TRANSLATION IN MEASURE ALGEBRAS AND THE CORRESPONDENCE TO FOURIER TRANSFORMS VANISHING AT INFINITY

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Let G denote a locally compact (not necessarily abelian) group and M(G) the collection of finite regular Borel measures on G. The set M(G) is a semisimple Banach algebra with identity under convolution *. It can be identified with the dual space of $C_0(G)$, the space of continuous complex-valued functions on G that vanish at infinity, with the sup-norm. The group G has a left-invariant regular Borel measure dm(x) that is unique up to a constant and is called the left Haar measure of G. Let $C^B(G)$ denote the space of bounded continuous functions on G. For each $x \in G$, we define on $C^B(G)$ the left-translation operator by the relation

$$L(x)f(y) = f(x^{-1}y) \qquad (f \in C^B(G)).$$

We say that $f \in C^B(G)$ is right uniformly continuous if $L(x_\alpha)f \xrightarrow{\alpha} L(x)f$ uniformly, whenever $x_\alpha \xrightarrow{\alpha} x$. Let $C^B_{ru}(G)$ denote the subspace of $C^B(G)$ of right uniformly continuous functions. For $\mu \in M(G)$, define $L(x)\mu \in M(G)$ by the condition

$$\int_G f(t)\,dL(x)\mu(t) = \int_G L(x^{-1})f(t)\,d\mu(t),$$

where $f \in C_0(G)$. We wish to study for which $\mu \in M(G)$ the map $x \mapsto L(x)\mu$ is continuous from G into M(G), where M(G) will be equipped with an L(x)-invariant metric topology. In particular, we shall characterize $M_0(G)$, the algebra of measures whose Fourier transform vanishes at infinity.

Let $A \subset C^B_{ru}(G)$ be a linear subspace with sufficiently many elements to separate the points of M(G); in other words, if $\mu \in M(G)$ and if

$$\int_G f(t)\,d\mu(t) = 0$$

for all $f \in A$, then $\mu = 0$. We are then able to pair A and M(G) by the relation

$$\langle f, \mu \rangle = \int_G f(t)\,d\mu(t) \qquad (f \in A;\ \mu \in M(G)).$$

Let $\sigma(A, M(G))$ denote the weak topology on A induced by this pairing. Suppose A can be written as $\bigcup_{k=1}^\infty A_k$, where each A_k is a subset of A that is L(x)-invariant for all $x \in G$ and where each A_k is $\sigma(A, M(G))$-bounded. Note that A_k is

Received December 24, 1969.
This research was supported in part by NSF contract number GP 8981.

Michigan Math. J. 17 (1970).

$\sigma(A, M(G))$-bounded if and only if A_k is bounded in sup-norm. We let $\mathscr{T}(A_k)$ denote the topology on $M(G)$ of uniform convergence on the sets A_k. Note that $\mathscr{T}(A_k)$ gives an $L(x)$-invariant metric topology on $M(G)$. For $k \geq 1$, let

$$\tau_k(\mu) = \sup \{ | \langle f, \mu \rangle | : f \in A_k \} .$$

Then τ_k is an $L(x)$-invariant seminorm on $M(G)$.

Definition. For $\mu \in M(G)$, we say that μ has *separable orbit in* $(M(G), \mathscr{T}(A_k))$ if there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset G$ such that for each $x \in G$, $k \geq 1$, and $\varepsilon > 0$, there exists an x_n such that $\tau_k(L(x)\mu - L(x_n)\mu) < \varepsilon$.

PROPOSITION 1. *Let* $\mu \in M(G)$ *have separable orbit in* $(M(G), \mathscr{T}(A_k))$. *Then* $s \mapsto L(s)\mu$ *is continuous from* G *to* $(M(G), \mathscr{T}(A_k))$.

