^{(1)}$ of (4.4b, c) with $j = 1$ is given by

$$f^{(1)} = 0. \tag{4.10}$$

We note that (4.4a) with $j \ge 2$ is an inhomogeneous form of (4.4a) with $j = 1$. A necessary condition for the existence of a bounded solution for $j \ge 2$ is that the functions s_j satisfy the orthogonality condition

$$\{s_j , w^{(1)}\} \equiv \lim_{T \to \infty} \frac{1}{T} \int_0^T \int_0^l \int_0^1 s_j w^{(1)} \, dy \, dx \, dt = 0, \qquad j = 2, 3 \cdots . \tag{4.11}$$

To determine the second term in the expansion for w we set $j = 2$ in (4.4a), (4.4c), (4.4d) and (4.4e) with $f^{(1)}$ given by (4.10). Substituting $w^{(1)}$, which is given by (4.6a) and (4.7a), in the expression for s_2 , we find that

$$s_2 = \sum_{m,n=1}^{\infty} \omega_{mn}[-(2(a_{1mn})_\theta + \gamma a_{1mn}) \cos \omega_{mn} t + (2(b_{1mn})_\theta + \gamma b_{1mn}) \sin \omega_{mn} t]\phi_{m,n} . \tag{4.12}$$

Then the orthogonality condition (4.11) implies that

$$\begin{aligned} (a_{1mn})_\theta + \tfrac{1}{2}\gamma a_{1mn} &= 0, \\ (b_{1mn})_\theta + \tfrac{1}{2}\gamma b_{1mn} &= 0, \qquad m \ne i \text{ and } n \ne 1 \text{ simultaneously.} \end{aligned} \tag{4.13a}$$

Solving these equations and using the initial conditions, we obtain

$$w^{(1)} = b(\theta)\phi_{i,1}(x, y) + \sum_{m,n=1}^{N_1} \gamma_{mn}(\theta, t)\phi_{m,n}(xy) \tag{4.13b}$$

where

$$\gamma_{mn}(\theta, t) \equiv \exp\left(-\tfrac{1}{2}\gamma\theta\right)\left\{\frac{J_{1mn}}{\omega_{mn}} \sin \omega_{mn} t + G_{1mn} \cos \omega_{mn} t\right\} \tag{4.13c}$$

with

$$b(0) = G_{1i1} , \quad b_\theta(0) = J_{2i1} . \tag{4.13d}$$

Then $w^{(2)}$ is given by

$$w^{(2)} = b_{2i1}(\theta)\phi_{i,1}(x, y) + \sum_{m,n=1}^{\infty} [a_{2mn}(\theta) \sin \omega_{mn}t + b_{2mn}(\theta) \cos \omega_{mn}t]\phi_{m,n}(x, y). \tag{4.14}$$

Note that $w^{(1)}$ is still not completely determined since we do not know $b(\theta)$. To find the equation satisfied by $b(\theta)$, we consider (4.4a) with $j = 3$. Since $f^{(2)}$ appears in that equation, we first solve for it. Setting $j = 2$ in (4.4b) and (4.4d), with $w^{(1)}$ given by (4.13b), we solve for $f^{(2)}$ as

$$\begin{aligned} f^{(2)} = {} & (b(\theta))^2 \sum_{k,r=1}^{\infty} P_{i1i1kr}\alpha_{kr}\phi_{k,r}(x, y) + 2b(\theta) \sum_{k,r=1}^{\infty} \sum_{m,n=1}^{N_1} \gamma_{mn}P_{mni1kr}\alpha_{kr}\phi_{k,r}(x, y) \\ & + \sum_{k,r=1}^{\infty} \sum_{\substack{m,n=1 \\ p,q=1}}^{N_1} P_{mnpqkr}\gamma_{mn}\gamma_{pq}\alpha_{kr}\phi_{k,r}(x, y) \end{aligned} \tag{4.15}$$

where α_{kr} is defined in (3.9), and

$$P_{mnpqkr} \equiv -\tfrac{1}{2}([\phi_{mn} , \phi_{p,q}], \phi_{k,r}). \tag{4.16}$$

