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TRANSITIVE SEMIGROUP ACTIONS(1)

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Following Wallace [15], we define an act to be a continuous function $\mu: S\times X \to X$ such that (i) S is a topological semigroup, (ii) X is a topological space, and (iii) $\mu(s, \mu(t, x))=\mu(st, x)$ for all $s, t\in S$ and $x\in X$. We call (S, X, μ) an action triple, X the state space of the act, and we say S acts on X. We assume all spaces are Hausdorff and write sx for $\mu(s, x)$. S is said to act transitively if $Sx=X$ for all $x\in X$ and effectively if $sx=tx$ for all $x\in X$ implies that $s=t$. The first section of this paper deals with transitive actions and especially with the case where the semigroup is simple. We obtain as a corollary that if S is a compact connected semigroup acting transitively and effectively on a space X that contains a cut point, then K, the minimal ideal of S, is a left zero semigroup and X is homeomorphic to K.

A C-set is a subset, Y, of X with the property that if M is any continuum contained in X with $M\cap Y\neq\varnothing$, then either $M\subset Y$ or $Y\subset M$. In the second section, we consider the position of C-sets in the state space and prove as a corollary that if S is a compact connected semigroup with identity acting effectively on the metric indecomposable continuum, X, such that $SX=X$, then S must be a group.

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Definitions and notation. The notation is generally that of Wallace [16] for semigroups and Stadtlander [12] for actions. Let S be a topological semigroup then we denote by $K(S)$ the unique minimal ideal (if it exists) of S and by $E(S)$ the set of idempotents of S. When the semigroup referred to is clear, the above will be shortened to K and E respectively. We recall that if S is compact then $K(S)$ exists and is closed and $E(S)\neq\varnothing$. For each $e\in E(S)$, $H(e)$ denotes the maximal subgroup of S containing e. S is a left zero semigroup if $xy=x$ for all $x, y\in S$. A left group is a semigroup that is left simple and right cancellative; it is isomorphic to $E\times G$ where E is a left zero semigroup, G is a group and multiplication is coordinate wise [2]. An algebraic isomorphism that is simultaneously a topological homeomorphism is called an iseomorphism.

The Q-set of the action triple (S, X, μ) is the set $Q=\{x\in X \mid Sx=X\}$, thus if $Q=X$ the action is transitive. The action triple (S, X, μ) is said to be equivalent to

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the action triple (T, Y, ν) if there is an iseomorphism $\phi: S \to T$ and a homeomorphism $\psi: X \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} S \times X & \xrightarrow{\mu} & X \\ \phi\times\psi \downarrow & & \downarrow \psi \\ T \times Y & \xrightarrow{\nu} & Y \end{array}$$

We say that $s \in S$ acts as a constant if sX is a point. Finally X^* denotes the topological closure of X. Examples of actions include topological transformation groups, semigroups acting on their underlying space by multiplication and the following: let X be a locally compact space and $M(X)$ the set of all continuous functions of X into X. With the compact open topology and composition of maps as multiplication, $M(X)$ is a topological semigroup. Defining $\mu: M(X) \times X \to X$ by $\mu(f, x) = f(x)$ makes $(M(X), X, \mu)$ an action triple.

Transitive action. It follows from a result of Stadtlander [10] that if a compact semigroup, S, acts transitively on X then the restriction of the act to $K(S) \times X$ is still a transitive action. Thus we use the transitive actions of compact simple ($K(S) = S$) semigroups as a tool to study the transitive actions of arbitrary compact semigroups.

We first show that for compact simple semigroups transitive action results from a seemingly weaker assumption.

THEOREM 1.1. *Let S be a compact simple semigroup acting on X such that $Q \neq \varnothing$. Then S acts transitively on X.*

Proof. Let $x \in Q$ and y be any member of X. Since $S = \bigcup \{H(f) \mid f \in E\}$ [1], $X = Sx = \bigcup \{H(f)x \mid f \in E\}$ so that $x \in H(f)x$ for some $f \in E$. Then $X = Sx = Sfx = \bigcup \{H(e)x \mid e \in Sf \cap E\}$. Thus $y \in H(e)x$ for some $e \in Sf \cap E$; say $y = px$ where $p \in H(e)$. Then $x = fx = fex = fp^{-1}px = fp^{-1}y \in Sy$ and we have $X = Sx \subset Sy \subset X$, that is $Sy = X$. Since y is arbitrary, the action is transitive.

The author wishes to thank the referee for pointing out the above proof which is more concise than the original one.

A band is a semigroup S such that $E(S) = S$, that is, every element is an idempotent. We now characterize the transitive actions of a compact simple band.

THEOREM 1.2. *Let S be a compact simple band acting transitively and effectively on X. Then S must be a left zero semigroup, X and S are homeomorphic and the action is equivalent to multiplication in S.*

Two lemmas are necessary to complete the proof.