Proof. Let $s_\alpha \xrightarrow{\alpha} s$. Choose $k \geq 1$ and $\varepsilon > 0$. We need to show there exists an α_0 such that for $\alpha \geq \alpha_0$, we have the inequality $\tau_k(L(s_\alpha)\mu - L(s)\mu) < \varepsilon$. Note that for $f \in C_{ru}^B(G)$, $L(y_\beta^{-1})f \xrightarrow{\beta} L(y^{-1})f$ uniformly as $y_\beta \xrightarrow{\beta} y$ (and hence as $y_\beta^{-1} \xrightarrow{\beta} y^{-1}$). Thus

$$\langle f, L(y_\beta)\mu \rangle = \langle L(y_\beta^{-1})f, \mu \rangle \xrightarrow{\beta} \langle L(y^{-1})f, \mu \rangle = \langle f, L(y)\mu \rangle .$$

Let $S(n) = \{y \in G: \tau_k(L(y)\mu - L(x_n)\mu) \leq \varepsilon/3\}$. We wish to show that $S(n)$ is closed. Let $y_\beta \in S(n)$ be such that $y_\beta \xrightarrow{\beta} y$. Thus

$$\tau_k(L(y)\mu - L(x_n)\mu) = \sup \{\lim_\beta | \langle f, L(y_\beta)\mu - L(x_n)\mu \rangle | : f \in A_k\} \leq \varepsilon/3 .$$

Hence $S(n)$ is closed.

By hypothesis, $G = \bigcup_{n=1}^{\infty} S(n)$. By the Baire category theorem for locally compact groups, there exists n_0 such that $S(n_0)$ has an interior. Thus there exists an open set U about s such that $t_0 s^{-1} U \subset S(n_0)$ for some $t_0 \in S(n_0)$. Let α_0 be such that $s_\alpha \in U$ for $\alpha \geq \alpha_0$. We now show that for $\alpha \geq \alpha_0$, the inequality

$$\tau_k(L(s_\alpha)\mu - L(s)\mu) < \varepsilon$$

holds. For $\alpha \geq \alpha_0$, we have that

$$\begin{aligned} \tau_k(L(s_\alpha)\mu - L(s)\mu) &= \tau_k(L(t_0 s^{-1})L(s_\alpha)\mu - L(t_0 s^{-1})L(s)\mu) \\ &\leq \tau_k(L(t_0 s^{-1} s_\alpha)\mu - L(x_{n_0})\mu) + \tau_k(L(x_{n_0})\mu - L(t_0)\mu) \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon, \end{aligned}$$

since $t_0, t_0 s^{-1} s_\alpha \in t_0 s^{-1} U \subset S(n_0)$. ■

PROPOSITION 2. *Let* G *be* σ*-compact. If* $x \mapsto L(x)\mu$ *is continuous from* G *to* $(M(G), \mathscr{T}(A_k))$, *then* μ *has separable orbit in* $(M(G), \mathscr{T}(A_k))$.

Proof. Note that $(M(G), \mathscr{T}(A_k))$ is a metric space. Let $G = \bigcup_{n=1}^{\infty} K_n$, where K_n is compact. The image of K_n under $x \to L(x)\mu$ is a compact metric space and hence is separable. Thus the image of G is separable. ■

If G is not σ-compact and M(G) has the measure norm topology, then no nonzero measure has a separable orbit.

We now show that $\mu \in M(G)$ has the property that $x \mapsto L(x)\mu$ is continuous from G to $(M(G), \mathcal{T}(A_k))$ if and only if μ is in the $\mathcal{T}(A_k)$-closure of $L^1(G)$, denoted by $\overline{L^1(G)}^A$.

THEOREM 3. *Let* $\mu \in M(G)$ *be such that* $x \mapsto L(x)\mu$ *is continuous from* G *to* $(M(G), \mathcal{T}(A_k))$. *Then* $\mu \in \overline{L^1(G)}^A$.

Proof. Let $\{f_\alpha\}$ be an approximate identity in $L^1(G)$, indexed over a neighborhood base of e; in other words, support $(f_\alpha) \subset \alpha$, $f_\alpha \geq 0$, and $\|f_\alpha\|_1 = 1$. Choose $k_0 \geq 1$ and $\varepsilon > 0$. It suffices to show that $\tau_k(f_\alpha * \mu - \mu) \leq \varepsilon$ for $\alpha \geq \alpha_0$, for some α_0. Pick U to be a symmetric neighborhood of e in G such that

$$\tau_k(L(x)\mu - \mu) < \varepsilon$$

for $x \in U$. Choose α_0 such that the inequality $\alpha \geq \alpha_0$ implies that support $(f_\alpha) \subset U$. Now for $\alpha \geq \alpha_0$,

