

**TRANSLATIONAL ADDITION THEOREMS
FOR SPHERICAL LAPLACIAN FUNCTIONS
AND THEIR APPLICATION
TO BOUNDARY-VALUE PROBLEMS**

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Abstract. General translational addition theorems are presented for spherical scalar Laplacian functions, and their application to boundary value problems is illustrated. By these theorems, the eigenfunction solutions in a system of spherical coordinates are expressed in terms of the spherical coordinates in another system, translated with respect to the first one. This allows for a rigorous analytic solution to be obtained for Laplacian and Poissonian fields in the presence of arbitrary configurations of spheres by imposing the exact boundary conditions. Complete formulations and solutions are presented for systems of electrically charged spheres and for arrays of perfect conductor spheres in external electric and magnetic fields. Illustrative computation examples are given for three-sphere systems. Numerical results of specified accuracy are generated, which are useful for validating various approximate numerical methods.

1. Introduction. Addition theorems for spherical functions are needed in order to obtain scalar or vector field problem solutions of controllable accuracy relative to multiple-sphere systems. To impose the exact boundary conditions at the surface of each sphere, the fields due to the presence of the other spheres, which can easily be expressed in terms of coordinates attached to those spheres, have to be ‘translated’ to the system of coordinates attached to the sphere at whose surface the boundary condition is to be imposed. Such theorems for spherical scalar wave functions were derived in Friedman and Russek, 1954, and for spherical vector wave functions in Stein, 1961, and Cruzan, 1962. The latter were used to analyze the electromagnetic wave scattering by systems of

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conducting and dielectric spheres (Hamid, Ciric and Hamid (1990)). The general expressions of the addition theorems for the special case of spherical Laplacian functions are not available in the literature. In this paper, we obtain the general form of the translational addition theorems for scalar spherical Laplacian functions and illustrate their application to the solution of electric and magnetic field problems.

2. Translational addition theorems for spherical scalar Laplacian functions.

The addition theorems for scalar wave functions given in Cruzan, 1962, for the translation from the system of spherical coordinates r, θ, ϕ to the system r', θ', ϕ' shown in Figure 1 is, for $r' \leq r_0$,

$$z_n(kr)P_n^m(\cos \theta) \exp(im\phi) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \sum_p (-1)^\mu i^{\nu-n+p} (2\nu + 1) a_{\nu\mu p}^{nm} \cdot j_\nu(kr') z_p(kr_0) P_\nu^\mu(\cos \theta') P_p^{m-\mu}(\cos \theta_0) \exp(i\mu\phi') \exp[i(m - \mu)\phi_0], \tag{2.1}$$

where $i \equiv \sqrt{-1}$, k is the wave number, n and m are integers, $n \geq 0$, $|m| \leq n$, z_n stands for any of the spherical Bessel function j_n , Neumann function n_n , or Hankel functions, P_n^m is the associated Legendre function of the first kind of degree n and order m (Smythe (1989)), and $a_{\nu\mu p}^{nm}$ are expansion coefficients defined in the Appendix. The sum over p is over all of the values

$$p = n + \nu, n + \nu - 2, n + \nu - 4, \dots, |n - \nu|, \tag{2.2}$$

for which the coefficient $a_{\nu\mu p}^{nm}$ is different from zero. For $r' \geq r_0$, the addition theorem is obtained from (2.1) by interchanging r', θ', ϕ' and r_0, θ_0, ϕ_0 , respectively.

The general form of the solution of the scalar Laplace equation in spherical coordinates r, θ, ϕ , for regions with $0 \leq \phi \leq 2\pi$ and which contain sections of the negative z -axis, can be constructed in terms of two elementary solutions, i.e.,

$$f_{nm}^{(1)}(r, \theta, \phi) = r^n P_n^m(\cos \theta) \exp(im\phi), \quad f_{nm}^{(2)}(r, \theta, \phi) = r^{-(n+1)} P_n^m(\cos \theta) \exp(im\phi). \tag{2.3}$$