The initial boundary-value problem for $w^{(3)}$ is obtained by considering (4.4a), (4.4c), (4.4d) and (4.4e) with $j = 3$ and with $f^{(1)}$ given by (4.10). The inhomogeneous term s_3 depends on $b(\theta)$, $a_{1mn}(\theta)$, $a_{2mn}(\theta)$, $b_{1mn}(\theta)$ and $b_{2mn}(\theta)$. Applying the orthogonality condition (4.11), we find that $b(\theta)$ must satisfy the nonlinear ordinary differential equation

$$\begin{aligned} b_{\theta\theta} + \gamma b_\theta + b\Bigg[& -\frac{i^2\Pi^2}{l^2} + \sum_{k,r=1}^{\infty} \sum_{m,n=1}^{N_1} \left(\frac{J_{1mn}}{\omega_{mn}}\right)^2 + (G_{1mn})^2 \\ & \cdot \exp(-\gamma\theta)\alpha_{kr}(2P_{mnmnkr}P_{i1i1kr} - 4P_{mni1kr}{}^2)\Bigg] + b^3 \sum_{k,r=1}^{\infty} 2\alpha_{kr}P_{i1i1kr}{}^2 = 0 \end{aligned} \tag{4.19a}$$

with initial conditions given by

$$b(0) = G_{1i1} , \quad b(0) = J_{2i1} . \tag{4.19b}$$

We refer to the differential equation (4.19a) as the amplitude equation and the initial-value problem (4.19) as the amplitude problem. When this problem is solved, $w^{(1)}$ is completely determined.

Applying the orthogonality condition (4.11) to the other terms in s_3 yields equations which determine $a_{2mn}(\theta)$ and $b_{2mn}(\theta)$. To evaluate $b_{2i1}(\theta)$ and thus completely determine $w^{(2)}$, it is necessary to go to the fourth term in the expansion of w. We will not present these results. Instead, we will study the approximation given by $w^{(1)}$.

We call the term $G_{1i1}\phi_{i,1}(x, y)$ in the initial data the primary data. The remaining terms are called the secondary data. The slow-time standing wave $b(\theta)\phi_{i,1}(x, y)$ is called the primary motion. The rest of $w^{(1)}$ will be called the secondary motion.

5. Amplitude problem. The amplitude problem (4.19) is similar to one analyzed in [4]. The two problems differ only in the coefficients of the equation and the initial data. Therefore the conclusions of [4] hold with the obvious modifications due to changed constants. Here we shall only summarize the results.

We first consider the amplitude problem with $\gamma = 0$ and monochromatic data

$$b_{\theta\theta} + K_i(-1 + \mu_i{}^2b^2)b = 0, \tag{5.1a}$$

$$b(0) = G_{1i1}\,, \quad b_\theta(0) = J_{2i1}\,, \tag{5.1b}$$

where

$$\mu_i^{\,2} \equiv \frac{l^2}{i^2\Pi^2} \sum_{k,r=1}^{\infty} 2\alpha_{kr}(P_{i1i1kr})^2, \qquad K_i \equiv \frac{i^2\Pi^2}{l^2} \tag{5.1c}$$

and i is given by (3.11). We find that there are three singular points in the phase plane (b, b_θ). They are

$$b = b_\theta = 0, \tag{5.2a}$$

$$b = 0, \qquad b_\theta = \pm\frac{1}{\mu_1}\,, \tag{5.2b}$$

which represent the unbuckled state and the two static buckled states respectively. The singular point (5.2a) is a saddle point and the singular points (5.2b) are centers. We define the constant H as

$$H \equiv J_{2i1}^{\,2} + K_i(-1 + \tfrac{1}{2}G_{1i1}^{\,2}\mu_i^{\,2})G_{1i1}^{\,2} \tag{5.3}$$

and note that the curve with $H = 0$, which is usually referred to as the separatrix, passes through the origin. If $H < 0$ then there are two branches of closed curves. Each curve contains one of the singular points (5.2b). Therefore for $H < 0$, $b(\theta)$ is a periodic function of one sign. The plate will then oscillate periodically about one of the static buckled states that branch from $\lambda_c(l)$. This is referred to as the polarized mode of vibration.

For each $H > 0$, we have a closed curve that contains the three singular points (5.2). Thus $b(\theta)$ is a periodic function that changes sign twice in each period. Therefore the plate sways between a neighborhood of each of the static buckled states that branch from $\lambda_c(l)$ and passes twice in each period through the unbuckled state. This is referred to as the swaying mode. If $H = 0$, then $\lim_{t\to\infty} w(x, y, t) = 0$.