$$\begin{aligned}
\tau_k(f_\alpha * \mu - \mu) &= \sup\{|\langle \phi, f * \mu - \mu\rangle| : \phi \in A_k\} \\
&= \sup\left\{\left|\int_G \phi(x)\,df_\alpha * \mu(x) - \int_G \phi(y)\,d\mu(y)\right| : \phi \in A_k\right\} \\
&= \sup\left\{\left|\int_G\int_G \phi(xy)d\mu(y)f_\alpha(x)dx - \int_G\int_G f_\alpha(x)dx\,\phi(y)d\mu(y)\right| : \phi \in A_k\right\} \\
&= \sup\left\{\left|\int_G f_\alpha(x)\,dx\left[\int_G \phi(y)\,dL(x)\mu(y) - \int_G \phi(y)\,d\mu(y)\right]\right| : \phi \in A_k\right\} \\
&\leq \sup_{x \in U} \tau_k(L(x)\mu - \mu) \leq \varepsilon. \blacksquare
\end{aligned}$$

THEOREM 4. *Let* $\mu \in \overline{L^1(G)}^A$. *Then* $x \mapsto L(x)\mu$ *is continuous from* G *to* $(M(G), \mathcal{T}(A_k))$.

Proof. We note first that since A_k is $\sigma(A, M(G))$-bounded and L(x)-invariant, A_k is a sup-norm bounded set in $C^B(G)$; in fact, for all $x \in G$, we have that

$$\sup_{f \in A_k} |f(x)| = \sup_{f \in A_k} \left|\int_G L(x^{-1})f\,d\delta_e\right| = \sup_{f \in A_k}\left|\int_G f\,d\delta_e\right| = M < \infty,$$

where δ_e is the unit mass at e. Now $x \mapsto L(x)\mu$ is continuous from G to M(G) in the measure norm, for $\mu \in L^1(G)$. Thus, since A_k is a sup-norm bounded set, $x \mapsto L(x)\mu$ is continuous from G to $(M(G), \mathcal{T}(A_k))$ for $\mu \in L^1(G)$. Choose $\mu \in \overline{L^1(G)}^A$, and let $x_\alpha \overset{\alpha}{\to} x$. Let $k \geq 1$ and $\varepsilon > 0$. We need to find an α_0 such that if $\alpha_0 \leq \alpha$, then $\tau_k(L(x_\alpha)\mu - L(x)\mu) < \varepsilon$. First pick $f \in L^1(G)$ such that $\tau_k(f - \mu) < \varepsilon/3$. Now choose α_0 such that for $\alpha \geq \alpha_0$, $\tau_k(L(x_\alpha)f - L(x)f) < \varepsilon/3$. Thus for $\alpha \geq \alpha_0$,

$$\tau_k(L(x_\alpha)\mu - L(x)\mu) \leq \tau_k(L(x_\alpha)\mu - L(x_\alpha)f) + \tau_k(L(x_\alpha)f - L(x)f) + \tau_k(L(x)f - L(x)\mu)$$

$$< \tau_k(\mu - f) + \frac{\varepsilon}{3} + \tau_k(f - \mu) < \varepsilon . \blacksquare$$

Remark. The two theorems above also hold if A is a space of bounded Borel functions, rather than a subspace of $C^B_{ru}(G)$.

For $\mu \in M(G)$, let $\|\mu\|$ denote the measure norm of μ, that is, the norm of μ as a linear functional on $C_0(G)$ with sup-norm $\|f\|_\infty = \sup\{|f(x)|: x \in G\}$. If we let $A_k = \{f \in C_0(G): \|f\|_\infty < k\}$, then $\mathscr{T}(A_k)$ is the measure norm topology. Thus we have the following corollaries.

COROLLARY 5. *Let* $\mu \in M(G)$. *If* μ *has separable orbit in* $(M(G), \|\cdot\|)$, *then* $x \mapsto L(x)\mu$ *is continuous from* G *to* $(M(G), \|\cdot\|)$.

Suppose G *is* σ*-compact. If* $x \mapsto L(x)\mu$ *is continuous from* G *to* $(M(G), \|\cdot\|)$, *then* μ *has separable orbit in* $(M(G), \|\cdot\|)$.

COROLLARY 6. *Let* $\mu \in M(G)$. *The measure* μ *is absolutely continuous if and only if* $x \mapsto L(x)\mu$ *is continuous from* G *to* $(M(G), \|\cdot\|)$.

