Transmission Eigenvalues in Inverse Scattering Theory

Fioralba Cakoni

Department of Mathematical Sciences
University of Delaware
Newark, DE 19716, USA
email: cakoni@math.udel.edu

Jointly with D. Colton, D. Gintides, H. Haddar and A. Kirsch

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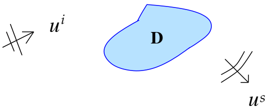
Scattering by an Inhomogeneous Media



$$\Delta u + k^2 n(x) u = 0 \quad \text{in } \mathbb{R}^d, \quad d = 2,3$$

$$u = u^s + u^i$$

$$\lim_{r\to\infty} r^{\frac{d-1}{2}} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0$$

We assume that $n - 1$ has compact support $\overline{D}$ and $n \in L^\infty(D)$ is such that $\Re(n) \geq \gamma > 0$ and $\Im(n) \geq 0$ in $\overline{D}$. Here $k > 0$ is the wave number proportional to the frequency ω.

Question: Is there an incident wave u^i that does not scatter?

The answer to this question leads to the transmission eigenvalue problem.

Transmission Eigenvalues

If there exists a nontrivial solution to the homogeneous interior transmission problem

$$\begin{aligned} \Delta w + k^2 n(x) w = 0 \quad & \text{in} \quad D \\ \Delta v + k^2 v = 0 \quad & \text{in} \quad D \\ w = v \quad & \text{on} \quad \partial D \\ \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad & \text{on} \quad \partial D \end{aligned}$$

such that v can be extended outside D as a solution to the Helmholtz equation $\tilde{v}$, then the scattered field due to $\tilde{v}$ as incident wave is identically zero.

Values of k for which this problem has non trivial solution are referred to as transmission eigenvalues and the corresponding nontrivial solution w, v as eigen-pairs.

Transmission Eigenvalues

In general such an extension of v does not exits!

Since Herglotz wave functions

$$v_g(x) := \int_{\Omega} e^{ikx\cdot d} g(d)ds(d), \qquad \Omega := \{x : |x| = 1\},$$

are dense in the space

$$\left\{v \in L^2(D) : \ \Delta v + k^2 v = 0 \quad \text{in } D\right\}$$

at a transmission eigenvalue there is an incident field that produces arbitrarily small scattered field.

Motivation

Two important issues:

- Real transmission eigenvalues can be determined from the scattered data.
- Transmission eigenvalues carry information about material properties.

Therefore, transmission eigenvalues can be used

- to quantify the presence of abnormalities inside homogeneous media and use this information to test the integrity of materials.

How are real transmission eigenvalues seen in the scattering data?

Measurements

We assume that $u^i(x) = e^{ikx\cdot d}$ and the far field pattern $u_\infty(\hat{x}, d, k)$ of the scattered field $u^s(x, d, k)$ is available for $\hat{x}, d \in \Omega$, and $k \in [k_0,\ k_1]$

$$\text{where} \qquad u^s(x, d, k) = \frac{e^{ikr}}{r^{\frac{d-1}{2}}} u_\infty(\hat{x}, d, k) + O\left(\frac{1}{r^{3/2}}\right)$$

as $r \to \infty$, $\hat{x} = x/|x|$, $r = |x|$.

Define the far field operator $F : L^2(\Omega) \to L^2(\Omega)$ by

$$(Fg)(\hat{x}) := \int_\Omega u_\infty(\hat{x}, d, k) g(d) ds(d), \qquad \left(S = I + \frac{ik}{\sqrt{2\pi k}} e^{-i\pi/4} F\right)$$

The Far Field Operator

Theorem

The far field operator $F : L^2(\Omega) \to L^2(\Omega)$ is injective and has dense range if and only if k is not a transmission eigenvalue such that for a corresponding eigensolution (w, v), v takes the form of a Herglotz wave function.

For $z \in D$ the far field equation is

$$(Fg)(\hat{x}) = \Phi_\infty(\hat{x}, z, k), \quad g \in L^2(\Omega)$$

where $\Phi_\infty(\hat{x}, z, k)$ is the far field pattern of the fundamental solution $\Phi(x, z, k)$ of the Helmholtz equation $\Delta v + k^2 v = 0$.