Employing the definitions of strong (weak) nonlinear stability and nonlinear instability used in [4], we see that the unbuckled state is nonlinearly unstable when $H < 0$. For $H > 0$, the buckled state has weak nonlinear stability with respect to the data (5.1b). For $H = 0$, we see that the buckled state has strong nonlinear stability with respect to data (5.1b). It is thus obvious that when $\gamma = 0$ the nonlinear stability of the buckled state depends on the sign of H.

We now consider monochromatic data with $\gamma > 0$. The singular points are again given by (5.2). The origin is a saddle and the singularities (5.2b) are stable spirals (stable nodes) if $\gamma^2 - (B_i^{\,2}\Pi^2/l^2) < 0$ (≥ 0). For simplicity, we consider only stable spirals. For any initial data not on the separatrix, the solution is "captured" by one of the singular points (5.2b) when $t \to \infty$. Thus the unbuckled state is nonlinearly unstable. For $|G_{1i1}|$ and $|J_{2i1}|$ sufficiently small, the notion is polarized about one of the unbuckled states for all $t > 0$ and is captured by this state as $t \to \infty$. For $|G_{1i1}|$ and $|J_{2i1}|$ sufficiently large, the motion consists in a swaying between the two static buckled states for a finite time and then becomes damped to one of the static unbuckled states. The final state need not be the one nearest the initial point.

We next consider the case of nonmonochromatic data with zero damping. We define the quantity

$$\xi_i = -K_i + \sum_{k,r=1}^{\infty} \sum_{m,n=1}^{N_1} \left[\left(\frac{J_{1mn}}{\omega_{mn}}\right)^2 + (G_{1mn})^2\right](2P_{mnmnkr}P_{i1i1kr} - 4P_{mni1kr}{}^2). \tag{5.4}$$

If $\xi_i \geq 0$ then the origin $b = b_\theta = 0$ of the phase plane is the only singularity and it is a center. Under these circumstances $b(\theta)$ is a periodic function. Thus the primary motion is a periodic motion of the slow time θ about the unbuckled state. If ϵ and the amplitude of the secondary motion are sufficiently small, we conclude that the unbuckled state is nonlinearly stable with respect to the data (5.1b).

If $\xi_i \leq 0$, then the three singular points are

$$b = b_\theta = 0, \tag{5.5a}$$

$$b = \pm b_0 , \qquad V = 0, \qquad b_0 = (1/\mu_i)(\xi_i/K_i)^{1/2} \tag{5.5b}$$

The singular point (5.5a) is a saddle point and the singular points (5.5b) are centers. The origin corresponds to the unbuckled state and the points (5.5b) correspond to two buckled states.

If $|G_{1i1}| < (1/\mu_i)\{-2\xi_i/K_i\}^{1/2}$ the primary motion is a slow-time periodic standing wave. It is polarized about the static buckled state $b = b_0(-b_0)$ if $G_{1i1} > 0$ (< 0). According to our definitions, the unbuckled state is nonlinearly unstable. If $|G_{1i1}| > (1/\mu_i)\{-2\xi_i/K_I\}^{1/2}$, the primary motion is a slow-time periodic wave in the swaying mode. Thus if ϵ and the amplitude of the secondary motion are sufficiently small, the unbuckled state is nonlinearly stable with respect to that data. On the separatrix the primary motion monotonically approaches the unbuckled state as $\theta \to \infty$.

For arbitrary initial data and $\gamma > 0$, the secondary motion is a slowly damped superposition of fast-time periodic standing waves for θ sufficiently large and $\xi_i < 0$. Thus, for large θ, the qualitative behavior is similar to that discussed when $\gamma > 0$ with monochromatic data. Depending on the magnitude of G_{1i1}, the primary motion for small θ will be either in the polarized mode or the swaying mode. For sufficiently large θ, the primary motion will be in the polarized mode and $\lim_{\theta\to\infty} b(\theta) = \pm b_0$. The final state need not be the same as that attained in the monochromatic case. Thus, the unbuckled state is nonlinearly unstable for $\lambda > \lambda_c(l)$ and each of the buckled states branching from $\lambda_c(l)$ is also unstable even though these states are linearly stable, i.e. stable to infinitesimal perturbations. The static buckled states considered as a pair are dynamically stable for sufficiently small disturbances since the solution always approaches one of them as $t \to \infty$. It should be noted that the final state need not be the one nearest to its initial value.