LEMMA 1.3. *Let S be a compact simple band acting transitively on X. Then every element of S acts as a constant.*

Proof. It is shown in [10] that if T is a compact semigroup acting transitively on X and $e \in E \cap K$, then $(H(e), eX)$ is a topological transformation group which is transitive on eX and $H(e)x = eX$ for each $x \in X$. Since S is a band, $S = E \cap K$ and $H(e) = e$. Therefore eX is a point for each $e \in S$.

The proof of Theorem 1.2 as stated could now follow from Lemma 1.3 and a result of Day and Wallace [4], however we choose to present the following lemma to cover the noneffective case. Let S be compact, ρ a closed left congruence on S, and also let ρ denote the natural map from S onto S/ρ. If $\nu\colon S \times S/\rho \to S/\rho$ is defined by $\nu(s, \rho(t)) = \rho(st)$, then ν is an act called the canonical act [10]. Stadtlander has shown that if $Y = Sx$ is an orbit of the action triple (S, X, μ) such that $SY = Y$ and if ρ is defined as $\{(s, t) \in S \times S \mid sx = tx\}$ then (S, Y, μ) is equivalent to $(S, S/\rho, \nu)$ where ν is the canonical act.

LEMMA 1.4. *Let S be a compact simple band acting transitively on X by the function μ and let $x_0 \in X$ and define $\rho = \{(s, t) \in S \times S \mid sx_0 = tx_0\}$. Then ρ is a two-sided congruence, (S, X, μ) is equivalent to $(S, S/\rho, \nu)$ where ν is the canonical action and S/ρ is a left zero semigroup.*

Proof. By Lemma 1.3, every element of S acts as a constant, thus ρ is a two-sided congruence and since $X = Sx_0$ is an orbit, we know (S, X, μ) is equivalent to $(S, S/\rho, \nu)$ by Stadtlander's result. Because every element of S acts as a constant, we have $\nu(s, \rho(t)) = \nu(s, \rho(s)) = \rho(s^2) = \rho(s)$ for all $s, t \in S$. Now let $t_1, t_2 \in S/\rho$ then $t_1 = \rho(s_1)$, $t_2 = \rho(s_2)$ for $s_1, s_2 \in S$. But then $t_1 t_2 = \rho(s_1)\rho(s_2) = \rho(s_1 s_2) = \nu(s_1, \rho(s_2)) = \rho(s_1) = t_1$ which shows that S/ρ is a left zero semigroup.

Proof of Theorem 1.2. We have only to note, since every element acts as a constant and S acts effectively, that $\rho = \Delta$ the diagonal of S. Thus $S = S/\rho$ and an application of Lemma 1.4 completes the proof.

The following lemma is a partial converse to Lemma 1.3 to be used in the proof of Corollary 1.9.

LEMMA 1.5. *Let S be a compact simple semigroup acting effectively on X such that some element of S acts as a constant then S is a band.*

Proof. Since S is simple, we know S is iseomorphic to $(Se \cap E) \times eSe \times (eS \cap E)$ when the latter is endowed with the Rees multiplication and $e \in E$ [17]. We will show that $eSe = e$ thus making S iseomorphic to the band $(Se \cap E) \times \{e\} \times (eS \cap E)$. Since S is simple every element acts as a constant, thus $e(SeX) = y$ for some $y \in X$. Let $g \in eSe$, then $gx = egex = e(gex) = y = ex$ for all $x \in X$, but S acts effectively, therefore $g = e$, thus $eSe = e$.

We now investigate the effect a cut point in the state space has in a transitive action by a compact connected semigroup. First recall that if G is a compact connected group acting transitively on X then X is homogeneous [9], that is, for every $x, y \in X$, there is a homeomorphism $h\colon X \to X$ such that $h(x) = y$. Furthermore X is a continuum and if nondegenerate must contain at least two noncut points which together with the fact that X is homogeneous implies that every point of X is a noncut point. Thus in the group case X cannot contain a cut point. This does not follow for semigroups however as the following example illustrates. Let $S=[-1,1]$ with the usual topology and for $s_1, s_2 \in [-1,0]$ and $t_1, t_2 \in [0,1]$ define multiplication in S as follows: $s_1s_2=s_1$, $s_1t_1=s_1$, $t_1t_2=$ the usual product of the real numbers t_1 and t_2, $t_1s_1=$ the usual product of the real numbers t_1 and s_1. Then S is a compact connected topological semigroup with identity. Now let $X=[0,1]$ with the usual topology. Define $\mu: S \times X \to X$ as follows where s_1 and t_1 are as above and $x \in X: \mu(s_1, x) = -s_1$ and $\mu(t_1, x)=$ the usual product of the real numbers t_1 and x, then μ is a transitive and effective act. Thus, the state space of a transitive act by a compact connected semigroup may contain cut points, however, in Corollary 1.9 already mentioned in the introduction, it is shown that this has a profound effect on the multiplication of S. We begin with the following lemma.