Computation of Real TE

Theorem (Cakoni-Colton-Haddar, *Comp. Rend. Math. 2010*)

Assume that either $n > 1$ or $n < 1$ and $z \in D$.

- *If k^2 is not a transmission eigenvalue then for every $\epsilon > 0$ there exists $g_{z,\epsilon,k} \in L^2(\Omega)$ satisfying $\|Fg_{z,\epsilon,k} - \Phi_\infty\|_{L^2(\Omega)} < \epsilon$ and*

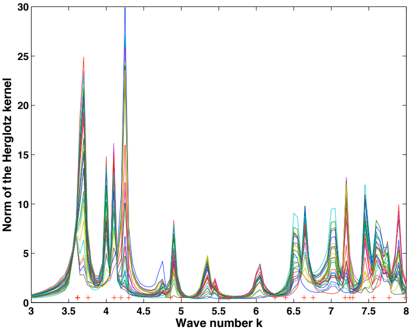
$$\lim_{\epsilon\to 0} \|v_{g_{z,\epsilon,k}}\|_{L^2(D)} \qquad \textit{exists.}$$

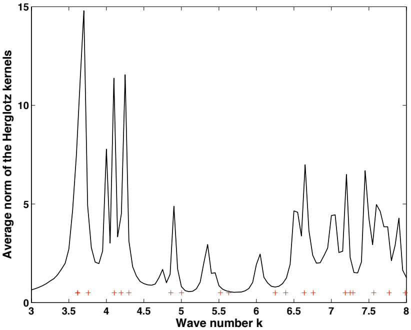
- *If k^2 is a transmission eigenvalue for any $g_{z,\epsilon,k} \in L^2(\Omega)$ satisfying $\|Fg_{z,\epsilon,k} - \Phi_\infty\|_{L^2(\Omega)} < \epsilon$ and for almost every $z \in D$*

$$\lim_{\epsilon\to 0} \|v_{g_{z,\epsilon,k}}\|_{L^2(D)} = \infty.$$

If g is the computed Tikhonov regularized solution, the second part still holds, whereas the first part is proven only for the scalar case *Arens, Inverse Problems (2004)*.

Computation of Real TE





A composite plot of $\|g_{z_i}\|_{L^2(\Omega)}$ against k for 25 random points $z_i \in D$

The average of $\|g_{z_i}\|_{L^2(\Omega)}$ over all choices of $z_i \in D$.

Computation of the transmission eigenvalues from the far field equation for the unit square D.

Transmission Eigenvalue Problem

Recall the transmission eigenvalue problem

$$\begin{aligned} \Delta w + k^2 n(x) w = 0 & \quad \text{in} \quad D \\ \Delta v + k^2 v = 0 & \quad \text{in} \quad D \\ w = v & \quad \text{on} \quad \partial D \\ \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} & \quad \text{on} \quad \partial D \end{aligned}$$

It is a nonstandard eigenvalue problem

$$\int_D \left(\nabla w \cdot \nabla \overline{\psi} - k^2 n(x) w \overline{\psi} \right) \, dx = \int_D \left(\nabla v \cdot \nabla \overline{\phi} - k^2 v \, \overline{\phi} \right) \, dx$$

- If $n = 1$ the interior transmission problem is degenerate
- If $\Im(n) > 0$ in $\overline{D}$, there are no real transmission eigenvalues.

Historical Overview

- The transmission eigenvalue problem in scattering theory was introduced by *Kirsch (1986)* and *Colton-Monk (1988)*
- Research was focused on the discreteness of transmission eigenvalues for variety of scattering problems: *Colton-Kirsch-Päivärinta (1989), Rynne-Sleeman (1991), Cakoni-Haddar (2007), Colton-Päivärinta-Sylvester (2007), Kirsch (2009), Cakoni-Haddar (2009), Hickmann (to appear).*

  In the above work, it is always assumed that either $n - 1 > 0$ or $1 - n > 0$.