6. The effect of greater damping. To see what the effect of damping is on the solution, we now consider the case when

$$\Gamma = 0(1). \tag{6.1}$$

Otherwise, the assumptions are the same as those of Sec. 4. We will see that this causes a qualitative change in the motion of the plate.

We again assume that the solution depends upon two time scales; however, the longer-time large-amplitude time scale θ is now defined by

$$\theta = \epsilon^2 t. \tag{6.2}$$

We seek a formal asymptotic solution of problem (2.1), valid as ϵ approaches zero, in the form (4.3a, b) with θ given by (6.2). Substituting (6.1), (6.2) and (4.3) into (2.1) and equating coefficients of each power of ϵ, we obtain a system of equations for the coefficients $w^{(i)}(x, y, t, \theta)$ and $f^{(i)}(x, y, t, \theta)$. We shall not present the calculations here and shall merely state some of the results. The calculations may be found in [5].

The leading term in the expansion of w is given by (4.6a) where now the coefficients A_{1mn} satisfy

$$(A_{1mn})_{tt} + \Gamma(A_{1mn})_t + \frac{m^2\Pi^2}{l^2}(\lambda_{mn} - \lambda_c(l))A_{1mn} = 0, \tag{6.3}$$

$$B_{lmn}(0, 0) = G_{lmn} \quad (B_{lmn})_t(0, 0) = J_{lmn}.$$

subject to the initial data (4.6c). Thus we find that

$$\begin{aligned} A_{1mn}(t, \theta) &= a_{i1}(\theta) \exp(-\Gamma t) + b_{i1}(\theta) \quad \text{if} \quad m = n = 1, \\ &= \exp(-\tfrac{1}{2}\Gamma t)(a_{1mn}(\theta) \sin \omega_{mn} t + b_{1mn}(\theta) \cos \omega_{mn} t), \end{aligned} \tag{6.4a}$$

$$a_{i1}(0) = (-J_{1i1}/\Gamma), \; b_{i1}(0) = G_{1i1} + (J_{1i1}/\Gamma), \tag{6.4b}$$

$$\begin{aligned} a_{1mn}(0) &= \frac{J_{1mn} + \frac{1}{2}\Gamma G_{1mn}}{\omega_{mn}} && \text{if} \quad m \neq i, n \neq 1, \quad m, n \leq N_1, \\ &= 0 && \text{if} \quad m, n > N_1, \\ b_{1mn}(0) &= G_{1mn} && \text{if} \quad m \neq i, n \neq 1, \quad m, n \leq N_1, \\ &= 0 && \text{if} \quad m, n > N_1, \end{aligned}$$

where ω_{mn} is defined in (4.7b) and i is defined in (3.11). The coefficients $a_{1mn}(\theta)$, $b_{1mn}(\theta)$, $a_{i1}(\theta)$ and $b_{i1}(\theta)$ are to be determined. Since all of these coefficients except $b_{i1}(\theta)$ are exponentially damped, it will be our principal task to determine $b_{i1}(\theta)$.

We find that $b_{i1}(\theta)$ satisfies

$$(b_{i1})_\theta - \frac{M^2\Pi^2}{l^2}(b_{i1} - \lambda_i^{(2)} b_{i1}^3) = 0 \tag{6.5}$$

where $\lambda_i^{(2)}$ is the $O(\epsilon^2)$ term in (3.8). The solution of this first-order ordinary differential equation subject to the initial conditions (6.4b) is

$$b_{i1}(\theta) = \frac{G_{1i1} \exp\left(\frac{M^2\Pi^2}{l^2\Gamma}\theta\right)}{|G_{1i1}| \left[\mu - \lambda_i^{(2)}\left(1 - \exp\left(\frac{2M^2\Pi^2}{l^2\Gamma}\theta\right)\right)\right]^{1/2}} \tag{6.6}$$

where

$$\mu = 1/G_{1i1}^2. \tag{6.7}$$

Thus b_{i1} is a monotonic function which, as θ becomes infinite, approaches the constant $\pm(\lambda_i^{(2)})^{-1/2}$ which are the equilibrium points of (6.5). These represent the static buckled states when $\Gamma = O(1)$. Thus the leading term in the expansion describes a stationary state whose dependence on the initial data is one of sign only.

Finally we mention that the buckling problem for a circular plate was also considered and results similar to those above were obtained. The detailed calculations can be found in [5].

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