Remarks. Propositions 1 and 2 are similar in spirit to a theorem of K. Shiga [8] in the compact case. Corollary 5 was obtained by R. Larsen [5] for the case where G is second countable and by K. W. Tam [9] in the general case. Corollary 6 was obtained by W. Rudin [7].

We now study M(G) under its sup-norm $\|\cdot\|_\infty$. We shall give first the abelian case for motivation. We then treat the compact nonabelian case and finally the general case.

Let G be abelian, and let $\hat{G}$ denote the character group of G. For $\mu \in M(G)$, define $\hat{\mu}(\gamma) = \int_G \overline{\gamma(x)}\, d\mu(x)$, for $\gamma \in \hat{G}$. Then $\hat{\mu}$ is the Fourier transform of μ. For $\mu \in M(G)$, let

$$\|\mu\|_\infty = \sup\{|\hat{\mu}(\gamma)|: \gamma \in \hat{G}\}.$$

Let $M_0(G) = \{\mu \in M(G): \hat{\mu} \in C_0(\hat{G})\}$.

COROLLARY 7. *Let* G *be abelian. The map* $x \mapsto L(x)\mu$ *is continuous from* G *to* $(M(G), \|\cdot\|_\infty)$ *if and only if* $\mu \in M_0(G)$.

Proof. Let $A_k = \{\hat{f}: f \in L^1(\hat{G})$ with $\|f\|_1 < k\}$. Then $\mathscr{T}(A_k)$ is the topology of $(M(G), \|\cdot\|_\infty)$. $\blacksquare$

Remark. Corollary 7 was obtained by R. Goldberg and A. Simon [3]. They used the following result: If U is a relatively compact neighborhood of 0 in G (where G is abelian), there exists a compact subset K of $\hat{G}$ such that for $\gamma \in \hat{G}\setminus K$, there exists an $x \in U$ with $\Re\gamma(x) \leq 0$. To see this, let $\sqrt{2} \leq \delta < \sqrt{3}$, and define $U^0 = \{\gamma \in \hat{G}: |\gamma(x) - 1| < \delta$ for all $x \in U\}$. Note that U^0 is relatively compact in $\hat{G}$ (K. H. Hofmann and P. S. Mostert [4, p. 284] or Pontryagin [6, p. 237]). Let K be the closure of U^0 in $\hat{G}$. We now prove the analogous result for the case where G is compact and nonabelian. This result is independent of the rest of this paper. We use the notation of Dunkl and Ramirez [1, Chapters 7 and 8], where proofs of unproved statements below may be found.

Let G be a compact, nonabelian group. We let $\hat{G}$ denote the set of equivalence classes of continuous, unitary irreducible representations of G. For $\alpha \in \hat{G}$, let T_α be an element of α. Then T_α is a homomorphism of G into $U(n_\alpha)$, the group of unitary $n_\alpha \times n_\alpha$ matrices, where n_α is the dimension of α. We use $T_\alpha(x)_{ij}$ to denote the matrix entries of $T_\alpha(x)$ $(1 \leq i, j \leq n)$ and $T_{\alpha ij}$ to denote the function $x \mapsto T_\alpha(x)_{ij}$. Clearly

$$T_\alpha(xy)_{ij} = \sum_{k=1}^{n_\alpha} T_\alpha(x)_{ik} T_\alpha(y)_{kj} \quad \text{and} \quad T_\alpha(y^{-1})_{ij} = \overline{T_\alpha(y)_{ji}}.$$

Furthermore, $T_{\alpha ij} \in C(G)$, where $C(G)$ denotes the set of continuous functions on G. For $\alpha \in \hat{G}$, let

$$\chi_\alpha(x) = \operatorname{trace}(T_\alpha(x)) = \sum_{i=1}^{n_\alpha} T_\alpha(x)_{ii}.$$