Historical Overview, cont.

- The first proof of existence of at least one transmission eigenvalues for large enough contrast is due to *Päivärinta-Sylvester (2009)*.
- The existence of an infinite set of transmission eigenvalues is proven by *Cakoni-Gintides-Haddar (2010)* under only assumption that either $n - 1 > 0$ or $1 - n > 0$. The existence has been extended to other scattering problems by *Kirsch (2009)*, *Cakoni-Haddar (2010) Cakoni-Kirsch (2010)*, *Bellis-Cakoni-Guzina (2011)*, *Cossonniere (Ph.D. thesis)* etc.
- *Hitrik-Krupchyk-Ola-Päivärinta (2010)*, in a series of papers have extended the transmission eigenvalue problem to a more general class of differential operators with constant coefficients.

Historical Overview, cont.

- *Finch* has connected the discreteness of the transmission spectrum to a uniqueness question in thermo-acoustic imaging for which $n - 1$ can change sign.
- *Cakoni-Colton-Haddar (2010)* and then *Cossonniere-Haddar (2011)* have investigated the case when $n = 1$ in $D_0 \subset D$ and $n - 1 > \alpha > 0$ in $D \setminus \overline{D}_0$.
- Recently *Sylvester (to appear)* has shown that the set of transmission eigenvalues is at most discrete if $n - 1$ is positive (or negative) only in a neighborhood of ∂D but otherwise could changes sign inside D. A similar result is obtained by *Bonnet Ben Dhia - Chesnel - Haddar (2011)* using T-coercivity and *Lakshtanov-Vainberg (to appear)*, for the case when there is contrast in both the main differential operator and lower term.

Scattering by a Spherically Stratified Medium

We consider the interior eigenvalue problem for a ball of radius a with index of refraction $n(r)$ being a function of $r := |x|$

$$\begin{aligned} \Delta w + k^2 n(r) w = 0 &\qquad \text{in } B \\ \Delta v + k^2 v = 0 &\qquad \text{in } B \\ w = v &\qquad \text{on } \partial B \\ \frac{\partial w}{\partial r} = \frac{\partial v}{\partial r} &\qquad \text{on } \partial B \end{aligned}$$

where $B := \{x \in \mathbb{R}^3 : |x| < a\}$.

Scattering by a Spherically Stratified Medium

Look for solutions in polar coordinates (r, θ, φ)

$$v(r,\theta) = a_\ell j_\ell(kr) P_\ell(\cos\theta) \quad \text{and} \quad w(r,\theta) = a_\ell Y_\ell(kr) P_\ell(\cos\theta)$$

where j_ℓ is a spherical Bessel function and Y_ℓ is the solution of

$$Y_\ell'' + \frac{2}{r} Y_\ell' + \left(k^2 n(r) - \frac{\ell(\ell+1)}{r^2}\right) Y_\ell = 0$$

such that $\lim_{r \to 0} \left(Y_\ell(r) - j_\ell(kr)\right) = 0$. There exists a nontrivial solution of the interior transmission problem provided that

$$d_\ell(k) := det \begin{pmatrix} Y_\ell(a) & -j_\ell(ka) \\ Y_\ell'(a) & -kj_\ell'(ka) \end{pmatrix} = 0.$$

Values of k such that $d_\ell(k) = 0$ are the transmission eigenvalues. $d_\ell(k)$ are entire function of k of finite type and bounded for $k > 0$.

Transmission Eigenvalues

Assume that $\Im(n) = 0$ and $n \in C^2[0, a]$.

- If either $n(a) \neq 1$ or $n(a) = 1$ and $\int_0^a \sqrt{n(\rho)}d\rho \neq a$.
  - The set of all transmission eigenvalue is discrete.
  - There exists an infinite number of real transmission eigenvalues accumulating only at $+\infty$.
- For a subclass of $n(r)$ there exist infinitely many complex transmission eigenvalues, *Leung-Colton, (to appear)*.

