Transmit Beamforming to Multiple Cochannel Multicast Groups

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To my parents

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Abstract

The major contribution of this thesis is on the problem of transmit beamforming to multiple cochannel multicast groups. Two viewpoints are considered: i) minimizing total transmission power while guaranteeing a prescribed minimum signal-to-interference-plus-noise ratio (SINR) at each receiver; and ii) a “fair” approach maximizing the overall minimum SINR under a total power budget. The core problem is a multicast generalization of the multiuser downlink beamforming problem; the difference is that each transmitted stream is directed to multiple receivers, each with its own channel. Such generalization is relevant and timely, e.g., in the context of the emerging WiMAX and UMTS-LTE wireless networks. The joint multicast beamforming problem is in general NP-hard, motivating the pursuit of computationally efficient quasi-optimal solutions. In chapter 1, it is shown that semidefinite relaxation coupled with suitable randomization / cochannel multicast power control yield computationally efficient high-quality approximate solutions.

The multicast beamforming problem is revisited in chapter 2 for the important special case when the channel vectors are Vandermonde. This arises when a uniform linear antenna array is used at the transmitter under far-field line-of-sight propagation conditions, as provisioned in 802.16e and related wireless backhaul scenarios. It is shown that for Vandermonde channel vectors it is possible to recast the optimization in terms of the autocorrelation sequences of the sought beamvectors, yielding an equivalent convex reformulation. This affords efficient optimal solution using modern interior point methods. The optimal beamvectors can then be recovered using spectral factorization. Robust extensions for the case of partial channel state information, where the direction of each receiver is known to lie in an interval, are also developed. Interestingly, these also admit convex reformulation.

Chapter 3 considers the joint scheduling, admission, and power control problem under quality-of-service (QoS) constraints and a general formulation that incorporates multi-casting, cochannel or orthogonal transmission modalities, and access point selection. Several special cases are well-known to be NP-hard, yet important for QoS provisioning and bandwidth-efficient operation of existing and emerging cellular and overlay / underlay networks. Approximate solutions to the original problem are generated following a disciplined approach. The general problem is first concisely formulated as constrained optimization. A geometric programming approximation is then developed, which forms the core of a heuristic, yet well-motivated centralized algorithm.

Chapter 4 considers the throughput maximization problem in the context of wireless networks. One is given a directed multi-hop network between a source and a destination, with edge capacities that are a function of transmission powers. Each node may split its aggregate incoming flow to multiple outgoing flows, and the objective is to select flows to maximize the end-to-end flow from source to destination. Power control can be used to obtain a favorable 'topology' from the throughput maximization viewpoint. This suggests a joint max-flow power control problem that is basic, yet has not been considered in the cross-layer network optimization literature. Alternatively, power control may be coupled with dynamic routing by means of differential queue lengths information. Both approaches are sketched and convex approximations, in the high SINR regime, are provided for these cross-layer power control problems.

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Chapter 1

Transmit Beamforming to Multiple Cochannel Multicast Groups

The problem of transmit beamforming to multiple cochannel multicast groups is considered, when the channels are known at the transmitter. Two viewpoints are considered: i) minimizing total transmission power while guaranteeing a prescribed minimum signal-to-interference-plus-noise ratio (SINR) at each receiver; and ii) a "fair" approach maximizing the overall minimum SINR under a total power budget. The core problem is a multicast generalization of the multiuser downlink beamforming problem; the difference is that each transmitted stream is directed to multiple receivers, each with its own channel. Such generalization is relevant and timely, e.g., in the context of the emerging WiMAX and UMTS-LTE wireless networks. The joint problem also contains single-group multicast beamforming as a special case. The latter (and therefore also the former) is NP-hard. This motivates the pursuit of computationally efficient quasi-optimal solutions. It is shown that Lagrangian relaxation coupled with suitable randomization / cochannel multicast power control yield computationally efficient high-quality approximate solutions. For a significant fraction of problem instances, the solutions generated this way are exactly optimal. Extensive numerical results using both simulated and measured wireless channels are presented to corroborate the main findings.

1.1 Introduction

The proliferation of streaming media (digital audio, video, IP radio), peer-to-peer services, large-scale software updates, and profiled newscasts over the wireline Internet has brought renewed interest in multicast routing protocols. These protocols were originally conceived and have since evolved under the “wireline premise”: the physical network is a graph comprising point-to-point links that do not interfere with each other at the physical layer. Today, multicast routing protocols operate at the network or application layer, using either controlled flooding or minimum spanning tree access.

As wireless networks become ever more ubiquitous, and wireless becomes the choice for not only the “last hop” but also suburban- and metropolitan-area backbones, wireless multicasting solutions are needed to account for and exploit the idiosyncracies of the wireless medium. Wireless is inherently a broadcast medium, where it is possible to reach multiple destinations with a single transmission; different cochannel transmissions are interfering with one-another at the intended destination(s); and links are subject to fading and shadowing, in addition to cochannel interference.

The broadcast advantage of wireless has of course been exploited since the early days of radio. The interference problem was dealt with by allocating different frequency bands to the different stations, and transmission was mostly isotropic or focused towards a specific service area.

Today, the situation with wireless networks is much different. First, transmissions need not be “blind”. Many wireless network standards provision the use of transmit antenna arrays. Using baseband beamforming, it is possible to steer energy in the direction(s) of the intended users, whose locations (or, more generally, channels) can often be accurately estimated. Second, the push towards higher capacity and end-user rates necessitates cochannel transmission which exploits the spatial diversity in the user population (*spatial multiplexing*). Third, quality of service (QoS) is an important consideration, especially in wireless backhaul solutions like 802.16e. Finally, due to cochannel interference, wireless multicasting cannot be dealt with in isolation, one group at a time; a joint solution is needed.

The problem of transmit beamforming towards a single group of users was first considered in the Ph.D. dissertation of Lopez [28], using the averaged (over all users in the group) received signal-to-noise ratio (SNR) as the design criterion. The solution boils down to a relatively simple eigenvalue problem, but no SNR guarantee is provided this way: some users may get really poor SNR [34]. This is not acceptable in multicasting applications, because it is the worst SNR that determines the *common* information rate. QoS (providing a guaranteed minimum received SNR to every user) and max-min fair (MMF) (maximizing the smallest received SNR) designs were first proposed in [33, 34], where it was shown that the core problem is NP-hard, yet high-quality approximate solutions can be obtained using relaxation techniques based on semidefinite programming (SDP). The latter is a class of convex optimization problems which can be solved in polynomial time by powerful interior point methods.

In this chapter a new and interesting problem is formulated: transmit beamforming for multicasting to *multiple* cochannel groups under QoS and MMF criteria. The joint design problem is considered, since designing a transmit beamformer separately for each multicast group can be far from optimal, due to intergroup interference. By simultaneously serving several cochannel groups, the spectral efficiency is much higher than in the single-group case. The extension is nontrivial in many ways:

- The multigroup QoS problem can be infeasible.
- The QoS and MMF versions are different, unlike the single-group case.
- The approximation step is much more involved: randomization is coupled with multicast power control, which is of interest in its own right.

Two solid and well-motivated (Lagrange dual) algorithms are proposed. In addition to semidefinite relaxation (SDR) ideas, the proposed solutions entail a cochannel multicast power control component, which can be viewed as a generalization of multiuser power control ideas for the cellular downlink (see, e.g., [14] and references therein). It is important to note that the problem formulation considered here contains as special cases the single-group multicasting (broadcasting) problem [34], as well as the multiuser downlink beamforming problem (see, e.g., [2, 11] and references therein), where each multicast group consists of a single receiver. Extensive numerical results, including experiments with measured channels, are presented and show that in the multigroup case as well, the proposed SDR-based algorithms work remarkably well.

Notation Boldface uppercase and lowercase letters denote matrices and column vectors, respectively. Integer sets are denoted by calligraphic uppercase letters, whereas constrained optimization problems are referenced with sans serif uppercase letters. The superscripts $(\cdot)^*$, $(\cdot)^T$, and $(\cdot)^H$ denote complex conjugate, transpose, and Hermitian (conjugate) transpose matrix operators, respectively. $\mathrm{tr}(\cdot)$, $\mathrm{rank}(\cdot)$, $|\cdot|$, and $\|\cdot\|_2$ denote the trace, the rank, the absolute value, and the Euclidean norm operators, respectively. By $\mathbf{W} \succeq \mathbf{0}$ we denote that $\mathbf{W}$ is a Hermitian positive semidefinite matrix. Finally, $\mathbf{I}_N$ and $\mathbf{1}_G$ denote the $N \times N$ identity matrix and the $G \times 1$ all ones vector.

1.2 System Model

Consider a communication scenario where an access point employing an antenna array of N elements is used to feed content, simultaneously and over the same frequency channel, to K single-antenna[1] receivers. Let $\mathbf{h}_k$ denote the $N \times 1$ complex vector that models the propagation loss and phase shift of the frequency-flat quasi-static channel from each transmit antenna to the receive antenna of user $k \in \mathcal{K} := \{1, \ldots, K\}$. Each receiver listens to a single multicast stream $i \in \mathcal{G} := \{1, \ldots, G\}$, where $1 \leq G \leq K$ is the total number of multicasts. Each multicast group, denoted by $\mathcal{G}_i$, is formed by the indices of the participating receivers. These sets are non-overlapping and collectively contain all the users; thus, $\mathcal{G}_i \cap \mathcal{G}_j = \emptyset$, $i \neq j$, $\cup_i \mathcal{G}_i = \mathcal{K}$, and, denoting $G_i := |\mathcal{G}_i|$, $\sum_{i=1}^{G} G_i = K$. Note that $G = 1$ corresponds to the case of (selective) broadcasting [34], whereas $G = K$ corresponds to the case of individual information transmission to each receiver (the multiuser downlink problem, see, e.g., [2, 11].

Let $\mathbf{w}_i^* \in \mathbb{C}^N$ denote the beamforming weight vector applied to the N transmitting antenna elements to generate the spatial channel for transmission to group i (see Fig. 1.1). Then the signal transmitted by the antenna array is equal to $\sum_{i=1}^{G} \mathbf{w}_i^* s_i(t)$, where $s_i(t)$ is the temporal information-bearing signal directed to receivers in multicast group i. If each $s_i(t)$ is zero-mean, temporally white with unit variance, and the waveforms

[1]Single-antenna receives are assumed for brevity of exposition; the designs can be generalized to account for multi-antenna receivers.

$\{s_i(t)\}_{i=1}^G$ are mutually uncorrelated, then the total power radiated by the transmitting antenna array is equal to $\sum_{i=1}^G \|\mathbf{w}_i\|_2^2$. The signal received by receiver $k \in \mathcal{G}_i$ is equal to

$$r_k(t) = s_i(t)\mathbf{w}_i^H\mathbf{h}_k + \sum_{\substack{j=1\\ j\neq i}}^G s_j(t)\mathbf{w}_j^H\mathbf{h}_k + \eta_k, \tag{1.1}$$

where the three terms comprising the sum account for the useful signal, interference and noise, respectively.

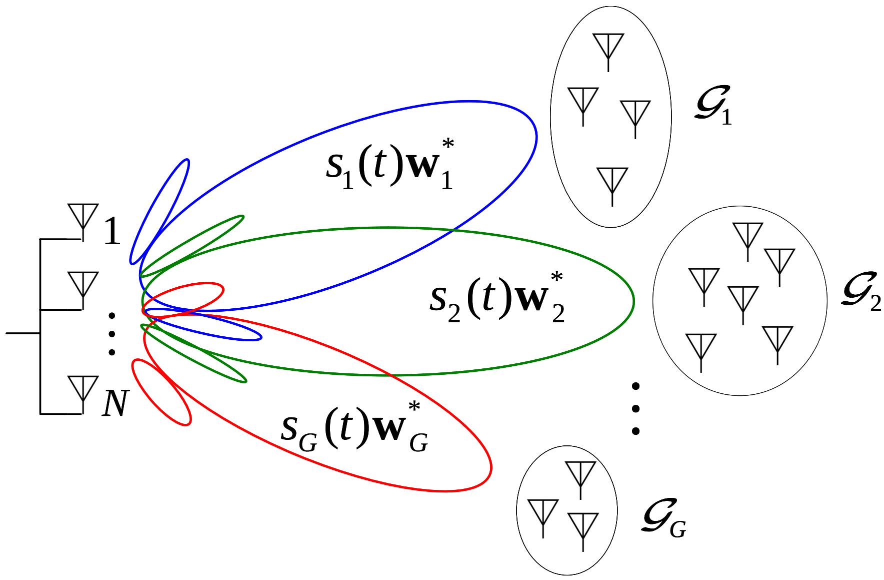

Figure 1.1: Cochannel multicast beamforming concept (note that groups need not be spatially clustered)

1.3 Quality of Service Multicast Beamforming

The joint design of transmit beamformers can be posed as the constrained optimization problem of minimizing the total radiated power subject to meeting a prescribed SINR constraint γ_k for each receiver $k \in \mathcal{K}$

$$\mathcal{Q}: \quad \min_{\{\mathbf{w}_i \in \mathbb{C}^N\}_{i=1}^G} \sum_{i=1}^G \|\mathbf{w}_i\|_2^2$$

$$\text{s.t.}: \quad \frac{|\mathbf{w}_i^H\mathbf{h}_k|^2}{\sum_{\substack{j=1\\ j\neq i}}^G |\mathbf{w}_j^H\mathbf{h}_k|^2 + \sigma_k^2} \geq \gamma_k \qquad \forall k \in \mathcal{G}_i \quad \forall i \in \mathcal{G}.$$

When formulating the QoS problem of interest Q, it is assumed that all channel vectors $\{\mathbf{h}_k\}_{k=1}^K$ and corresponding noise variances $\{\sigma_k^2\}_{k=1}^K$ are accurately known at the transmitter. Contrary to the single-group case [34], problem Q can be infeasible due to interference, if the SINR requirements are too stringent and / or the channels of users listening to different multicasts are highly correlated. Then, in order to render the problem feasible it is necessary to loosen some of the QoS thresholds or deny service to some users in the specific frequency tone / time slot, by means of proper admission control [30]. Beyond feasibility concerns, it is important to note that each beamformer must serve multiple users listening to the same information. When a feasible solution exists, at least one SINR constraint per group will be satisfied with equality at the optimum, whereas the others may be inactive (i.e., over-satisfied); this is contrary to the case of independent information transmission, where all SINR constraints are tight at the optimum [2].

Using the fact that the denominator of the SINR is positive, problem Q is *equivalently* reformulated to

$$\mathsf{Q} \qquad \min_{\{\mathbf{w}_i \in \mathbb{C}^N\}_{i=1}^G} \sum_{i=1}^G \mathbf{w}_i^H \mathbf{w}_i$$

$$\text{s.t. :} \qquad \gamma_k \sum_{\substack{j=1 \\ j \neq i}}^G \mathbf{w}_j^H \mathbf{h}_k \mathbf{h}_k^H \mathbf{w}_j - \mathbf{w}_i^H \mathbf{h}_k \mathbf{h}_k^H \mathbf{w}_i + \gamma_k \sigma_k^2 \leq 0 \qquad \forall k \in \mathcal{G}_i \;\; \forall i \in \mathcal{G}.$$

Problem Q is a quadratically constrained quadratic programming (QCQP) problem. It is seen that the constraints are *nonconvex*, since all quadratic terms comprising the sum are convex, apart from the last term (corresponding to the useful signal power) which is added with a negative sign.

Problem Q contains as a special case the associated broadcasting problem ($G = 1$), which was proved to be NP-hard in [34]. Hence, it immediately follows that

Claim 1. *Problem* Q *is NP-hard.*

Claim 1 reveals the fundamental difference of the general multicast QoS problem to its other special case, i.e., when $G = K$. The QoS multiuser downlink beamforming problem can be optimally solved in polynomial time by the alternating power control /

beamforming algorithm of [11]. Moreover, it admits an equivalent *convex*; specifically, second order cone programming (SOCP) reformulation [2]. Note that there exist more special cases of problem Q that are not NP-hard, e.g., a restriction to Vandermonde channel vectors enables convex reformulation [24, 25]. Optimal and efficient solutions for this case will be presented in Section 2.2. Claim 1 motivates (cf. [16]) the pursuit of sensible approximate solutions to the QoS problem Q.

1.3.1 Semidefinite Relaxation

The first step in the pursuit of approximate solutions is to change the optimization variables to $\{\mathbf{W}_i := \mathbf{w}_i\mathbf{w}_i^H\}_{i=1}^G$. Note that for some $\mathbf{w}_i \in \mathbb{C}^N$

$$\mathbf{W}_i = \mathbf{w}_i\mathbf{w}_i^H \Leftrightarrow \begin{cases} \mathbf{W}_i \succeq \mathbf{0}, \\ \mathrm{rank}(\mathbf{W}_i) = 1. \end{cases} \tag{1.2}$$

Defining $\{\mathbf{H}_k := \mathbf{h}_k\mathbf{h}_k^H\}_{k=1}^K$ and using that $\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$ for matrices $\mathbf{A}, \mathbf{B}$ of compatible dimensions, the signal power user k receives by multicast i can be expressed as

$$|\mathbf{w}_i^H\mathbf{h}_k|^2 = \mathbf{h}_k^H\mathbf{w}_i\mathbf{w}_i^H\mathbf{h}_k = \mathrm{tr}(\mathbf{h}_k^H\mathbf{w}_i\mathbf{w}_i^H\mathbf{h}_k) = \mathrm{tr}(\mathbf{h}_k\mathbf{h}_k^H\mathbf{w}_i\mathbf{w}_i^H) = \mathrm{tr}(\mathbf{H}_k\mathbf{W}_i). \tag{1.3}$$

Likewise, the power of each beamforming vector can be written as

$$\|\mathbf{w}_i\|_2^2 = \mathbf{w}_i^H\mathbf{w}_i = \mathrm{tr}(\mathbf{w}_i^H\mathbf{w}_i) = \mathrm{tr}(\mathbf{w}_i\mathbf{w}_i^H) = \mathrm{tr}(\mathbf{W}_i). \tag{1.4}$$

It follows that due to (1.2), (1.3), and (1.4) problem Q can be *equivalently* reformulated, with respect to the variables $\{\mathbf{W}_i\}_{i=1}^G$, to

$$\mathsf{Q} \qquad \min_{\{\mathbf{W}_i \in \mathbb{C}^{N\times N}\}_{i=1}^G} \sum_{i=1}^G \mathrm{tr}(\mathbf{W}_i)$$

$$\text{s.t.}: \quad \gamma_k \sum_{\substack{j=1\\ j\neq i}}^G \mathrm{tr}(\mathbf{H}_k\mathbf{W}_j) - \mathrm{tr}(\mathbf{H}_k\mathbf{W}_i) + \gamma_k\sigma_k^2 \leq 0 \qquad \forall k \in \mathcal{G}_i \;\; \forall i \in \mathcal{G},$$

$$\mathbf{W}_i \succeq \mathbf{0} \qquad \forall i \in \mathcal{G},$$

$$\mathrm{rank}(\mathbf{W}_i) = 1 \qquad \forall i \in \mathcal{G}.$$

Note that if the *instantaneous* channel vectors $\{\mathbf{h}_k\}_{k=1}^K$ are unknown, the channel correlation matrices can be used instead as input parameters $\{\mathbf{H}_k\}_{k=1}^K$. However, in this case the resulting design can only guarantee *average* received SINR's.

