TRANSITIVITY, TWO-SIDED LIMIT SHADOWING PROPERTY AND DENSE ω-CHAOS

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ABSTRACT. We consider ω-chaos as defined by S. H. Li in 1993. We show that c-dense ω-scrambled sets are present in every transitive system with two-sided limit shadowing property (TSLmSP) and that every transitive map on topological graph has a dense Mycielski ω-scrambled set. As a preliminary step, we provide a characterization of dynamical properties of maps with TSLmSP.

1. Introduction

In 1993, S. H. Li introduced in [15] the notion of ω-chaos for interval maps. Since then, only a small progress was made in the topic. The authors of [29] show that transitive map on the unit interval has a dense Mycielski ω-scrambled set, and later Bobok presented in [5] how this result can be extended to produce a map on the unit n-dimensional cube with ω-scrambled set. A few special cases of construction of uncountable ω-scrambled sets for systems on Cantor set are dealt in [13, 24] (see also [15] and [29]) and not much more beyond these results is known. The main difficulty is that standard techniques working for residual relations (an effective tool for most of definitions of chaos stated in terms of pairs or tuples) cannot be employed here. This makes proving ω-chaos in concrete cases a quite challenging problem.

In this paper we use two tools when proving ω-chaos. First of all, we extend results of [29] to show that on topological graphs transitivity implies ω-chaos (with dense Mycielski ω-scrambled set), provided that entropy is positive (or equivalently, the map is not an irrational relation on the unit circle). Our construction highly rely on the recent results from [10] (and of course, some ideas from [29]).

Second tool, applicable in a more general case is two-sided limit shadowing property (abbrev. TSLmSP). This property was probably first introduced by Pilyugin in his book [25] and is a version of standard shadowing property (SP

for short), where emphasis is put on asymptotic aspects of tracing. It can be proved that all structurally stable diffeomorphisms exhibit TSLmSP, while the class of diffeomorphisms with TSLmSP is larger (see Theorems 2 and 3 in [26]). Sakai proved in [27] that C^1-generic diffeomorphism has a combination of SP and TSLmSP, called s-limit shadowing property, if and only if it satisfies Axiom A and no-cycle condition (see [27] for more details). It was later proved in [17] that for C^1-generic diffeomorphism the above condition is actually equivalent to TSLmSP (called weak shadowing property in [17]). Carvahlo characterized in [6] the C^1-interior of the set of diffeomorphisms with TSLmSP on closed manifolds as exactly the set of transitive Anosov diffeomorphisms. In the case of expansive homeomorphism, shadowing property and TSLmSP are mutually equivalent properties [14]. However, it is not hard to verify (e.g. taking an infinite Cartesian product) that TSLmSP can exist in systems which are not expansive. It is also a consequence of results of [26] mentioned above. Beyond the above mentioned results, not much more is known about TSLmSP. Recently, an interesting connection between a modified version of TSLmSP (so-called two-sided limit shadowing property with a gap) and various mixing properties for homeomorphism has been addressed in [7].

This paper is organized as follows. The next section collects a few most important facts and definitions. As we could see in previous paragraph, the most work done on TSLmSP was devoted to special classes of diffeomorphisms. In particular, no general properties of maps with TSLmSP were studied. To fill this gap in the literature, first we prove in Section 3 a few theorems for a better understanding of dynamics of maps with TSLmSP (e.g. a kind of Smale and Bowen spectral decompositions). Next we prove that ω-chaos can be a consequence of TSLmSP. In the last section of the paper we show that on topological graphs transitivity (no shadowing assumption) is enough to induce ω-chaos with dense Mycielski ω-scrambled set.

2. Preliminaries

Denote by $\mathbb{N}$ the set of positive integers, by $\mathbb{Z}$ the set of integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. A set $Q \subset X$ is *c-dense* if for any nonempty open set U the set $U \cap Q$ is uncountable. A point $x \in A$ is *isolated* (*in A*) if there is an open neighborhood U of x such that $U \cap A = \{x\}$. A set is *prefect* if it is nonempty, closed and does not have isolated points. If there is a basis of clopen (i.e., closed and open at the same time) sets, then we say that X is zero dimensional. A set A is *Cantor* if it is perfect and zero dimensional and *Mycielski* if it can be presented as a countable union of Cantor sets.

A pair (X, f) is a *dynamical system* if it consists of a compact metric space (X, d) and a continuous map $f \colon X \to X$. We denote the (*positive*) *orbit* of a point $x \in X$ by $\mathrm{Orb}^+(x, f) = \{x, f(x), f^2(x), \ldots\}$ and an ω-limit set of x is defined by $\omega(x, f) = \bigcap_{n=1}^{\infty} \overline{\mathrm{Orb}^+(f^n(x), f)}$. A set M is *minimal* if $M = \overline{\mathrm{Orb}^+(x, f)}$ for every $x \in M$.

The following lemma is a special case of theorem proved by Mycielski in [19]. Result of [19] is a more powerful tool, however we do not need it here in such generality.

Lemma 1. *Let X be a compact perfect set. If R is a residual subset of $X \times X$, then there exists a dense Mycielski set $M \subset X$ such that $M \times M \subset R \cup \Delta$.*

2.1. Shadowing

Let $\delta > 0$ be a positive constant. A sequence $\{y_n\}_{n \in \mathbb{Z}} \subset X$ is a δ-pseudo-orbit of f if

$$d(f(y_n), y_{n+1}) \leq \delta \ \ \text{for every } n \in \mathbb{Z}.$$

Note that 0-pseudo-orbit of f is simply its genuine (full) orbit. A δ-pseudo-orbit $\{y_n\}$ is *two-sided limit δ-pseudo-orbit of f* if it satisfies

(1)
$$\lim_{|n| \to \infty} d(f(y_n), y_{n+1}) = 0.$$

If only condition (1) is satisfied (i.e., no particular $\delta > 0$ is established), we will say that $\{y_n\}_{n \in \mathbb{Z}}$ is a *two-sided limit pseudo-orbit of f*.

We say that f posses the *shadowing property (SP for short)* if for every $\varepsilon > 0$ there exists $\delta > 0$ satisfying the following condition: given a δ-pseudo-orbit $\{y_n\}_{n \in \mathbb{Z}}$ we can find a corresponding 0-pseudo-orbit $\{x_n\}_{n \in \mathbb{Z}}$ which ε-traces $\{y_n\}_{n \in \mathbb{Z}}$, i.e.,

$$d(x_n, y_n) \leq \varepsilon \ \ \text{for every } n \in \mathbb{Z}.$$

If there is $\beta > 0$ such that for any two-sided limit β-pseudo-orbit $\{y_n\}_{n \in \mathbb{Z}}$ we can find a 0-pseudo-orbit $\{x_n\}_{n \in \mathbb{Z}}$ which traces $\{y_n\}_{n \in \mathbb{Z}}$ in the limit sense, i.e.,

$$\lim_{|n| \to \infty} d(x_n, y_n) = 0,$$

then we say that f possess the *two-sided limit shadowing property (abbrev. TSLmSP)*.