Inverse spectral problem

- All transmission eigenvalues uniquely determine $n(r)$ provide $n(0)$ is given and either $n(r) > 1$ or $n(r) < 1$. *Cakoni-Colton-Gintides, SIAM Journal Math Analysis, (2010)*.
- If $n(r) < 1$ then transmission eigenvalues corresponding to spherically symmetric eigenfunctions uniquely determine $n(r)$ *Aktosun-Gintides-Papanicolaou, Inverse Problems, (2011)*.

Transmission Eigenvalue Problem

Recall the transmission eigenvalue problem

$$\begin{aligned} \Delta w + k^2 n(x) w = 0 & \quad \text{in} \quad D \\ \Delta v + k^2 v = 0 & \quad \text{in} \quad D \\ w = v & \quad \text{on} \quad \partial D \\ \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} & \quad \text{on} \quad \partial D \end{aligned}$$

Let $u = w - v$, we have that

$$\Delta u + k^2 n u = k^2 (n - 1) v.$$

Then eliminate v to get an equation only in terms of u by applying $(\Delta + k^2)$

Transmission Eigenvalues

Let $n \in L^\infty(D)$, and denote $n^* = \sup_{x \in D} n(x)$ and $0 < n_* = \inf_{x \in D} n(x)$.
To fix our ideas assume $n_* > 1$ (similar analysis if $n^* < 1$).

Let $u := w - v \in H_0^2(D)$. The transmission eigenvalue problem can be written for u as an eigenvalue problem for the fourth order equation:

$$(\Delta + k^2)\frac{1}{n-1}(\Delta + k^2 n)u = 0$$

i.e. in the variational form

$$\int_D \frac{1}{n-1}(\Delta u + k^2 n u)(\Delta \varphi + k^2 \varphi)\, dx = 0 \qquad \text{for all } \varphi \in H_0^2(D)$$

Definition: $k \in \mathbb{C}$ is a transmission eigenvalue if there exists a nontrivial solution $v \in L^2(D)$, $w \in L^2(D)$, $w - v \in H_0^2(D)$ of the homogeneous interior transmission problem.

Transmission Eigenvalues

Obviously we have

$$0 = \int_D \frac{1}{n-1} \left|(\Delta u + k^2 n u)\right|^2 \, dx + k^2 \int_D \left(|\nabla u|^2 - k^2 n |u|^2\right) dx.$$

Poincare inequality yields the Faber-Krahn type inequality for the first transmission eigenvalue (not isoperimetric)

$$k^2_{1,D,n} > \frac{\lambda_1(D)}{n^*}.$$

where $\lambda_1(D)$ is the first Dirichlet eigenvalue of $-\Delta$ in D.

In particular there are no real transmission eigenvalues in the interval $(0, \lambda_1(D)/n^*)$.

Transmission Eigenvalues

Letting $k^2 := \tau$, the transmission eigenvalue problem can be written as a quadratic pencil operator

$$u - \tau K_1 u + \tau^2 K_2 u = 0, \qquad u \in H_0^2(D)$$

with selfadjoint compact operators $K_1 = T^{-1/2} T_1 T^{-1/2}$ and $K_2 = T^{-1/2} T_2 T^{-1/2}$ where

$$(Tu, \varphi)_{H^2(D)} = \int_D \frac{1}{n-1} \Delta u \, \Delta\varphi \, \mathrm{d}x \qquad \text{coercive}$$

$$(T_1 u, \varphi)_{H^2(D)} = -\int_D \frac{1}{n-1} (\Delta u \, \varphi + n u \, \Delta\varphi) \, \mathrm{d}x$$

$$(T_2 u, \varphi)_{H^2(D)} = \int_D \frac{n}{n-1} u \, \varphi \, \mathrm{d}x \qquad \text{non-negative.}$$

Transmission Eigenvalues

The transmission eigenvalue problem can be transformed to the eigenvalue problem

$$(\mathbb{K} - \xi \mathbb{I})U = 0, \qquad U = \begin{pmatrix} u \\ \tau K_2^{1/2} u \end{pmatrix}, \qquad \xi := \frac{1}{\tau}$$

for the non-selfadjoint compact operator $\mathbb{K} \colon H_0^2(D) \times H_0^2(D) \to H_0^2(D) \times H_0^2(D)$ given by

$$\mathbb{K} := \begin{pmatrix} K_1 & -K_2^{1/2} \\ K_2^{1/2} & 0 \end{pmatrix}.$$

However from here one can see that the transmission eigenvalues form a discrete set with $+\infty$ as the only possible accumulation point.