Problem Q consists of a linear objective function and linear inequality, positive semidefinite, and rank constraints. Positive semidefinite constraints are convex [4], but rank constraints are not, since the sum of two rank-1 matrices has generic rank 2. Hence, the reformulation of problem Q with respect to the variables $\{\mathbf{W}_i\}_{i=1}^G$ has allowed us to pin down the source of nonconvexity. Disregarding the problematic rank constraints, problem Q is *relaxed* to

$$\mathsf{Q}_\mathbf{r}$$

$$\min_{\{\mathbf{W}_i \in \mathbb{C}^{N\times N}\}_{i=1}^G} \sum_{i=1}^G \mathrm{tr}(\mathbf{W}_i)$$

$$\text{s.t.}: \quad \gamma_k \sum_{\substack{j=1 \\ j\neq i}}^G \mathrm{tr}(\mathbf{H}_k\mathbf{W}_j) - \mathrm{tr}(\mathbf{H}_k\mathbf{W}_i) + \gamma_k\sigma_k^2 \leq 0 \qquad \forall k \in \mathcal{G}_i \;\; \forall i \in \mathcal{G},$$

$$\mathbf{W}_i \succeq \mathbf{0} \qquad \forall i \in \mathcal{G}.$$

The resulting problem $\mathsf{Q}_\mathbf{r}$ is convex; specifically, an SDP problem [4]. SDR $\mathsf{Q} \rightarrow \mathsf{Q}_\mathbf{r}$ is well-motivated, since it is the tightest relaxation, in the Lagrangian sense, for QCQP problems [29, 34, 44].

Modern SDP solvers, such as SeDuMi [35] and SDPT3 [40], use interior point methods to efficiently find an optimum solution to problem $\mathsf{Q}_\mathbf{r}$, if it is feasible; otherwise, they return a certificate of infeasibility. The SDP problem $\mathsf{Q}_\mathbf{r}$ has G matrix variables of size $N \times N$, and K linear constraints. Interior point methods will take $O(\sqrt{GN}\log(1/\epsilon))$ iterations, with each iteration requiring at most $O(G^3N^6 + KGN^2)$ arithmetic operations [48], where the parameter ϵ represents the solution accuracy at the algorithm's termination. Actual runtime complexity will usually scale far slower with G, N, and K than this worst-case bound, as observed during the preparation of the numerical results presented in Section 1.5.

1.3.2 Approximate Solution

Problem Q may not admit a feasible solution, but if it does, the aforementioned approach will yield a solution to problem $\mathsf{Q}_\mathbf{r}$. However, due to the relaxation, this solution will not, in general, consist of rank-1 matrices. This is because the *convex* feasible set of $\mathsf{Q_r}$ is a superset of the *nonconvex* feasible set of Q. In addition, the optimum objective value of $\mathsf{Q_r}$ is merely a lower bound on the transmission power required by the rank-1 transmit beamforming scheme. An *approximate* solution to the original QoS problem Q can be found using a *randomization* technique (see, e.g., [13, 49]). The idea is to generate candidate sets of beamforming vectors $\{\tilde{\mathbf{w}}_i\}_{i=1}^G$ from the optimum solution matrices $\{\mathbf{W}_i^{\text{opt}}\}_{i=1}^G$ of problem $\mathsf{Q_r}$ and choose the one that can be scaled to satisfy the SINR constraints of problem Q with the minimum total power cost.

The Gaussian randomization method (see, e.g., [13,49]) is proposed for the generation of the candidate beamformers, motivated by its successful application in related QCQP problems and especially in the single-group multicast beamforming problem [29, 34]. Initially, the eigenvalue decomposition $\mathbf{W}_i^{\text{opt}} = \mathbf{U}_i \mathbf{\Sigma}_i \mathbf{U}_i^H$ of each optimal solution matrix is calculated. Then, the respective candidate beamformers are generated as $\tilde{\mathbf{w}}_i := \mathbf{U}_i \mathbf{\Sigma}_i^{1/2} \mathbf{u}$, where $\mathbf{u} \in \mathbb{C}^N \sim \mathcal{CN}(\mathbf{0}, \mathbf{I})$, so that $\mathrm{E}[\tilde{\mathbf{w}}_i \tilde{\mathbf{w}}_i^H] = \mathbf{W}_i^{\text{opt}}$. The main difference relative to the broadcasting case considered in [34], is that here we cannot simply "scale up" the candidate beamforming vectors to satisfy the SINR constraints of problem Q. The reason is that, in contrast to [34], we herein deal with an interference scenario, and boosting the beamforming vector of one group also increases interference to nodes in other groups. Whether it is feasible to satisfy the constraints for a given set of candidate beamforming vectors is also an issue here.

Let $\alpha_{k,i} := |\tilde{\mathbf{w}}_i^H \mathbf{h}_k|^2$ denote the signal power user k receives when beamvector $\tilde{\mathbf{w}}_i$ is used to serve multicast stream i. Let $\beta_i := \|\tilde{\mathbf{w}}_i\|_2^2$ denote the power of the candidate beamvector for multicast stream i and p_i the respective sought power boost (or reduction) factor. Then the following *multicast power control* (MPC) problem emerges in converting the candidate beamforming vectors to a candidate solution of problem Q.

$$
\mathsf{P^Q} \qquad \min_{\{p_i \geq 0\}_{i=1}^G} \sum_{i=1}^G \beta_i p_i
$$

$$
\text{s.t.}: \quad \frac{\alpha_{k,i} p_i}{\sum_{\substack{j=1 \\ j \neq i}}^G \alpha_{k,j} p_j + \sigma_k^2} \geq \gamma_k \qquad \forall k \in \mathcal{G}_i \quad \forall i \in \mathcal{G}.
$$

Since the denominator of the SINR is positive, problem $\mathsf{P^Q}$ can be equivalently reformulated to

$$
\mathsf{P^Q} \qquad \min_{\{p_i \geq 0\}_{i=1}^{G}} \sum_{i=1}^{G} \beta_i p_i
$$

$$
\text{s.t. :} \qquad \gamma_k \sum_{\substack{j=1 \\ j \neq i}}^{G} \alpha_{k,j} p_j - \alpha_{k,i} p_i + \gamma_k \sigma_k^2 \leq 0 \qquad \forall k \in \mathcal{G}_i \ \ \forall i \in \mathcal{G}.
$$

$\mathsf{P^Q}$ is a linear programming (LP) problem with G real nonnegative variables and K linear inequality constraints. For a feasible instance of the MPC problem $\mathsf{P^Q}$, interior point methods can generate an ϵ-optimal solution in $O(\sqrt{G}\log(1/\epsilon))$ iterations, each requiring at most $O(G^3 + KG)$ arithmetic operations [48]. Otherwise, they yield an infeasibility certificate. This is a useful property in determining the feasibility of a candidate beamforming configuration. The simplex method could also be used and it will typically be more efficient for small problem sizes.

As noted already, for $G = K$ (independent information transmission to each receiver), problem $\mathsf{Q}_\mathbf{r}$ is in fact *equivalent to* (not a relaxation of) problem Q, see [2]. Likewise, problem $\mathsf{P^Q}$ reduces to the well-known multiuser downlink power control problem, which can be solved using simpler means (see, e.g., [14]): matrix inversion and iterative descent algorithms. In this special case, (in)feasibility can be determined from the spectral radius of a certain "connectivity" matrix. Similar simplifications for the general instance of MPC are perhaps possible, but nontrivial. In fact, an iterative MPC algorithm based on the concept of interference functions was proposed in [15]. However, the power iterations advocated therein are only guaranteed to converge when the problem is feasible. Keeping in mind that the MPC problem emerges in the context of randomization, it is clear that effective detection of infeasibility is an important issue. Furthermore, even when the problem is feasible, it is not clear whether the power iterations in [15] require smaller overall complexity to find an optimum solution than the available LP routines, which are highly efficient.

The overall algorithm for generating an approximate solution to the original QoS problem Q can be summarized as follows:

1. **Relaxation:** Solve the SDP problem $\mathsf{Q}_{\mathbf{r}}$ and denote the solution $\{\mathbf{W}_i^{\mathrm{opt}}\}_{i=1}^G$.

2. **Randomization / Power Control Loop:** Generate candidate beamforming vectors using the Gaussian randomization technique. If, for some $i \in \mathcal{G}$, $\mathrm{rank}(\mathbf{W}_i^{\mathrm{opt}}) = 1$, then use the principal component instead. Next, feed the resulting set of candidate beamforming vectors $\{\tilde{\mathbf{w}}_i\}_{i=1}^G$ into the MPC problem P^{Q} and solve it using an LP solver. If the particular instance of problem P^{Q} is infeasible or yields larger objective value than the previously checked candidates, discard the proposed set of candidate beamforming vectors; else, record the set of beamforming vectors, the associated power factors $\{p_i\}_{i=1}^G$ and the objective value. Repeat for a predetermined number N_{rand} of randomizations.

Assuming that the randomization / power control loop yields at least one feasible solution, let $\{\tilde{\mathbf{w}}_i^{\mathrm{opt}}, p_i^{\mathrm{opt}}\}_{i=1}^G$ denote the recorded beamvectors and power factors at algorithm's termination. Then, the approximate solution of problem Q is given by $\{\sqrt{p_i^{\mathrm{opt}}}\tilde{\mathbf{w}}_i^{\mathrm{opt}}\}_{i=1}^G$.

The overall complexity of this solution is that of solving the SDP problem $\mathsf{Q}_{\mathbf{r}}$ once and the LP problem P^{Q} N_{rand} times. The choice of N_{rand} is a trade-off between the extent of suboptimality of the final solution and the overall complexity of the algorithm. The *quality* of the approximate solution to problem Q can be measured by the ratio of the minimum objective value of problem P^{Q}, attained in the randomization / power control loop, to the lower bound on the transmission power, obtained by the solution of problem $\mathsf{Q}_{\mathbf{r}}$. The numerical results reported in Section 1.5 show that a few hundred randomizations are adequate, in most scenarios considered, to yield a solution which is at most 3–4 dB away from this lower bound; hence, even less from the (NP-hard to find) optimum. The lower bound obtained by solving problem $\mathsf{Q}_{\mathbf{r}}$ can be further motivated from a duality perspective; that is, the aforementioned relaxation lower bound is in fact the tightest lower bound on the optimum value of problem Q attainable via Lagrangian duality [4]. This follows from arguments in [44] (see also the single-group case in [34]), due to the fact that Q is a QCQP problem. For theoretical *a priori* bounds on the extent of the suboptimality of the solution in [34] for the single-group case see [29].

1.4 Max-Min Fair Multicast Beamforming

In this section, we consider the related problem of maximizing the minimum SINR, received by any of the K intended users, irrespective of the multicast group they belong to, subject to an upper bound P on the total transmission power. Actually, a more general problem is considered, in which each received SINR is scaled by a predetermined positive real constant weight factor $1/\gamma_k$, to account for the possibility of different classes of service. The joint weighted MMF multicast beamformer design can be formulated as the constrained optimization problem

F

$$\max_{\{\mathbf{w}_i \in \mathbb{C}^N\}_{i=1}^G} \min_{i \in \mathcal{G}} \min_{k \in \mathcal{G}_i} \frac{1}{\gamma_k} \frac{|\mathbf{w}_i^H \mathbf{h}_k|^2}{\sum_{\substack{j=1 \\ j \neq i}}^G |\mathbf{w}_j^H \mathbf{h}_k|^2 + \sigma_k^2}$$

$$\text{s.t. :} \quad \sum_{i=1}^G \|\mathbf{w}_i\|_2^2 \leq P.$$

This problem formulation is important for systems required to comply with a strict upper bound on the total transmission power, e.g., due to regulation. It is straightforward to see that the power bound will be met with equality at the optimum. Otherwise, if there is power budget left, one could distribute it evenly, i.e., multiply all beamformers by a constant $c > 1$, thereby increasing the minimum SINR (note that $\sigma_k^2 > 0$), thus contradicting optimality. We may therefore focus on the equality constrained problem and denote this as F from now on. Introducing an auxiliary positive real variable g to lower bound the worst-case scaled SINR, problem F can be *equivalently* rewritten as

F

$$\max_{\{\mathbf{w}_i \in \mathbb{C}^N\}_{i=1}^G,\ g \geq 0} g$$

$$\text{s.t. :} \quad \frac{1}{\gamma_k} \frac{|\mathbf{w}_i^H \mathbf{h}_k|^2}{\sum_{\substack{j=1 \\ j \neq i}}^G |\mathbf{w}_j^H \mathbf{h}_k|^2 + \sigma_k^2} \geq g \qquad \forall k \in \mathcal{G}_i \ \ \forall i \in \mathcal{G},$$

$$\sum_{i=1}^G \|\mathbf{w}_i\|_2^2 = P.$$

The design criterion seeks to maximize the worst-case *scaled* SINR, so as to ensure *weighted* fairness among the received multicasts. Let $\mathbf{g} := [\gamma_1, \ldots, \gamma_K]^T$. Obviously, *equal* fairness is a special case that corresponds to the choice of $\mathbf{g} = \mathbf{1}_K$. Contrary to the QoS approach, discussed in Section 1.3, problem F always admits a feasible solution, apart from the trivial case of zero channel vectors. Denoting as g^{opt} the optimum value of F, the optimum beamformers guarantee SINR levels equal to $g^{\mathrm{opt}}\mathbf{g}$. Interpreting the weight factors $\mathbf{g}$ as target SINR's, these are achieved, with total transmission power P, if and only if $g^{\mathrm{opt}} \geq 1$. In this sense, MMF is more flexible formulation than QoS and it can be used to determine whether, in a power-constrained system, it is possible to satisfy a specific set of SINR targets $\mathbf{g}$. Moreover, it determines the exact level of (under-)over-satisfaction g^{opt}.

Formulation F is a generalization of the respective MMF multiuser downlink beamforming problem (see, e.g., [32, 43] and references therein). As in the QoS case, not all SINR inequalities will be in general tight at the optimum. Regarding complexity, problem F contains as special case the respective broadcasting problem ($G = 1$), which is NP-hard [34]. Hence, it immediately follows that

Claim 2. *Problem* F *is NP-hard.*

As for problem Q, there exist special cases of problem F that are not NP-hard, e.g., for independent data transmission ($G = K$) it can be reformulated as a generalized eigenvalue problem [43]. Furthermore, for Vandermonde channel vectors an efficient solution, by means of bisection over SDP problems, will be presented in Section 2.3 [24, 25]. Claim 2 motivates the pursuit of sensible approximate solutions to the MMF problem F.

1.4.1 Relation to QoS Problem

Before proceeding in proposing an algorithm to find such an approximate solution, let us have a closer look at the relation between problem formulation F and Q. For a given set of channels and noise powers, F is parameterized by $\mathbf{g}$ and P. We will use the notation $\mathsf{F}(\mathbf{g}, P)$ to capture this dependence, and, with slight abuse of notation[2], $g = \mathsf{F}(\mathbf{g}, P)$ to denote the associated optimum value (maximum worst-case scaled SINR). Likewise, Q is parameterized by the vector $\mathbf{g}$ of QoS constraints; we will use the notation $\mathsf{Q}(\mathbf{g})$ to account for this, and $P = \mathsf{Q}(\mathbf{g})$ to denote the associated optimum value (minimum power).

[2]The meaning will be clear from context.

Generalizing the respective results for the two extreme cases of the multicast beamforming problem ($G = K$ [43] and $G = 1$ [34]), we have the following result:

Claim 3. *The QoS problem* Q *and the MMF problem* F *are related as follows:*

$$g = \mathsf{F}(\mathbf{g}, \mathsf{Q}(g\mathbf{g})) \tag{1.5}$$

$$P = \mathsf{Q}(\mathsf{F}(\mathbf{g}, P)\mathbf{g}) \tag{1.6}$$

Proof: Contradiction can be used to prove (1.5). Let $\{\mathbf{w}_i^{\mathsf{Q}}\}_{i=1}^G$ and P^{Q} denote an optimal solution and the associated optimal value to a feasible instance of problem $\mathsf{Q}(g\mathbf{g})$, where $g\mathbf{g}$ are the required SINR targets. Consider the problem instance $\mathsf{F}(\mathbf{g}, P^{\mathsf{Q}})$. The set $\{\mathbf{w}_i^{\mathsf{Q}}\}_{i=1}^G$ is a feasible solution with objective value g. Assume the existence of another feasible solution $\{\mathbf{w}_i^{\mathsf{F}}\}_{i=1}^G$ with associated optimal value $g^{\mathsf{F}} > g$. Then, it is possible to find a constant $c < 1$ to scale down this solution set, while still fulfilling the SINR constraints of problem $\mathsf{Q}(g\mathbf{g})$. The resulting set $\{c\mathbf{w}_i^{\mathsf{F}}\}_{i=1}^G$has smaller objective value (total transmission power) than P^{Q}, which contradicts optimality of $\{\mathbf{w}_i^{\mathsf{Q}}\}_{i=1}^G$.