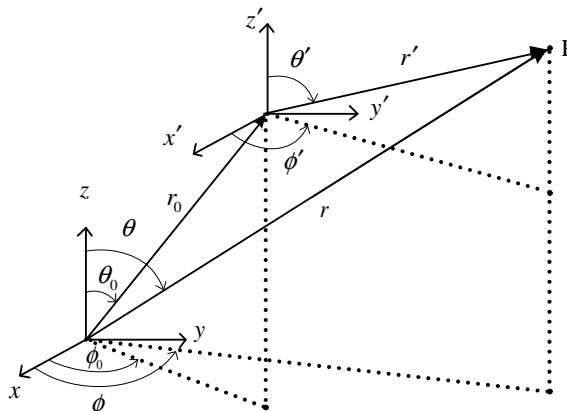


FIG. 1. Translation of the coordinate system

To express $f_{nm}^{(1)}$ and $f_{nm}^{(2)}$ in the coordinates r', θ', ϕ' in a translated system of coordinates, we notice that the elementary solutions in (2.3) are simply related to $z_n(kr)P_n^m(\cos \theta) \exp(im\phi)$. Indeed, taking z_n to be j_n and using the limiting value as $k \rightarrow 0$ (Abramovitz and Steigun (1965)),

$$(kr)^{-n} j_n(kr) \rightarrow 1/(2n + 1)!!, \tag{2.4}$$

where $()!!$ denotes the double factorial, (2.1) yields

$$r^n P_n^m(\cos \theta) \exp(im\phi) = \lim_{k \rightarrow 0} (2n + 1)!! \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \sum_p (-1)^\mu i^{\nu-n+p} a_{\nu\mu p}^{nm} \cdot \frac{r'^\nu r_0^p k^{\nu-n+p}}{(2\nu - 1)!!(2p + 1)!!} P_\nu^\mu(\cos \theta') P_p^{m-\mu}(\cos \theta_0) \exp(i\mu\phi') \exp[i(m - \mu)\phi_0], \quad r' \leq r_0. \tag{2.5}$$

Since $p \geq |n - \nu| \geq 0$ (see (2.2)), we have $\nu - n + p \geq 0$, and the only contributions to the summations are from the terms with the power of k equal to zero, $\nu - n + p = 0$, i.e., $p = n - \nu \geq 0$ and $\nu \leq n$. Using the expression of $a_{\nu\mu, n-\nu}^{nm}$ (see Appendix, (A.7)), (2.5) becomes

$$r^n P_n^m(\cos \theta) \exp(im\phi) = (n + m)! r_0^n \sum_{\nu=0}^n \sum_{\mu=-\nu}^{\nu} \left(\frac{r'}{r_0}\right)^\nu \frac{1}{(\nu + \mu)!(n + m - \nu - \mu)!} \cdot P_\nu^\mu(\cos \theta') P_{n-\nu}^{m-\mu}(\cos \theta_0) \exp(i\mu\phi') \exp[i(m - \mu)\phi_0], \quad r' \leq r_0. \tag{2.6}$$

The translational theorem for $r' \geq r_0$ can be obtained directly from (2.6) by interchanging r', θ', ϕ' and r_0, θ_0, ϕ_0 , respectively. It should be remarked that the series in these two theorems have a finite number of terms and that each of the two expressions, for $r' \leq r_0$ or $r' \geq r_0$, can be employed for either $r' \leq r_0$ or $r' \geq r_0$ without restriction on the relative size of r' and r_0 (Stein (1961)).

Taking now z_n to be n_n and using the limiting value as $k \rightarrow 0$ (Abramovitz and Steigun (1965)),

$$(kr)^{n+1} n_n(kr) \rightarrow -(2n - 1)!!, \tag{2.7}$$

substitution in (2.1) yields

$$r^{-(n+1)} P_n^m(\cos \theta) \exp(im\phi) = \lim_{k \rightarrow 0} \frac{1}{(2n - 1)!!} \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \sum_p (-1)^\mu i^{\nu-n+p} a_{\nu\mu p}^{nm} r'^\nu \cdot r_0^{-(p+1)} k^{\nu+n-p} \frac{(2p - 1)!!}{(2\nu - 1)!!} P_\nu^\mu(\cos \theta') P_p^{m-\mu}(\cos \theta_0) \exp(i\mu\phi') \exp[i(m - \mu)\phi_0], \quad r' \leq r_0. \tag{2.8}$$

Since $p \leq n + \nu$ (see (2.2)), in the limit, only the terms with $p = n + \nu$ (when the power of k is equal to zero) contribute to the summations. With the expression of $a_{\nu\mu, n+\nu}^{nm}$ in the Appendix (see (A.10)), (2.8) becomes

$$r^{-(n+1)} P_n^m(\cos \theta) \exp(im\phi) = \frac{r_0^{-(n+1)}}{(n - m)!} \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} (-1)^{\nu+\mu} \left(\frac{r'}{r_0}\right)^\nu \frac{(n - m + \nu + \mu)!}{(\nu + \mu)!} \cdot P_\nu^\mu(\cos \theta') P_{n+\nu}^{m-\mu}(\cos \theta_0) \exp(i\mu\phi') \exp[i(m - \mu)\phi_0], \quad r' \leq r_0. \tag{2.9}$$

The translational addition theorem for $r' \geq r_0$ is obtained in a similar manner or directly from (2.9) by interchanging r', θ', ϕ' and r_0, θ_0, ϕ_0 , respectively.