This trace χ_α is called the character of α, and it is independent of the choice of T_α in α. Let X be an n-dimensional, complex inner-product space. Let $\mathscr{B}(X)$ denote the space of linear maps from X into X. We define the operator norm of $A \in \mathscr{B}(X)$ by

$$\|A\|_\infty = \sup\{|A\xi|: \xi \in X, |\xi| \leq 1\}.$$

For the trace of A, we find that $\operatorname{Tr} A = \sum_{i=1}^n (A\xi_i, \xi_i)$, where $\{\xi_i\}_{i=1}^n$ is some orthonormal basis for X and $(\cdot, \cdot)$ denotes the inner product in X. Let $|A|$ denote $(A^*A)^{1/2}$. The operator norm of A is $\|A\|_\infty$, that is, $\max\{\lambda_i: 1 \leq i \leq n\}$, where the λ_i are the eigenvalues of $|A|$. For each $A \in \mathscr{B}(X)$, we have the inequality $|\operatorname{Tr} A| \leq n\|A\|_\infty$.

PROPOSITION 8. *Let* G *be a compact group. Suppose* $0 < \delta < \sqrt{3}$, *and let* U *be a neighborhood of* e *in* G. *Let* $U^0 = \{\alpha \in \hat{G}: \|T_\alpha(x) - I\|_\infty < \delta$ *for all* $x \in U\}$. *Then* U^0 *is finite.*

Proof. We show that U^0 is an equicontinuous set of representations of G. Choose $\varepsilon > 0$. Let K be a positive constant such that for $0 \leq \theta \leq 2\pi/3$, we have the inequality $|e^{i\theta} - 1| \leq K\theta$ (for example, let $K = 3\pi\sqrt{3}/2$). Define

$$V_m = \{x \in G: x, x^2, \cdots, x^m \in U\}.$$

Clearly, V_m is a neighborhood of e in G. Pick m such that $K\delta/m < \varepsilon$. Then for $x_1, x_2 \in G$ with $x = x_1^{-1}x_2 \in V_m$, we have that

$$\begin{aligned}\|T_\alpha(x_1) - T_\alpha(x_2)\|_\infty &= \|I - T_\alpha(x_1^{-1}x_2)\|_\infty = \|I - T_\alpha(x)\|_\infty \\ &= \sup\{|1 - e^{i\theta_j}|: 1 \leq j \leq n_\alpha\} \quad (\alpha \in U^0),\end{aligned}$$

by diagonalizing $T_\alpha(x)$. Thus

$$\|I - T_\alpha(x^r)\|_\infty = \sup\{|1 - e^{ir\theta_j}|: 1 \leq j \leq n_\alpha\} < \delta$$

for $1 \leq r \leq m$. Therefore

$$\|I - T_\alpha(x)\|_\infty = \sup\{|1 - e^{i\theta_j}|: 1 \le j \le n_\alpha\} < \frac{K\delta}{m} < \varepsilon .$$

Thus U^0 is an equicontinuous set of representation of G.

Let $\chi_\alpha = \operatorname{Tr} T_\alpha$. We claim that $\{\chi_\alpha/n_\alpha: \alpha \in U^0\}$ is an equicontinuous, uniformly bounded set of functions. This is the case since

$$|\operatorname{Tr}(I - T_\alpha)| \le n_\alpha \|I - T_\alpha\|_\infty .$$

Further $\|\chi_\alpha/n_\alpha\|_\infty \le 1$, and hence $\{\chi_\alpha/n_\alpha: \alpha \in U^0\}$ is relatively compact, by the Arzelà-Ascoli theorem. Since the $\{\chi_\alpha/n_\alpha\}$ are orthogonal in $L^2(G)$, either U^0 is finite or $\{\chi_\alpha/n_\alpha: \alpha \in U^0\}$ has 0 as a uniform cluster point. This latter condition cannot happen, since $\chi_\alpha(e)/n_\alpha = 1$. ■

Let G be as above (that is, compact and nonabelian). We shall give the analogue to Corollary 7. Let the set $\phi = \{\phi_\alpha: \alpha \in \hat{G}$, where $\phi_\alpha \in \mathscr{B}(\mathbf{C}^{n_\alpha})\}$ be such that $\sup\{\|\phi_\alpha\|_\infty: \alpha \in \hat{G}\} < \infty$. The set of all such ϕ is denoted by $\mathscr{L}^\infty(\hat{G})$. It is a Banach algebra under the norm $\|\phi\|_\infty = \sup\{\|\phi_\alpha\|_\infty: \alpha \in \hat{G}\}$ and under co-ordinatewise operations. Let

$$\mathscr{C}_0(\hat{G}) = \{\phi \in \mathscr{L}^\infty(\hat{G}): \lim_{\alpha \to \infty} \|\phi_\alpha\|_\infty = 0\} .$$