A similar procedure can be used to prove (1.6). Specifically, let $\{\mathbf{w}_i^{\mathsf{F}}\}_{i=1}^G$ and g^{F} denote an optimal solution and the associated optimal value to a problem instance $\mathsf{F}(\mathbf{g}, P)$. Consider the problem instance $\mathsf{Q}(g^{\mathsf{F}}\mathbf{g})$. The set $\{\mathbf{w}_i^{\mathsf{F}}\}_{i=1}^G$ is a feasible solution with objective value P. Assume the existence of another feasible solution $\{\mathbf{w}_i^{\mathsf{Q}}\}_{i=1}^G$ with associated optimal value $P^{\mathsf{Q}} < P$. As noted before, this contradicts optimality of $\{\mathbf{w}_i^{\mathsf{F}}\}_{i=1}^G$ for $\mathsf{F}(\mathbf{g}, P)$, since the power budget $P - P^{\mathsf{Q}}$ can be distributed evenly to yield an objective value larger than g^{F}. ■

Another useful property of both formulations is the following

Claim 4. *The optimum objective values of the QoS problem* $\mathsf{Q}(g\mathbf{g})$ *and the MMF problem* $\mathsf{F}(\mathbf{g}, P)$ *are monotonically nondecreasing in* g *and* P*, respectively, for given* $\mathbf{g}$.

Proof: The feasible set of $\mathsf{Q}(g\mathbf{g})$ is decreasing in g. For $\mathsf{F}(\mathbf{g}, P)$, any additional power can be evenly distributed, thereby increasing all SINR's, provided that $\{\sigma_k^2 > 0\}_{k=1}^K$. ■

Corollary 1. *A solution to* $\mathsf{F}(\mathbf{g}, P)$ *can be found by iteratively solving* $\mathsf{Q}(g\mathbf{g})$ *for varying values of* g*. Claim 3 guarantees optimality of the solution for* $P = \mathsf{Q}(g\mathbf{g})$ *and Claim 4 enables the use of a simple one-dimensional bisection search for the sought* g *(see [43] for* *the special case of multiuser downlink beamforming). Similarly, bisection of* $g = \mathsf{F}(\mathbf{g}, P)$ *over* P *can be used to solve* $\mathsf{Q}(g\mathbf{g})$.

Corollary 1 suggests a solution to the MMF problem, provided that the QoS problem can be solved *optimally*. However, Q is NP-hard and we can only find an approximate solution, as proposed in Section 1.3.2. Due to this, and keeping in mind that F is NP-hard (Claim 2), we again pursue a respective sensible approximate solution.

1.4.2 Approximate Solution

Using the notation of Section 1.3.1 and following similar steps as for the relaxation $\mathsf{Q} \rightarrow \mathsf{Q}_{\mathbf{r}}$, the following relaxation of the original MMF problem F is obtained by dropping the nonconvex rank constraints, associated with the matrices $\{\mathbf{W}_i\}_{i=1}^{G}$.

$$\mathsf{F}_{\mathbf{r}}$$

$$\max_{\{\mathbf{W}_i \in \mathbb{C}^{N\times N}\}_{i=1}^{G},\ g \geq 0} \quad g$$

$$\text{s.t. :} \quad g\gamma_k \sum_{\substack{j=1 \\ j\neq i}}^{G} \operatorname{tr}(\mathbf{H}_k \mathbf{W}_j) - \operatorname{tr}(\mathbf{H}_k \mathbf{W}_i) + g\gamma_k \sigma_k^2 \leq 0 \qquad \forall k \in \mathcal{G}_i \ \ \forall i \in \mathcal{G},$$

$$\sum_{i=1}^{G} \operatorname{tr}(\mathbf{W}_i) = P,$$

$$\mathbf{W}_i \succeq \mathbf{0} \qquad \forall i \in \mathcal{G}.$$

At first glance, problem $\mathsf{F}_{\mathbf{r}}$ may seem to be of the same form as $\mathsf{Q}_{\mathbf{r}}$ of Section 1.3.1, except for the extra linear equality constraint on the total transmission power, the nonnegativity constraint on g, and the different (yet still linear) objective function. However, contrary to $\mathsf{Q}_{\mathbf{r}}$, $\mathsf{F}_{\mathbf{r}}$ does not admit an equivalent SDP reformulation, because the K inequality constraints on the received SINR's are nonlinear (note that g is a variable).[3]

Claim 5. *The relaxed QoS problem* $\mathsf{Q}_{\mathbf{r}}$ *and the relaxed MMF problem* $\mathsf{F}_{\mathbf{r}}$ *are related as follows:*

$$g = \mathsf{F}_{\mathbf{r}}(\mathbf{g}, \mathsf{Q}_{\mathbf{r}}(g\mathbf{g})) \tag{1.7}$$

$$P = \mathsf{Q}_{\mathbf{r}}(\mathsf{F}_{\mathbf{r}}(\mathbf{g}, P)\mathbf{g}) \tag{1.8}$$

[3] In the single-group multicast beamforming case, $\mathsf{F}_{\mathbf{r}}$ is an SDP problem [34].

Proof: Verbatim to the proof of Claim 3, denoting the problem solutions as $\{\mathbf{W}_i\}_{i=1}^G$ instead of $\{\mathbf{w}_i\}_{i=1}^G$. ■

Claim 6. *The optimum objective values of the relaxed QoS problem* $\mathsf{Q}_{\mathbf{r}}(g\mathbf{g})$ *and the relaxed MMF problem* $\mathsf{F}_{\mathbf{r}}(\mathbf{g}, P)$ *are monotonically nondecreasing in* g *and* P*, respectively, for a given* $\mathbf{g}$*.*

Proof: Verbatim to the proof of Claim 4. ■

Corollary 2. *Due to Claims 5 and 6, the relaxed problem* $\mathsf{F}_{\mathbf{r}}(\mathbf{g}, P)$ *can be solved by a one-dimensional bisection search over* g*, to attain* $P = \mathsf{Q}_{\mathbf{r}}(g\mathbf{g})$*.*

Specifically, let $g^{\text{opt}} = \mathsf{F}_{\mathbf{r}}(\mathbf{g}, P)$. A feasible solution of $\mathsf{F}_{\mathbf{r}}(\mathbf{g}, P)$ that is at most $\epsilon > 0$ away from g^{opt} can be generated as follows. Let $[L, U]$ be an interval containing g^{opt}. Due to the nonnegativity of g^{opt}, the lower bound is initialized as $L = 0$. Assuming that the total available power is directed towards a single group and using the Cauchy-Schwartz inequality, the upper bound is initialized as $U = \min_{k \in \mathcal{K}} \frac{P\|\mathbf{h}_k\|_2^2}{\gamma_k \sigma_k^2}$. Given $[L, U]$, the SDP problem $\mathsf{Q}_{\mathbf{r}}(g\mathbf{g})$ is solved at the midpoint $g := (L+U)/2$ of the interval. If it is feasible for the given choice of g and its objective value is lower than P, the solution is stored and $L := g$; otherwise $U := g$. The use of interior point SDP solvers, such as SeDuMi [35] or SDPT3 [40], is useful in this context, because they do not only yield an efficient solution to problem $\mathsf{Q}_{\mathbf{r}}(g\mathbf{g})$ when it is feasible, but they otherwise provide a certificate of infeasibility. The aforementioned steps are repeated until $U - L \leq \epsilon$. Since in each iteration the interval is halved, the algorithm requires only $N_{\text{iter}} = \lceil \log_2((U - L)/\epsilon) \rceil$ iterations. In practice, 10–12 iterations are usually enough for typical problem setups. Building on [43], a similar bisection search algorithm was also proposed in [15].

When the algorithm terminates, the resulting matrices $\{\mathbf{W}_i^{\text{opt}}\}_{i=1}^G$ are an ϵ-optimal solution of $\mathsf{F}_{\mathbf{r}}(\mathbf{g}, P)$. The associated optimal value, which is approximately equal to g^{opt}, is merely an upper bound on the scaled SINR that can be guaranteed to every user, for the specific power budget P. This bound can only be met when all matrices $\{\mathbf{W}_i^{\text{opt}}\}_{i=1}^G$ are rank-1, so that their principal components can be chosen as optimum beamforming vectors. However, due to the relaxation, this is generally not the case. As in the QoS approach, post-processing of the relaxed solution is needed, when the solution matrices $\{\mathbf{W}_i^{\text{opt}}\}_{i=1}^G$ are not all rank-1, to yield an approximate solution to the original MMF problem F. This can be accomplished by a combined randomization / power control procedure, similar to the one described in Section 1.3.2. Specifically, Gaussian randomization may be used in a first step to create candidate sets of beamforming vectors $\{\tilde{\mathbf{w}}_i\}_{i=1}^G$ in the span of the respective transmit covariance matrices (the optimum solution matrices $\{\mathbf{W}_i^{\text{opt}}\}_{i=1}^G$ of problem $\mathsf{F}_\mathbf{r}$). In a second step, the total available transmission power P is optimally allocated to the candidate beamforming vectors, by means of an appropriate MPC problem, as explained in the rest of this section. The set $\{\tilde{\mathbf{w}}_i^{\text{opt}}, p_i^{\text{opt}}\}_{i=1}^G$ of beamforming vectors and respective power boost (or back-off) factors, that yields the highest objective value is chosen among all solutions generated this way. The approximate solution to the original MMF problem F is then equal to $\{\sqrt{p_i^{\text{opt}}}\tilde{\mathbf{w}}_i^{\text{opt}}\}_{i=1}^G$.

Given a candidate set of beamforming vectors $\{\tilde{\mathbf{w}}_i\}_{i=1}^G$, the power budget P can be optimally allocated among them by solving the following MPC problem

$$\mathsf{P}^\mathsf{F}$$

$$\max_{\{p_i \geq 0\}_{i=1}^G,\ g \geq 0} \quad g$$

$$\text{s.t.}: \quad \frac{1}{\gamma_k} \frac{\alpha_{k,i} p_i}{\sum_{\substack{j=1 \\ j \neq i}}^G \alpha_{k,j} p_j + \sigma_k^2} \geq g \qquad \forall k \in \mathcal{G}_i \quad \forall i \in \mathcal{G},$$

$$\sum_{i=1}^G \beta_i p_i = P,$$

where the coefficients $\alpha_{k,i}$ and β_i have been introduced in Section 1.3.2.

Unlike P^Q, P^F does not admit an equivalent LP reformulation, because the K inequality constraints are nonlinear; this can be easily seen by multiplying both sides of the inequalities with the SINR denominator and remembering that g is a variable.

Claim 7. *The QoS MPC problem* P^Q *and the MMF MPC problem* P^F *are related as follows:*

$$g = \mathsf{P}^\mathsf{F}(\mathbf{g}, \mathsf{P}^\mathsf{Q}(g\mathbf{g})) \tag{1.9}$$

$$P = \mathsf{P}^\mathsf{Q}(\mathsf{P}^\mathsf{F}(\mathbf{g}, P)\mathbf{g}) \tag{1.10}$$

Proof: Verbatim to the proof of Claim 3, denoting the problem solutions as $\{p_i\}_{i=1}^G$ instead of $\{\mathbf{w}_i\}_{i=1}^G$. ∎

Claim 8. *The optimum objective values of the QoS MPC problem* $\mathsf{P}^{\mathsf{Q}}(g\mathbf{g})$ *and the MMF MPC problem* $\mathsf{P}^{\mathsf{F}}(\mathbf{g}, P)$ *are monotonically nondecreasing in* g *and* P*, respectively, for a given* $\mathbf{g}$*.*

Proof: Verbatim to the proof of Claim 4. ■

Corollary 3. *Due to Claims 7 and 8, the MMF MPC problem* $\mathsf{P}^{\mathsf{F}}(\mathbf{g}, P)$ *can be solved by a one-dimensional bisection search over* g*, to attain* $P = \mathsf{P}^{\mathsf{Q}}(g\mathbf{g})$*.*

The bisection algorithm, described earlier in this section, can be used again to obtain a solution to problem $\mathsf{P}^{\mathsf{F}}(\mathbf{g}, P)$. The search interval is bounded below by $L = 0$, as before. However, the upper bound may now be further restricted to $U = g^{\text{opt}}$ (the optimal objective value of $\mathsf{F}_{\mathbf{r}}(\mathbf{g}, P)$). The difference is that for each iteration (value of g), the LP problem $\mathsf{P}^{\mathsf{Q}}(g\mathbf{g})$ is solved instead of the SDP problem $\mathsf{Q}_{\mathbf{r}}(g\mathbf{g})$.

The overall complexity of finding an approximate solution to the original MMF problem F is that of solving N_{iter} times the SDP problem $\mathsf{Q}_{\mathbf{r}}$ and $N_{\text{rand}}N'_{\text{iter}}$ times the LP problem P^{Q}, where N_{iter} and N'_{iter} denote the number of bisection iterations required for the solution of $\mathsf{F}_{\mathbf{r}}$ and P^{F}, respectively and N_{rand} is the number of Gaussian randomization trials. The quality of the final approximate solution to problem F can be measured by the ratio of the upper bound obtained by the solution of the relaxed problem $\mathsf{F}_{\mathbf{r}}$ to the maximum value of problem P^{F} attained in the randomization / power control loop.

Note that P^{F} also admits the following equivalent reformulation

$$
\boxed{
\begin{array}{lll}
\mathsf{P}^{\mathsf{F}} & & \\
& \displaystyle\max_{\{p_i \geq 0\}_{i=1}^{G},\ g \geq 0} \ g & \\
\text{s.t. :} & \displaystyle\sum_{\substack{j=1 \\ j \neq i}}^{G} \gamma_k \alpha_{k,j} g p_j + \gamma_k \sigma_k^2 g \leq \alpha_{k,i} p_i & \forall k \in \mathcal{G}_i \ \ \forall i \in \mathcal{G}, \\
& \displaystyle\sum_{i=1}^{G} \beta_i p_i \leq P. &
\end{array}
}
$$

Since all optimization variables are nonnegative, problem P^{F} is a standard form geometric programming (GP) problem, i.e., maximizing a monomial subject to posynomial upper bounded by monomial inequality constraints [5, 7]. It is straightforward to bring it to convex form by means of logarithmic change of variables and hence efficiently solve it in one step, using modern interior point methods, bypassing the need for bisection over LP problems. However, the number of bisection steps is rather small in general (at most 12 in all cases considered in the results reported in Section 1.5), so this does not appear to be a big issue.

1.5 Monte-Carlo Simulation Results

In Section 1.3.2, a two-step polynomial-time algorithm has been proposed to generate an approximate solution to the joint QoS multicast beamforming problem Q. The first step of the proposed algorithm consists of the SDR $\mathsf{Q} \rightarrow \mathsf{Q}_{\mathbf{r}}$. Problem Q may or not be feasible; if it is, then so is problem $\mathsf{Q}_{\mathbf{r}}$. If $\mathsf{Q}_{\mathbf{r}}$ is infeasible, then so is Q. The converse is generally not true; i.e., if $\mathsf{Q}_{\mathbf{r}}$ is feasible, Q need not be feasible. In order to establish feasibility of Q in this case, the randomization / power control loop should yield at least one feasible solution. This is most often the case, as verified by the numerical results presented in the sequel. If the randomization / power control loop fails to return at least one feasible solution, then the (in)feasibility of Q cannot be determined. There is, therefore, a relatively small proportion of problem instances for which (in)feasibility of Q cannot be decided using the proposed approach. It is evident from the above discussion that feasibility is a key aspect of problem Q and its proposed solution via problem $\mathsf{Q}_{\mathbf{r}}$ and the randomization / power control loop. Feasibility depends on a number of factors; namely, the number of transmit antenna elements, the number and the populations of the multicast groups, the channel characteristics, the noise variances, and the sought SINR targets.

Beyond feasibility, there are two key issues of interest. The first has to do with cases when the solution of the relaxed problem $\mathsf{Q}_{\mathbf{r}}$ yields an optimum solution to the original problem Q. This happens when the solution matrices $\{\mathbf{W}_i^{\text{opt}}\}_{i=1}^G$ turn out all being rank-1. Then, the associated principal components solve optimally the original problem Q, i.e., in such a case $\mathsf{Q}_{\mathbf{r}}$ is not a relaxation after all. It is interesting to find the frequency of occurrence of such an event, whose benefit is twofold: the problem is solved optimally and at a smaller complexity, since the randomization step and the repeated solution of the ensuing MPC problem P^{Q} is avoided. The second issue of interest is the quality of the final approximate solution to problem Q. A practical figure of merit is the power ratio discussed in Section 1.3.2.

We first consider the standard i.i.d. Rayleigh fading model, i.e., the elements of each channel vector are i.i.d. circularly symmetric zero-mean complex Gaussian random variables of variance 1. The results presented in this section are obtained by averaging over 300 different channel snapshots, using 300 Gaussian randomization samples in each Monte-Carlo run. Tables 1.1 and 1.2 summarize these results, for number N of transmit antenna elements set to 4 and 8, respectively. The proposed two-step algorithm is tested for a variety of choices for the total number K of single-antenna receivers and the number G of multicast groups, which index the rows in the tables (columns 1 and 2, respectively). The users are considered to be evenly distributed among the multicast groups, i.e., $\{G_i = K/G\}_{i=1}^{G}$. For each such configuration, the same SINR targets are requested for all users (in the 6–20 dB range, see column 3). The noise variance is set to 1 for all channels.