In the special case when the translation is along the z -axis, $\phi' = \phi, \theta_0 = 0$ or $\pi, (r_0 = z_0)$, and (2.6) and (2.9) reduce, respectively, to

$$r^n P_n^m(\cos \theta) = (n+m)! r_0^n \sum_{\nu=|m|}^n \left(\frac{r'}{r_0}\right)^\nu \frac{1}{(n-\nu)!(m+\nu)!} P_\nu^m(\cos \theta') P_{n-\nu}(\cos \theta_0), \quad r' \leq r_0, \tag{2.10}$$

$$r^{-(n+1)} P_n^m(\cos \theta) = \frac{r_0^{-(n+1)}}{(n-m)!} \sum_{\nu=|m|}^\infty (-1)^{m+\nu} \left(\frac{r'}{r_0}\right)^\nu \frac{(n+\nu)!}{(m+\nu)!} P_\nu^m(\cos \theta') P_{n+\nu}(\cos \theta_0), \tag{2.11}$$

$r' \leq r_0.$

3. Application to Laplacian scalar potential problems. Consider a system of S spheres arbitrarily located, the spheres p and q being shown in Figure 2. The Cartesian axes of all the spheres are, respectively, parallel. The center of the sphere q has the spherical coordinates $r_{pq}, \theta_{pq}, \phi_{pq}$ in the system of coordinates attached to the sphere p , and the spherical coordinates of the center of p in the system attached to q are $r_{qp} = r_{pq}, \theta_{qp} = \pi - \theta_{pq}, \phi_{qp} = \pi + \phi_{pq}$.

The scalar potential due to the presence of the sphere p can be written as (see (2.3))

$$\Phi_p(r_p, \theta_p, \phi_p) = \sum_{n=0}^\infty \sum_{m=-n}^n A_{nm}^{(p)} r_p^{-(n+1)} P_n^m(\cos \theta_p) \exp(im\phi_p), \quad r_p \geq a_p, \quad p = 1, 2, \dots, S, \tag{3.1}$$

where r_p, θ_p, ϕ_p are the spherical coordinates of the observation point, of position vector $\mathbf{r} \equiv \mathbf{r}_p$, in the system p and $A_{nm}^{(p)}$ are constants of integration to be determined. The resultant potential at any point outside the spheres due to the presence of all of them and of an external applied field is

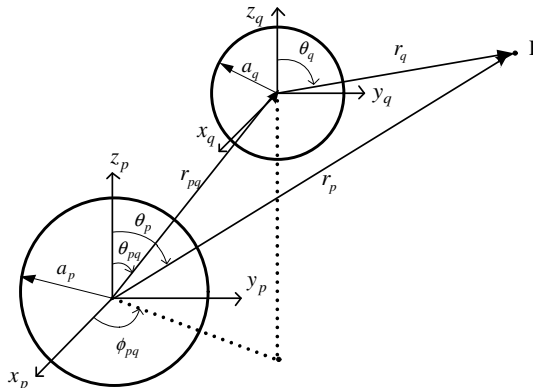


FIG. 2. Pair of spheres p and q

$$\Phi(\mathbf{r}) = \sum_{p=1}^S \Phi_p(r_p, \theta_p, \phi_p) + \Phi_0(\mathbf{r}), \tag{3.2}$$

where Φ_0 is the potential due to the applied field.

When the material inside a sphere p is homogeneous and field-penetrable, then the potential here is

$$\Phi_{in}^{(p)}(r_p, \theta_p, \phi_p) = \sum_{n=0}^{\infty} \sum_{m=-n}^n B_{nm}^{(p)} r_p^n P_n^m(\cos \theta_p) \exp(im\phi_p), \quad r_p \leq a_p, \tag{3.3}$$

with the constants $B_{nm}^{(p)}$ to be determined.