Sometimes it is more convenient to work with finite pseudo-orbits when speaking about SP. A finite sequence $x_0, \ldots, x_n$ is a *(finite) δ-pseudo-orbit* (or a *δ-chain from x_0 to x_n*) if $d(f(x_i), x_{i+1}) < \delta$ for $i = 0, \ldots, n-1$. A point z is *ε-tracing* a pseudo-orbit $\{x_i\}_{i=0}^{n}$ if $d(f^i(z), x_i) < \varepsilon$ for $i = 0, \ldots, n$. When X is compact and (X, f) is onto, it is not hard to verify that (X, f) has SP if and only if for every $\varepsilon > 0$ there is $\delta > 0$ such that every finite δ-pseudo-orbit is ε-traced by a point in X.

Fix $x, y \in X$. If for every $\delta > 0$ there are finite δ-pseudo-orbits from x to y and from y to x, then we write $x \sim y$. A point x is *chain-recurrent* if $x \sim x$. The set of all chain recurrent points is denoted $CR(f)$. It is not hard to see that $\sim$ is an equivalence relation on $CR(f)$. If $x \in CR(f)$, then $[x]_\sim$ is well defined, and we call it a *chain recurrent class (of x)*.

Given a 0-pseudo-orbit $\{x_n\}_{n \in \mathbb{Z}}$ we define

$$\omega(\{x_n\}_{n \in \mathbb{Z}}, f) = \bigcap_{k=1}^{\infty} \overline{\{x_n : n > k\}} \ \text{ and } \ \alpha(\{x_n\}_{n \in \mathbb{Z}}, f) = \bigcap_{k=1}^{\infty} \overline{\{x_{-n} : n > k\}}.$$

Note that $\omega(\{x_n\}_{n \in \mathbb{Z}}, f) = \omega(x_k, f)$ for every $k \in \mathbb{Z}$.

2.2. Transitivity, equicontinuity and ω-chaos

A dynamical system (X, f) is *equicontinuous* if for every $\epsilon > 0$, there is $\delta > 0$ with the property that $d(x, y) < \delta$ implies $d(f^n(x), f^n(y)) < \epsilon$ for every $x, y \in X$ and every $n \in \mathbb{N}$. A point $x \in X$ is *equicontinuous* if for every $\epsilon > 0$, there is $\delta > 0$ with the property that $d(x, y) < \delta$ implies $d(f^n(x), f^n(y)) < \epsilon$ for every $y \in X$ and every $n \in \mathbb{N}$. By compactness of X, a system (X, f) is equicontinuous if and only if every point in X is equicontinuous.

A dynamical system (X, f) is *sensitive* if there exists $\delta > 0$ such that for any nonempty open subset U of X, we have $\text{diam}(f^n(U)) > \delta$ for some $n \in \mathbb{N}$. A point $x \in X$ is *sensitive* if it is not equicontinuous, that is, there exists a $\delta > 0$ such that for any neighborhood U of x there is $n > 0$ such that $\text{diam}(f^n(U)) > \delta$. Clearly, if (X, f) is sensitive, then every $x \in X$ is sensitive.

The following definition of ω-chaos was introduced by S. H. Li in [15].

Definition 2. A set $S \subset X$ containing at least two points is called an ω-*scrambled set for f* if, for any two $x \neq y$ in S,

   (1) $\omega_f(x) \setminus \omega_f(y)$ is uncountable,
   (2) $\omega_f(x) \cap \omega_f(y) \neq \emptyset$ and
   (3) $\omega_f(x) \setminus Per(f) \neq \emptyset$.

The map f is ω-*chaotic* if there is an uncountable ω-scrambled set. By an ω-chaotic pair we mean any ω-scrambled set of cardinality 2.

In minimal set every point has dense orbit. It is not hard to verify that if a minimal set M has an isolated point, then M is a single periodic orbit, and if M does not have isolated points, then it is uncountable, as is every perfect compact set.

Remark 3. In most of the known constructions it is possible to replace conditions (2), (2) by the following stronger condition:

(1') $\omega_f(x) \setminus \omega_f(y)$ contains an infinite minimal set.

In this paper, when we prove ω-chaos, we construct ω-scrambled sets satisfying this stronger condition.

2.3. Subshifts

We denote $\Sigma_2^+ = \{0, 1\}^{\mathbb{N}}$ endowed with the metric $d(x, y) = 2^{-k}$ when $x \neq y$, where $k = \min\{i : x_i \neq y_i\}$. The space (Σ_2^+, d) is compact and the *shift map* $\sigma \colon \Sigma_2^+ \to \Sigma_2^+$, given by $\sigma(x)_i = x_{i+1}$ for all $i \in \mathbb{N}$, is continuous. For $x \in \Sigma_2^+$ we usually write $x = x_1 x_2 x_3 \cdots$ instead of $x = (x_1, x_2, x_3, \ldots)$. We denote by 0^{∞}

the constant sequence $0^\infty = 0000\cdots$. By a *shift* or *subshift* we mean any closed and σ-invariant subset of Σ_2^+. For $1 \leq i \leq j$ we denote $x_{[i,j]} = x_i x_{i+1} \cdots x_j$ and $x_{[i,j)} = x_{[i,j-1]}$. The set of words of length n in a subshift X is denoted $B_n(X) = \{x_{[i,i+n)} : x \in X, i \in \mathbb{N}\}$.

3. TSLmSP and transitivity

We say that regularly closed sets $D_0, \ldots, D_{n-1}$ (i.e., $\overline{\operatorname{Int} D_i} = D_i$) form a *periodic decomposition* of X if they cover X, $D_i \cap \operatorname{Int} D_j = \emptyset$ for $i \neq j$, and $f(D_i) = D_{i+1}$ for $i = 0, \ldots, n-1$, where for technical reasons we put $D_n = D_0$.

The proof of the following fact is standard. We present it for completeness.