The percentage of the 300 Monte-Carlo runs for which $\mathsf{Q}_{\mathbf{r}}$ is feasible is shown in column 4. Column 5 reports the percentage of feasible solutions to problem $\mathsf{Q}_{\mathbf{r}}$, for which the solution matrices turn out all being essentially rank-1; defined by the second largest eigenvalue being smaller than 10^{-3} of the sum of all eigenvalues. Column 6 reports the percentage of problem instances for which, once a feasible solution to problem $\mathsf{Q}_{\mathbf{r}}$ is found, the proposed randomization / power control loop yields at least one feasible solution to the original problem Q. Columns 7 and 8 hold the mean and the standard deviation of the ratio of the total transmission power corresponding to the final approximate solution over the lower bound obtained from the SDR solution. This ratio equals 1 when the relaxation is tight, and the reported statistics depend on the frequency (see column 5) of this event. In order to obtain additional insight on the quality of the approximation step, conditional statistics are also reported in columns 9 and 10 after excluding optimum solutions from the calculation. The $\mathsf{Q}_{\mathbf{r}}$ feasibility percentages, stored in column 4 of Tables 1.1 and 1.2, are also plotted in Figs. 1.2 and 1.3, respectively. In all configurations considered, the higher the target SINR, the less likely it is that $\mathsf{Q}_{\mathbf{r}}$ is feasible, which is intuitive. Furthermore, $\mathsf{Q}_{\mathbf{r}}$ is getting more difficult to solve as the number G of multicast groups increases and/or as more users are added in each group, since in both cases interference is higher. Finally, it is seen that increasing the number of transmit antenna elements improves service, i.e., higher received SINR can be attained by more users in more multicast groups.

The $\mathsf{Q}_\mathbf{r}$ optimality percentages are also plotted in Figs. 1.4 and 1.5, for the case of 4 and 8 transmit antennas, respectively. The most interesting observation is that the optimality percentage increases as the number of users per multicast group decreases; percentages are significant especially when the number of users per group is smaller or equal to the number of transmit antennas. This can be seen in two ways: either by holding the number of groups fixed while decreasing their populations, or by fixing the total number of users and distributing them in more multicast groups. Trying to interpret this fact, note that in both cases the problem is pushed towards the multiuser (independent information) downlink problem, where each user forms a multicast group by itself. The latter is known to be convex, and the associated SDP relaxation has been shown to be tight [2]. In addition, the $\mathsf{Q}_\mathbf{r}$ optimality percentage also increases with target SINR. It seems as if rank-1 solutions are more likely when operating close to infeasibility boundary.

Regarding the approximation step of the proposed algorithm, we can distinguish two cases. In most of the scenarios considered, the number of users per multicast group was kept smaller or equal to the number of transmit antenna elements, so that a realistic value of the received SINR could be guaranteed, for a significant fraction of the different channel instances. There, the randomization / power control loop yields a feasible solution with a probability higher than 90% in most cases where $\mathsf{Q}_\mathbf{r}$ is feasible, as shown in Fig. 1.6 which illustrates the contents of column 6 of Table 1.2. The approximate solution entails transmission power that is under two times (3 dB from) the possibly unattainable lower bound, on average. The actual numbers for each configuration depend on the number of the Gaussian randomization samples; 300 have proved adequate for most configurations. However, when a (relatively low) target SINR is to be guaranteed to a number of users per group larger than the number of antennas, the feasibility of the approximation decreases and the power penalty increases. This can be appreciated by looking at the lowest sub-matrices of Tables 1.1 and 1.2. Using 1000, instead of

300, Gaussian random samples for these configurations, a small improvement has been observed in the quality of the approximation.

The Monte-Carlo simulations have been repeated, under the same setup, in order to validate the performance of the MMF algorithm presented in Section 1.4. The results are very similar to the ones presented so far for the QoS case and are therefore skipped for brevity. The sole difference is that feasibility is not an issue in the MMF case. Specifically, there is a considerable percentage of problem instances for which the proposed relaxation is tight, so that the optimum solution is found. For all other instances, the proposed algorithm finds a high-quality approximate solution at manageable complexity cost. An interesting observation is that the quality of the approximation for the multi-group case is consistently better than the respective single-group case [34] and that it becomes better as a given number of users is distributed among a larger number of multicast groups; again, moving closer to the multiuser downlink problem.

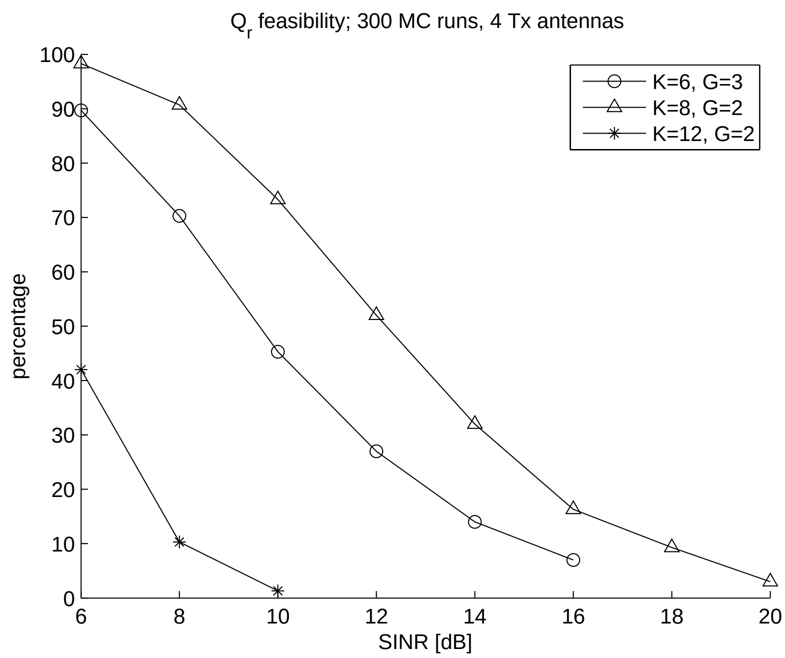

Figure 1.2: $\mathsf{Q}_{\mathbf{r}}$ feasibility; 300 MC runs, 4 Tx antennas

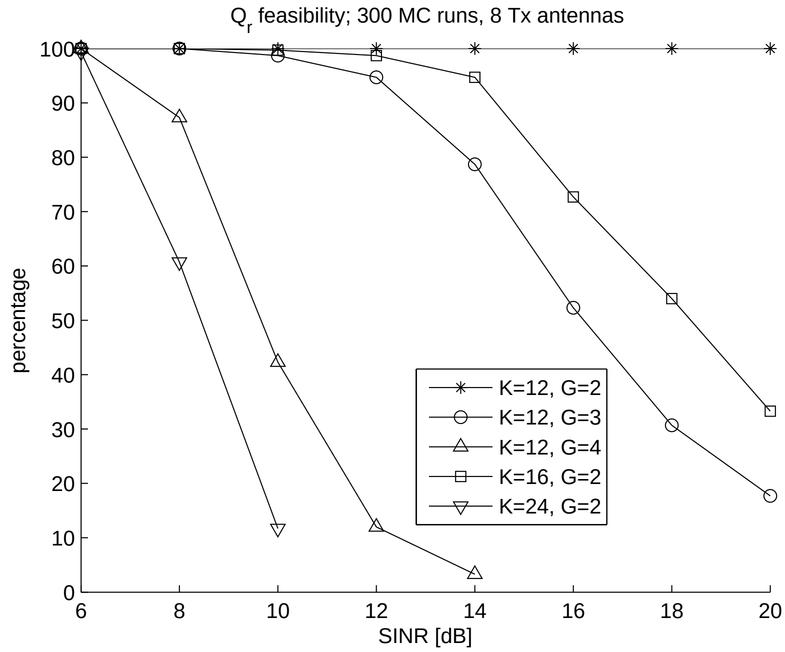

Figure 1.3: $\mathsf{Q}_{\mathbf{r}}$ feasibility; 300 MC runs, 8 Tx antennas

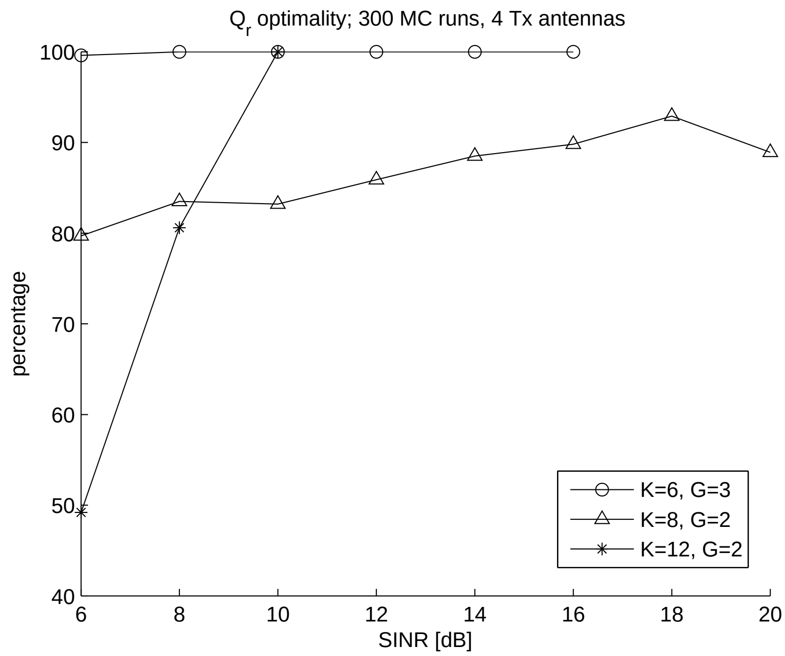

Figure 1.4: $\mathsf{Q}_{\mathbf{r}}$ optimality; 300 MC runs, 4 Tx antennas

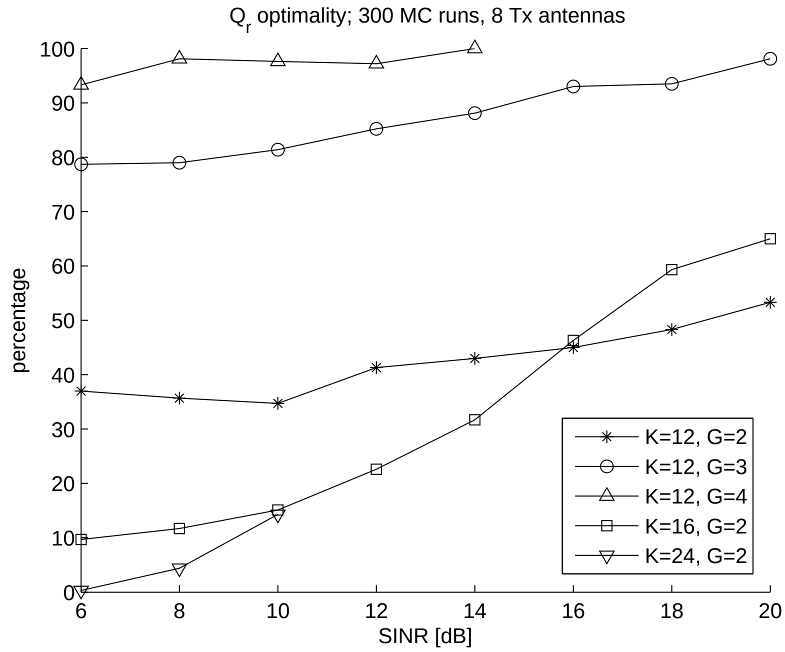

Figure 1.5: $\mathsf{Q}_\mathbf{r}$ optimality; 300 MC runs, 8 Tx antennas

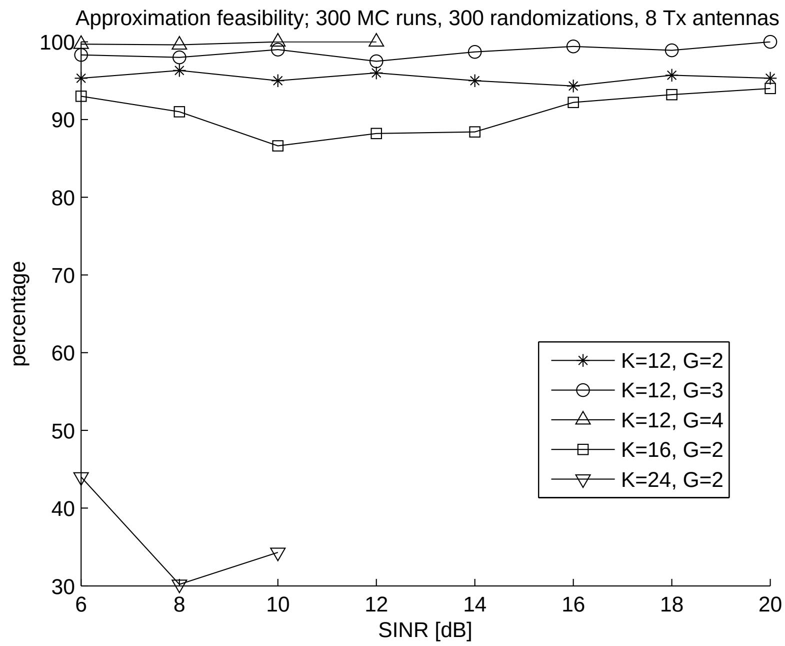

Figure 1.6: Approximation feasibility; 300 MC runs, 300 randomizations, 8 Tx antennas

Table 1.1: MC simulation results (Rayleigh); 4 Tx antennas

K	G	SINR	feas. $\mathsf{Q}_\mathbf{r}$	opt. $\mathsf{Q}_\mathbf{r}$	feas. appr.	all solutions mean	all solutions std	appr. solutions mean	appr. solutions std
6	3	6	90	100	-	1	0	-	-
6	3	8	70	100	-	1	0	-	-
6	3	10	45	100	-	1	0	-	-
6	3	12	27	100	-	1	0	-	-
6	3	14	14	100	-	1	0	-	-
6	3	16	7	100	-	1	0	-	-
8	2	6	98	80	98	1.06	0.17	1.29	0.30
8	2	8	91	84	99	1.08	0.38	1.54	0.83
8	2	10	73	83	98	1.19	1.81	2.27	4.54
8	2	12	52	86	99	1.20	2.12	2.55	5.84
8	2	14	32	89	100	1.01	0.06	1.11	0.15
8	2	16	16	90	96	1.04	0.19	1.67	0.44
8	2	18	9	93	100	1.02	0.07	1.22	0.19
8	2	20	3	89	100	1.05	0.16	1.49	0
12	2	6	42	49	79	1.69	1.89	2.82	2.73
12	2	8	10	81	94	1.19	0.51	2.39	0.47
12	2	10	1	100	-	1	0	-	-

Table 1.2: MC simulation results (Rayleigh); 8 Tx antennas

K	G	SINR	feas. $\mathsf{Q_r}$	opt. $\mathsf{Q_r}$	feas. appr.	all solutions mean	all solutions std	appr. solutions mean	appr. solutions std
12	2	6	100	37	95	1.18	0.25	1.30	0.27
12	2	8	100	36	96	1.17	0.24	1.28	0.25
12	2	10	100	35	95	1.17	0.23	1.27	0.24
12	2	12	100	41	96	1.15	0.21	1.26	0.22
12	2	14	100	43	95	1.15	0.22	1.27	0.23
12	2	16	100	45	94	1.13	0.20	1.25	0.21
12	2	18	100	48	96	1.12	0.23	1.25	0.28
12	2	20	100	53	95	1.10	0.18	1.23	0.21
12	3	6	100	79	98	1.04	0.11	1.19	0.17
12	3	8	100	79	98	1.04	0.11	1.19	0.18
12	3	10	99	81	99	1.05	0.14	1.25	0.24
12	3	12	95	85	98	1.04	0.15	1.31	0.29
12	3	14	79	88	99	1.06	0.29	1.52	0.74
12	3	16	52	93	99	1.02	0.11	1.38	0.26
12	3	18	31	94	99	1.03	0.14	1.53	0.37
12	3	20	18	98	100	1.01	0.04	1.29	0
12	4	6	100	93	100	1.01	0.03	1.11	0.08
12	4	8	87	98	100	1.00	0.04	1.24	0.17
12	4	10	42	98	100	1.01	0.06	1.32	0.33
12	4	12	12	97	100	1.01	0.06	1.36	0
12	4	14	3	100	-	1	0	-	-
16	2	6	100	10	93	1.88	1.63	1.99	1.69
16	2	8	100	12	91	2.00	2.27	2.14	2.40
16	2	10	100	15	87	1.88	1.32	2.06	1.38
16	2	12	99	23	88	1.70	1.57	1.94	1.76
16	2	14	95	32	88	1.80	2.30	2.24	2.78
16	2	16	73	46	92	1.71	3.86	2.42	5.40
16	2	18	54	59	93	1.33	1.04	1.91	1.56
16	2	20	33	65	94	1.27	0.82	1.86	1.31
24	2	6	99	0	44	6.79	8.74	6.84	8.76
24	2	8	61	4	30	4.87	6.23	5.53	6.52
24	2	10	12	14	34	3.64	5.20	5.53	6.30

1.6 Experiments with Measured Channel Data

The performance of the proposed multicast beamforming algorithms has also been tested on measured channel data courtesy of iCORE HCDC Lab, University of Alberta, Edmonton, Canada. Measurements had been carried out using a portable 4×4 multiple-input multiple-output (MIMO) testbed that operates in the 902–928 MHz (ISM) band. The transmitter (Tx) and the receiver (Rx) were equipped with antenna arrays, each comprising 4 vertically polarized dipole antennas spaced $\lambda/2$ (≈ 16 cm) apart. The chip rate used for sounding was low enough to safely assume that the channel is not frequency selective. More details on the testbed configuration and the procedure used to estimate the channel gains of the MIMO channel matrix can be found in [19]. Datasets and a detailed description of many measurement campaigns in typical propagation environments are available at the iCORE HCDC Lab website (http://www.ece.ualberta.ca/∼mimo/). The most pertinent scenario is the stationary outdoor one, called “Quad” and illustrated in Figure 1.7. Quad is a 150 by 60 meters lawn surrounded by buildings with heights from approximately 15 to 30 meters. The Tx location was fixed, whereas the Rx was placed in 6 different locations (no measurements are actually provided for location 4) as indicated in Figure 1.7. For every Rx location, 9 different measurements were taken by shifting the Rx antenna array on a 3×3 square grid with $\lambda/4$ spacing. Each measurement contains about 100 4×4 channel snapshots, recorded 3 per second; thus for each location there are about 900 MIMO channel gain matrices available. The multicast groups are formed by considering each receive antenna at each location as a separate terminal, and grouping terminals in 1–3 locations. The results reported in Tables 1.3–1.5 for the QoS problem were obtained by averaging over the 900 channel instances. Tens of different configurations have been tried and only representative results are presented for each scenario considered. All channel gains are normalized before use *by the same constant* (average amplitude over all channels and all snapshots), in order to facilitate the comparison with the simulated Rayleigh case. Note that this normalization maintains the differences in path loss. 300 Gaussian samples are employed in the randomization / power control loop. The main findings regarding the performance of the proposed algorithms when applied to the measured channel data can be summarized as follows:

- For 2 multicast groups and number of users per group equal to the number of Tx antennas ($N = 4$), the SDR $\mathsf{Q} \rightarrow \mathsf{Q_r}$ is tight very frequently (70–100%) and the power penalty paid by the approximation step very small. These hold irrespective of the distribution of the users of each multicast group in 1, 2, or even 3 locations (see Table 1.3).

