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TRANSLATING THE HYPERGAME PARADOX: REMARKS ON THE SET OF FOUNDED ELEMENTS OF A RELATION

ABSTRACT. In Zwicker (1987) the hypergame paradox is introduced and studied. In this paper we continue this investigation, comparing the hypergame argument with the diagonal one, in order to find a proof schema. In particular, in Theorems 9 and 10 we discuss the complexity of the set of founded elements in a recursively enumerable relation on the set N of natural numbers, in the framework of reduction between relations. We also find an application in the theory of diagonalizable algebras and construct an undecidable formula.

1. INTRODUCTION

One feature shared by the majority of logical and set-theoretical paradoxes is the liar's symmetry 'p is true iff p is not true'; as is well known, the same structure, translated into the diagonal schema, becomes a useful tool to produce proofs.

A few years ago, Zwicker (1987) introduced an asymmetric paradox, the so-called *hypergame paradox*, which also can be used as a schema to produce proofs by contradiction. Consider games between two players A and B. We use the word 'game' in two different meanings, to denote both a game in general and any particular competition between A and B. Given a game G, we will say 'a game of G', to denote a single competition which proceeds according to the rules of G. Call a game G *founded* if every game of G must terminate after finitely many moves, that is, following the rules, it is impossible for a game of G to go on for ever, even if a priori there is no fixed bound on the lengths of games of G.

Now, define the hypergame as follows: player A chooses a founded game G, then player B makes the first move in G and the game continues according to the rule of G. (Of course, there are simple winning strategies for A, but we are not concerned with them – in fact, we only specify legal moves, while the result of the game, who wins and who loses, has no importance.) Does this game have an end? Since G is founded, the answer is obviously 'yes'. In other words, hypergame is a founded game. As a consequence, in a game of hypergame, player A can choose the hypergame itself as a founded game, giving B the right to choose the

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founded game. But, if B decides to be as bizarre as A and also chooses hypergame, and A in turn repeats 'let's play hypergame', and so on, we get an unfounded game in which both players move according to the rules of a founded game.

Note that here the argument which leads to the paradox has an asymmetric pattern which consists of two parts: (1) 'p is true' (hypergame is a founded game); (2) 'if p is true then $\neg p$ is true' (if hypergame is founded, then it is not founded). In other words, we arrive to a contradiction by showing 'p' and '$p \to \neg p$', rather than '$p \leftrightarrow \neg p$'. (The hypergame argument involves the classical inference rule 'from $p \to \neg p$ deduce $\neg p$' which was called *consequentia mirabilis* in the Renaissance.)

The structure of the hypergame paradox may be adapted and proposed in several contexts. Some earlier statements of the paradox (even if not completely clear) can be found in non-technical papers, as Gardner (1984), Smullyan (1983). A more interesting earlier version is due to Shen-Yuting and dates back to 1953 (Shen-Yuting, 1953): the argument is essentially the same, but it is only concerned with naive set theory. Here the notion of a grounded set is considered: a set x is *grounded* if there is no sequence $(x_n)_{n \in N}$ with $x_0 = x$ and $x_{n+1} \in x_n$ for each natural number n. Consider the set G of all grounded sets. It is easy to see that G is grounded, because if (x_n) were a sequence such that $x_0 = G$ and $x_{n+1} \in x_n$ for each n, the sequence $y_n = x_{n+1}$ would show that x_1 was ungrounded; but this is in contradiction with $x_1 \in x_0 = G$. As G is grounded, we have $G \in G$, and the constant sequence $\cdots \in G \in G \in G$ proves that G is ungrounded, a contradiction.