Lemma 4. *Let $D_0, \ldots, D_{n-1}$ form a periodic decomposition for a dynamical system (X, f) and assume that (D_i, f^n) is transitive for every i. If (X, f) has SP, then $D_i \cap D_j = \emptyset$ for $i \neq j$.*

Proof. Suppose that there are $i \neq j$ such that $D_i \cap D_j \neq \emptyset$. By the definition of decomposition $D_i = \overline{\operatorname{Int} D_i}$, hence $\operatorname{Int} D_i \neq \emptyset$. Recall also that $\operatorname{Int} D_i \cap D_j = \emptyset$.

Fix points $z_1 \in \operatorname{Int} D_i$, $z_2 \in \operatorname{Int} D_j$ and take $\varepsilon > 0$ such that $B(z_1, 2\varepsilon) \subset \operatorname{Int} D_i$, $B(z_2, 2\varepsilon) \subset \operatorname{Int} D_j$. Let δ be provided to ε by SP.

Suppose that there is $z \in D_i \cap D_j$. By transitivity of f^n on D_i and on D_j there are finite δ-pseudo-orbits γ, ξ of length $|\gamma| = sn$, $|\xi| = tn+1$ for some $s, t > 0$, such that $\gamma_1 = z_1$, $d(f(\gamma_{sn}), z) < \beta/2$ and $d(\xi_1, z) < \beta/2$, $\xi_{tn+1} = z_2$. If x is a point which is ε-tracing the δ-pseudo-orbit $\zeta = \gamma\xi$ formed by concatenation of γ with ξ, then $d(x, z_1) < \varepsilon$ and $d(f^{(t+s)n}, \zeta_{(t+s)n+1}) = d(f^{(t+s)n}, \xi_{tn+1}) < \varepsilon$. This immediately implies that

$$f^{(s+t)n}(x) \in f^{(s+t)n}(\operatorname{Int} D_i) \cap \operatorname{Int} D_j \subset D_i \cap \operatorname{Int} D_j = \emptyset.$$

This is a contradiction, which finishes the proof. $\square$

Lemma 5. *If (X, f) is a transitive dynamical system with TSLmSP, then there is $N > 0$ such that if $D_0, \ldots, D_{n-1}$ is a periodic decomposition of X, then $n \leq N$.*

Proof. Let β be provided by TSLmSP. Let $U_1, \ldots, U_N$ be a cover of X by open sets of diameter smaller than $\beta/3$. If $D_0, \ldots, D_n$ is a periodic decomposition, and $n \geq N$, then there are $i \neq j$ and k such that $D_i \cap U_k \neq \emptyset$ and $D_j \cap U_k \neq \emptyset$. We will repeat argument from the proof Lemma 4 to obtain that $\operatorname{Int} D_i \cap D_j \neq \emptyset$ which will lead to a contradiction.

Take recurrent points $z_1 \in \operatorname{Int} D_i \cap U_k$ and $z_2 \in \operatorname{Int} D_j \cap U_k$. It is well known that point recurrent for f is also recurrent for f^n. Therefore, there are increasing sequences r_i, s_i such that $d(f^{nr_i}(z_1), z_1) < \beta 2^{-i}$ and $d(f^{ns_i}(z_2), z_2) < \beta 2^{-i}$. Let $\gamma_i = \{z_1, f(z_1), \ldots, f^{nr_i-1}(z_1)\}$ and $\xi_i = \{z_2, f(z_2), \ldots, f^{ns_i-1}(z_2)\}$ and consider the sequence obtained as the following two-sided infinite concatenation

$$\zeta = \bigcup_{n=-\infty}^{1} \{\gamma_n\} \cup \bigcup_{n=1}^{\infty} \{\xi_n\} = \cdots \gamma_n \cdots \gamma_2 \gamma_1 \xi_1 \xi_2 \cdots \xi_n \cdots .$$

Note that ζ is an limit β-pseudo orbit because

$$d(f(f^{nr_1-1}(z_1)), z_2) \leq d(f^{nr_1}(z_1), z_1) + d(z_1, z_2) < \beta/2 + \beta/3 < \beta.$$

Let $\{x_n\}_{n \in \mathbb{Z}}$ be a 0-pseudo-orbit provided to ζ by TSLmSP. Note that both sets $\{l > 0 : \zeta_{-nl} = z_1\}$ and $\{t > 0 : \zeta_{nt} = z_2\}$ are infinite. Hence there are $l, t > 0$ such that $y = x_{-ln} \in B(z_1, \varepsilon)$ and $x_{tn} \in B(z_2, \varepsilon) \subset \text{Int } D_j$, which in turn implies that

$$f^{(t+l)n}(y) = x_{tn} \in f^{(t+l)n}(D_i) \cap \text{Int } D_j = \emptyset.$$

This is a contradiction, hence the proof is finished. $\square$

Lemma 6. *If (X, f) has TSLmSP, then there are finitely many chain recurrent classes, that is $CR(f) = [x_1]_\sim \cup \cdots \cup [x_n]_\sim$ for some points $x_1, \ldots, x_n$.*

Proof. Assume that there are infinitely many chain-recurrent classes. Since $[x]_\sim$ is closed for each $x \in CR(f)$ there is a sequence of points $\{x_i\}_{i=1}^\infty$ such that all $[x_i]_\sim$ are distinct and are convergent in the Hausdorff metric $\mathcal{H}_d$ induced by d onto the space of all nonempty compact subsets of X. In particular $\{[x_i]_\sim\}_{i=1}^\infty$ is a Cauchy sequence in the Hausdorff metric. Let $\beta > 0$ be provided by TSLmSP. There are distinct $x, y \in \{x_i\}_{i=1}^\infty$ with

$$\text{dist}([x]_\sim, [y]_\sim) \leq \mathcal{H}_d([x]_\sim, [y]_\sim) < \beta.$$

Therefore, without loss of generality (replacing x, y by other points in their equivalence classes if necessary) we may assume that $d(x, y) < \beta$. But $x, y \in CR(f)$, hence we can easily construct a two-sided limit β-pseudo-orbit $\xi = \{\xi_i\}_{i \in \mathbb{Z}}$ such that $\{\xi_i\}_{i \geq 0} \subset [y]_\sim$, $\{\xi_i\}_{i < 0} \subset [x]_\sim$ and both sets $\{i < 0 : \xi_i = x\}$ and $\{i > 0 : \xi_i = y\}$ are infinite. Then as an immediate consequence of TSLmSP we obtain that for every $\varepsilon > 0$ there is an ε-chain from x to y. Similarly, we can construct two-sided limit β-pseudo-orbit $\gamma = \{\gamma_i\}_{i \in \mathbb{Z}}$ such that $\{\gamma_i\}_{i < 0} \subset [y]_\sim$, $\{\xi_i\}_{i \geq 0} \subset [x]_\sim$ and both sets $\{i > 0 : \xi_i = x\}$ and $\{i < 0 : \xi_i = y\}$ are infinite. Consequently, for every $\varepsilon > 0$ there is an ε-chain from y to x. We have just proved that $x \sim y$ which is a contradiction. $\square$