LEMMA 1.6. *Let S be a compact connected simple semigroup acting transitively on X such that no element of S acts as a constant. Then X has no cut points.*

Proof. $Sx=X$ implies that X is a continuum and since no element acts as a constant, fX is a nondegenerate continuum for all $f \in E$. But then fX contains at least two noncut points of fX and since $(H(f), fX)$ is a transitive topological transformation group [10] making fX homogeneous [9], we have that every element of fX is a noncut point of fX. We now show for every $s \in S$, $sX=fX$ for some $f \in E$. Let $s \in S$. Because S is simple, $S=\bigcup \{H(e) \mid e \in E\}$ [1], thus $s \in H(f)$ for some $f \in E$ and since $(H(f), fX)$ is a topological transformation group, $s(fX)=fX$. Hence, $fX=s(fX) \subset sX=(fsf)X \subset fX$, whence $fX=sX$. Thus, for each $s \in S$, no point of sX is a cut point of sX.

Suppose $p \in X$ cuts X, then $X \backslash \{p\} = Y \cup Z$ where Y and Z are mutually separated. Let $A=\{s \in S \mid sX \subset Y \cup \{p\}\}$ and $B=\{s \in S \mid sX \subset Z \cup \{p\}\}$, then $S=A \cup B$. For let $s \in S$ and suppose $p \notin sX$, then since sX is connected, $sX \subset Y$ or $sX \subset Z$, thus $s \in A \cup B$. Now suppose $p \in sX$, then since p is a noncut point of sX, $sX \backslash \{p\}$ is connected which implies that $sX \backslash \{p\} \subset Y$ or $sX \backslash \{p\} \subset Z$ and $s \in A \cup B$. Therefore $S=A \cup B$. Now suppose that $t \in A \cap B$, then $tX \subset (Y \cup \{p\}) \cap (Z \cup \{p\}) = \{p\}$ which is impossible since no element acts as a constant, hence $A \cap B = \varnothing$. It is easy to show that A and B are both closed and thus contradict the fact that S is connected. Therefore X has no cut points.

Since a left group that is not left zero always acts transitively on itself with no element acting as a constant, we have the following corollary.

COROLLARY 1.7. *A compact connected left group that is not a left zero semigroup contains no cut points.*

It follows from a result of Stadtlander [10] that if S acts transitively on X then $K(S)$ acts transitively on X and since $K(S)$ is connected whenever S is [13] we can apply Lemma 1.6 to the action of $K(S)$ on X to obtain the following theorem.

THEOREM 1.8. *Let S be a compact connected semigroup acting transitively on X such that no element of $K(S)$ acts as a constant. Then X has no cut points.*

It is easy to see that if S acts effectively then $K(S)$ does also, thus we can put together Lemma 1.5 and Theorems 1.2 and 1.8 to obtain the following result, first proved for semigroups by Faucett [5].

COROLLARY 1.9. *Let S be a compact connected semigroup acting transitively and effectively on X. Then either* (i) *X has no cut points or* (ii) *$K(S)$ is a left zero semigroup and X is homeomorphic to $K(S)$.*

C-sets in the state space. Let $Y=\{(0, y) \mid -1 \leqq y \leqq 1\}$ and let

$$X = \{(x, \sin(1/x)) \mid 0 < x \leqq 1\} \cup Y,$$

then Y is a C-set in X and the complement of Y is an open dense half line in X. C-sets of this type have been studied independently by Day and Wallace [4] and Stadtlander [19]. It follows from their results, for example, that a compact connected semigroup with identity cannot act on the space X defined above such that $\varnothing \neq Q \neq X$. This also follows from the results to be given below.

In [8], Hunter has shown that if S is a compact connected semigroup with identity and if Y is a nondegenerate C-set contained in S, then $Y^* = K(S)$ and $K(S)$ is a group. We use the techniques of Hunter as an important tool in the proof of the following theorem.

THEOREM 2.1. *Let S be a compact connected semigroup with identity acting on the continuum X with $SX=X$ and suppose Y is a nondegenerate C-set in X. Then $Y \subset eX$ for some $e \in E(S) \cap K(S)$.*

We need the preliminary result that follows.