In order to impose the boundary conditions at the surface of the sphere p , the potential in (3.2) is transformed to

$$\Phi^{(p)}(r_p, \theta_p, \phi_p) = \Phi_p(r_p, \theta_p, \phi_p) + \sum_{\substack{q=1 \\ q \neq p}}^S \Phi_q^{(p)}(r_p, \theta_p, \phi_p) + \Phi_0^{(p)}(r_p, \theta_p, \phi_p), \quad r_p \geq a_p, \tag{3.4}$$

$p = 1, 2, \dots, S,$

where $\Phi_0^{(p)}$ is Φ_0 expressed in coordinates r_p, θ_p, ϕ_p and $\Phi_q^{(p)}$ is the potential $\Phi_q(r_q, \theta_q, \phi_q)$ (see (3.1)) in the system of coordinates r_p, θ_p, ϕ_p obtained by translation from the system of coordinates r_q, θ_q, ϕ_q , i.e. (see (2.9)),

$$\begin{aligned} \Phi_q^{(p)}(r_p, \theta_p, \phi_p) = & \sum_{n=0}^{\infty} \sum_{m=-n}^n A_{nm}^{(q)} \frac{r_{qp}^{-(n+1)}}{(n-m)!} \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} (-1)^{\nu+\mu} \left(\frac{r_p}{r_{qp}}\right)^{\nu} \frac{(n-m+\nu+\mu)!}{(\nu+\mu)!} \\ & \cdot P_{\nu}^{\mu}(\cos \theta_p) P_{n+\nu}^{m-\mu}(\cos \theta_{qp}) \exp(i\mu\phi_p) \exp[i(m-\mu)\phi_{qp}], \quad r_p \leq r_{qp}. \end{aligned} \tag{3.5}$$

The unknown constants of integration $A_{nm}^{(p)}, B_{nm}^{(p)}, p = 1, 2, \dots, S$, are determined by imposing the boundary conditions at the surfaces $r_p = a_p$ of all the spheres. When a surface $r_p = a_p$ is equipotential, for instance, then the (Dirichlet) boundary condition is (with (3.4) and (3.5)) $\Phi^{(p)}(a_p, \theta_p, \phi_p) = V_p$, where V_p is the given potential of the sphere p . When Neumann boundary conditions are to be imposed at a surface $r_p = a_p$, then one needs the normal derivative $\partial\Phi^{(p)}/\partial r_p$ at $r_p = a_p$ calculated from (3.4) and (3.5).

In the case of a system of spheres with a common z -axis and axisymmetric boundary conditions and applied external field, there is no dependence on the variable ϕ and, thus, $m = 0$ in (3.1), (3.3), and $m = 0, \mu = 0$ in (3.5) (see examples in sections 4.2 and 4.3 below).

4. Illustrative examples. In what follows, the theorems in (2.9) and (2.11) are applied to calculate the electrostatic field of systems of metallic spheres electrically charged or immersed in external fields, as well as the magnetic field in the presence of configurations of magnetic spheres.

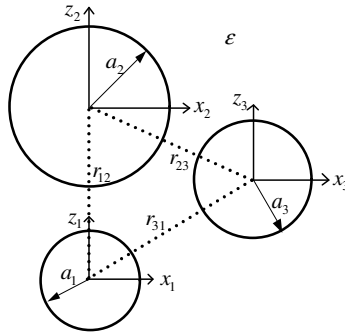


FIG. 3. System of spheres in arbitrary configuration

4.1. *Electrically charged metallic spheres in arbitrary configurations.* Consider for illustration a system of S conducting spheres at given electric potentials V_p (as shown in Figure 3 for $S = 3$), with the medium outside the spheres being homogeneous, of permittivity ϵ . The electrostatic potential is given by (3.1), (3.2), with $\Phi_0 \equiv 0$. The boundary conditions at the surfaces of the S spheres are (see (3.4))

$$\Phi_p(a_p, \theta_p, \phi_p) + \sum_{\substack{q=1 \\ q \neq p}}^S \Phi_q^{(p)}(a_p, \theta_p, \phi_p) = V_p, \quad p = 1, 2, \dots, S. \tag{4.1}$$

Multiplying both sides of the equations (4.1) with $P_n^m(\cos \theta_p) \sin \theta_p \exp(-im\phi_p)$ and applying the orthogonality properties of the spherical harmonics (Smythe (1989)) yields an infinite system of linear algebraic equations for the unknown constants of integration $A_{nm}^{(p)}, p = 1, 2, \dots, S$, which can be written in the form

$$A_{n'm'}^{(p)} + (-1)^{n'+m'} a_p^{n'+1} \frac{1}{(n'+m')!} \sum_{\substack{q=1 \\ q \neq p}}^S \sum_{n=0}^{\infty} \sum_{m=-n}^n A_{nm}^{(q)} r_{qp}^{-(n+1)} \left(\frac{a_p}{r_{qp}}\right)^{n'} \cdot \frac{(n-m+n'+m')!}{(n-m)!} P_{n+n'}^{m-m'}(\cos \theta_{qp}) \exp[i(m-m')\phi_{qp}] = V_p a_p \delta_{n'0},$$

$$p = 1, 2, \dots, S; \quad n' = 0, 1, 2, \dots; \quad m' = -n', -n' + 1, \dots, n', \tag{4.2}$$

where $\delta_{n'0}$ is the Kronecker symbol.