- For 2 multicast groups of 6 users each, evenly distributed in 2 locations, the SDR $\mathsf{Q} \rightarrow \mathsf{Q_r}$ is tight for more than half of the occasions (see Table 1.4). There exist channel instances for which SINR up to 14 dB can be guaranteed; such high SINR values are unattainable under the corresponding i.i.d. Rayleigh fading scenario. The quality of approximation is good, even though the number of users per group is larger than the number of transmit antenna elements. When the 6 users of each group are evenly distributed in 3 locations, the problem is feasible only up to about 10 dB and the feasibility of the approximation step can drop $< 80\%$.

- For 3 multicast groups (see Table 1.5) of 3 collocated users each, the SDR $\mathsf{Q} \rightarrow \mathsf{Q_r}$ is almost always tight ($> 90\%$) and feasible up to SINR of 10 dB. For 4 users per group it becomes infeasible for SINR values larger than about 8 dB.

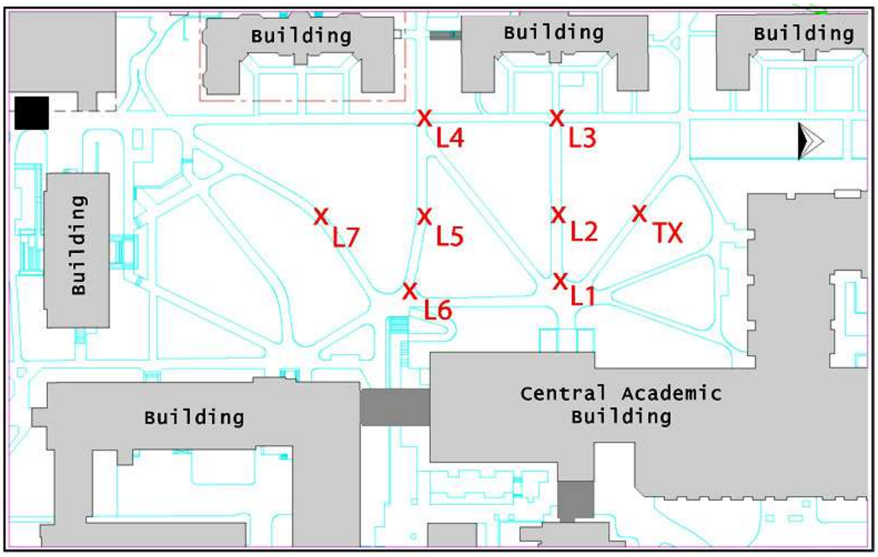

Figure 1.7: Wireless channel measurement scenario from University of Alberta

Table 1.3: Measured channel results; 2 multicast groups of 4 users each

	feas.	opt.	feas.	all solutions		appr. solutions	
SINR	$\mathsf{Q_r}$	$\mathsf{Q_r}$	appr.	mean	std	mean	std
Group 1 (4 at L5) & Group 2 (4 at L7)							
6	98	82	98	1.05	0.19	1.30	0.39
8	91	84	88	1.05	0.23	1.33	0.53
10	82	87	98	1.04	0.21	1.40	0.53
12	50	92	99	1.04	0.29	1.61	0.95
14	19	91	98	1.02	0.08	1.23	0.19
16	9	94	100	1.01	0.06	1.16	0.22
18	3	92	96	1.00	0.00	1.01	0
Group 1 (2 at L2 & 2 at L6) & Group 2 (2 at L5 & 2 at L7)							
6	100	81	99	1.05	0.18	1.29	0.34
8	100	81	99	1.05	0.18	1.30	0.33
10	96.1	86	99	1.04	0.17	1.32	0.36
12	83	90	99	1.04	0.26	1.43	0.80
14	57	93	99	1.15	2.89	3.73	12.10
16	31	93	99	1.02	0.12	1.36	0.33
18	14	95	99	1.01	0.07	1.31	0.21
20	6	92	100	1.03	0.18	1.44	0.53
Group 1 (1 at L1, 1 at L3 & 2 at L6) Group 2 (1 at L2, 1 at L5 & 2 at L7)							
6	100	72	98	1.12	0.54	1.45	0.98
8	99	75	98	1.09	0.31	1.39	0.55
10	93	80	97	1.18	2.71	2.03	6.44
12	73	87	97	1.05	0.25	1.44	0.63
14	44	89	98	1.07	0.71	1.78	2.19
16	23	93	99	1.03	0.18	1.52	0.59

Table 1.4: Measured channel results; 2 multicast groups of 6 users each

SINR	feas. Q_r	opt. Q_r	feas. appr.	all solutions mean	all solutions std	appr. solutions mean	appr. solutions std
Group 1 (3 at L1 & 3 at L3) & Group 2 (3 at L2 & 3 at L6)							
6	100	73	98	1.19	1.50	1.76	2.91
8	90	68	94	1.39	2.45	2.38	4.49
10	61	66	92	1.33	1.06	2.15	1.73
12	18	72	92	1.33	1.15	2.56	2.12
Group 1 (2 at L1, 2 at L2 & 2 at L6) Group 2 (2 at L3, 2 at L5 & 2 at L7)							
6	70	24	82	2.25	3.51	2.78	4.06
8	33	41	81	2.02	3.58	3.08	4.90
10	7	48	65	1.18	0.45	1.67	0.69

Table 1.5: Measured channel results; 3 multicast groups of 3 or 4 collocated users each

SINR	feas. Q_r	opt. Q_r	feas. appr.	all solutions mean	all solutions std	appr. solutions mean	appr. solutions std
Group 1 (3 at L1), Group 2 (3 at L2) & Group 3 (3 at L3)							
6	72	98	100	1.01	0.10	1.36	0.66
8	37	99	100	1.00	0.01	1.10	0.07
10	14	100	-	1	0	-	-
Group 1 (4 at L1), Group 2 (4 at L2) & Group 3 (4 at L3)							
6	29	95	99	1.02	0.11	1.36	0.40
8	8	100	-	1	0	-	-

1.7 Conclusions

The transmit beamforming problem has been considered for the general case of multiple cochannel multicast groups, under two design criteria: QoS, in which we seek to minimize the total transmission power while guaranteing a prescribed minimum SINR at all receivers; and a fair objective, in which we seek to maximize the minimum received SINR under a total power constraint. Both formulations contain single-group multicast beamforming as a special case, and are therefore NP-hard [34]. Computationally efficient quasi-optimal solutions have been proposed by means of SDR and a combined randomization / power control loop. Extensive numerical results have been presented, using both simulated (i.i.d. Rayleigh) and measured outdoor wireless channel data, showing that the proposed algorithms yield high-quality approximate solutions at a moderate complexity cost. Interestingly, the numerical findings indicate that the solutions generated by the proposed algorithms are often exactly optimal, especially in the case of measured channels. In certain cases this optimality can be proven beforehand, and alternative convex reformulations of lower complexity can be constructed [24, 25]; this is the topic of the subsequent chapter. In other cases, a theoretical worst-case bound on approximation accuracy can be derived, and shown to be tight; on this issue, see [29].

Chapter 2

Far-field Multicast Beamforming for Uniform Linear Antenna Arrays

The problem of transmit beamforming to multiple cochannel multicast groups is considered for the important special case when the channel vectors are Vandermonde. This arises when a uniform linear antenna (ULA) array is used at the transmitter under far-field line-of-sight propagation conditions, as provisioned in 802.16e and related wireless backhaul scenarios. Two design approaches are pursued: i) minimizing the total transmit power subject to providing a prescribed received SINR to each intended receiver; and ii) maximizing the minimum received SINR under a total transmit power budget. It has been seen in the previous chapter that these problems are NP-hard in general; however, it is proven here that for Vandermonde channel vectors it is possible to recast the optimization in terms of the autocorrelation sequences of the sought beamvectors, yielding an equivalent convex reformulation [24, 25]. This affords efficient optimal solution using modern interior point methods. The optimal beamvectors can then be recovered using spectral factorization. Robust extensions for the case of partial channel state information (CSI), where the direction of each receiver is known to lie in an interval, are also developed. Interestingly, these also admit convex reformulation. The various optimal designs are illustrated and contrasted in a suite of pertinent numerical experiments.

2.1 Introduction

As network technology evolves towards seamless interconnection and *triple-play*[1] services, multicasting techniques become increasingly important in delivering batch updates and streaming media content. Multicasting is a network layer issue for wired and optical networks, where multicast routing has received considerable attention, and associated tools (e.g., MBONE) have long been available for the Internet.

In recent years, there is a clear trend and emerging consensus that wireless is the access method of choice for the last hop, or even the last few hops. This is partially due to accessibility and cost issues, but, perhaps more importantly, for ease of use and mobility considerations. This is evident in the proliferation of wireless local area and wireless backhaul solutions, in addition to the convergence of cellular phones and wireless-enabled handheld computers.

Wireless is an inherently broadcast medium, thus opening the door to multicasting *at the physical layer*, in addition to multicast routing at the network layer. Access points nowadays are typically equipped with antenna arrays. Baseband beamforming can be used to create suitable beampatterns to serve multiple multicast groups simultaneously over the same bandwidth. Physical layer multicasting can yield significant rate, energy, and latency advantages over network layer multicast routing, in which duplicate transmissions are unavoidable. However, wireless multicasting requires some level of physical CSI to be effective, and it obviously cannot be employed over the optical or wired backbone. Thus, physical layer multicasting and network layer multicast routing are complementary techniques.

The general problem of jointly designing beamformers for several cochannel multicast groups has been considered in the previous chapter under QoS and MMF criteria. These problems are NP-hard in general; however computationally efficient, high-quality approximate solutions have been proposed using SDR coupled with randomization and multicast power control [23, 26]. Interestingly, extensive numerical experiments have shown that the proposed algorithms attain close to optimal performance for both simulated and measured channel data. These numerical findings suggest that, for Vander-

[1] Voice, Internet, video on-demand, video broadcasting / multicasting.

monde channel vectors, exact solutions are often generated with remarkable consistency. Vandermonde channel vectors arise when a ULA is used at the transmitter under far-field, line-of-sight propagation conditions. Such conditions are quite realistic in wireless backhaul scenarios, such as the line-of-sight mode of 802.16e. In this chapter, it is proven that, indeed, the aforementioned design problems are convex (and thus "easy" to solve exactly) under such conditions. Then, the assumption of perfect CSI is loosened to the pragmatic case that the user angles are known only within a certain tolerance. Robust design problems are formulated under both QoS and MMF service criteria and are shown to be convex. Hence, they can be optimally and efficiently solved using modern interior point methods. The chapter is concluded with several illustrative simulation results for all proposed formulations.

2.2 Quality of Service Multicast Beamforming

When the transmitter employs a ULA under far-field line-of-sight propagation conditions, the $N \times 1$ complex vectors that model the phase shift from each transmit antenna element to the receive antenna of user $k \in \mathcal{K}$ are Vandermonde

$$\mathbf{h}_k = \mathbf{v}(\theta_k) := [1 \; e^{j\theta_k} \; e^{j2\theta_k} \; \cdots \; e^{j(N-1)\theta_k}]^T, \tag{2.1}$$

where the angles θ_k are given by $\theta_k = -2\pi d \sin(\phi_k)/\lambda$. Here, d denotes the spacing between successive antenna elements, λ is the carrier wavelength, and the angles ϕ_k define the directions of the receivers.

In such a propagation scenario, it was observed from the simulation results of [23,26] that there exist receiver configurations for which the optimal solution blocks $\{\mathbf{W}_i^{\text{opt}}\}_{i=1}^G$ of the relaxed problem $\mathsf{Q}_\mathbf{r}$ turn out all being rank-1. Then, the second step of the algorithm proposed in Section 1.3.2 (comprising the randomization / multicast power control loop) is no longer needed and the set of optimum beamforming vectors $\{\mathbf{w}_i^{\text{opt}}\}_{i=1}^G$ can be formed by the principal components of the blocks $\{\mathbf{W}_i^{\text{opt}}\}_{i=1}^G$. In such an occasion, problem $\mathsf{Q}_\mathbf{r}$ is equivalent to, and not a relaxation of, the original problem Q. In Section 2.2.1, this fact is proven to hold for any feasible configuration, in the case of Vandermonde channel vectors, and it suggests that the original problem Q is no longer NP-hard, but may be equivalently posed as a convex optimization problem. A suitable convex reformulation, in terms of the autocorrelation of the beamforming vectors, is developed in Section 2.2.2.

2.2.1 Tightness of Semidefinite Relaxation

Claim 9. *The SDR* $\mathsf{Q} \rightarrow \mathsf{Q}_{\mathbf{r}}$ *is always tight for Vandermonde channels, i.e., if* $\mathsf{Q}_{\mathbf{r}}$ *is feasible, then it always admits an equivalent solution whose blocks are all rank-1.*

Proof : Let $\mathbf{W}_i^{\text{opt}} \in \mathbb{C}^{N\times N}$ be one of the G blocks comprising the optimal solution of the relaxed problem $\mathsf{Q}_{\mathbf{r}}$. Let $\rho_i \geq 1$ denote the rank of $\mathbf{W}_i^{\text{opt}}$, and consider the outer product decomposition[2] $\mathbf{W}_i^{\text{opt}} = \sum_{\ell=1}^{\rho_i} \mathbf{w}_{i,\ell}^{\text{opt}}(\mathbf{w}_{i,\ell}^{\text{opt}})^H$. The signal power user k receives from multicast i can then be written as

$$
\begin{aligned}
\text{tr}\left(\mathbf{W}_i^{\text{opt}}\mathbf{H}_k\right) &= \text{tr}\left(\mathbf{W}_i^{\text{opt}}\mathbf{h}_k\mathbf{h}_k^H\right) \\
&= \text{tr}\left[\sum_{\ell=1}^{\rho_i}\mathbf{w}_{i,\ell}^{\text{opt}}(\mathbf{w}_{i,\ell}^{\text{opt}})^H\mathbf{h}_k\mathbf{h}_k^H\right] \\
&= \sum_{\ell=1}^{\rho_i}\text{tr}\left[\mathbf{w}_{i,\ell}^{\text{opt}}(\mathbf{w}_{i,\ell}^{\text{opt}})^H\mathbf{h}_k\mathbf{h}_k^H\right] \\
&= \sum_{\ell=1}^{\rho_i}\text{tr}\left[\mathbf{h}_k^H\mathbf{w}_{i,\ell}^{\text{opt}}(\mathbf{w}_{i,\ell}^{\text{opt}})^H\mathbf{h}_k\right] \\
&= \sum_{\ell=1}^{\rho_i}\left|\mathbf{h}_k^H\mathbf{w}_{i,\ell}^{\text{opt}}\right|^2 \\
&= \sum_{\ell=1}^{\rho_i}\left|\mathbf{v}(\theta_k)^H\mathbf{w}_{i,\ell}^{\text{opt}}\right|^2,
\end{aligned}
\tag{2.2}
$$

using the linearity of the trace operator and the property that $\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA})$, for any matrices $\mathbf{A}$ and $\mathbf{B}$ of appropriate dimensions. The last equality comes from the assumption that channel vectors are Vandermonde, cf. (2.1).

The result of (2.2) is a real-valued complex trigonometric polynomial, which is non-negative for any value of $\theta_k \in [0, 2\pi)$. Thus, according to the Riesz-Féjer theorem [36],

[2] Such decomposition is not unique, but this is irrelevant for our purposes; we simply use one such decomposition.

there exists a vector $\mathbf{w}_i^{\text{opt}} \in \mathbb{R} \times \mathbb{C}^{N-1}$ that is independent of θ_k, such that for all θ_k

$$\begin{aligned}\sum_{\ell=1}^{\rho_i} \left|\mathbf{v}(\theta_k)^H \mathbf{w}_{i,\ell}^{\text{opt}}\right|^2 &= \left|\mathbf{v}(\theta_k)^H \mathbf{w}_i^{\text{opt}}\right|^2 \\ &= \operatorname{tr}\left(\left|\mathbf{h}_k^H \mathbf{w}_i^{\text{opt}}\right|^2\right) \\ &= \operatorname{tr}\left[\mathbf{w}_i^{\text{opt}}(\mathbf{w}_i^{\text{opt}})^H \mathbf{h}_k \mathbf{h}_k^H\right] \\ &= \operatorname{tr}\left(\bar{\mathbf{W}}_i^{\text{opt}} \mathbf{H}_k\right),\end{aligned} \tag{2.3}$$

where $\bar{\mathbf{W}}_i^{\text{opt}} := \mathbf{w}_i^{\text{opt}}(\mathbf{w}_i^{\text{opt}})^H$. Combining the results of (2.2) and (2.3), we obtain

$$\operatorname{tr}\left(\mathbf{W}_i^{\text{opt}} \mathbf{H}_k\right) = \operatorname{tr}\left(\bar{\mathbf{W}}_i^{\text{opt}} \mathbf{H}_k\right), \tag{2.4}$$

which shows that for every optimum (generally high-rank) beamforming matrix $\mathbf{W}_i^{\text{opt}}$, there exists a rank-1 positive semidefinite matrix $\bar{\mathbf{W}}_i^{\text{opt}}$, which is equivalent with respect to the power received at each node. Therefore, the blocks $\{\bar{\mathbf{W}}_i^{\text{opt}}\}_{i=1}^G$ form a feasible solution set of problem $\mathsf{Q}_{\mathbf{r}}$.