Lemma 7. *Suppose that Λ is a chain-recurrent class and that (Λ, f) has TSLmSP. Then (Λ, f) is transitive.*

Proof. Let $\beta > 0$ be provided by TSLmSP. Fix any $x, y \in \Lambda$. Repeating arguments from the proof of Lemma 6 we can find a limit β-pseudo-orbit $\{x_n\}_{n \in \mathbb{Z}}$ such that both sets $\{i > 0 : \xi_i = y\}$ and $\{i < 0 : \xi_i = x\}$ are infinite. Let $\{z_n\}_{n \in \mathbb{Z}}$ be a 0-pseudo-orbit which is tracing $\{x_n\}_{n \in \mathbb{Z}}$ in the limit sense. Then for every $N > 0$ there are indexes $i, j > N$ such that $d(z_{-i}, x) < 1/N$ and $d(z_j, y) < 1/N$. This immediately implies that for every $\varepsilon > 0$ and every $n \in \mathbb{Z}$ there are ε-chains from x to z_n and from z_n to y. But Λ is chain recurrent class, hence $x \sim y$ and therefore $x \sim z_n$ for every $n \in \mathbb{Z}$, in particular $\{z_n\}_{n \in \mathbb{Z}} \subset \Lambda$. But if we take any open sets $x \in U$, $y \in V$, then there are $n < m$ such that $z_n \in U$, $z_m \in V$, proving that $f^{m-n}(U) \cap V \neq \emptyset$. Indeed (Λ, f) is transitive. $\square$

Theorem 8. *If (X, f) has TSLmSP, then $f|_{CR(f)}$ has shadowing and there are closed invariant sets $\Lambda_1, \ldots, \Lambda_n$ such that $CR(f) = \bigcup_{i=1}^n \Lambda_i$ and $f|_{\Lambda_i}$ is transitive and has SP for each $i = 1, \ldots, n$. Furthermore, for every i there is $m = m(i)$ and periodic decomposition $D_0, \ldots, D_{m-1}$ of Λ_i such that sets D_j are disjoint and (D_j, f^m) is mixing for $j = 0, 1, \ldots, m-1$.*

Proof. Let $\beta > 0$ be provided by TSLmSP and let $\Lambda_1, \ldots, \Lambda_n$ be a partition of $CR(f)$ into chain-recurrent classes provided by Lemma 6. Denote $\varepsilon = \frac{1}{2} \min \{ \min_{i \neq j} \operatorname{dist}(\Lambda_i, \Lambda_j), \beta \}$. Clearly, if we take any two-sided limit ε-pseudo-orbit contained completely in Λ_i and $\{x_n\}_{n \in \mathbb{Z}}$ is a full orbit (i.e., 0-pseudo-orbit) which traces it in the limit sense, then $\omega(\{x_n\}_{n \in \mathbb{Z}}, f) \cap \Lambda_i \neq \emptyset$ and $\alpha(\{x_n\}_{n \in \mathbb{Z}}, f) \cap \Lambda_i \neq \emptyset$. But then, since Λ_i is chain-recurrent class, we clearly have that $\{x_n\}_{n \in \mathbb{Z}} \subset \Lambda_i$. This shows that (Λ_i, f) has TSLmSP for each i and therefore is transitive by Lemma 7. By results of [11] we obtain that (Λ_i, f) has SP.

If (Λ_i, f) is totally transitive, then it must be mixing (e.g. see [12]). Otherwise there exists a periodic decomposition of Λ_i (see [3]), and by Lemma 5 length of every such decomposition is bounded. But then, again by results of [3], we obtain that there are sets $D_0, \ldots, D_{m-1}$ ($m \geq 1$) forming periodic decomposition of Λ_i and such that (D_j, f^m) is totally transitive. By Lemma 4 sets D_j are disjoint, and hence it is clear that (D_j, f^m) has SP. Repeating previous argument, we see that each (D_j, f^m) is mixing completing the proof. $\square$

Corollary 9. *If (X, f) is a minimal system with TSLmSP, then it is a periodic orbit.*

Proof. By results of [11] we obtain that (X, f) has SP. Then it is a consequence of [18] that (X, f) is equicontinuous and by Theorem 8 there is a periodic decomposition $D_0, \ldots, D_{n-1}$ of X such that (D_i, f^n) is mixing for every i. But the only possibility for equicontinuous system to be mixing is that D_i is a singleton. The proof is finished. $\square$

4. TSLmSP and ω-scrambled sets

Lemma 10. *Let (X, f) be a transitive dynamical system with TSLmSP. Let M_1, M_2 be two minimal sets. There exists z and at most countable set Z such that $\omega_f(z) = M_1 \cup M_2 \cup Z$.*

Proof. Fix any $\varepsilon > 0$ and let $\beta > 0$ be provided by TSLmSP. By transitivity, there exist points $x, y \in X$ and integers $n, m > 0$ such that $x, f^m(y) \in B(M_1, \beta/2)$ and $y, f^n(x) \in B(M_2, \beta/2)$. Since M_1, M_2 are minimal, there are sequences $\xi = \{\xi_i\}_{i=-\infty}^0 \subset M_1$, $\zeta = \{\zeta_i\}_{i=-\infty}^0 \subset M_2$ and points $p \in M_1$, $q \in M_2$ such that $d(p, f^m(y)) < \beta$, $d(q, f^n(x)) < \beta$, $d(\xi_0, x) < \beta$, $d(\zeta_0, y) < \beta$ and $f(\xi_i) = \xi_{i+1}$, $f(\zeta_i) = \zeta_{i+1}$ for all $i < 0$.