Transmission Eigenvalues

To obtain existence of transmission eigenvalues and isoperimetric Faber-Krahn type inequalities we rewrite the transmission eigenvalue problem in the form

$$(\mathbb{A}_\tau - \tau\mathbb{B})u = 0 \quad \text{in } H_0^2(D)$$

$$(\mathbb{A}_\tau u, \varphi)_{H^2(D)} = \int_D \frac{1}{n-1}(\Delta u + \tau u)(\Delta\varphi + \tau\varphi)\,dx + \tau^2 \int_D u \cdot \varphi\,dx$$

$$(\mathbb{B}u, \varphi)_{H^2(D)} = \int_D \nabla u \cdot \nabla\varphi\,dx$$

Observe that

- The mapping $\tau \to \mathbb{A}_\tau$ is continuous from $(0, +\infty)$ to the set of self-adjoint coercive operators from $H_0^2(D) \to H_0^2(D)$.
- $\mathbb{B} : H_0^2(D) \to H_0^2(D)$ is self-adjoint, compact and non-negative.

Transmission Eigenvalues

Now we consider the generalized eigenvalue problem

$$(\mathbb{A}_\tau - \lambda(\tau)\mathbb{B})u = 0 \quad \text{in} \;\; H_0^2(D)$$

Note that $k^2 = \tau$ is a transmission eigenvalue if and only if satisfies $\lambda(\tau) = \tau$

For a fixed $\tau > 0$ there exists an increasing sequence of eigenvalues $\lambda_j(\tau)_{j\geq 1}$ such that $\lambda_j(\tau) \to +\infty$ as $j \to \infty$.

These eigenvalues satisfy

$$\lambda_j(\tau) = \min_{W\subset\mathcal{U}_j} \left(\max_{u\in W\setminus\{0\}} \frac{(\mathbb{A}_\tau u, u)}{(\mathbb{B}u, u)} \right).$$

Transmission Eigenvalues

Hence, if there exists two positive constants $\tau_0 > 0$ and $\tau_1 > 0$ such that

- $\mathbb{A}_{\tau_0} - \tau_0\mathbb{B}$ is positive on $H_0^2(D)$,
- $\mathbb{A}_{\tau_1} - \tau_1\mathbb{B}$ is non positive on a m dimensional subspace of $H_0^2(D)$

then each of the equations $\lambda_j(\tau) = \tau$ for $j = 1, \dots, m$, has at least one solution in $[\tau_0, \tau_1]$ meaning that there exists m transmission eigenvalues (counting multiplicity) within the interval $[\tau_0, \tau_1]$.

It is now obvious that determining such constants τ_0 and τ_1 provides the existence of transmission eigenvalues as well as the desired isoperimetric inequalities.

Transmission Eigenvalues

- $(\mathbb{A}_\tau u - \tau \mathbb{B} u, u)_{\mathcal{U}_0(D)} \geq \alpha \|u\|_{\mathcal{U}_0(D)}$ for all $0 < \tau < \dfrac{\lambda_1(D)}{n^*}$.
- Take $\tau_1 := k^2(B_r)$ the first eigenvalue a ball $B_r \subset D$ and $n(x) = n_*$ constant, u_r the corresponding eigenfunction and denote $\tilde{u}_r \in H_0^2(D)$ its extension by zero to the whole of D. Then

$$(\mathbb{A}_{\tau_1} \tilde{u}_r - \tau_1 \mathbb{B} \tilde{u}_r, \tilde{u}_r)_{\mathcal{U}_0(D)} \leq 0.$$

  If the radius of the ball is such that $m(r)$ disjoint balls can be included in D, the above condition is satisfied in a $m(r)$-dimensional subspace of $H_0^2(D)$

Thus there exists $m(r)$ transmission eigenvalues (counting multiplicity). As $r \to 0$, $m(r) \to \infty$ and since the multiplicity of an eigenvalue is finite we prove the existence of an infinite set of real transmission eigenvalues.