A not-too-different structure can be found in other known paradoxes, for instance in Girard's one in λ-calculus. In any case, asymmetric paradoxes belonging to the hypergame family can be readily constructed. Let us see an example in topology. Call a topological space *Noetherian* if it contains no infinite strictly descending chain of closed sets (with respect to the relation of inclusion). Given a family $(X_i)_{i \in I}$ of Noetherian spaces, we introduce a topology on the disjoint union X of the spaces X_i: the open sets are the empty set and the subsets A of X such that $A \cap X_i$ is open in X_i for each $i \in I$, and moreover $A \cap X_i = X_i$ for almost all i. It is easy to show that X is in turn a Noetherian space. Consider now a family $(X_i)_{i \in I}$ of spaces that contains (a homeomorphic copy of) every Noetherian space, and let X be the space obtained as before. By hypothesis, there is an index i_0 such that X_{i_0} is homeomorphic to X by means of a function f: $X \to X_{i_0}$. We have $f(X) = X_{i_0} \subset X$, where the inclusion is strict and X_{i_0} is closed in X. It follows that $f^{n+1}(X) \subset f^n(X)$ for each natural n, a contradiction, because each $f^n(X)$ is closed.

In natural language, the most popular version of a symmetric paradox is obtained by considering a barber that shaves every man that does not shave himself; but, if one consider a barber that shaves every man that does not shave himself, *except* for the barber himself, the paradox is avoided. Van Benthem (1978) showed that a similar construction is not enough to avoid Russell's paradox in naive set theory: if we assume that a set d exists such that, for every set n different from d, $(n \in d \leftrightarrow n \notin n)$, we again arrive at a contradiction, just considering the set $d - \{d\}$ if $d \in d$, and the set $d \cup \{d\}$ otherwise. The same observation applies to the Shen-Yuting paradox: if we assume the existence of a set i such that for every set n different from i ($n \in i \leftrightarrow n$ is grounded), we again arrive at a contradiction. Indeed, starting from i, we can construct a set h that contains exactly all grounded sets, regardless of whether $n = i$ or not; it is enough to define $h = i \cup \{i\}$ if i is grounded or $h = i - \{i\}$ if i is not grounded. In both cases, we get a contradiction following the same argument as in Shen-Yuting's paradox.

In this paper we investigate the hypergame paradox, comparing it with diagonal paradoxes, and showing that also the hypergame paradox can be translated into a proof schema. Our aim, first of all, is to give insight into the situation; to this end, some related questions are discussed and some applications are provided.

2. THE HYPERGAME SCHEMA

The core of the diagonal method consists of a binary relation R on a set X, and the observation that there is no element d of X such that xRd iff not xRx (otherwise, by replacing x by d we would have dRd iff not dRd, a symmetric contradiction). This proof schema, adapted to various situations by choosing appropriately a set X and a relation R, is used in different contexts of mathematical logic, such as recursion theory, proof theory, and set theory.

Let us come to the task of extracting a proof schema from the hypergame paradox. Again, the core of the method is a binary relation R on a set X; we say that an element $x \in X$ is *founded* if there is no sequence $(x_n)_{n \in N}$ with $x_0 = x$ and $x_{n+1} R x_n$ for each natural number n. The *hypergame method* consists of the following observation: there is no element i in X such that xRi iff x is founded (w.r.t. R). If there were such an i, it would be a founded element: every sequence (x_n) contradicting the foundedness of i is such that x_1 is founded, while the sequence $y_n = x_{n+1}$ shows that x_1 is not founded. But if i is founded, then iRi and i is not founded, a contradiction.

Let us compare the diagonal method with the hypergame method. First, remark that, while negation has a crucial rôle in the diagonal method, in the sentence 'xRi iff x is founded' negation does not appear (at least if 'infinite' is regarded as negation of 'finite', and not conversely).

EXAMPLE 1. We can prove the uncountability of the set of real numbers both in the usual way, that fits the schema of a diagonal proof, and using a hypergame argument. In both cases $X = N$; if $(a_n)_{n \in N}$ is a sequence that contains all real numbers, for each n fix a decimal representation of a_n, and call $a_{n,m}$ the mth digit of a_n. Then define the binary relation R on N as mRn iff $a_{n,m} = 1$. Up to this point, the diagonal and the hypergame proofs follow the same path. Now, the classical diagonal proof refers to the number $z = 0, b_1 b_2 \ldots$, where $b_n = 1$ if $a_{n,n} \neq 1$, and $b_n = 2$ otherwise, and gets a contradiction.