Note that the following two-sided sequences

$$\phi = \{\phi_n\}_{n \in \mathbb{Z}} = \left\{ \ldots, \xi_{-2}, \xi_{-1}, x, \ldots, f^{n-1}(x), q, f(q), f^2(q), \ldots \right\},$$

$$\psi = \{\psi_n\}_{n \in \mathbb{Z}} = \left\{ \ldots, \zeta_{-2}, \zeta_{-1}, y, \ldots, f^{m-1}(y), p, f(p), f^2(p), \ldots \right\}$$

form two-sided limit pseudo-orbits. By TSLmSP there are 0-pseudo-orbits $\hat{\phi}, \hat{\psi}$ (i.e., actual orbits) which are tracing ϕ and ψ, respectively, in the limit sense, i.e., $\lim_{|n| \to \infty} d(\phi_n, \hat{\phi}_n) = 0$ and $\lim_{|n| \to \infty} d(\psi_n, \hat{\psi}_n) = 0$. Since $\alpha\left(\left\{\hat{\phi}_n\right\}_{n \in \mathbb{Z}}, f\right)$ $= \alpha(\{\phi_n\}_{n \in \mathbb{Z}}, f) = M_1$ and $\omega\left(\left\{\hat{\psi}_n\right\}_{n \in \mathbb{Z}}, f\right) = \omega(\{\psi_n\}_{n \in \mathbb{Z}}, f) = M_1$, there is a decreasing sequence $\{k_i\}_{i=0}^{\infty} \subset (-\infty, 0)$ and an increasing sequence $\left\{\hat{k}_i\right\}_{i=0}^{\infty} \subset (0, \infty)$ such that $d(\hat{\phi}_{k_i}, \hat{\psi}_{\hat{k}_i}) < 2^{-i}\beta$ and $\lim_{i \to \infty} d(\hat{\phi}_{k_i}, M_1) = 0$. Similarly, there is a decreasing sequence $\left\{\hat{l}_i\right\}_{i=0}^{\infty} \subset (-\infty, 0)$ and an increasing sequence $\{l_i\}_{i=0}^{\infty} \subset (0, \infty)$ such that $d(\hat{\phi}_{l_i}, \hat{\psi}_{\hat{l}_i}) < 2^{-i}\beta$ and $\lim_{i \to \infty} d(\hat{\phi}_{l_i}, M_2) = 0$.

Denote by γ the following two-sided sequence:

$$\gamma = \{\gamma_n\}_{n \in \mathbb{Z}} = \left\{\hat{\psi}_i\right\}_{i=-\infty}^{\hat{k}_1 - 1} \cup \bigcup_{j=1}^{\infty} \left(\left\{\hat{\phi}_i\right\}_{i=k_j}^{l_j - 1} \cup \left\{\hat{\psi}_i\right\}_{i=\hat{l}_j}^{\hat{k}_{j+1} - 1} \right)$$

$$= \{\ldots, \hat{\psi}_{\hat{k}_1 - 3}, \hat{\psi}_{\hat{k}_1 - 2}, \hat{\psi}_{\hat{k}_1 - 1}, \hat{\phi}_{k_1}, \ldots, \hat{\phi}_{l_1 - 1}, \hat{\psi}_{\hat{l}_1}, \ldots, \hat{\psi}_{\hat{k}_2 - 1}, \hat{\phi}_{k_2},$$

$$\ldots, \hat{\phi}_{l_2 - 1}, \hat{\psi}_{\hat{l}_2}, \ldots, \hat{\psi}_{\hat{k}_3 - 1}, \hat{\phi}_{k_3}, \ldots\}.$$

Observe that γ is a two-sided limit β-pseudo-orbit. Now, let $\{z_n\}_{n \in \mathbb{Z}}$ be a 0-pseudo-orbit provided by TSLmSP to γ and denote $z = z_0$. We claim that

$$\omega_f(z) = M_1 \cup M_2 \cup \hat{\psi} \cup \hat{\phi}.$$

First of all, let us observe that by the definition of sequences k_i, l_i there are increasing sequences n_i, m_i such that the following limits exist: $\lim_{i \to \infty} \gamma_{n_i} = p \in M_1$ and $\lim_{i \to \infty} \gamma_{m_i} = q \in M_2$. Therefore, $M_1 \cap \omega_f(z) \neq \emptyset$ and $M_2 \cap \omega_f(z) \neq \emptyset$ which gives $M_1 \cup M_2 \subset \omega_f(z)$. Clearly, for any $s \in \mathbb{Z}$ we can find increasing sequences $r_i, \hat{r}_i$ such that $\gamma_{r_i} = \hat{\phi}_s$ and $\gamma_{\hat{r}_i} = \hat{\psi}_s$ and hence $\hat{\phi} \cup \hat{\psi} \subset \omega_f(z)$.

Next, fix any increasing sequence s_i such that $\lim_{i \to \infty} f^{s_i}(z)$ exists. Without loss of generality, going to a subsequence if necessary, we may assume that all points γ_{s_i} are elements of exactly one of the sequences $\hat{\phi}$ or $\hat{\psi}$. Let us assume that the first possibility takes place (the second case is symmetric), that is there there exists a sequence $\hat{s}_i$ such that $\gamma_{s_i} = \hat{\phi}_{\hat{s}_i}$ for all $i \in \mathbb{N}$. Assume first, that the sequence $\hat{s}_i$ is bounded. Then, going to a subsequence again, we may assume that it is constant, say $\hat{s}_i = s$ for all i, and then $\lim_{i \to \infty} f^{s_i}(z) = \lim_{i \to \infty} \hat{\phi}_{\hat{s}_i} = \hat{\phi}_s$. If $\hat{s}_i$ is unbounded, then we may assume that it is monotone and then $\lim_{i \to \infty} f^{s_i}(z) \in M_1$ or $\lim_{i \to \infty} f^{s_i}(z) \in M_2$, depending whether $\hat{s}_i$ is decreasing or increasing. The proof is completed. $\square$

Lemma 11. *Let (X, f) be a transitive dynamical system with TSLmSP and let $M_1, M_2, \ldots, M_n \subset X$ be minimal sets. There exist z and at most countable set Z such that $\omega_f(z) = Z \cup \bigcup_{i=1}^{n} M_i$.*

Proof. Let β be provided by TSLmSP. We prove the statement of the lemma by induction on n. The case $n = 2$ is solved by Lemma 10. Now suppose that the lemma holds with some $n = N \geq 2$. We are going to show that it also holds with $n = N + 1$.

By Lemma 10 and our assumptions, there are points z_1, z_2 and countable sets Z_1, Z_2 such that

$$\omega_f(z_1) = Z_1 \cup \bigcup_{i=1}^{N} M_i, \quad \omega_f(z_2) = Z_2 \cup M_1 \cup M_{N+1}.$$

Note that $M_1 \subset \omega_f(z_1) \cap \omega_f(z_2)$, which yields that there are increasing sequences k_i, s_i such that

(1) $d(f^{k_i}(z_1), f^{k_{i+1}}(z_1)) < 2^{-i}\beta$ and $d(f^{s_i}(z_2), f^{s_{i+1}}(z_2)) < 2^{-i}\beta$,

(2) $d(f^{k_i}(z_1), f^{s_i}(z_2)) < 2^{-i}\beta$ and $d(f^{k_i}(z_1), M_1) < 2^{-i}\beta$.