Faber-Krahn Inequalities

Theorem (Cakoni-Gintides-Haddar, SIMA (2010))

Assume that $1 < n_*$*. Then, there exists an infinite discrete set of real transmission eigenvalues accumulating at infinity* $+\infty$*. Furthermore*

$$k_{1,n^*,B_1} \leq k_{1,n^*,D} \leq k_{1,n(x),D} \leq k_{1,n_*,D} \leq k_{1,n_*,B_2}.$$

where $B_2 \subset D \subset B_1$*.*

One can prove that, for n constant, the first transmission eigenvalue $k_{1,n}$ is continuous and strictly monotonically decreasing with respect to n. In particular, this shows that the first transmission eigenvalue determine uniquely the constant index of refraction, provided that it is known a priori that either $n > 1$.

Similar results can be obtained for the case when $0 < n^* < 1$.

Detection of Anomalies in an Isotropic Medium

What does the first transmission eigenvalue say about the inhomogeneous media $n(x)$?

We find the constant n_0 such that the first transmission eigenvalue of

$$
\begin{aligned}
\Delta w + k^2 n_0 w &= 0 && \text{in} \quad D \\
\Delta v + k^2 v &= 0 && \text{in} \quad D \\
w &= v && \text{on} \quad \partial D \\
\frac{\partial w}{\partial \nu} &= \frac{\partial v}{\partial \nu} && \text{on} \quad \partial D
\end{aligned}
$$

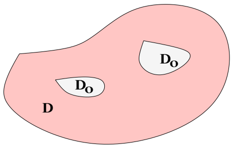
is $k_{1,n(x)}$ (which can be determined from the measured data).

Then from the previous discussion we have that $n_* \leq n_0 \leq n^*$.

Open Question: Find an exact formula that connect n_0 to $n(x)$ and D.

The Case with Cavities

Can the assumption $n > 1$ or $0 < n < 1$ in D be relaxed?



$n = 1$ in D_0

$n - 1 \geq \delta > 0$ in $D \setminus D_0$

The case when there are regions D_0 in D where $n = 1$ (i.e. cavities) is more delicate. The same type of analysis can be carried through by looking for solutions of the transmission eigenvalue problem

$v \in L^2(D)$ and $w \in L^2(D)$ such that $w - v$ is in

$$V_0(D, D_0, k) := \{u \in H_0^2(D) \text{ such that } \Delta u + k^2 u = 0 \text{ in } D_0\}.$$

Cakoni-Colton-Haddar, SIMA (2010)

The Case with Cavities

In particular if $n > 1$ and $k(D_0, n(x))$ is the first eigenvalue for a fixed D, one has the following properties:

- The Faber Krahn inequality

$$0 < \frac{\lambda_1(D)}{n^*} \leq k(D_0, n(x)).$$

- Monotonicity with respect to the index of refraction

$$k(D_0, n(x)) \leq k(D_0, \tilde{n}(x)), \qquad \tilde{n}(x) \leq n(x).$$

- Monotonicity with respect to voids

$$k(D_0, n(x)) \leq k(\tilde{D}_0, n(x)), \qquad D_0 \subset \tilde{D}_0.$$

where $\lambda_1(D)$ is the first Dirichlet eigenvalue of $-\Delta$ in D.

The Case of $n-1$ Changing Sign

Recently, progress has been made in the case of the contrast $n-1$ changing sign inside D with state of the art result by *Sylvester (to appear)*. Roughly speaking he shows that transmission eigenvalues form a discrete (possibly empty) set provided $n-1$ has fixed sign **only in a neighborhood** of ∂D. There are two aspects in the proof:

- Fredholm property. Sylvester consideres the problem in the form

  $$\Delta u+k^2 nu = k^2(n-1)v, \ \Delta v+k^2 v = 0, \quad u \in H_0^2(D), \ v \in H^1(D)$$

  and uses the concept of upper-triangular compact operators. This property can also be obtained via variational formulation (*Kirsch*) or integral equation formulation (*Cossonniere-Haddar*).
- Find a k that is not a transmission eigenvalues. This requires careful estimates for the solution inside D in terms of its values in a neighborhood of ∂D.