The total charge of the sphere p is obtained as

$$Q_p = 4\pi\epsilon A_{00}^{(p)}, \quad p = 1, 2, \dots, S. \tag{4.3}$$

Numerical results are generated by truncating appropriately the infinite system (4.2). To give a concrete example for the system of three spheres ($S = 3$) in Figure 3, with $a_1 = 3\text{cm}$, $a_2 = 5\text{cm}$, $a_3 = 4\text{cm}$ and $r_{12} = r_{23} = r_{31} = 10\text{cm}$, a 5-digit accuracy was achieved for the charge values when the infinite series were truncated to $n = 10$. For a free space outside the spheres, the following values in pF have been obtained for the capacitance coefficients (Smythe (1989)):

$c_{11} = 4.4710, c_{22} = 8.4680, c_{33} = 6.6793, c_{12} = c_{21} = -1.7474, c_{23} = c_{32} = -3.0661, c_{31} = c_{13} = -1.0902.$

4.2. *Electric field intensification in the presence of coaxial conducting spheres.* Consider now an array of S coaxial metallic spheres in a uniform external electric field of intensity \mathbf{E}_0 oriented along the common z -axis of the spheres. Each of the spheres has a total electric charge equal to zero, and the medium outside the spheres is homogeneous.

There is no dependence on the variable ϕ , and θ_{qp} is either 0 or π ; the potential in (3.4) becomes (with $m = 0, \mu = 0$ in (3.1) and (3.5))

$$\begin{aligned} \Phi^{(p)}(r_p, \theta_p) &= \sum_{n=0}^{\infty} A_n^{(p)} r_p^{-(n+1)} P_n(\cos \theta_p) + \sum_{q=1}^S \sum_{\substack{n=0 \\ q \neq p}}^{\infty} A_n^{(q)} \frac{r_{qp}^{-(n+1)}}{n!} \sum_{\nu=0}^{\infty} (-1)^\nu \left(\frac{r_p}{r_{qp}}\right)^\nu \\ &\cdot \frac{(n+\nu)!}{\nu!} P_\nu(\cos \theta_p) P_{n+\nu}(\cos \theta_{qp}) - E_0 z + C_p, \quad r_p \geq a_p, \quad p = 1, 2, \dots, S, \end{aligned} \tag{4.4}$$

where $-E_0 z$ is the potential corresponding to \mathbf{E}_0 with the origin of the z -axis chosen arbitrarily, for instance at the center of one of the spheres, the additive constants C_p being fixed in terms of the reference point for the potential of the external field. From the condition $Q_p = 0, p = 1, 2, \dots, S, S$ constants of integration are determined, namely (see (4.3)) $A_0^{(p)} = 0, p = 1, 2, \dots, S.$

The spheres are equipotential, i.e., $\Phi^{(p)}(a_p, \theta_p) = V_p, p = 1, 2, \dots, S,$ but the potentials V_p are now unknown. With $E_0 z = E_0 r_p \cos \theta_p \equiv E_0 r_p P_1(\cos \theta_p)$ in (4.4) and by using the orthogonality properties of the Legendre polynomials (Smythe (1989)), we get an infinite system of linear equations for the unknowns $(V_p - C_p)$ and $A_n^{(p)}, p = 1, 2, \dots, S, n = 1, 2, 3, \dots,$ in the form

$$\begin{aligned} (V_p - C_p) - \sum_{\substack{q=1 \\ q \neq p}}^S \sum_{n=1}^{\infty} A_n^{(q)} r_{qp}^{-(n+1)} P_n(\cos \theta_{qp}) &= 0, \\ A_{n'}^{(p)} + (-1)^{n'} a_p^{n'+1} \frac{1}{n'!} \sum_{\substack{q=1 \\ q \neq p}}^S \sum_{n=1}^{\infty} A_n^{(q)} r_{qp}^{-(n+1)} \left(\frac{a_p}{r_{qp}}\right)^{n'} \frac{(n+n')!}{n!} P_{n+n'}(\cos \theta_{qp}) &= \frac{2}{3} a_p^3 E_0 \delta_{n'1}, \\ p = 1, 2, \dots, S; n' = 1, 2, 3, \dots \end{aligned} \tag{4.5}$$