Let $\{\xi_i\}_{i=0}^{\infty} \subset M_1$ be any sequence satisfying $f(\xi_i) = \xi_{i+1}$ for all $i \leq 0$ and $d(\xi_1, f^{k_1}(z_1)) < \beta$. Now define a two-sided limit β-pseudo-orbit γ putting

$$\gamma = \{\gamma_n\}_{n \in \mathbb{Z}} = \{\xi_i\}_{i=-\infty}^{0} \cup \bigcup_{j=1}^{\infty} \left(\{f^i(z_1)\}_{i=k_j}^{k_{j+1}-1} \cup \{f^i(z_2)\}_{i=s_j}^{s_{j+1}-1} \right)$$

$$= \{\ldots, \xi_{-2}, \xi_{-1}, \xi_0, f^{k_1}(z_1), \ldots, f^{k_2-1}(z_1), f^{s_1}(z_2), \ldots,$$

$$f^{s_2-1}(z_2), f^{k_2}(z_1), \ldots, f^{k_3-1}(z_1), f^{s_2}(z_2), \ldots\}.$$

Clearly γ is a limit β-pseudo-orbit. Now let $\{z_i\}_{i \in \mathbb{Z}}$ be a 0-pseudo-orbit which traces γ in the limit sense, and put $z = z_0$. Since $\{\gamma_j\}_{j=0}^{\infty}$ contains infinite forward segments of both orbits of z_1 and z_2 we immediately obtain that

$$Z_1 \cup Z_2 \cup \bigcup_{i=1}^{N+1} M_i = \omega_f(z_1) \cup \omega_f(z_2) \subset \omega_f(z).$$

But conversely, if we fix any increasing sequence t_i, then, after passing to a subsequence if necessary, there is a sequence $\hat{t}_i$ such that $\gamma_{t_i} = f^{\hat{t}_i}(z_1)$ for all $i \geq 0$ or $\gamma_{t_i} = f^{\hat{t}_i}(z_2)$ for all $i \geq 0$. But $\{\gamma_j\}_{j=0}^{\infty}$ consists of blocks $f^{k_i}(z_1), \ldots, f^{k_{i+1}-1}(z_1)$ separated by blocks $f^{s_i}(z_2), \ldots, f^{s_{i+1}-1}(z_2)$ which implies we may assume that $\hat{t}_i$ is increasing, since s_i is increasing. But this implies that

$$\omega_f(z) \subset \omega_f(z_1) \cup \omega_f(z_2)$$

finishing the proof. $\square$

Theorem 12. *Let (X, f) be a transitive but not minimal dynamical system with TSLmSP. Then there exists a c-dense ω-scrambled set $Q \subset X$.*

Proof. It can be proved that transitive system with shadowing is either sensitive or equicontinuous [20]. But it is easy to see that if we take a minimal system $M \subset X$ and (X, f) is equicontinuous, then $X = M$, hence (X, f) is sensitive.

By results of [11] we know that transitive map with limit shadowing property (a one-sided version of TSLmSP) has SP.

By Theorem 5.2 in [22], for every nonempty open set U there is $m > 0$, a closed set $\Lambda \subset U$ invariant for f^m and a factor map $\pi \colon (\Lambda, f^m) \to (\Sigma_2^+, \sigma)$. Since Σ_2^+ contains an uncountable family of disjoint minimal and infinite sets (thus uncountable), there exist an uncountable set C_U and pairwise disjoint minimal sets $\hat{M}_c \subset \Lambda$, $c \in C_U$ for the dynamical system (Λ, f^m). Note that for each c the set $M_c = \bigcup_{i=0}^{\infty} f^i(\hat{M}_c) = \bigcup_{i=0}^{m-1} f^i(\hat{M}_c)$ is minimal and if for some $c, d \in C_U$ and $i \geq 0$ we have $f^i(\hat{M}_c) \cap \hat{M}_d \neq \emptyset$, then $f^i(\hat{M}_c) = \hat{M}_d$. Therefore, if we define $c \approx d$ when $f^i(\hat{M}_c) = \hat{M}_d$ or $f^i(\hat{M}_d) = \hat{M}_c$ for some $i \geq 0$, then $\approx$ is an equivalence relation on C_U with finite equivalence classes. This implies that we may assume (selecting exactly one element of each class) that minimal sets M_c are pairwise disjoint for every $c \in C_U$. By the construction each set M_c is uncountable.

Let us select a countable basis $\{U_i\}_{i=1}^{\infty}$ of the topology on X and for each U_i let us perform the above construction to obtain a family of minimal sets M_c with $c \in C_{U_i}$, where each C_{U_i} is uncountable. We claim that there are uncountable sets $D_i \subset C_{U_i}$ such that $M_c \cap M_d = \emptyset$ provided that $c \in D_i$, $d \in D_j$ and $i \neq j$.

First, for each i denote $C_i^{(1)} = C_{U_i}$. Up to a bijection (not necessarily continuous), we can identify $C_1^{(1)}$ with $[0,1]^2$. Put $W = \emptyset$. Now suppose that for some $r \in [0,1]$ there is $j > 1$ such that the set of indexes $c \in C_j^{(1)}$ such that $M_c \neq M_{(r,d)}$ for all $d \in [0,1]$ is at most countable (i.e., all but countably many minimal sets with indexes in $C_j^{(1)}$ are in the fibre of $C_1^{(1)}$ at the level r). In such a case we include r in W. We perform this operation for each $j = 2, 3, \ldots$. As a result of this (infinite) procedure we obtain at most countable set W such that if we fix any $s \in [0,1] \setminus W$, then the set of $c \in C_j^{(1)}$ such that $M_c \neq M_{(s,d)}$ for all $s \in [0,1]$ is uncountable for each $j > 1$. Fix an index $s \in [0,1] \setminus W$ and denote $D_1 = \{s\} \times [0,1] \subset C_1^{(1)}$. Now, if we remove from each $C_j^{(1)}$ indexes defining set M_c for some $c \in D_1$, and denote the result of this operation by $C_j^{(2)}$, for each $j > 1$, then by our construction every set $C_j^{(2)}$ is uncountable. We can perform this construction inductively, using family $\left\{ C_j^{(n)} \right\}_{j \geq n}$ for the construction of D_n the same way as it was done with D_1. This proves the claim.