If we prefer a hypergame proof, instead, we define the number $t = 0, b_1 b_2 \ldots$ as the real number such that $b_n = 1$ iff n is founded w.r.t. R, $b_i = 2$ otherwise. Let i be such that $a_i = t$. We conclude that nRi iff $a_{i,n} = 1$, iff $b_n = 1$ iff n is founded, reaching in this way a contradiction.

EXAMPLE 2. A classical application of the diagonal method is Cantor's proof of the non-existence of a surjective function from X to the power set of X. Let us compare diagonal and hypergame methods in this context. Suppose that there is such a function h; then define a relation R on X as xRy iff $x \in h(y)$. When using a diagonal argument, we consider an element d such that $h(d) = \{x \in X/ \text{ not } xRx\}$; if we choose the hypergame method, the set of non-reflexive elements of the relation R is replaced by the set of founded elements of R. If i is such that $h(i) = \{n/n \text{ is founded w.r.t. } R\}$, then mRi iff $m \in h(i)$ iff m is founded, and we get a contradiction.

EXAMPLE 3. Here we work in recursion theory. Let xEy iff $x \in W_y$ for natural numbers x and y, where $(W_x)_{x \in N}$ is the standard enumeration of recursively enumerable sets. The diagonal method on E gives us a proof that the set $\overline{K} = \{x \in N/x \notin W_x\}$ is not r.e. By using a hypergame argument, we show that the set $F = \{n/n \text{ is } E\text{-founded}\}$ is not r.e. Indeed, if this were not the case, there would be an i such that $F = W_i$, and again we would have nEi iff $n \in W_i$ iff n is founded, a contradiction.

For any binary relation R on a set X, the set $F_R = \{n/n \text{ is founded}\}$ is a subset of the set $\overline{D} = \{n/ \text{ not } nRn\}$, even if these sets can differ from each other. In fact, F_R and $\overline{D}$ present some analogies, but do not share all properties. Let us compare, in particular, the sets F_E and $\overline{K}$ of Example 3. They are both productive in a very natural way: if $W_x \subseteq F_E$, then x is founded, and therefore cannot belong to W_x. However, F_E is much more complicated than $\overline{K}$: in fact, $\overline{K}$ is co-r.e. and Π^0_1-complete, while we will show that F_E is Π^1_1-complete.

In general, we may appreciate the difference between F_R and $\overline{D}$ by considering the operator *dom*: $P(X) \to P(X)$ that maps the set A to $dom\, A = \{x \in X/\{y/yRx\} \subseteq A\}$ (this definition is from recursion theory – see for instance (Rogers, 1967)). It can be shown that F_R is the least fixed point of the operator *dom*. Moreover, if $I = \{A \in P(X)/domA \subseteq A \text{ and } A \cap D = \varnothing\}$, the set F_R is the least element in I, with regard to the inclusion relation, while $\overline{D}$ is the greatest element. Note that the hypergame argument works for every A in I: there exists no i in X such that xRi iff $x \in A$. Indeed, if $\{x/xRi\} \subseteq A$, then $i \in domA \subseteq A$ but $i \notin \{x/xRi\}$ because $A \cap D = \varnothing$. Adapted to the relation E of Example 3, this argument proves that every set in I is productive, with the identity map as a productive function. Actually, in recursion theory it is not hard to prove more: every set A containing all indices of the empty set and no index of N is productive*.

For instance, call a number x *finite* if there exists an n that bounds the length of any descending E-chain starting from x: the set of finite numbers is productive, but does not belong to the family I (that means that the hypergame argument does not work for this set).

We end this section by fitting into the hypergame schema a paradox and a proof already known in set theory.