Integrating out θ_k in the first equality of (2.3) yields (cf. Parseval's theorem)

$$\begin{aligned}\sum_{\ell=1}^{\rho_i} \left\|\mathbf{w}_{i,\ell}^{\text{opt}}\right\|^2 &= \left\|\mathbf{w}_i^{\text{opt}}\right\|^2 \Leftrightarrow \\ \operatorname{tr}\left[\sum_{\ell=1}^{\rho_i} \mathbf{w}_{i,\ell}^{\text{opt}}(\mathbf{w}_{i,\ell}^{\text{opt}})^H\right] &= \operatorname{tr}\left[\mathbf{w}_i^{\text{opt}}(\mathbf{w}_i^{\text{opt}})^H\right] \Leftrightarrow \\ \operatorname{tr}\left(\mathbf{W}_i^{\text{opt}}\right) &= \operatorname{tr}\left(\bar{\mathbf{W}}_i^{\text{opt}}\right).\end{aligned} \tag{2.5}$$

Hence, the feasible set of rank-1 blocks $\{\bar{\mathbf{W}}_i^{\text{opt}}\}_{i=1}^G$ is an optimum solution of the problem $\mathsf{Q}_{\mathbf{r}}$, since it has the same objective value as $\{\mathbf{W}_i^{\text{opt}}\}_{i=1}^G$. ■

2.2.2 Convex Reformulation

In this section, the nonconvex quadratic inequality constraints of the original QoS multicast beamforming problem Q (cf. Section 1.3) are reformulated with respect to the autocorrelation of the beamforming vectors. Towards this end, the signal power received at each user k by multicast i can be equivalently written, for Vandermonde channel vectors (2.1), as

$$
\begin{aligned}
\left|\mathbf{w}_i^H \mathbf{h}_k\right|^2 &= \left(\mathbf{w}_i^H \mathbf{h}_k\right)\left(\mathbf{h}_k^H \mathbf{w}_i\right) \\
&= \sum_{n=1}^{N} w_{i,n}^* h_{k,n} \sum_{m=1}^{N} w_{i,m} h_{k,m}^* \\
&= \sum_{n=1}^{N}\sum_{m=1}^{N} w_{i,m} w_{i,n}^* h_{k,n} h_{k,m}^* \\
&= \sum_{n=1}^{N}\sum_{m=1}^{N} w_{i,m} w_{i,n}^* e^{j\theta_k (n-1)} e^{-j\theta_k (m-1)} \\
&= \sum_{n=1}^{N}\sum_{m=1}^{N} w_{i,m} w_{i,n}^* e^{-j\theta_k (m-n)} \\
&= \sum_{m=1}^{N}\sum_{\ell=m-1}^{m-N} w_{i,m} w_{i,m-\ell}^* e^{-j\theta_k \ell} \\
&= \sum_{\ell=-(N-1)}^{N-1}\sum_{m=\max(1+\ell,1)}^{\min(N+\ell,N)} w_{i,m} w_{i,m-\ell}^* e^{-j\theta_k \ell} \\
&= \sum_{\ell=-(N-1)}^{N-1} r_{i,\ell} e^{-j\theta_k \ell},
\end{aligned}
\tag{2.6}
$$

where

$$
\begin{aligned}
\mathbf{w}_i &:= [w_{i,1}\ w_{i,2}\ \cdots\ w_{i,N}]^T \quad \text{and} \\
\mathbf{h}_k &:= [h_{k,1}\ h_{k,2}\ \cdots\ h_{k,N}]^T.
\end{aligned}
\tag{2.7}
$$

In (2.6), we have denoted $\ell := m - n$ and for all $\ell \in \{-N+1, \ldots, N-1\}$

$$
r_{i,\ell} := \sum_{m=\max(1+\ell,1)}^{\min(N+\ell,N)} w_{i,m} w_{i,m-\ell}^*.
\tag{2.8}
$$

It is easy to see that $r_{i,\ell}$ is conjugate-symmetric about the origin, which allows us to rewrite the received signal power in terms of $\{r_{i,\ell}\}_{\ell=0}^{N-1}$ only

$$
\begin{aligned}
\left|\mathbf{w}_i^H \mathbf{v}(\theta_k)\right|^2 &= r_{i,0} + \sum_{\ell=1}^{N-1}\left(r_{i,\ell} e^{-j\theta_k \ell} + r_{i,-\ell} e^{j\theta_k \ell}\right) \\
&= r_{i,0} + \sum_{\ell=1}^{N-1}\left(r_{i,\ell} e^{-j\theta_k \ell} + r_{i,\ell}^* e^{j\theta_k \ell}\right) \\
&= r_{i,0} + 2\sum_{\ell=1}^{N-1} \operatorname{Re}\left[r_{i,\ell} e^{-j\theta_k \ell}\right] \\
&= \operatorname{Re}\left[\mathbf{v}(\theta_k)^H \tilde{\mathbf{I}} \mathbf{r}_i\right],
\end{aligned}
\tag{2.9}
$$

where we have defined the autocorrelation vectors $\mathbf{r}_i \in \mathbb{R} \times \mathbb{C}^{N-1} \quad \forall i \in \mathcal{G}$ as

$$\mathbf{r}_i := [r_{i,0} \; r_{i,1} \; \cdots \; r_{i,N-1}]^T, \tag{2.10}$$

and the $N \times N$ diagonal matrix

$$\tilde{\mathbf{I}} = \begin{bmatrix} 1 & 0 \\ 0 & 2\mathbf{I}_{N-1} \end{bmatrix}, \tag{2.11}$$

where $\mathbf{I}_{N-1}$ denotes the $(N-1) \times (N-1)$ identity matrix. Furthermore, note that for $\ell = 0$ (2.8) yields $r_{i,0} = \sum_{m=1}^{N} w_{i,m} w_{i,m}^* = ||\mathbf{w}_i||_2^2$.

It therefore follows that problem Q can be equivalently reformulated, with respect to the optimization variables $\{\mathbf{r}_i\}_{i=1}^G$, to

V^{Q}

$$\min_{\{\mathbf{r}_i \in \mathbb{R} \times \mathbb{C}^{N-1}\}_{i=1}^G} \sum_{i=1}^{G} r_{i,0}$$

$$\text{s.t. :} \quad \frac{\text{Re}\left[\mathbf{v}(\theta_k)^H \tilde{\mathbf{I}} \mathbf{r}_i\right]}{\sum_{\substack{j=1 \\ j \neq i}}^{G} \text{Re}\left[\mathbf{v}(\theta_k)^H \tilde{\mathbf{I}} \mathbf{r}_j\right] + \sigma_k^2} \geq \gamma_k \qquad \forall k \in \mathcal{G}_i \;\; \forall i \in \mathcal{G},$$

$$\mathbf{r}_i \;\; \text{is an autocorrelation vector} \qquad \forall i \in \mathcal{G}.$$

Each of the K inequality constraints can be written as

$$\begin{aligned} \text{Re}\left[\mathbf{v}(\theta_k)^H \tilde{\mathbf{I}} \mathbf{r}_i\right] - \gamma_k \sum_{\substack{j=1 \\ j \neq i}}^{G} \text{Re}\left[\mathbf{v}(\theta_k)^H \tilde{\mathbf{I}} \mathbf{r}_j\right] \geq \gamma_k \sigma_k^2 \Leftrightarrow \\ \text{Re}\left[\mathbf{v}(\theta_k)^H \tilde{\mathbf{I}} \left(\mathbf{r}_i - \gamma_k \sum_{\substack{j=1 \\ j \neq i}}^{G} \mathbf{r}_j\right)\right] \geq \gamma_k \sigma_k^2 \Leftrightarrow \\ \text{Re}\left[\mathbf{v}(\theta_k)^H \tilde{\mathbf{I}} \mathbf{A}_k \mathbf{r}\right] \geq \gamma_k \sigma_k^2 \Leftrightarrow \\ \mathbf{v}(\theta_k)^H \tilde{\mathbf{I}} \mathbf{A}_k \mathbf{r} + j\xi_k \geq \gamma_k \sigma_k^2, \end{aligned} \tag{2.12}$$

where the fact that the denominator is nonnegative was taken into account in the first step. In the third step, the $N \times GN$ matrix $\mathbf{A}_k = \mathbf{a}_k \otimes \mathbf{I}_N$ was introduced, where $\mathbf{a}_k = (\gamma_k + 1)\mathbf{e}_i^T - \gamma_k \mathbf{1}_G^T$ is the $1 \times G$ vector whose ith element is equal to 1, whereas all others are set to $-\gamma_k$. Here, $\mathbf{1}_G$ is the $G \times 1$ all-1 vector, $\mathbf{e}_i$ is the $G \times 1$ vector indicating the multicast group i that user k belongs to, and $\otimes$ denotes the Kronecker product. Furthermore, the autocorrelation vectors are stacked in the $GN \times 1$ optimization variable vector $\mathbf{r} = [\mathbf{r}_1^T \;\; \cdots \;\; \mathbf{r}_G^T]^T$. In the fourth step, the real "slack" variables $\{\xi_k\}_{k=1}^K$ were inserted to compensate the terms $\mathrm{Im}\left[\mathbf{v}(\theta_k)^H\tilde{\mathbf{I}}\mathbf{A}_k\mathbf{r}\right]$. Apparently, the last inequality in (2.12) is linear.

Each of the G autocorrelation constraints admits an equivalent linear matrix inequality representation (see, e.g., [1, 8]). Note that (2.8) can be rewritten $\forall \ell \in \mathcal{N} := \{0, \ldots, N-1\}$ as

$$r_{i,\ell} = \sum_{m=1+\ell}^{N} w_{i,m} w_{i,m-\ell}^{*} = \mathbf{w}_i^H \mathbf{E}_N^{\ell} \mathbf{w}_i = \mathrm{tr}\left[\mathbf{E}_N^{\ell} \mathbf{w}_i \mathbf{w}_i^H\right] = \mathrm{tr}\left[\mathbf{E}_N^{\ell} \mathbf{W}_i\right], \tag{2.13}$$

where $\mathbf{W}_i$ is a rank-1 positive semidefinite matrix by definition, cf. (1.2), and $\mathbf{E}_N^{\ell}$ is the $N \times N$ Toeplitz matrix with ones in the ℓth upper sub-diagonal and zeros elsewhere (note that $\mathbf{E}_N^0 = \mathbf{I}_N$). Remarkably, it is proven in [1] that the result of (2.13) holds even when the associated rank constraint on the auxiliary matrix is relaxed. Specifically, $\{r_{i,\ell}\}_{\ell=0}^{N-1}$ belongs to the set of finite autocorrelation sequences if and only if $\left\{r_{i,\ell} = \mathrm{tr}\left[\mathbf{E}_N^{\ell} \mathbf{W}_i\right]\right\}_{\ell=0}^{N-1}$ for some positive semidefinite matrix $\mathbf{W}_i$. Thus, introducing an $N \times N$ auxiliary matrix $\mathbf{W}_i$ for each autocorrelation vector $\mathbf{r}_i$, the G autocorrelation constraints are equivalently converted to GN linear equalities and G positive semidefinite constraints.

Replacing the constraints of problem V^{Q}, with the equivalent representations of (2.12) and (2.13), the QoS multicast beamforming problem for Vandermonde channels is reformulated to

$$\mathsf{V}^{\mathsf{Q}}$$

$$\begin{aligned}
&\min_{\{\mathbf{r}_i \in \mathbb{R}\times\mathbb{C}^{N-1},\ \mathbf{W}_i \in \mathbb{C}^{N\times N}\}_{i=1}^{G},\ \{\xi_k \in \mathbb{R}\}_{k=1}^{K}} \sum_{i=1}^{G} r_{i,0} \\
\text{s.t. :}\quad & \mathbf{v}(\theta_k)^H \tilde{\mathbf{I}}\mathbf{A}_k \mathbf{r} + j\xi_k \geq \gamma_k \sigma_k^2 \qquad \forall k \in \mathcal{G}_i \ \ \forall i \in \mathcal{G}, \\
& r_{i,\ell} - \mathrm{tr}\left[\mathbf{E}_N^{\ell} \mathbf{W}_i\right] = 0 \qquad \forall \ell \in \mathcal{N} \ \ \forall i \in \mathcal{G}, \\
& \mathbf{W}_i \succeq \mathbf{0} \quad \forall i \in \mathcal{G}.
\end{aligned}$$

Problem V^{Q} is an SDP problem; it can therefore be efficiently solved by means of interior point methods. Problem V^{Q} consists of G vector variables of size $N \times 1$, G matrix variables of size $N \times N$, and $K + GN$ linear constraints. Interior point methods will take $O[\sqrt{GN}\log(1/\epsilon)]$ iterations, with each iteration requiring at most $O[G^3N^6 + (K+GN)GN^2]$ arithmetic operations [48], where the parameter ϵ represents the desired solution accuracy at the algorithm's termination. Actual runtime complexity will usually scale far slower with G, N, and K than this worst-case bound[3]. The optimum autocorrelation sequences $\{\mathbf{r}_i^{\text{opt}}\}_{i=1}^G$ are obtained solving V^{Q}; then, the respective optimum beamforming vectors $\{\mathbf{w}_i^{\text{opt}}\}_{i=1}^G$ can be found using spectral factorization techniques (see, e.g., [45]).

The QoS multicast beamforming problem for Vandermonde channels can thus be solved equivalently in two distinct ways. First, by the principal components of the optimum solution blocks of the SDP problem $\mathsf{Q}_{\mathbf{r}}$, when these turn out all being rank-1. Second, by spectral factorization of the optimum autocorrelation sequences solving the SDP problem V^{Q}. However, problem $\mathsf{Q}_{\mathbf{r}}$ is not guaranteed to *consistently* yield rank-1 solutions for Vandermonde channel vectors; Claim 9 only proves the existence of such a solution, and counter-examples in which SDP yields higher-rank solutions do arise in practice. Post-processing via spectral factorization is needed in such cases in order to obtain an equivalent rank-1 solution. The first approach (via $\mathsf{Q}_{\mathbf{r}}$) is computationally cheaper when general-purpose interior point SDP software is used, because V^{Q} involves a higher number of optimization variables and associated constraints. However, the dual of V^{Q} involves significantly fewer variables and can be solved via application-specific interior point methods which can drop the arithmetic operations per iteration by two to three orders of magnitude (see, e.g., [1]). Finally, and perhaps most importantly, the reformulation of the QoS constraints in terms of autocorrelation sequences as inequalities on (the real part of) trigonometric polynomials, cf. (2.12), enables us to extend the multicast beamforming problem to the case where there is partial knowledge of the angles $\{\theta_k\}_{k=1}^K$ determining the Vandermonde channel vectors. The respective robust design is considered in Section 2.4.1.

2.3 Max-Min Fair Multicast Beamforming

This section treats the related multicast beamforming problem of maximizing the minimum (weighted) received SINR, subject to an upper bound P on the total transmission power. The general MMF problem has been considered in Section 1.4; the important

[3]This is true for all problems considered here.

special case when a ULA is used at the transmitter under far-field line-of-sight propagation conditions is studied here. In such a scenario, the channel vectors are Vandermonde, cf. (2.1), and Claim 9, proved in Section 2.2.1 for the QoS formulation, holds for the MMF formulation too, i.e., the relaxation $\mathsf{F} \to \mathsf{F}_\mathbf{r}$ is always tight. The proof is similar to the QoS case; specifically, using (2.4) and (2.5), it is seen that the set of rank-1 blocks $\{\bar{\mathbf{W}}_i^{\text{opt}}\}_{i=1}^G$ forms a feasible solution of problem $\mathsf{F}_\mathbf{r}$ with the same objective value as the (generally higher-rank) optimum solution set $\{\mathbf{W}_i^{\text{opt}}\}_{i=1}^G$.

In the following, an approach similar to Section 2.2.2 is followed in order to formulate the MMF multicast beamforming problem for the case of Vandermonde channels. Using the representation of (2.9) for the signal power received by user k from multicast i, problem F may be equivalently written, in terms of the autocorrelation of the sought beamforming vectors, as

V^F

$$\max_{\{\mathbf{r}_i \in \mathbb{R}\times\mathbb{C}^{N-1}\}_{i=1}^G,\ g\geq 0} \quad g$$

$$\text{s.t.}: \quad \frac{1}{\gamma_k} \frac{\operatorname{Re}\left[\mathbf{v}(\theta_k)^H \tilde{\mathbf{I}} \mathbf{r}_i\right]}{\sum_{\substack{j=1 \\ j\neq i}}^G \operatorname{Re}\left[\mathbf{v}(\theta_k)^H \tilde{\mathbf{I}} \mathbf{r}_j\right] + \sigma_k^2} \geq g \qquad \forall k \in \mathcal{G}_i \ \ \forall i \in \mathcal{G},$$

$$\sum_{i=1}^G r_{i,0} = P,$$

$$\mathbf{r}_i \ \text{ is an autocorrelation vector} \qquad \forall i \in \mathcal{G}.$$

The optimization problem V^F consists of a linear cost function and K inequality, 1 equality, and G autocorrelation constraints. The latter can be replaced by linear matrix inequalities, with the introduction of G auxiliary positive semidefinite matrices, as in Section 2.2.2. Hence, the interest is focused on the inequality constraints; the remainder is an SDP problem. Following the same steps as in (2.12) and replacing every occurrence of γ_k by the term $g\gamma_k$, the inequalities can be equivalently written as

$$\mathbf{v}(\theta_k)^H \tilde{\mathbf{I}} \mathbf{A}_k(g) \mathbf{r} + j\xi_k \geq g\gamma_k\sigma_k^2 \qquad \forall k \in \mathcal{G}_i \ \ \forall i \in \mathcal{G}. \tag{2.14}$$

The fundamental difference of (2.14) to (2.12) is that the auxiliary matrices $\mathbf{A}_k(g)$ are not constant factors anymore but variables, since they depend on g. Hence, contrary to the QoS case, the inequalities (2.14) are nonlinear.