For $\mu \in M(G)$, the Fourier transform $\hat{\mu}$ of μ is a matrix-valued function, defined for $\alpha \in \hat{G}$ by the relation

$$\alpha \mapsto \hat{\mu}_\alpha = \int_G T_\alpha(x^{-1})\, d\mu(x) .$$

Note that $\hat{\mu} \in \mathscr{L}^\infty(\hat{G})$. Thus for $\mu \in M(G)$, let $\|\mu\|_\infty = \sup\{\|\hat{\mu}_\alpha\|_\infty: \alpha \in \hat{G}\}$. We define $M_0(G)$ to be the set $\{\mu \in M(G): \hat{\mu} \in \mathscr{C}_0(\hat{G})\}$.

Let $A \in \mathscr{B}(X)$, where X is a finite-dimensional, complex inner-product space. We define the dual norm to $\|\cdot\|_\infty$ by $\|A\|_1 = \sup\{|\operatorname{Tr}(AB)|: \|B\|_\infty \le 1\}$. This norm can also be characterized by the condition $\|A\|_1 = \operatorname{Tr}(|A|)$. For $\phi \in \mathscr{L}^\infty(G)$, we put

$$\|\phi\|_1 = \sum_{\alpha \in \hat{G}} n_\alpha \|\phi_\alpha\|_1 .$$

Let $\mathscr{L}^1(\hat{G}) = \{\phi \in \mathscr{L}^\infty(\hat{G}): \|\phi\|_1 < \infty\}$. Then $\mathscr{L}^1(\hat{G})$ is a Banach space under $\|\cdot\|_1$. For $\phi \in \mathscr{L}^1(\hat{G})$, let $\operatorname{Tr}(\phi) = \sum_{\alpha \in \hat{G}} n_\alpha \operatorname{Tr}(\phi_\alpha)$. For $\psi \in \mathscr{L}^1(G)$ and $\phi \in \mathscr{L}^\infty(G)$, we obtain the inequality $|\operatorname{Tr}(\phi\psi)| \le \|\phi\|_\infty \|\psi\|_1$.

We now define A(G), the Fourier algebra of G, and we pair A(G) and M(G) to get the compact analogue of Corollary 7. Let A(G) be the set of $f \in C(G)$ for which $\hat{f} \in \mathscr{L}^1(\hat{G})$. We define a norm on A(G) by

$$\|f\|_A = \|\hat{f}\|_1 = \sum_{\alpha \in \hat{G}} n_\alpha \|\hat{f}_\alpha\|_1 < \infty .$$

Note that A(G) is isomorphic to $\mathscr{L}^1(\hat{G})$, because for each $\phi \in \mathscr{L}^1(\hat{G})$, the function $f(x) = \sum_{\alpha \in \hat{G}} n_\alpha \operatorname{Tr}(\phi_\alpha T_\alpha(x))$ is in A(G); further,

$$\|f\|_\infty = \sup_{x \in G} \left| \sum_{\alpha \in \hat{G}} n_\alpha \operatorname{Tr}(\phi_\alpha T_\alpha(x)) \right| \leq \sum_{\alpha \in \hat{G}} n_\alpha \|\phi_\alpha\|_1 = \|\phi\|_1 .$$

We note that for $f \in A(G)$, $\|L(x)f\|_A = \|f\|_A$.

THEOREM 9. *Let* G *be a compact (nonabelian) group, and let* $\mu \in M(G)$. *Then* $x \mapsto L(x)\mu$ *is continuous from* G *to* $(M(G), \|\cdot\|_\infty)$ *if and only if* $\mu \in M_0(G)$.