Finally, we may pick a minimal set M and guarantee that it is disjoint with all M_c, $c \in D = \bigcup_{i>0} D_i$ (e.g. we can remove one set from D_1). Now, for each $c \in D$ we use Lemma 10 to construct a point z_c such that $M_c \cup M \subset \omega_f(z_c)$ and $\omega_f(z_c) \setminus M_c \cup M$ is at most countable (and in particular $z_c \neq z_d$). This implies that $M_d \cap \omega_f(z_c) = \emptyset$ for each $c \neq d$ since M_d is uncountable and disjoint with M_c and M. We may also assume (replacing z_c by its forward iterate) that $z_c \in U_i$ for each $c \in D_i$ because $U_i \cap M_c \neq \emptyset$ for every $c \in D_i$.

Denote $Q = \{z_c : c \in \bigcup_{i=1}^{\infty} D_i\}$. Then Q is c-dense simply by the definition. But now, if we fix any distinct $x, y \in Q$, then there are distinct c, d such that $\omega_f(x) = M \cup M_c \cup Z_1$ and $\omega_f(y) = M \cup M_d \cup Z_2$, where Z_1 and Z_2 are at most countable. In particular $M_d \cap \omega_f(x) = \emptyset$ and $M_c \cap \omega_f(y) = \emptyset$. This shows that x, y form an ω-scrambled pair completing the proof. $\square$

It seems that there is no chance to expect that the set of all ω-scrambled pairs is G_δ in $X \times X$. So in practice there is no chance to use standard arguments of construction of Mycielski set. From that point of view the following question can be of interest.

Question 1. Is it possible to construct set Q in Theorem 12 in such a way that it is Borel?

By standard argument (e.g. see [28]), a positive answer to the above question immediately implies that there is a dense Mycielski ω-scrambled set in the system.

5. Topological graphs and ω-chaos

The aim of this section is to extend the results of [29] (proved for interval maps) onto the class of maps on topological graphs. While the results are more general, the proof is somehow shorter (which does not mean elementary).

The following fact completes [29, Proposition 1]. Among other things it allows to transfer Cantor ω-scrambled set from higher iterate f^m to f. Note that $\omega(x, f) = \bigcup_{i=0}^{n-1} \omega(f^i(x), f^n) = \bigcup_{i=0}^{n-1} f^i(\omega(x, f^n))$ and then (under certain conditions) in an ω-scrambled set S for f^m, there are at most m points $p, q \in S$ such that $\omega(p, f) \backslash \omega(q, f)$ is at most countable and we can define an equivalence relation in $S \times S$ with uncountably many equivalence classes (see the proof of Theorem 3.5 in [15]). This relation, however, is not necessarily Borel, hence we cannot ensure that we can pick elements of these classes in such a way that a perfect set is obtained. Therefore we need to perform a little more delicate construction, to remove this difficulty. For this purpose, we are going to adjust the argument in the proof of [29, Proposition 1].

Theorem 13. *Let (X, f) be a dynamical system and let Λ and $m > 0$ be such that (Λ, f^m) is conjugated with (Σ_2^+, σ). Then there exists a Cantor set $C subset \Lambda$ which is ω-scrambled for f.*

Proof. First, we will perform a construction from [29, Proposition 1] to show that there exists a perfect set $S subset \Sigma_2^+$ which is ω-scrambled for σ and satisfies the following conditions:

(1) $0^\infty \in \omega(x, \sigma)$ for every $x \in S$.
(2) $\omega(x, \sigma) = M_x \cup V_x \cup U_x \cup \{0^\infty\}$ where V_x, U_x are at most countable sets and M_x is a minimal set (and all these sets depend on x; are pairwise disjoint).

(3) for every $\varepsilon > 0$ there is $\delta > 0$ such that if $d(x,y) < \delta$, $x, y \in S$, then $\mathcal{H}_d(M_x, M_y) < \varepsilon$, where $\mathcal{H}_d$ is the Hausdorff metric induced by d.

Denote by $A \subset \Sigma_2^+$ the set containing all sequences with infinitely many occurrences of symbols zero and one, that is $a \in A$ provided that both sets $\{i : a_i = 0\}$ and $\{i : a_i = 1\}$ are infinite. Note that $\Sigma_2^+ \setminus A$ is at most countable, and hence A is a Borel set. Then there is a Cantor set $\hat{A} \subset A \subset \Sigma_2^+$ (see [28]). Denote by $\mathcal{N}$ a partition of $\mathbb{N}$ by the following sets $N_n = 2^{n-1}(1 + 2\mathbb{N}_0)$, e.g. $N_1 = \{1, 3, 5, \ldots\}$, $N_2 = \{2, 6, 10, \ldots\}$, etc. Using the partition $\mathcal{N}$, we define a map $\Phi \colon \Sigma_2^+ \to \Sigma_2^+$ putting $\Phi(x)_i = x_n$ where n is the unique index such that $i \in N_n$. Roughly speaking, $\Phi(x)$ has the following structure

$$\Phi(x) = x_1 x_2 x_1 x_3 x_1 x_2 x_1 x_4 x_1 x_2 x_1 x_3 x_1 x_2 x_1 x_5 x_1 \cdots.$$

Clearly Φ is continuous and one-to-one, hence the set $D = \Phi(\hat{A})$ is a Cantor set. For every $\beta \in D$ denote $D_\beta = \omega(\beta, \sigma)$. It follow directly by the construction that D_β is a minimal set and $\beta \in D_\beta$. Note that if $x \in D_\beta$, then $x_n = \beta_1$ for all $n \in N_1$ or all $n \in N_1 + 1$ and if $x_n = \beta_1$ for all $n \in N_1$, then we can find $i, j \in N_1 + 1$ such that $x_i \neq x_j$ (or $i, j \in N_1$ in the second possible case). We can repeat this observation for N_2, N_3, etc. showing that $D_\beta \neq D_\alpha$ when $\beta \neq \alpha$.

Note that if for some $k > 0$ we define $\Phi_k(x)_i = x_n$ where n is the unique index such that $i \in N_n$ and $n \leq k$ and $\Phi_k(x) = *$ otherwise, where $*$ is an additional symbol other than $0, 1$, then $\Phi_k(x)_{[i, i+k]}$ contains at most one occurrence of $*$. Furthermore, observe that if we fix any i and put 0 or 1 in place of $*$ in the word $w = \Phi_k(x)_{[i, i+k]}$, then w can be located somewhere in $\Phi(x)$. This implies that if $x_{[1,k]} = y_{[1,k]}$, then $B_{k+1}(D_{\Phi(x)}) = B_{k+1}(D_{\Phi(y)})$ and hence $\mathcal{H}_d(D_{\Phi(x)}, D_{\Phi(y)}) \leq 2^{-k}$.