This difficulty can be tackled by appreciating two key observations: i) by fixing g the inequalities (2.14) become linear in the remaining variables; and ii) the objective of V^{F} is to maximize g. It follows that V^{F} can be solved by bisection over SDP problems. Specifically, let $[L, U]$ denote the interval containing the optimum value g^{opt} of problem V^{F}. Due to the nonnegativity of g^{opt} and the Cauchy-Schwartz inequality, we may set $L = 0$ and $U = \min_{k\in\mathcal{K}} \frac{PN}{\sigma_k^2}$ for the lower and the upper bound, respectively. Given $[L, U]$, the SDP feasibility problem F_{f}, shown in the box below, is solved at the midpoint $g := (L + U)/2$ of the interval. If problem F_{f} is feasible for the given choice of g, set $L := g$; otherwise $U := g$. Thus, in each iteration the interval is halved. The algorithm terminates when the interval length becomes smaller than the desired accuracy.

$\mathsf{V}_{\text{f}}^{\mathsf{F}}$

$$
\begin{aligned}
&\text{find } \mathbf{x}\\
\text{s.t. :}\quad &\mathbf{v}(\theta_k)^H \tilde{\mathbf{I}}\mathbf{A}_k(g)\mathbf{r} + j\xi_k \geq g\gamma_k\sigma_k^2 \qquad \forall k \in \mathcal{G}_i \;\; \forall i \in \mathcal{G},\\
&r_{i,\ell} - \text{tr}\left[\mathbf{E}_N^{\ell}\mathbf{W}_i\right] = 0 \qquad \forall \ell \in \mathcal{N} \;\; \forall i \in \mathcal{G},\\
&\sum_{i=1}^{G} r_{i,0} = P,\\
&\mathbf{W}_i \succeq \mathbf{0} \qquad \forall i \in \mathcal{G}.
\end{aligned}
$$

In $\mathsf{V}_{\text{f}}^{\mathsf{F}}$, $\mathbf{x}$ denotes the optimization vector

$$\mathbf{x} := [\boldsymbol{\xi}^T \; \mathbf{r}^T \; \text{vec}(\mathbf{W}_1)^T \; \cdots \; \text{vec}(\mathbf{W}_G)^T]^T, \tag{2.15}$$

where $\boldsymbol{\xi} := [\xi_1 \; \cdots \; \xi_K]^T$. The solution vector obtained by the last feasible iteration of the algorithm contains the optimum autocorrelation sequences $\{\mathbf{r}_i^{\text{opt}}\}_{i=1}^G$. The sought beamforming vectors $\{\mathbf{w}_i^{\text{opt}}\}_{i=1}^G$ can then be found using spectral factorization techniques (see, e.g., [45]).

In every iteration, the algorithm tries to find a solution to the SDP feasibility problem $\mathsf{V}_{\text{f}}^{\mathsf{F}}$. Problems of this form can be efficiently solved by general-purpose interior point SDP solvers. The use of interior point methods is convenient, because they not only yield a solution to problem $\mathsf{V}_{\text{f}}^{\mathsf{F}}$ when the latter is feasible, but also provide a certificate of infeasibility otherwise. Similar to problem V^{Q}, problem $\mathsf{V}_{\text{f}}^{\mathsf{F}}$ consists of G vector variables of size $N \times 1$, G matrix variables of size $N \times N$, and $K + GN + 1$ linear constraints. Computing an optimal solution of tolerance ϵ using an interior point method will have an overall iteration count of $O[\sqrt{GN}\log(1/\epsilon)]$, with each iteration costing $O[G^3N^6 + (K+GN)GN^2]$ in the worst case [48].

As for the QoS problem considered in Section 2.2, the MMF multicast beamforming problem for Vandermonde channels can be solved optimally by two algorithms, both employing bisection over SDP problems. Again, going through $\mathsf{F}_{\mathbf{r}}$ entails lower complexity when a general-purpose interior point solver is used, but the solution matrices are not guaranteed to be rank-1 . On the other hand, the dual of $\mathsf{V}_{\mathrm{f}}^{\mathsf{F}}$ is cheaper to solve using custom interior point methods developed specifically for problems involving autocorrelation constraints (see, e.g., [1]). Furthermore, problem $\mathsf{V}_{\mathrm{f}}^{\mathsf{F}}$ forms the basis for the robust extension considered in Section 2.4.2.

2.4 Robust Multicast Beamforming

In the multicast beamforming problems presented so far, the beamformers are designed under the assumption that full CSI is available at the transmitter. For the case of Vandermonde channel vectors, considered throughout this chapter, accurate CSI boils down to exact knowledge of users' directions $\{\phi_k\}_{k=1}^K$ (equivalently, of the angles $\{\theta_k\}_{k=1}^K$). In a more realistic scenario, only partial CSI is practically available at the design center, due to errors in the estimation of the angles $\{\theta_k\}_{k=1}^K$. It is often reasonable to assume that errors are bounded, e.g., due to quantization, in which case each θ_k lies in some interval $[\alpha_k, \beta_k]$. In the following, robust formulations for the QoS and MMF multicast beamforming problems are presented, using a worst-case approach, i.e., the constraints account for all possible channel vectors.

2.4.1 Robust QoS Formulation

A robust extension of the QoS multicast beamforming problem V^{Q}, considered in Section 2.2.2, would be to jointly design the beamforming vectors so that for every user $k \in \mathcal{K}$ the received SINR target γ_k is met for all possible values of the angle $\theta_k \in [\alpha_k, \beta_k]$ that determines the Vandermonde channel vector. In such a scenario, each of the K SINR constraints are posed, cf. (2.12), as

$$\mathrm{Re}\left[\mathbf{v}(\theta_k)^H \tilde{\mathbf{I}} \mathbf{A}_k \mathbf{r}\right] \geq \gamma_k \sigma_k^2 \qquad \forall \theta_k \in [\alpha_k, \beta_k]. \tag{2.16}$$

An interpretation of (2.16) is that it requires (the real part of) certain trigonometric polynomial to be nonnegative over a segment of the unit circle. As proved in [8], constraints of this form can be equivalently reformulated to the linear matrix inequality constraints

$$\tilde{\mathbf{I}}\mathbf{A}_k\mathbf{r} + j\xi_k\mathbf{e}_1 - \gamma_k\sigma_k^2\mathbf{e}_1 = \mathbf{y}(\mathbf{Y}_k) + \mathbf{z}(\mathbf{Z}_k;\alpha_k,\beta_k), \tag{2.17}$$

where $\mathbf{Y}_k \in \mathbb{C}^{N\times N} \succeq \mathbf{0}$, $\mathbf{Z}_k \in \mathbb{C}^{(N-1)\times(N-1)} \succeq \mathbf{0}$, $\xi_k \in \mathbb{R}$ are auxiliary variables, and $\mathbf{e}_1$ denotes the $N \times 1$ indicator vector whose first element is 1 and all others are 0. The vector $\mathbf{y}(\mathbf{Y}) := [y_1\ y_2\ \cdots\ y_N]^T \in \mathbb{R}\times\mathbb{C}^{N-1}$ is defined [8] by the equations

$$\begin{aligned} y_1 &:= \left\langle \mathbf{E}_N^0, \mathbf{Y} \right\rangle, \\ y_{\ell+1} &:= 2\left\langle \mathbf{E}_N^\ell, \mathbf{Y} \right\rangle \qquad \forall \ell \in \{1,\ldots,N-1\}, \end{aligned} \tag{2.18}$$

where $\langle \mathbf{A},\mathbf{B}\rangle := \mathrm{tr}\left(\mathbf{A}^H\mathbf{B}\right)$ denotes the inner product between two (generally complex) matrices $\mathbf{A}$ and $\mathbf{B}$. The vector $\mathbf{z}(\mathbf{Z};\alpha,\beta) := [z_1\ z_2\ \cdots\ z_N]^T \in \mathbb{C}^N$ is defined [8] by the equations

$$\begin{aligned} z_1 &:= d_1(\alpha,\beta)\left\langle \mathbf{E}_{N-1}^0, \mathbf{Z}\right\rangle + d_2^*(\alpha,\beta)\left\langle \mathbf{E}_{N-1}^1, \mathbf{Z}\right\rangle, \\ z_{\ell+1} &:= 2d_1(\alpha,\beta)\left\langle \mathbf{E}_{N-1}^\ell, \mathbf{Z}\right\rangle + d_2(\alpha,\beta)\left\langle \mathbf{E}_{N-1}^{\ell-1}, \mathbf{Z}\right\rangle + d_2^*(\alpha,\beta)\left\langle \mathbf{E}_{N-1}^{\ell+1}, \mathbf{Z}\right\rangle \\ &\qquad\qquad \forall \ell \in \{1,\ldots,N-3\}, \\ z_{N-1} &:= 2d_1(\alpha,\beta)\left\langle \mathbf{E}_{N-1}^{N-2}, \mathbf{Z}\right\rangle + d_2(\alpha,\beta)\left\langle \mathbf{E}_{N-1}^{N-3}, \mathbf{Z}\right\rangle, \\ z_N &:= d_2(\alpha,\beta)\left\langle \mathbf{E}_{N-1}^{N-2}, \mathbf{Z}\right\rangle, \end{aligned} \tag{2.19}$$

where, for given $\alpha,\beta \in [0,2\pi)$, the vector $\mathbf{d}(\alpha,\beta) = [d_1(\alpha,\beta)\ d_2(\alpha,\beta)]^T \in \mathbb{R}\times\mathbb{C}$ is defined by

$$\mathbf{d}(\alpha,\beta) := \begin{bmatrix} \cos\alpha + \cos\beta - \cos(\beta-\alpha) - 1 \\ [1-\exp(j\alpha)]\,[\exp(j\beta)-1] \end{bmatrix}. \tag{2.20}$$

Hence, introducing the auxiliary matrices $\{\mathbf{Y}_k \in \mathbb{C}^{N\times N},\ \mathbf{Z}_k^{(N-1)\times(N-1)}\}_{k=1}^K$ and replacing the conservative SINR constraints of (2.16) with the representation of (2.17), the robust version of the QoS multicast beamforming problem V^{Q} is equivalently written as

R^{Q}

$$\min_{\{\mathbf{r}_i,\ \mathbf{W}_i\}_{i=1}^G,\ \{\mathbf{Y}_k,\ \mathbf{Z}_k,\ \xi_k\}_{k=1}^K} \sum_{i=1}^{G} r_{i,0}$$

$$\begin{aligned}
\text{s.t.}: \quad & \tilde{\mathbf{I}}\mathbf{A}_k\mathbf{r} + j\xi_k\mathbf{e}_1 - \mathbf{y}(\mathbf{Y}_k) - \mathbf{z}(\mathbf{Z}_k;\alpha_k,\beta_k) = \gamma_k\sigma_k^2\mathbf{e}_1 \qquad \forall k \in \mathcal{G}_i \ \ \forall i \in \mathcal{G},\\
& r_{i,\ell} - \operatorname{tr}\left[\mathbf{E}_N^{\ell}\mathbf{W}_i\right] = 0 \qquad \forall \ell \in \mathcal{N} \ \ \forall i \in \mathcal{G},\\
& \mathbf{W}_i \succeq \mathbf{0} \qquad \forall i \in \mathcal{G},\\
& \mathbf{Y}_k \succeq \mathbf{0} \qquad \forall k \in \mathcal{K},\\
& \mathbf{Z}_k \succeq \mathbf{0} \qquad \forall k \in \mathcal{K}.
\end{aligned}$$

Problem R^{Q} consists of a linear cost and $KN + GN$ linear equality, and $G + 2K$ positive semidefinite constraints; it is therefore an SDP problem. An ϵ-optimal solution, by means of general-purpose interior point methods, entails $O[\sqrt{(G+2K)N}\log(1/\epsilon)]$ iterations, each of complexity $O[(G+2K)^3N^6+(K+G)(G+2K)N^3]$ [48]. More efficient solutions, of significantly lower complexity, can be found by means of application-specific interior point methods (see, e.g., [1]). The optimum beamforming vectors $\{\mathbf{w}_i^{\text{opt}}\}_{i=1}^G$ can be computed from the solution of problem R^{Q} using spectral factorization techniques (see, e.g., [45]).

It is interesting to note that the original QoS multicast beamforming problem V^{Q} and its robust counterpart R^{Q} are convex (specifically, SDP) problems. The price paid for the extension to the partial CSI case is higher computational complexity due to the larger number of the optimization variables and constraints.

2.4.2 Robust MMF Formulation

A robust extension of the MMF multicast beamforming problem V^{F} can be found in a manner similar to the respective QoS problem considered in the previous section. Each of the K SINR inequality constraints that must be fulfilled are now posed as

$$\operatorname{Re}\left[\mathbf{v}(\theta_k)^H\tilde{\mathbf{I}}\mathbf{A}_k(g)\mathbf{r}\right] \geq g\gamma_k\sigma_k^2 \qquad \forall\theta_k \in [\alpha_k,\beta_k], \tag{2.21}$$

where, as noted in Section 2.3, $\mathbf{A}_k(g)$ depends on the variable g. Exactly as for (2.16), inequalities (2.21) can be equivalently reformulated [8] to

$$\tilde{\mathbf{I}}\mathbf{A}_k(g)\mathbf{r} + j\xi_k\mathbf{e}_1 - g\gamma_k\sigma_k^2\mathbf{e}_1 = \mathbf{y}(\mathbf{Y}_k) + \mathbf{z}(\mathbf{Z}_k;\alpha_k,\beta_k). \tag{2.22}$$

Fixing the value of the variable g, (2.22) represents linear matrix inequality constraints. Thus, as with the MMF multicast beamforming problem V^{F} in Section 2.3, the robust version can be efficiently solved, by means of bisection over the following SDP feasibility problem $\mathsf{R}_{\mathrm{f}}^{\mathsf{F}}$. The optimization variables of $\mathsf{R}_{\mathrm{f}}^{\mathsf{F}}$ are stacked in the vector

$$\tilde{\mathbf{x}} := [\mathbf{x}^T \ \mathrm{vec}(\mathbf{Y}_1)^T \ \cdots \ \mathrm{vec}(\mathbf{Y}_K)^T \ \mathrm{vec}(\mathbf{Z}_1)^T \ \cdots \ \mathrm{vec}(\mathbf{Z}_K)^T]^T, \tag{2.23}$$

where $\mathbf{x}$ is defined by (2.15).

$\mathsf{R}_{\mathrm{f}}^{\mathsf{F}}$

$$\begin{aligned}
& \text{find } \tilde{\mathbf{x}} \\
\text{s.t. :} \quad & \tilde{\mathbf{I}}\mathbf{A}_k(g)\mathbf{r} + j\xi_k\mathbf{e}_1 - \mathbf{y}(\mathbf{Y}_k) - \mathbf{z}(\mathbf{Z}_k;\alpha_k,\beta_k) = g\gamma_k\sigma_k^2\mathbf{e}_1 \qquad \forall k \in \mathcal{G}_i \ \ \forall i \in \mathcal{G}, \\
& r_{i,\ell} - \mathrm{tr}\left[\mathbf{E}_N^{\ell}\mathbf{W}_i\right] = 0 \qquad \forall \ell \in \mathcal{N} \ \ \forall i \in \mathcal{G}, \\
& \sum_{i=1}^{G} r_{i,0} = P, \\
& \mathbf{W}_i \succeq \mathbf{0} \qquad \forall i \in \mathcal{G}, \\
& \mathbf{Y}_k \succeq \mathbf{0} \qquad \forall k \in \mathcal{K}, \\
& \mathbf{Z}_k \succeq \mathbf{0} \qquad \forall k \in \mathcal{K}.
\end{aligned}$$

Feasibility problem $\mathsf{R}_{\mathrm{f}}^{\mathsf{F}}$ is an SDP problem comprising $KN + GN + 1$ linear equality and $G + 2K$ positive semidefinite constraints. As in the QoS case, it has the same form as the original (full CSI) problem $\mathsf{V}_{\mathrm{f}}^{\mathsf{F}}$, but higher dimensionality. An ϵ-optimal solution, by means of general-purpose interior point methods, entails $O[\sqrt{(G+2K)N}\log(1/\epsilon)]$ iterations, each of complexity $O[(G{+}2K)^3N^6{+}(K{+}G)(G{+}2K)N^3]$ [48]. The complexity can be significantly reduced with the use of application-specific interior point methods (see, e.g., [1]). Again, the optimum beamforming vectors $\{\mathbf{w}_i^{\mathrm{opt}}\}_{i=1}^{G}$ can be computed using spectral factorization techniques (see, e.g., [45]).

2.5 Numerical Results

In this section, we provide some representative numerical results illustrating and contrasting the various multicast beamformer designs for Vandermonde channels presented in Sections 2.2–2.4. For each design, the resulting optimized transmit beam pattern in the plane of the ULA is plotted in linear scale. All patterns are symmetric with respect to the vertical axis, due to the inherent radiation symmetry of the ULA. The Vandermonde channel vector of each user is calculated by plugging the respective angle in (2.1). The directions of the users comprising each multicast group are shown in the caption of each plot, for ease of reference. The noise variance of all channels is set to 1, except for the scenario in Fig. 2.8. The basic parameters for all simulation configurations considered are gathered in Table 2.1. Columns 2, 3, and 4 contain the number N of transmit antenna elements (spaced $d = \lambda/2$ apart), the total number K of users served, and the number G of multicasts, respectively. Column 5 stores the SINR values in dB, which are input parameters for the QoS problems in Figs. 2.1–2.6 and attained objective values for the MMF problems in Figs. 2.7–2.9. Column 6 reports the total transmission powers, which are objective values and input parameters for the QoS and MMF problems, respectively. Finally, column 7 lists the time in seconds spent on a typical desktop computer to solve the core SDP problem for each design, using SeDuMi [35]. The reported values are averages values over 10 problem instances.