The electric field intensity at points on the sphere p is evaluated as $E(\theta_p) = -\partial\Phi^{(p)}/\partial r_p$ at $r_p = a_p.$ Numerical results are presented in Table 1 a) for a system of three identical metallic spheres as shown in Figure 4. To generate results of a specified accuracy, the infinite series in (4.5) are truncated to a finite number of terms, $n = n' = N,$ which increases when the relative value of the gap g decreases. At the points A, B and C in Figure 4, results of 5-digit accuracy are obtained with $N = 8$ for $g/a \geq 1, N = 40$ for $g/a = 0.1, N = 120$ for $g/a = 0.01$ and $N = 170$ for $g/a = 0.005.$ A 3-digit accuracy is obtained with $N = 30, 75$ and $100,$ respectively, for $g/a = 0.1, 0.01$ and $0.005.$ One notices the strong intensification of the electric field in the gap regions of the spheres when they are very close to each other.

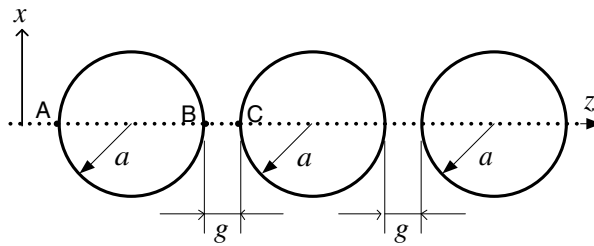


FIG. 4. Three coaxial spheres

TABLE 1. Field intensification at the points A, B and C in Figure 4 for three identical spheres in a uniform field:

a) conducting spheres in an axial electric field, $\mathbf{E}_0 = E_0 \hat{z}$

Point	$\frac{g/a}{\text{Field}}$	5.00	1.00	0.50	0.20	0.10	0.05	0.02	0.01	0.005
A	E_z/E_0	3.01460	3.15137	3.27008	3.45386	3.59462	3.72704	3.88319	3.98623	4.07715
B	E_z/E_0	3.02885	3.80677	5.22587	9.49097	16.1108	28.3915	62.0759	113.922	210.911
C	E_z/E_0	3.03875	3.85497	5.25679	9.49459	16.1109	28.3915	62.0759	113.922	210.911

b) perfect conductor spheres in a transverse magnetic field, $\mathbf{H}_0 = H_0 \hat{x}$

Point	$\frac{g/a}{\text{Field}}$	5.00	1.00	0.50	0.20	0.10	0.05	0.02	0.01	0.005
A	H_x/H_0	1.50161	1.51295	1.51971	1.52663	1.52990	1.53186	1.53320	1.53369	1.53395
B	H_x/H_0	1.50399	1.61580	1.79735	2.21176	2.65806	3.22789	4.20205	5.14121	6.29495
C	H_x/H_0	1.50504	1.61936	1.79923	2.21193	2.65807	3.22789	4.20205	5.14121	6.29495

4.3. *Arrays of coaxial perfect conductor spheres in transverse magnetic fields.* For a system of S perfect conductor spheres, the Laplacian scalar magnetic potential satisfies the general equations (3.1)-(3.5) with a zero boundary value for the Neumann condition at the surface of each sphere, i.e., $\partial\Phi^{(p)}/\partial r_p |_{r_p=a_p} = 0, p = 1, 2, \dots, S$. When the spheres are coaxial and in the presence of a uniform transverse magnetic field $\mathbf{H}_0 = H_0 \hat{x}$ (see Figure 4), the resultant potential in spherical coordinates attached to the sphere p has the form (see (3.4), (3.5) and (2.11))

$$\Phi^{(p)}(r_p, \theta_p, \phi_p) = \sum_{n=1}^{\infty} A_n^{(p)} r_p^{-(n+1)} P_n^1(\cos \theta_p) \cos \phi_p + \sum_{\substack{q=1 \\ q \neq p}}^S \sum_{n=1}^{\infty} A_n^{(q)} \frac{r_{qp}^{-(n+1)}}{(n-1)!} \sum_{\nu=1}^{\infty} (-1)^{\nu+1} \cdot \left(\frac{r_p}{r_{qp}}\right)^{\nu} \frac{(n+\nu)!}{(\nu+1)!} P_{n+\nu}^1(\cos \theta_{qp}) P_{\nu}^1(\cos \theta_p) \cos \phi_p - H_0 r_p \sin \theta_p \cos \phi_p, r_p \geq a_p,$$