As usual, consider an ordinal number α as the set of all the ordinal numbers that are less than α. Now, for every ordinal number α we can construct a founded game as follows: player A chooses an ordinal β in α, that is, a β less than α, then B chooses a γ in β and so on (as before, we are not concerned in winning strategies). For every α the game is founded because the ordinal number α is a well-ordered set. If only these games are considered, the hypergame is started by choosing an element of the 'set' Ω of all ordinal numbers. In some sense, we can think of Ω as the set of all well-ordered sets. But since Ω is in turn a well-ordered set (we are in the realm of naive-set theory), we can write $\Omega \in \Omega$, and we conclude that Ω is not well ordered. From this point of view, the hypergame paradox is closely related to Burali-Forti's paradox (cf. (Burali-Forti, 1897)).

An example of a proof that follows a schema not too different from the hypergame paradox occurs in (axiomatic) set theory when proving, without the axiom of choice, that there are uncountably many ordinal numbers. One proves that the collection $\Gamma(\omega)$ of all ordinals which are embeddable in ω is in turn an ordinal number, called the Hartogs number of ω (this is the affirmative part of the hypergame schema); then that if $\Gamma(\omega)$ were countable one would have $\Gamma(\omega) \in \Gamma(\omega)$, a contradiction (this is the part '$p \to \neg p$' of the hypergame).

3. THE COMPLEXITY OF THE SET OF FOUNDED ELEMENTS

Let us restrict ourselves to relations on natural numbers. If a relation R is r.e., the set $\overline{D}$ of non-reflexive elements of R is co-r.e., while the set F_R is Π^1_1. In fact, F_R is Π^1_1 also if R is an arithmetical relation: let $A(x)$ be a formula that defines the relation R in the set N, and let

$$\begin{aligned} V &= \{u/ \text{ there is a number } n \text{ such that } u = \langle x_1, \ldots, x_n \rangle \\ &\qquad \text{and } x_1 R x_2 \cdots R x_n\} \end{aligned}$$

where $\langle -, \ldots, - \rangle$ denotes a code for finite sequences. V is still an arithmetical set, and $x \in F_R$ iff $\forall f\ (f(0) = x \to \exists n \langle f(n), \ldots, f(0) \rangle \notin V)$.

At the end of this section we will find a global property of r.e. relations that implies the Π^1_1-completeness of the set of founded elements of R. For the moment, let us consider the case where the relation is the membership relation of Example 3. In this case the set F_E of founded elements is recursively isomorphic to the domain O of Kleene's ordinal notation. This fact is not surprising: an effective version of Burali-Forti construction moves from the 'set' Ω of all ordinal numbers to the set O. At the same time, F_E is related to the set T of indices of founded recursive trees, and it is known that this set is Π^1_1-complete.

Remark 4. A direct proof of the Π^1_1-completeness of F_E may be obtained as follows. Let A be a Π^1_1 set and let Q be a recursive relation such that

$$x \in A \quad \text{iff} \quad \forall f\, \exists y\, Q(x, \langle f(0), f(1), \ldots, f(y) \rangle)$$

(we are applying the normal form for Π^1_1 sets, cf. for example (Odifreddi, 1989)).

Let g be a recursive function such that

$$\begin{aligned} W_{g(x,u)} &= \{g(x,t)/u \text{ is a sequence number,} \\ &\qquad t = u * \langle k \rangle, k \in N, \neg Q(x,t)\}. \end{aligned}$$

If we define h as $h(x) = g(x, \langle\varnothing\rangle)$, one can prove that

$$h(x) \notin F_E \text{ iff there exists a function } f \text{ with } \neg Q(x, \langle f(0), f(1), \ldots, f(n)\rangle),$$

for each $n \in N$. It is easy to show that the function h reduces the set A to the set F_E and this proves that F_E is Π^1_1-complete.