The existence of transmission eigenvalues under such weaker assumptions is still open.

Complex Eigenvalues

Current results on complex transmission eigenvalues for media of general shape are limited to identifying eigenvalue free zones in the complex plane.

- The first result for homogeneous media is given in *Cakoni-Colton-Gintides SIMA (2010)*.
- The best result to date is due *Hitrik-Krupchyk-Ola-Päivärinta, Math. Research Letters (2011)*, where they show that almost all transmission eigenvalues are confined to a parabolic neighborhood of the positive real axis. More specifically they show

Theorem (Hitrik-Krupchyk-Ola-Päivärinta)

For $n \in C^\infty(\overline{D}, \mathbb{R})$ and $1 < \alpha \leq n \leq \beta$, there exists a $0 < \delta < 1$ and $C > 1$ both independent of α, β such that all transmission eigenvalues $\tau := k^2 \in \mathbb{C}$ with $|\tau| > C$ satisfies $\Re(\tau) > 0$ and $\Im(\tau) \leq C|\tau|^{1-\delta}$.

Absorbing-Dispersive Media

$$\Delta w + k^2\left(\epsilon_1 + i\frac{\gamma_1}{k}\right) w = 0 \qquad \text{in} \qquad D$$

$$\Delta v + k^2\left(\epsilon_0 + i\frac{\gamma_0}{k}\right) v = 0 \qquad \text{in} \qquad D$$

where $\epsilon_0 \geq \alpha_0 > 0$, $\epsilon_1 \geq \alpha_1 > 0$, $\gamma_0 \geq 0$, $\gamma_1 \geq 0$ are bounded functions.

For the corresponding spherically stratifies case we have:

Theorem

If

$$\frac{\gamma_0 a}{\sqrt{\epsilon_0}} = \int_0^a \frac{\gamma_1(r)}{\sqrt{\epsilon_1(r)}} dr \qquad \text{and} \qquad \sqrt{\epsilon_0}a \neq \int_0^a \sqrt{\epsilon_1(r)} dr$$

there exist an infinite number of real transmission eigenvalues. If the first condition is not met then there exist an infinite number of complex eigenvalues.

Absorbing-Dispersive Media

In the general case we have proven *Cakoni-Colton-Haddar (to appear)*:

- The set of transmission eigenvalues $k \in \mathbb{C}$ in the right half plane is discrete, provided $\epsilon_1(x) - \epsilon_0(x) > 0$.
- Using the stability of a finite set of eigenvalues for closed operators we have shown that if $\sup_D(\gamma_0 + \gamma_1)$ is small enough there exists at least $\ell > 0$ transmission eigenvalues each in a small neighborhood of the first ℓ real transmission eigenvalues corresponding to $\gamma_0 = \gamma_1 = 0$.
- For the case of $\epsilon_0, \epsilon_1, \gamma_0, \gamma_1$ constant, we have identified eigenvalue free zones in the complex plane

The existence of transmission eigenvalues for general media if absorption is present is still open.

Anisotropic Media

The corresponding transmission eigenvalue problem is to find $v, w \in H^1(D)$ such that

$$\begin{aligned} \nabla \cdot A \nabla w + k^2 n w &= 0 && \text{in} \quad D \\ \Delta v + k^2 v &= 0 && \text{in} \quad D \\ w &= v && \text{on} \quad \partial D \\ \nu \cdot A \nabla w &= \nu \cdot \nabla v && \text{on} \quad \partial D. \end{aligned}$$

This transmission eigenvalue problem has a more complicated nonlinear structure than quadratic.

The existence has been shown in *Cakoni-Gintides-Haddar, SIAM J. Math. Anal. (2010)* and *Cakoni-Kirsch, IJCSM (2010)*.