(4.6)

with $\theta_{qp} = 0$ or π . The following infinite system of equations is derived for the unknowns $A_n^{(p)}$, $p = 1, 2, \dots, S$, $n = 1, 2, 3, \dots$:

$$\begin{aligned}
 A_{n'}^{(p)} + (-1)^{n'} a_p^{n'+1} \frac{n'}{(n'+1)(n'+1)!} \sum_{\substack{q=1 \\ q \neq p}}^S \sum_{n=1}^{\infty} A_n^{(q)} r_{qp}^{-(n+1)} \left(\frac{a_p}{r_{qp}}\right)^{n'} \frac{(n+n')!}{(n-1)!} P_{n+n'}(\cos \theta_{qp}) \\
 = -\frac{1}{2} a_p^3 H_0 \delta_{n'1}, \quad p = 1, 2, \dots, S; \quad n' = 1, 2, 3, \dots
 \end{aligned}
 \tag{4.7}$$

In Table 1 b), we present numerical results for a system of three coaxial identical perfect conductor spheres shown in Figure 4. Again, the number $n = n' = N$ to be retained in the infinite series in (4.7) in order to obtain a desired accuracy of the numerical results increases as the relative gap, g/a , decreases. For the magnetic field intensity at the points A, B and C in Figure 4, a 5-digit accuracy is obtained with $N = 7$ for $g/a \geq 1$, $N = 35$ for $g/a = 0.1$, $N = 125$ for $g/a = 0.01$, and $N = 175$ for $g/a = 0.005$, while for a 3-digit accuracy one needs $N = 20, 100$ and 120 , respectively, for $g/a = 0.1, 0.01$ and 0.005 . An appreciable intensification of the magnetic field at the points B and C is observed for small gaps between the spheres though much less strong than the intensification of the electric field at these points in the example considered in Table 1 a).

5. Remarks and conclusion. The general expressions in (2.6) and (2.9) for $r' \leq r_0$, as well as the corresponding expressions for $r' \geq r_0$, take a simpler form in various particular situations. In the cases, for instance, when the centers of the spheres are on the same plane (the configuration with $S = 3$ in Figure 3, considered in section 4.1, is included in this category), then this plane is chosen to be the zx -plane such that $\phi_{qp} = 0$, $p, q = 1, 2, \dots, S$, and, as a consequence, all the entries and the unknowns (the constants of integration $A_{nm}^{(p)}$) in the system of equations (4.2) are real numbers. When all the spheres have a common z -axis (as in sections 4.2 and 4.3), the translational addition theorems only contain single series (see (2.10) and (2.11)) which yield much smaller truncated systems of equations (compare (4.5) with (4.2)) for an imposed accuracy of the numerical results. Comparing the classical expression in terms of Legendre polynomials of the inverse distance between two arbitrary points on a unit sphere with that obtained as a special case for $n = 0$, $r' = r_0 = 1$ of our formula in (2.9) yields directly the celebrated Legendre’s addition theorem (Hobson (1965)). A variety of simplified expressions for positive and negative powers of the distance between two points located at particular positions can be generated from (2.6), (2.9) and from their companions for $r' \geq r_0$.

The formulations presented in this paper particularized for a system of two spheres constitute an alternative to the classical formulations using the method of images in the case of equipotential spheres or employing the bispherical coordinates (Morse and Feshbach (1953), Ciric and Kotuwage (2011)). The application of the translational addition theorems to the field problems in section 4 can be performed in the same way for other scalar Laplacian fields relative to multiple-sphere systems, for example for temperature distribution and in fluid mechanics, thus extending the class of problems which can be

solved by exact analytic methods. This includes problems relative to spheres with non-homogeneous boundary conditions and also field-penetrable spheres. As well, the results in this paper are relevant to studies regarding the behavior of cloud water droplets in external electric fields, the determination of electric and magnetic field intensification, and of forces on particles in colloidal suspensions, the response of nanostructures to electromagnetic fields, where, at a first approximation, the intervening small bodies or particles are modeled as spheres. The formulation and solution for perfect conductor spheres in the presence of magnetic fields can also be used for calculating the field intensity, induced electric currents, losses and forces in problems of pronounced skin effect for solid conductors at sufficiently high frequencies, when the field depth of penetration is negligible with respect to the local radii of curvature of the bodies involved.