Obviously, the same argument applies to the case in which a relation E' is defined referring to an acceptable system of indices for partial recursive functions $(\psi_x)_{x\in N}$ as $xE'y$ iff $x \in domain \psi_y$. In fact, it is enough that the system satisfies the *parametrization* property: there is a recursive function f such that $\phi_x \sim \psi_{fx}$, for each number x. If this is the case, let $t(x, y)$ be a recursive injective function such that $W_{t(z,x)} = f\phi_z W_x$; by the s^n_m-Theorem in the standard system of indices, there exists a recursive function $h(z)$ such that $\phi_{hz}x = t(z, x)$. Let a be a number that verifies $\phi_{ha} \sim \phi_a$. We have:

$$f\phi_a(W_y) = W_{t(a,y)} = W_{\phi_{ha}(y)} = W_{\phi_a(y)} = domain \psi_{f\phi_a(y)}.$$

It follows that xEy implies $f\phi_a(x)E'f\phi_a(y)$, and that if $uE'f\phi_a(y)$, there is a number x such that xEy and $f\phi_a(x) = u$. Thus we have $x \in F_E$ iff $f\phi_a(x) \in F_{E'}$, and $F_{E'}$ is Π^1_1-complete.

Going back to the general case of a binary relation R, we note that the arithmetical complexity of the set of founded elements does not reflect the complexity of the relation itself. Indeed, R may be arbitrarily complex but have $F_R = \varnothing$ (it suffices to add a minimum reflexive element to a relation of the needed complexity), while the following relation R is recursive and the set of founded elements is Π^1_1-complete.

EXAMPLE 5. Let A be a Π^1_1-complete set, and let Q be a recursive relation such that

$$x \in A \quad \text{iff} \quad \forall f\, \exists y\, Q(x, \langle f(0), f(1), \ldots, f(y)\rangle).$$

Let R be

$$\langle x, a_0, \ldots, a_n\rangle R \langle x, a_0, \ldots, a_{n-1}\rangle \quad \text{iff} \quad \neg Q(x, \langle a_0, \ldots, a_n\rangle).$$

We have

$$x \in A \quad \text{iff} \quad \langle x\rangle \text{ is founded w.r.t. the relation } R.$$

Thus A reduces to F_R via the recursive function $h(x) = \langle x\rangle$.

Let us now look for a property of relations that implies the Π^1_1-completeness of F_R.

DEFINITION 6 (cf. for instance (Bernardi and Sorbi, 1983)). A relation S is said to be *reducible* to a relation R if there exists a recursive function f such that xSy iff $f(x)Rf(y)$. We say that the function f *reduces* S to R.

An r.e. relation R is said to be *m-complete* (respectively, 1-complete) if, for each r.e. relation S, there exists a recursive function (respectively, injective) f that reduces S to R.

LEMMA 7. *An m-complete relation is* 1*-complete.*

Proof. Let R be m-complete, and let S be an arbitrary r.e. relation. Embed N in $N \times \{0, 1\}$, by sending x to $(x, 0)$; define S' on $N \times N$ as: $(x, 0)S'(y, 0)$ iff xSy, and $(x, 0)S'(x, 1)$, for each $x, y \in N$. By codifying $N \times \{0, 1\}$ in N, we may look at S' as a relation on N. If f' is the recursive function that reduces S' to R, the function $f(x) = f'(x, 0)$ must be injective and reduces S to R. □

Since m-completeness and 1-completeness coincide, we will use only the term completeness.

DEFINITION 8. An r.e. relation S is said to be *strongly reducible* to an r.e. relation R if there is a recursive function f such that f reduces S to R, and moreover $zRf(x)$ implies $z \in Imf$. An r.e. relation R is said to be *strongly complete* if for each r.e. relation S there exists a recursive function f that strongly reduces S to R (as before, we may require that the function f is injective without affecting the generality of the definition).

THEOREM 9. *If R is strongly complete, then the set of founded elements of R is Π^1_1-complete.*

Proof. If f is a recursive function that strongly reduces the relation E to R, then F_E is reducible, via f, to the set F_R of founded elements of R. Since F_E is Π^1_1-complete, we conclude that F_R is Π^1_1-complete too. □

THEOREM 10. *The relation $E = \{(x, y)/x \in W_y\}$ is strongly complete.*

Proof. Given an r.e. relation S, let $\psi(x, y)$ be a recursive function such that $W_{\psi(x,y)} = \{\phi_x(t)/tSy\}$. By the s^n_m-Theorem we may assume that ψ is injective. By the Recursion Theorem, there exists a number n such that the partial recursive functions $\psi(n, -)$ and $\phi_n(-)$ are equal. Note that ϕ_n is in turn total and injective, since so is ψ; thus $W_{\phi_n(y)} = \{\phi_n(t)/tSy\}$, and the function ϕ_n strongly reduces the relation S to E. Since this holds for each r.e. relation S, we conclude that E is strongly complete. □

EXAMPLE 11. There exists a complete relation that is not strongly complete.