Existence of Transmission Eigenvalues

Set $u = w - v \in H_0^1(D)$. Find $v = v_u$ by solving a Neuman type problem: For every $\psi \in H^1(D)$

$$\int_D (A - I)\nabla v \cdot \nabla\overline{\psi} - k^2(n-1)v\overline{\psi}\, dx = \int_D A\nabla u \cdot \nabla\overline{\psi} - k^2 n u\overline{\psi}\, dx.$$

Having $u \to v_u$, we require that $v := v_u$ satisfies $\Delta v + k^2 v = 0$.

Thus we define $\mathbb{L}_k : H_0^1(D) \to H_0^1(D)$

$$(\mathbb{L}_k u, \phi)_{H_0^1(D)} = \int_D \nabla v_u \cdot \nabla\overline{\phi} - k^2 v_u \cdot \overline{\phi}\, dx, \qquad \phi \in H_0^1(D).$$

Then the transmission eigenvalue problem is equivalent to

$$\mathbb{L}_k u = 0 \quad \text{in} \quad H_0^1(D) \quad \text{which can be written}$$

$$(\mathbb{I} + \mathbb{L}_0^{-1/2}\mathbb{C}_k\mathbb{L}_0^{-1/2})u = 0 \quad \text{in} \quad H_0^1(D)$$

$\mathbb{L}_0$ self-adjoint positive definite and $\mathbb{C}_k$ self-adjoint compact.

Existence of Transmission Eigenvalues

- If $n(x) \equiv 1$ and the contrast $A - I$ is either positive or negative in D then there exists an infinite discrete set of real transmission eigenvalues accumulating at $+\infty$.
- If the contrasts $A - I$ and $n - 1$ have the same fixed sign, then there exists an infinite discrete set of real transmission eigenvalues accumulating at $+\infty$.
- If the contrasts $A - I$ and $n - 1$ have the opposite fixed sign, then there exits at least one real transmission eigenvalue providing that n is small enough.

Discreteness of Transmission Eigenvalues

The strongest result on the discreteness of transmission eigenvalues for this problem is due to *Bonnet Ben Dhia - Chesnel - Haddar, Comptes Rendus Math. (2011)* (using the concept of $\top$- coercivity).

In particular, the discreteness of transmission eigenvalues is proven under either one of the following assumptions (weaker than for the existence):

- Either $A - I > 0$ or $A - I < 0$ in D, and $\int_D (n-1)\,dx \neq 0$ or $n \equiv 1$.
- The contrasts $A - I$ and $n - 1$ have the same fixed sign only in a neighborhood of the boundary ∂D.

Numerical Example: Homogeneous Anisotropic Media

We consider D to be the unit square $[-1/2, 1/2] \times [-1/2, 1/2]$, $n \equiv 1$ and

$$A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix} \quad A_2 = \begin{pmatrix} 6 & 0 \\ 0 & 8 \end{pmatrix} \quad A_{2r} = \begin{pmatrix} 7.4136 & -0.9069 \\ -0.9069 & 6.5834 \end{pmatrix}$$

Matrix	Eigenvalues a_*, a^*	Predicted a_0
A_{iso}	4, 4	4.032
A_1	2, 8	5.319
A_2	6, 8	7.407
A_{2r}	6, 8	6.896

Cakoni-Colton-Monk-Sun, Inverse Problems, (2010)

Open Problem

- Can the existence of real transmission eigenvalues for non-absorbing media be established if the assumptions on the sign of the contrast are weakened?
- Do complex transmission eigenvalues exists for general non-absorbing media?
- Do real transmission eigenvalues exist for absorbing media?
- What would the necessary conditions be on the contrasts that guaranty the discreteness of transmission eigenvalues?
- Can Faber-Krahn type inequalities be established for the higher eigenvalues?
- Can an inverse spectral problem be developed for the general transmission eigenvalue problem? (Completeness of eigen-solutions?)

Cakoni - Haddar, Transmission Eigenvalues in Inverse Scattering Theory, in Inside Out 2, Uhlmann edt. MSRI Publication (to appear).