Proof. For $\mu \in M(G)$ and $f \in A(G)$, we define

$$\langle f, \mu \rangle = \int_G f(t)\, d\mu(t) = \operatorname{Tr}(\hat{\mu}\hat{h}),$$

where $h(t) = f(t^{-1})$. If $\check{f}$ is defined by $\check{f}(t) = f(t^{-1})$, then $\|\check{f}\|_A = \|f\|_A$. Thus $\langle \check{f}, \mu \rangle = \operatorname{Tr}(\hat{\mu}\hat{f})$. Let $A_k = \{f \in A(G): \|f\|_A < k\}$, and let $\mathscr{T}(A_k)$ be the topology on M(G) of uniform convergence on the sets A_k. Since

$$|\operatorname{Tr}(\hat{\mu}\hat{f})| \leq \|\hat{\mu}\|_\infty \|\hat{f}\|_1 = \|\hat{\mu}\|_\infty \|f\|_A = \|\mu\|_\infty \|f\|_A,$$

the topology $\mathscr{T}(A_k)$ is weaker than the $\|\cdot\|_\infty$-topology on M(G). However, since $\mathscr{L}^\infty(\hat{G})$ is identified with the dual space of $\mathscr{L}^1(\hat{G})$ by $\psi \to \operatorname{Tr}(\phi\psi)$ for $\phi \in \mathscr{L}^\infty(\hat{G})$ and $\psi \in \mathscr{L}^1(\hat{G})$, $\mathscr{T}(A_k)$ is the same as the $\|\cdot\|_\infty$-topology on M(G). Furthermore, A_k is L(x)-invariant, since $\|L(x)f\|_A = \|f\|_A$ for $f \in A(G)$. We now apply Theorems 3 and 4. ■

We conclude now with the general case. We shall use the machinery developed by P. Eymard [2], and we shall follow his conventions in the use of x in various formulae, where we used x^{-1} in the compact and abelian cases discussed above.

Let G be a locally compact group. Let Σ denote the equivalence classes of the continuous unitary representations on G. For $\pi \in \Sigma$, let $\mathscr{H}_\pi$ denote the representation space. We define $\hat{\mu}$ to be a function on Σ by $\pi \mapsto \hat{\mu}(\pi) = \int_G \pi(x)\, d\mu(x)$. For $\mathscr{S} \subset \Sigma$, let

$$\|\mu\|_{\mathscr{S}} = \sup\{\|\hat{\mu}(\pi)\|_\infty : \pi \in \mathscr{S}\},$$

where $\|\hat{\mu}(\pi)\|_\infty$ denotes the operator norm on $\mathscr{H}_\pi$. We define C*(G) to be the completion of $L^1(G)$ in $\|\cdot\|_\Sigma$ (see [2, Section 1.14]). Let $\{\rho\}$ denote the subset of Σ containing just the left regular representation of G on $L^2(G)$. Let $C^*_\rho(G)$ denote the completion of $L^1(G)$ in $\|\cdot\|_\rho$ (see [2, Section 1.16]).

For $\mu \in M(G)$, we let $\rho(\mu)$ denote the bounded operator on $L^2(G)$, defined by $h \mapsto \mu * h$ $(h \in L^2(G))$, with operator norm $\|\rho(\mu)\|_\rho$. Let $\mathscr{B}(L^2(G))$ denote the set of bounded operators on $L^2(G)$. Then $C^*_\rho(G)$ can be identified with the closure of $\rho(L^1(G)) = \{\rho(f): f \in L^1(G)\}$ in $\mathscr{B}(L^2(G))$. If G is abelian, then $C^*_\rho(G) = C_0(G)$. If G is compact, then $C^*(G) = \mathscr{C}_0(\hat{G})$.

Let VN(G) denote the von Neumann subalgebra of $\mathscr{B}(L^2(G))$ generated by the left translation operators (see [2, Section 3.9]). For $\mu \in M(G)$, we have that $\rho(\mu) \in VN(G)$. Further, $C^*_\rho(G) \subset VN(G)$. If G is abelian, then $VN(G) = L^\infty(\hat{G})$. If G is compact, then $VN(G) = \mathscr{L}^\infty(\hat{G})$.

Definition. $M_0(G) = \{\mu \in M(G): \rho(\mu) \in C^*_\rho(G)\}$.