Let R be the relation

$$R = (S+1) \cup \{\langle 0, n\rangle / n \in N\},$$

where S is a complete relation, for example E, and $S+1$ is $\{\langle x+1, y+1\rangle / \langle x, y\rangle \in S\}$. Since $0Rn$ for each $n \in N$, the set of founded elements in R is empty, and the relation R cannot be strongly complete. On the other hand, if a recursive injective function f reduces an r.e. relation to S, the recursive injective function $f'(x) = f(x)+1$ reduces the same relation to R.

Remark 12. Theorem 10 allows us to embed any r.e. relation R into E. We can get a more precise statement: if R is an r.e. relation on N, then there is a transitive element z (that is, an element z such that xEz and yEx implies yEz) such that R is recursively isomorphic to $E|_{W_z \times W_z}$; moreover, $z \in F_E$ iff R is founded. The proof follows along the same line as in Theorem 10.

Finally, we note that Myhill's Theorem does not hold for strong reduction between relations; there are relations which are strongly complete but not isomorphic. Indeed, we define a relation K_0 as $\langle x, z\rangle K_0 \langle y, z\rangle$ iff $\langle x, y\rangle \in W_z$; it is readily seen that K_0 is strongly complete. On the other hand, E and K_0 are not recursively isomorphic: if z is an index for the empty set, $\{y / \langle x, z\rangle K_0 y\} = \varnothing$, while for each $x \in N$ the set $\{y / xEy\}$ is not empty.

4. AN APPLICATION TO DIAGONALIZABLE ALGEBRAS

We shall see how the strong completeness of a relation R may be used to construct a generic Magari algebra. Recall that a *Magari algebra* (formerly called diagonalizable algebra) is a pair $(D, \Box)$, where D is a Boolean algebra with operations $0, 1, \wedge, \neg, \vee$, and $\Box$ is a map from D to D satisfying the following identities:

(1) $\Box 1 = 1$;

(2) $\Box(a \wedge b) = \Box a \wedge \Box b$;

(3) $\Box(\Box a \to a) = \Box a$ (where, as usual, $a \to b$ is $\neg a \vee b$).

The main example of a Magari algebra is the Lindenbaum algebra D_T of an r.e. theory T extending Peano Arithmetic PA, where the

$\Box$ is defined as $\Box[p] = [Theor_T(\bar{p})]$ (here $[p]$ denotes the equivalence class of the sentence p with respect to provable equivalence in T, while $Theor_T(x)$ is the standard provability predicate of T). It has been proved that this algebra is generic in the class of Magari algebras, in the sense that every identity true in D_T holds in all Magari algebras (cf. (Smorynski, 1982) for a general introduction to Magari algebras).

Given any binary relation R on a non empty set X, we may endow the power set $P(F_R)$ with a structure of Magari algebra by defining the $\Box$ operator on $P(F_R)$ as the operator *dom* of the transitive closure of R:

$$dom_{\overline{R}}A = \{x/\{y/\exists x_1 \cdots x_n \text{ with } yRx_nR \cdots Rx_1Rx\} \subseteq A\}.$$

The structure $(F_E, dom_{\overline{E}})$ is generic in the class of Magari algebras. More generally:

THEOREM 13. *If R is strongly complete, then the Magari algebra $(P(F_R), dom_{\overline{R}})$ is generic in the class of Magari algebras.*