THEOREM 2.2. *Let S be a compact connected semigroup with identity and zero acting on the continuum X with $SX=X$ and such that zero acts as a constant. Then X cannot contain a nondegenerate C-set.*

Proof. Let $OX=\theta \in X$. Once it has been shown that θ cannot be an element of a nondegenerate C-set in X, the proof of Theorem 2.2 proceeds almost exactly the same as the proof of Theorem 1 of [8], thus we will show only that θ cannot be an element of a nondegenerate C-set in X. In order to do this we will use the notion of an ideal in X. If the semigroup S acts on the space X and I is a subset of X such that $SI \subset I$, then I is called an ideal of X. For $A \subset X$, define

$$I_0(A) = \bigcup \{I \subset A \mid I \text{ is an ideal of } X\}.$$

If S is compact and A is an open set containing an ideal of X, then $I_0(A)$ is an open ideal of X. It is easy to see that under the conditions of this theorem, every ideal of X is connected.

Now, suppose $\theta \in Y$ a nondegenerate C-set in X and let U be open in X such that $\theta \in U$ and $Y \cap (X \backslash U) \neq \varnothing$. Let V be open in X such that $\theta \in V \subset V^* \subset U$ and let $W = I_0(V)$. Then W is an open connected set, W^* is a continuum and $\theta \in W \subset W^* \subset U$. But $W^* \cap Y \neq \varnothing$ and $W^* \not\supset Y$, hence $W \subset W^* \subset Y$, a contradiction since a C-set has empty interior.

Let S be a compact connected semigroup with identity and let T be a compact connected subsemigroup of S such that (i) $T \cap K(S) \neq \varnothing$, (ii) $1 \in T$ and (iii) if R is a compact connected subsemigroup of T satisfying (i) and (ii) then $R = T$. T is said to be algebraically irreducible from 1 to $K(S)$. In [7], Hofmann and Mostert show that if S is a compact connected semigroup with identity then S contains an algebraically irreducible semigroup and every algebraically irreducible semigroup is abelian.

We recall the Rees quotient [20]. Let S be a semigroup, I a closed ideal of S and define $\rho = \{(s, t) \in S \mid s = t \text{ or } s, t \in I\}$ then ρ is a closed congruence and we call the factor semigroup S/ρ the Rees quotient and denote it by S/I. We now use Theorem 2.2 to prove Theorem 2.1.

Proof of Theorem 2.1. Let T be a subsemigroup of S algebraically irreducible from 1 to $K(S)$, then T is a compact connected abelian semigroup with identity acting on X with $TX = X$. Let $T' = T/K(T)$ be the Rees quotient and $X' = X/K(T)X$ be the ordinary topological quotient and let $\eta: T \to T'$ and $\beta: X \to X'$ be the canonical maps, then T' acts on X' by $\eta(t)\beta(x) = \beta(tx)$ [10] and satisfies the hypothesis of Theorem 2.2. It is routine to show that if D is a continuum in X' and $E = \beta^{-1}(D)$ then E is a continuum in X.

We now show that $Y \subset K(T)X$. Suppose not then $\bar{Y} = \beta(Y)$ is a nondegenerate subset of X' which is a C-set. For let M be a continuum in X' with $M \cap \bar{Y} \neq \varnothing$ and consider the two cases (i) $Y \cap K(T)X = \varnothing$ and (ii) $Y \cap K(T)X \neq \varnothing$. In case (i), $\beta^{-1}(\bar{Y}) = Y$ since $\beta|_{X \backslash K(T)X}$ is a homeomorphism, and Y meets the continuum $\beta^{-1}(M)$, thus $\beta^{-1}(M) \subset Y$ or $Y \subset \beta^{-1}(M)$ which implies $M \subset \bar{Y}$ or $\bar{Y} \subset M$. In case (ii), $Y \cap K(T)X \neq \varnothing$ implies $K(T)X \subset Y$ since $K(T)X$ is a continuum hence $\beta^{-1}(\bar{Y}) = Y$ and the same argument as in case (i) shows that $\bar{Y}$ is a C-set. But this contradicts Theorem 2.2, therefore $Y \subset K(T)X$.

Since T is abelian, $K(T)$ is a group and $K(T) \subset K(S)$ which implies $K(T) \subset H(e)$ for some $e \in K(S) \cap E(S)$, thus $Y \subset K(T)X \subset H(e)X \subset eX$.

NOTE. We have actually proved a slightly stronger result than that stated since Y is contained in the state space of the abelian topological transformation group $(K(T), eX)$.

As an application of Theorem 2.1, we prove the following corollary, which is a special case of a more general theorem in [18].

COROLLARY 2.6. *Let S be a compact connected semigroup with identity acting effectively on the metric indecomposable continuum X with $SX = X$, then S is a group.*

Proof. Let Y be a composant of X, then, as is well known, Y is a C-set so

$Y \subset eX$ for some $e \in E \cap K$. But $Y^* = X$ [6], thus $X = eX$ and $1y = y = ey$ for all $y \in X$ which implies $1 = e$ since S acts effectively. But $1 \in K$ implies K is a group and $K = S$.

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