Let $\beta = \Phi(x)$ and define

$$\Psi(x) = \beta_1 0 \beta_1 \beta_2 0 0 \beta_1 \beta_2 \beta_3 0 0 0 \beta_1 \cdots \beta_{[1,n]} 0^n \cdots.$$

Denote $U_x = \left\{ \beta_{[1,n)} 0^\infty : n \geq 1 \right\}$ and $V_x = \{0^n \beta : n \geq 1\}$. Then U_x, V_x are countable sets and $\omega(\Psi(x), \sigma) = \{0^\infty\} \cup U_x \cup V_x \cup M_{\Phi(x)}$, e.g. by Lemma 2.2 in [15]. Combining all the previous remarks together and putting $S = \Phi(\hat{A})$ we obtain that conditions (1)–(3) are satisfied.

Since (Λ, f^m) is conjugated with (Σ_2^+, σ), without loss of generality we may assume that $S \subset \Lambda$. For every $x \in X$ denote $K_x = \bigcup_{i=0}^\infty f^i(M_x) = \bigcup_{i=0}^{m-1} f^i(M_x)$. Denote by R the set of pairs in $S \times S$ that are ω-scrambled.

This implies that for every $x \in S$ there are at most m other points $z \in S$ such that $K_x = K_y$. Clearly, for each $x \in S$ the set K_x is minimal and $\omega(x, f) \setminus K_x$ is at most countable. By (3), for every $\varepsilon > 0$ there is δ such that if $K_x \cap K_y = \emptyset$, then $K_p \cap K_q = \emptyset$ provided that $p \in B(x, \delta) \cap S$ and $q \in B(y, \delta) \cap S$. This proves that R is an open set. But if we fix any distinct $x, y \in S$ and any open neighborhood U of x, then $U \cap S$ is infinite, and so there is $z \in U \cap S$ such that $K_z \cap K_y = \emptyset$. This proves that the pair (z, y) is ω-scrambled and therefore R

is a residual subset of $S \times S$. By Lemma 1 there is a Cantor set $C \subset S$ such that $C \times C \subset R \cup \Delta$. The proof is completed. $\qquad\square$

The following fact was proved in [21] for interval maps, however the proof is the same in the context of graph maps. The only difference is that at first we have to use [16] to find a horseshoe for an iterate of map with positive entropy.

Lemma 14. *Let (G, f) be a dynamical system with positive topological entropy acting on a topological graph G. Then there exist $n > 0$, an f^n-invariant closed set Λ and a homeomorphism $\pi\colon \Lambda \to \Sigma_2^+$ such that $\pi \circ (f^n|_\Lambda) = \sigma \circ \pi$.*

Lemma 15. *Let G be a topological graph and let (G, f) be a topologically mixing dynamical system. Then there exists a dense ω-chaotic Mycielski set.*

Proof. It is well known that (G, f) has positive topological entropy (see [8] for proof of this fact in much wider context). Let Λ and n be provided by Lemma 14. By Theorem 13 there is an ω-scrambled Cantor set C for (G, f).

By Theorem 4.6 in [10] there is a finite set $I(f)$ such that for every nonempty open set U and every $\varepsilon > 0$ there is $n > 0$ such that $f^n(U) \supset G \setminus B(I(f), \varepsilon)$. Since $I(f)$ is finite, without loss of generality we may assume that $C \cap I(f) = \emptyset$ and hence there is $\gamma > 0$ such that $C \cap B(I(f), \gamma) = \emptyset$. Therefore, for every nonempty open set U there is $n > 0$ such that $C \subset f^n(U)$.

Since C is homeomorphic with $C \times C$, there exist a sequence of pairwise disjoint Cantor sets $C_1, C_2, \ldots \subset C$. If V is an open set and $n > 0$ is such that $C_i \subset f^n(\overline{V})$, then $A = f^{-n}(C_i) \cap \overline{V}$ is compact, and hence by [28, Remark 4.3.6] there is a Cantor set $D_i \subset \overline{V}$ such that f^n is injective on D_i and $f^n(D_i) \subset C_i$. Fix any base $U_1, U_2, \ldots$ of the topology of X and open sets V_i such that $\overline{V_i} \subset U_i$. For every i there is n such that $C_i \subset C \subset f^n(V_i) \subset f^n(\overline{V_i})$ and as a result there is also a Cantor set D_i such that $f^n(D_i) \subset C_i$ and if we fix distinct $x, y \in D_i$, then $f^n(x), f^n(y) \in C_i$ are also distinct. Clearly for every $x \in X$ and $n > 0$ we have $\omega(x, f) = \omega(f^n(x), f)$, thus $M = \bigcup_{i=1}^\infty D_i$ is a Mycielski ω-scrambled set. But $\emptyset \neq M \cap \overline{V_i} \subset M \cap U_i$ which proves that M is also dense, completing the proof. $\qquad\square$

Theorem 16. *Let G be a topological graph. If (G, f) is transitive dynamical system other than irrational rotation, then there exists a dense Mycielski ω-chaotic set for f.*

Proof. The only examples of transitive graph homeomorphisms are the irrational rotations of the circle, hence we may assume that (G, f) is not invertible. By periodic decomposition theorem (see [3] or [1]) there are graphs $G_0, \ldots, G_{s-1} \subset G$ with disjoint interiors, such that $f(G_i) = G_{i+1(\mathrm{mod}\ s)}$ and $f^s|_{G_i}$ is totally transitive (hence mixing, see [2, 4] or [9] for a proof).

By Lemma 15 there is a dense Mycielski set $M \subset G_0$ for $f^s|_{G_0}$. Graphs G_i have disjoint interiors, hence without loss of generality we may assume that $M \cap G_i = \emptyset$ for each $i \neq 0$. Since M is a dense Mycielski set, it is easy to find disjoint dense Mycielski sets $M_0, \ldots, M_{s-1} \subset M$. Additionally each M_i is dense

and $f^i(G_0) = G_i$, hence the set $f^i(M_i)$ is dense in G_i. But M_i is scrambled set, thus $f|_{M_i}$ is one-to-one. This proves that $f^i(M_i)$ is Mycielski set and so also $S = \bigcup_{i=0}^{s-1} f^i(M_i)$ is a dense Mycielski set. The proof is complete. $\square$

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