Proof. For the sake of brevity, we only sketch the proof. It is known that the class of finite Magari algebras is generic. Moreover, one can obtain any finite Magari algebra from a finite set X and from an irreflexive and transitive relation S on X, by considering $P(X)$ endowed with the operator dom_S as $\Box$. Thus to conclude that the algebra $(P(F_R), dom_{\overline{R}})$ is generic, we only have to show that, for each finite subset X of natural numbers and for each irreflexive and transitive relation S on X, there is an injective function f from X to F_R, such that:

(1) xSy implies $f(x)\overline{R}f(y)$;

(2) $z\overline{R}f(y)$ implies the existence of an $x \in X$ such that xSy and $z = f(x)$. Indeed, if this is the case, the function $f^*\colon P(F_R) \to P(X)$ defined as $f^*(A) = f^{-1}(A)$ is a surjective homomorphism of Magari algebras.

Being finite, the relation S is recursive. We may then apply Definition 8 to obtain an injective recursive function h such that xSy iff $h(x)Rh(y)$ and $\{t/tRh(y)\} \subseteq Imh$. This implies xSy iff $h(x)\overline{R}h(y)$, and, by considering the restriction of h to X, we obtain points (1) and (2) above. $\Box$

5. UNDECIDABLE FORMULAS BASED ON THE HYPERGAME

In this section we use the hypergame method to build undecidable formulas in ZF and in PA. In the proofs, we will refer to the standard models of these theories that we denote respectively by U and N, assuming in both cases the soundness hypothesis.

We identify natural numbers with formulas of ZF in one free variable, and we define $R \subseteq N \times N$ by

$$\beta R \alpha \quad \text{iff} \quad ZF \vdash \alpha(\bar{\beta})$$

(where $\bar{\beta}$ represents the natural number β in ZF).

The relation R is r.e. and can be represented in the universe U by means of a formula $R(x, y)$:

$$\beta R \alpha \quad \text{iff} \quad U \models R(\bar{\alpha}, \bar{\beta}).$$

Following the diagonal method, let $\overline{D} = \{\alpha / ZF \nvdash \alpha(\bar{\alpha})\}$. The formula $\delta(x) = \neg R(x, x)$ represents $\overline{D}$ in U, and $ZF \nvdash \delta(\bar{\delta})$ iff $U \models \delta(\bar{\delta})$. It follows that the formula $\delta(\bar{\delta})$ is undecidable: if $ZF \vdash \delta(\bar{\delta})$ then $U \models \delta(\bar{\delta})$ and $ZF \nvdash \delta(\bar{\delta})$; if $ZF \vdash \neg\delta(\bar{\delta})$, then $U \models \neg\delta(\bar{\delta})$, and $ZF \vdash \delta(\bar{\delta})$. In other words, assuming the completeness of ZF, we have a symmetric contradiction of type $p \leftrightarrow \neg p$.

Now with the hypergame: the set F_R of founded elements of the relation R is representable in U by the formula

$$\begin{aligned}\gamma(x) \;=\; & \forall f(f\colon \omega \rightarrow \omega \wedge f(0) = x) \\ & \rightarrow \exists n\big[n \in \omega \wedge \neg R(f(n+1), f(n))\big].\end{aligned}$$

THEOREM 14. *The formula $\gamma(\bar{\gamma})$ is undecidable in ZF.*

Proof. If $\alpha R \gamma$, then $ZF \vdash \gamma(\bar{\alpha})$ and $U \models \gamma(\bar{\alpha})$. From the definition of γ it follows that α is a founded formula with respect to R. We have $\{\alpha/\alpha R \gamma\} \subseteq F_R$ and so γ is founded. Thus $U \models \gamma(\bar{\gamma})$ and $ZF \nvdash \neg\gamma(\bar{\gamma})$. On the other hand, if $ZF \vdash \gamma(\bar{\gamma})$, then $\gamma R \gamma$, contradicting the foundedness of γ. We conclude that the formula $\gamma(\bar{\gamma})$ is undecidable in ZF. □