Appendix A. Expansion coefficients $a_{\nu\mu p}^{nm}$ in (2.5) and (2.8). The general expression of the expansion coefficients $a_{\nu\mu p}^{nm}$ in (2.1) is (Cruzan (1962))

$$a_{\nu\mu p}^{nm} = (-1)^{m-\mu} (2p+1) \left[\frac{(n+m)!(\nu-\mu)!(p-m+\mu)!}{(n-m)!(\nu+\mu)!(p+m-\mu)!} \right]^{1/2} \begin{bmatrix} n & \nu & p \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} n & \nu & p \\ m & -\mu & -m+\mu \end{bmatrix}, \tag{A.1}$$

where the Wigner “3j” symbol is calculated using the formula (Messiah (1999))

$$\begin{bmatrix} a & b & c \\ \alpha & \beta & \gamma \end{bmatrix} = (-1)^{a-b-\gamma} [(a+\alpha)!(a-\alpha)!(b+\beta)!(b-\beta)!(c+\gamma)!(c-\gamma)!\Delta(a,b,c)]^{1/2} \cdot \sum_s (-1)^s [s!(a+b-c-s)!(a-\alpha-s)!(b+\beta-s)!(c-b+\alpha+s)!(c-a-\beta+s)!]^{-1}, \tag{A.2}$$

with

$$\alpha + \beta + \gamma = 0, \quad |a - b| \leq c \leq a + b \tag{A.3}$$

and

$$\Delta(a,b,c) \equiv \frac{(a+b-c)!(b+c-a)!(c+a-b)!}{(a+b+c+1)!}. \tag{A.4}$$

\sum_s extends over all integral values of s for which the arguments of the factorials are positive or null ($0!=1$).

In the special case (see (2.5)) when $p = n - \nu \geq 0$, we obtain

$$\begin{bmatrix} n & \nu & n-\nu \\ m & -\mu & -m+\mu \end{bmatrix} = (-1)^{n+m} \left[\frac{(n+m)!(n-m)!}{(\nu+\mu)!(\nu-\mu)!(n+m-\nu-\mu)!(n-m-\nu+\mu)!} \right]^{1/2} \left[\frac{(2\nu)![2(n-\nu)]!}{(2n+1)!} \right]^{1/2}, \tag{A.5}$$

and, for $m = \mu = 0$,

$$\begin{bmatrix} n & \nu & n-\nu \\ 0 & 0 & 0 \end{bmatrix} = (-1)^n \frac{n!}{\nu!(n-\nu)!} \left[\frac{(2\nu)![2(n-\nu)]!}{(2n+1)!} \right]^{1/2}. \tag{A.6}$$

Thus,

$$a_{\nu\mu, n-\nu}^{nm} = (-1)^\mu \frac{n!(n+m)!(2\nu)! [2(n-\nu)+1]!}{(2n+1)!\nu!(\nu+\mu)!(n-\nu)!(n+m-\nu-\mu)!}, \quad |m-\mu| \leq n-\nu. \quad (\text{A.7})$$

In the case (see (2.8)) when $p = n + \nu$, we get

$$\begin{aligned} \begin{bmatrix} n & \nu & n+\nu \\ m & -\mu & -m+\mu \end{bmatrix} &= (-1)^{(n+m-\nu-\mu)} \left[\frac{(n+m+\nu-\mu)!(n-m+\nu+\mu)!}{(n+m)!(n-m)!(\nu+\mu)!(\nu-\mu)!} \right]^{1/2} \\ &\quad \cdot \left[\frac{(2n)!(2\nu)!}{[2(n+\nu)+1]!} \right]^{1/2}, \end{aligned} \quad (\text{A.8})$$

and, for $m = \mu = 0$,

$$\begin{bmatrix} n & \nu & n+\nu \\ 0 & 0 & 0 \end{bmatrix} = (-1)^{n-\nu} \frac{(n+\nu)!}{n!\nu!} \left[\frac{(2n)!(2\nu)!}{[2(n+\nu)+1]!} \right]^{1/2}. \quad (\text{A.9})$$

Thus,

$$a_{\nu\mu, n+\nu}^{nm} = \frac{(2n)!(2\nu)!(n+\nu)!(n-m+\nu+\mu)!}{n!(n-m)!\nu!(\nu+\mu)! [2(n+\nu)]!}. \quad (\text{A.10})$$

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