Let B(G) denote the linear subspace of $C^B(G)$ generated by the continuous positive-definite functions. Then B(G) can be identified with the dual space of $C^*(G)$ (see [2, Section 2.2]). For $f \in B(G)$, let $\|f\|_B$ denote the norm of f as a linear functional on $C^*(G)$. Finally, let A(G) be the closed subalgebra of B(G) generated by the continuous positive-definite functions with compact support (see [2, Section 3.4]). If G is abelian, then $A(G) = L^1(\hat{G})^\wedge$. If G is compact, then our previous definitions and those of Eymard are consistent. We have the inclusion $A(G) \subset C^B_{ru}(G)$, since $A(G) \subset C_0(G)$. We let $A_k = \{f \in A(G): \|f\|_B < k\}$. Now for $f \in A(G)$, $\|L(x)f\|_B = \|f\|_B$; hence each A_k is L(x)-invariant. We pair A(G) and M(G) by the relation

$$\langle f, \mu \rangle = \int_G f(t)\,d\mu(t) \qquad (f \in A(G) \text{ and } \mu \in M(G)).$$

Let $\mathscr{T}(A_k)$ be the topology on M(G) of uniform convergence on the sets A_k. We wish to apply Theorems 3 and 4 as we did in Theorem 9. To do this, it remains only to observe that VN(G) can be identified as the dual space of A(G) (see [2, Section 3.10]), and for $\mu \in M(G)$, the identification is given by the relation

$$f \mapsto \int_G f(x)\,d\mu(x) = \langle f, \mu \rangle,$$

where $f \in A(G)$. It follows now by Theorems 3 and 4 that $x \mapsto L(x)\mu$ is continuous from G to $(M(G), \|\cdot\|_\rho)$ if and only if $\rho(\mu) \in \rho(\overline{L^1(G)})$ (the closure in $\mathscr{B}(L^2(G))$). Hence we have the following result.

THEOREM 10. *Let* G *be a locally compact group. Let* $\mu \in M(G)$. *Then* $x \mapsto L(x)\mu$ *is continuous from* G *to* $(M(G), \|\cdot\|_\rho)$ *if and only if* $\mu \in M_0(G)$.

THEOREM 11. *Suppose* $A \subset C^B_{ru}(G)$ *has the further property that* A *is dense in* $L^1(|\mu|)$ *for each* $\mu \in M(G)$, *and that for each* $f \in A$ *we have inclusions* $fA_k \subset CA_{k'}$ $(k = 1, 2, \cdots)$, *where the constants* C *and* k' *depend on* f *and on* k. *Then* $\overline{L^1(G)}^A$ *is a band; in other words, if* $\mu \in \overline{L^1(G)}^A$ *and* $\nu \ll \mu$, *then* $\nu \in \overline{L^1(G)}^A$.

Proof. Let $\mu \in \overline{L^1(G)}^A$ and $\nu \ll \mu$; then $d\nu = g\,d\mu$, for some Borel function $g \in L^1(|\mu|)$. Now there exist functions $f_m \in A$ $(m = 1, 2, \cdots)$ such that

$$\int_G |f_m - g|\,d|\mu| < 1/m,$$

that is, $\|f_m d\mu - d\nu\|_{M(G)} \to 0$ as $m \to \infty$. We claim that each $f_m d\mu$ belongs to $\overline{L^1(G)}^A$. For if $\{g_n\} \subset L^1(G)$ and $g_n \xrightarrow{n} \mu$ in $\mathscr{T}(A_k)$, then $f_m g_n \xrightarrow{n} f_m d\mu$ (note that $f_m g_n \in L^1(G)$). In fact, for each k, we have the relations

$$\tau_k(f_m g_n - f_m d\mu) = \sup\left\{\left|\int_G \phi(x) f_m(x)[g_n(x)dx - d\mu(x)]\right| : \phi \in A_k\right\}$$

$$\leq C \sup\left\{\left|\int_G \phi(x)[g_n(x)dx - d\mu(x)]\right| : \phi \in A_{k'}\right\} = C\,\tau_{k'}(g_n - d\mu),$$

where C and k' depend on k and f_m. Thus $\tau_k(f_m g_n - f_m d\mu) \xrightarrow{n} 0$, and $f_m d\mu \in L^1(\overline{G})^A$.

Since $\mathscr{T}(A_k)$-closed sets are closed in the measure norm topology ($\sup\{\|\phi\|_\infty : \phi \in A_k\} < \infty$), we have that $\nu \in L^1(\overline{G})^A$. ■

COROLLARY 12. *For every locally compact group* G, $M_0(G)$ *is a band.*

Proof. Let A = A(G) as before, and recall that A(G) is a dense subalgebra of $C_0(G)$ (for the locally compact case, see [2, Section 3.4]). ■

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