Remark 15. A similar construction is not allowed in PA. Identify formulas with one free variable with the corresponding Gödel numbers, and denote with $\bar{\beta}$ the numeral of (the Gödel number of) β. Now define the relation R as follows: $\beta R \alpha$ iff $PA \vdash \alpha(\bar{\beta})$. The relations R and E are recursively isomorphic: if $V_\alpha = \{\beta / PA \vdash \alpha(\bar{\beta})\}$, then (V_α) is an acceptable numbering and, by a Theorem of Blum (see (Odifreddi, 1989), Th.II.5.8), there exists a recursive permutation h such that $x \in W_y$ iff $hx \in V_{hy}$. In particular, it follows that the set of founded elements with respect to $R = \{(\beta, \alpha)/PA \vdash \alpha(\bar{\beta})\}$ is not representable in ω, being Π^1_1-complete.

A not completely satisfactory way out is to require a kind of uniformity on descending sequences, in such a way to obtain an arithmetical set.

More precisely, we define the set of *uniformly founded* formulas UF:

$$\begin{aligned}&\alpha \in UF \text{ iff there is no formula } \phi(x,y) \text{ such that}\\ &\quad PA \vdash \forall y\, \phi(0,y) \leftrightarrow \alpha(y) \text{ and, for each natural } k,\\ &\quad PA \vdash \phi(k, \phi(k+1,y))\end{aligned}$$

(where the formula ϕ has at most two free variables, and we omitted signs for numerals). In other words, in the definition of founded elements we consider only the sequences $(\alpha_k)_{k\in N}$ of kind $\alpha_k(x) = \phi(k,x)$, for a suitable formula ϕ.

UF is an arithmetical set; thus there exists a formula $\gamma(x)$ such that

$$\alpha \in UF \quad \text{iff} \quad N \models \gamma(\bar{\alpha}).$$

THEOREM 16. *The formula $\gamma(\bar{\gamma})$ is undecidable in PA.*

Proof. We only sketch the proof. First one proves easily that the formula $\gamma(x)$ is uniformly founded; thus $\gamma(\bar{\gamma})$ is a true formula, and since we supposed PA to be sound, the formula $\neg\gamma(\bar{\gamma})$ is not provable in PA. From $\gamma \in UF$ it follows $PA \nvdash \gamma(\bar{\gamma})$: if $PA \vdash \gamma(\bar{\gamma})$, the formula $\phi(x,y) = \gamma(x)$ would prove that $\gamma \notin UF$. □

In spite of the use of uniformity in the last proof, we followed the hypergame method in a genuine way. The schema of the proof, indeed, is not symmetric: assuming the completeness of PA, the implication $PA \vdash \gamma(\bar{\gamma}) \Rightarrow PA \vdash \neg\gamma(\bar{\gamma})$ is still a part of the proof, but we do not have $PA \vdash \neg\gamma(\bar{\gamma}) \Rightarrow PA \vdash \gamma(\bar{\gamma})$. In other words, $PA \vdash \neg\gamma(\bar{\gamma})$ has to be excluded not because it implies $PA \vdash \gamma(\bar{\gamma})$, but because from it we deduce that $\gamma(x)$ is a non uniformly-founded formula, contradicting the first part of the proof.

6. A FINAL REMARK

Referring to the concept of *ungrounded sentence*, suggested by Kripke (1975), we can try to avoid paradoxes in natural languages in the following way. Given a sentence a, consider all the sentences b whose truth is mentioned in a, then the sentences c whose truth is mentioned in some sentence b, and so on; call the initial sentence a 'grounded' if this process terminates. Now, 'we can assign a truth value only to grounded sentences'. And, of course, since this last statement refers only to grounded sentences, it is in turn grounded, ... is it?

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NOTES

* Indeed, A is contraproductive with respect to a total recursive function g. To see this, for every n define the set W_{fx} to be the empty set if $x \notin W_n$, or N if $x \in W_n$. Choose an a such that $W_a = W_{fa}$ and put $gn = a$. Now, suppose $A \subseteq W_n$. From $gn \notin W_n$ we deduce $W_{gn} = \varnothing$ and $gn \in A \subseteq W_n$. Therefore $gn \in W_n$; we conclude $W_{gn} = N$ and $gn \notin A$.

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