In the simplest configuration considered (Fig. 2.1), the transmit ULA consists of 6 antenna elements and 30 intended receivers are clustered in 3 multicast groups. For the first multicast group, 10 users are evenly distributed in the range 26°–62° at 4° apart. This is henceforth compactly denoted as $\{26° : 4° : 62°\}$. The users of the second group are placed at $\{-18° : 4° : 18°\}$, while the third group is the reflection of the first, with respect to the horizontal axis. Exact knowledge of all user directions is assumed at the design center (the transmitter). A SINR threshold of 10 dB is prescribed for all users. This is a typical scenario (the angles of the users listening to the same multicast are close and the number of transmit antenna elements is small) under which each beamvector forms a single main lobe to serve all users in the respective multicast group. Then, it is natural to expect that the optimum solution blocks of $\mathbf{Q_r}$ will be rank-1, and this is indeed the case. The algorithm proposed in Section 1.3.2 (principal components of the optimum rank-1 solution blocks of the SDP problem $\mathsf{Q_r}$ [23, 26]) and the algorithm developed in Section 2.2.2 (spectral factorization of the autocorrelation sequences that solve problem $\mathsf{V^Q}$ [24, 25]) yield equivalent solutions, i.e., the beam pattern of Fig. 2.1.

It is apparent from the shape of the beams in Fig. 2.1, that the SINR constraint is over-satisfied for all users except the ones on the edges of the lobes. Note that, in such scenarios, the important design parameter is the direction span of each multicast group and not the actual number of users in the group. More subscribers can be added within the span of a given group without modifying the design. Repeating the design with the same parameters but this time using $N = 12$, the resulting beam pattern is plotted in Fig. 2.2. Due to the extra degrees of freedom, far less power is wasted in over-satisfying the constraints in the middle of the lobes. For proper interpretation of the results, it is important to note that the scale of the polar plots varies from figure to figure for better visualization. The same SINR threshold is guaranteed to all users with smaller cost in terms of total transmit power (cf. lines 1 and 2 of Table 2.1), at the expense of additional hardware and complexity, because of the larger number of design variables. Fig. 2.3 plots the robust QoS design (presented in Section 2.4.1) for the same parameters and $N = 12$. The tolerance in the user directions is $\delta = 1°$. Compared to Fig. 2.2, the beams are broader in Fig. 2.3, where higher total transmission power is required to assure the same minimum SINR level to wider ranges of directions. The runtime is also higher due to the additional auxiliary variables and constraints.

A more challenging scenario is considered in Figs. 2.4–2.6. The are 2 multicast groups and the users in each group are split in two separate direction spans. Furthermore, the SINR threshold is 10 and 6 dB for the users listening to the first and the second multicast, respectively. Figs. 2.4 and 2.5 depict the optimum beam patterns for N equal to 6 and 12, respectively and perfect CSI. Due to the interleaving of direction spans of the two groups, two main lobes are formed to serve each group. As expected, more power is transmitted towards the users of the first group which demand higher assured SINR. Again, the availability of more antenna elements at the transmitter results in less total radiated power. The respective robust design for $N = 6$ and maximum ambiguity $\delta = 0.5°$ is illustrated in Fig. 2.6. A comparison with Fig. 2.4 supports the findings discussed in the previous scenario.

Figs. 2.7–2.9 illustrate an example of the equally-fair max-min beamformer design for $N = 8$, $G = 2$, and two clusters of users per multicast group. The power budget is set to 10 and the relative accuracy of the bisection to 10^{-3}, resulting in 13 iterations. Figs. 2.7 and 2.8 show the optimized beam patterns of the MMF problem (presented in Section 2.3) under the assumption of perfect CSI. In Fig. 2.8, the noise variance is set to 2 for the users in directions $\geq 40°$. Note that more power is transmitted towards the users that suffer from larger noise variance (or, equivalently, from larger path loss) to ensure fairness. The respective robust design (presented in Section 2.4.2) for $\delta = 2°$, is shown in Fig. 2.9. Relative to Fig. 2.7, the SINR level assured to all users is smaller (cf. lines 7 and 9 of Table 2.1), since wider direction spans are served with the same power budget.

Table 2.1: Simulation Parameters

Fig.	N	K	G	P	γ [dB]	Time [s]
1	6	30	3	28.32	10	0.26
2	12	30	3	10.44	10	0.52
3	12	30	3	12.35	10	12
4	6	44	2	9.56	10/6	0.31
5	12	44	2	6.03	10/6	0.45
6	6	44	2	10.82	10/6	3.3
7	8	22	2	10	9.45	0.20
8	8	22	2	10	7.97	0.26
9	8	22	2	10	7.49	2.34

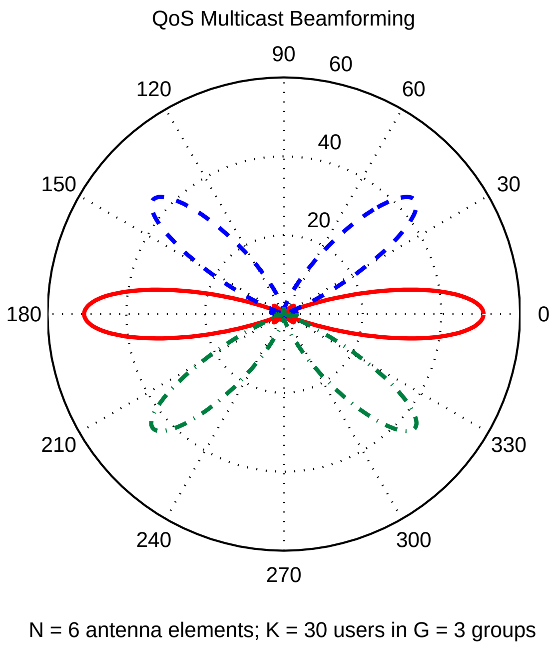

Figure 2.1: QoS, $\gamma = 10$ dB, $N = 6$; $\mathcal{G}_1 = \{26^\circ : 4^\circ : 62^\circ\}$, $\mathcal{G}_2 = \{-18^\circ : 4^\circ : 18^\circ\}$, $\mathcal{G}_3 = \{-62^\circ : 4^\circ : -26^\circ\}$

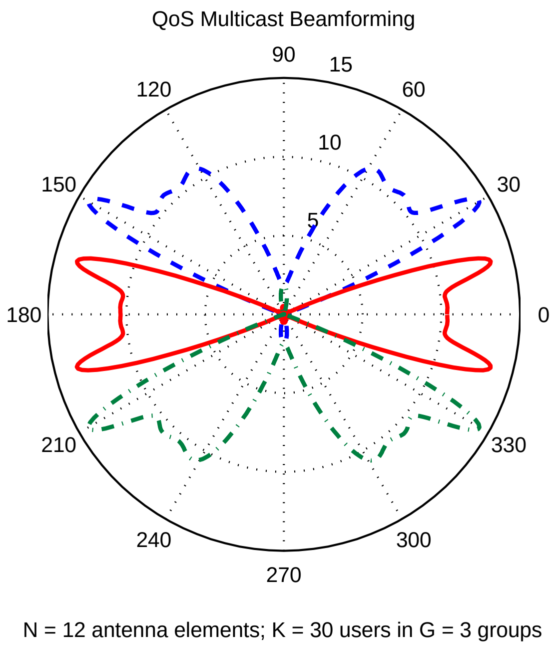

Figure 2.2: QoS, $\gamma = 10$ dB, $N = 12$; $\mathcal{G}_1 = \{26^\circ : 4^\circ : 62^\circ\}$, $\mathcal{G}_2 = \{-18^\circ : 4^\circ : 18^\circ\}$, $\mathcal{G}_3 = \{-62^\circ : 4^\circ : -26^\circ\}$

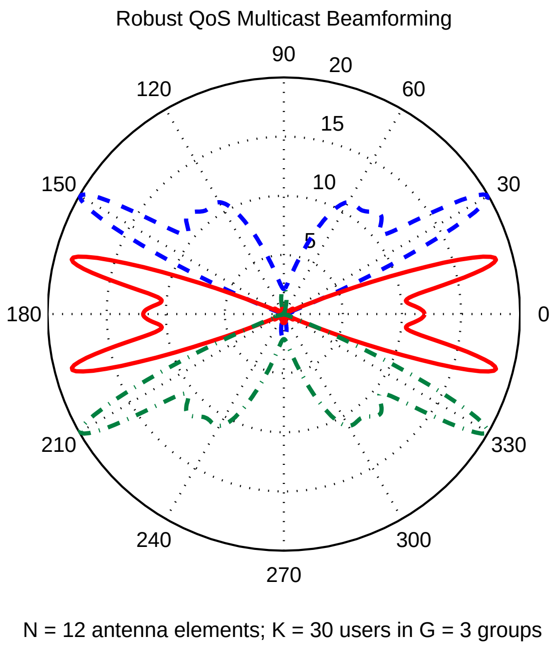

Figure 2.3: Robust QoS, $\delta = 1^\circ$, $\gamma = 10$ dB, $N = 12$; $\mathcal{G}_1 = \{26^\circ : 4^\circ : 62^\circ\}$, $\mathcal{G}_2 = \{-18^\circ : 4^\circ : 18^\circ\}$, $\mathcal{G}_3 = \{-62^\circ : 4^\circ : -26^\circ\}$

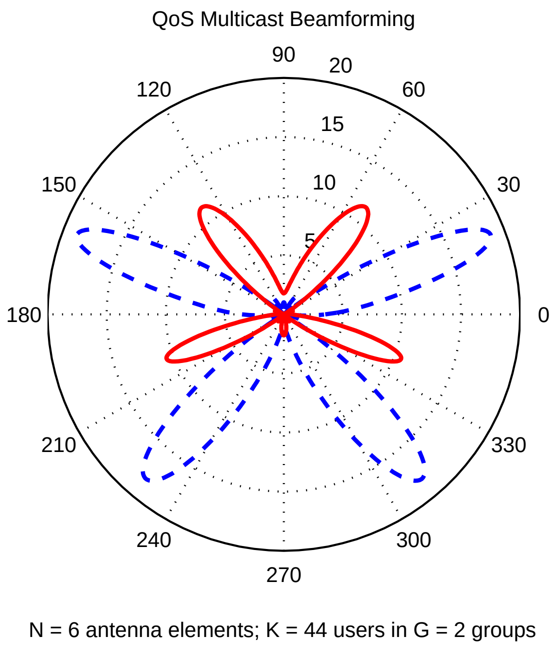

Figure 2.4: QoS, $\gamma = \{10, 6\}$ dB for $\{\mathcal{G}_1, \mathcal{G}_2\}$, $N = 6$; $\mathcal{G}_1 = \{-60^\circ : 2^\circ : -40^\circ,\ 10^\circ : 2^\circ : 30^\circ\}$, $\mathcal{G}_2 = \{-30^\circ : 2^\circ : -10^\circ,\ 40^\circ : 2^\circ : 60^\circ\}$

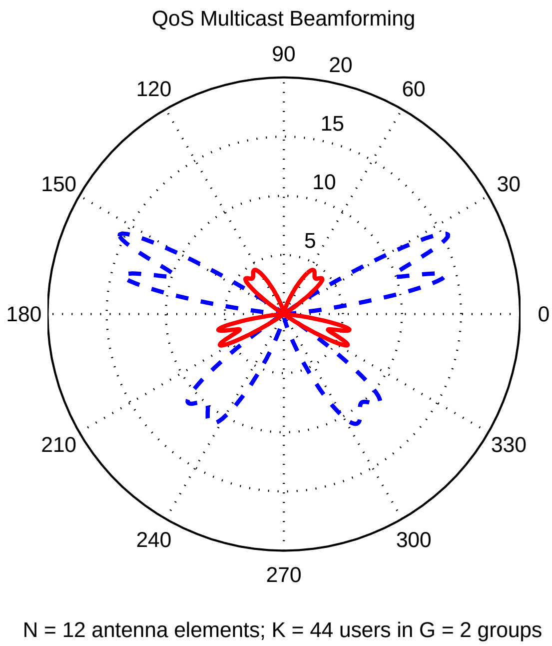

Figure 2.5: QoS, $\gamma = \{10, 6\}$ dB for $\{\mathcal{G}_1, \mathcal{G}_2\}$, $N = 12$; $\mathcal{G}_1 = \{-60° : 2° : -40°,\ 10° : 2° : 30°\}$, $\mathcal{G}_2 = \{-30° : 2° : -10°,\ 40° : 2° : 60°\}$

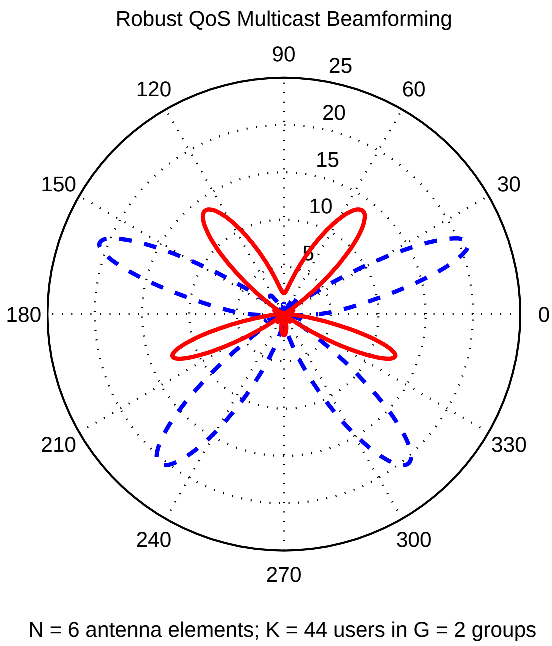

Figure 2.6: Robust QoS, $\delta = 0.5°$, $\gamma = \{10, 6\}$ dB for $\{\mathcal{G}_1, \mathcal{G}_2\}$, $N = 6$; $\mathcal{G}_1 = \{-60° : 2° : -40°,\ 10° : 2° : 30°\}$, $\mathcal{G}_2 = \{-30° : 2° : -10°,\ 40° : 2° : 60°\}$

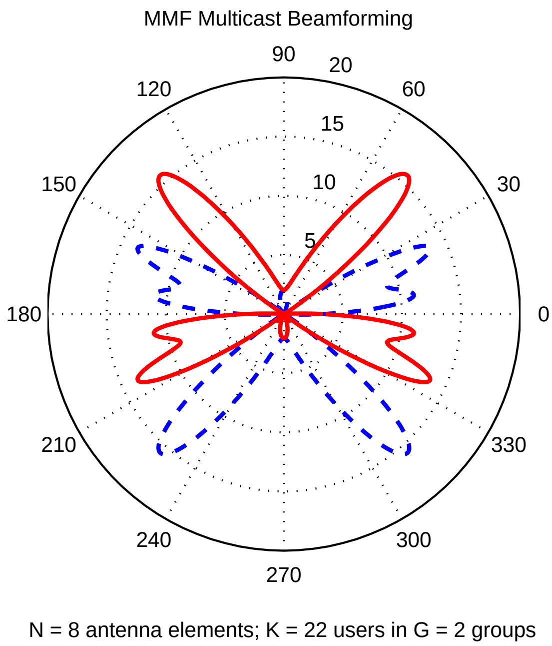

Figure 2.7: MMF, $P = 10$, $N = 8$; $\mathcal{G}_1 = \{-60° : 5° : -40°,\ 5° : 5° : 30°\}$, $\mathcal{G}_2 = \{-30° : 5° : -5°,\ 40° : 5° : 60°\}$

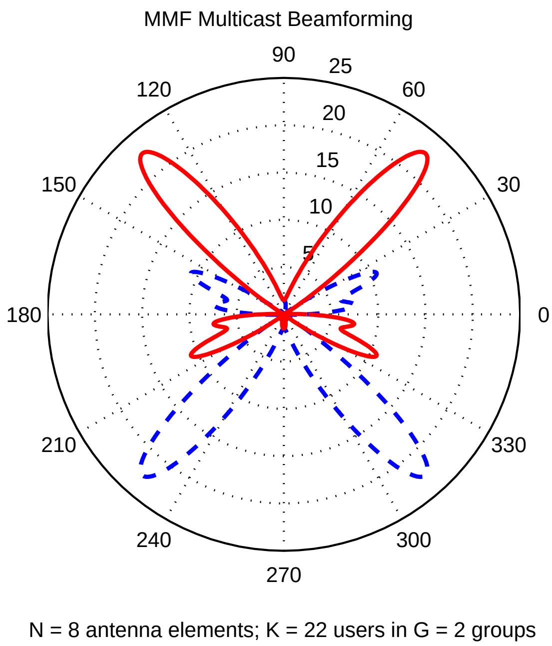

Figure 2.8: MMF, $P = 10$, $N = 8$; $\mathcal{G}_1 = \{-60° : 5° : -40°,\ 5° : 5° : 30°\}$, $\mathcal{G}_2 = \{-30° : 5° : -5°,\ 40° : 5° : 60°\}$; $\sigma^2 = 2$ for directions other than $\{-30° : 30°\}$

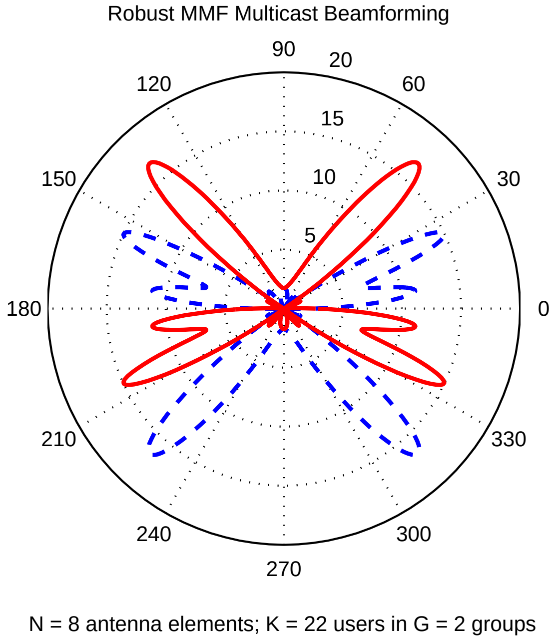

Figure 2.9: Robust MMF, $\delta = 2^\circ$, $P = 10$, $N = 8$; $\mathcal{G}_1 = \{-60^\circ : 5^\circ : -40^\circ,\ 5^\circ : 5^\circ : 30^\circ\}$, $\mathcal{G}_2 = \{-30^\circ : 5^\circ : -5^\circ,\ 40^\circ : 5^\circ : 60^\circ\}$

2.6 Conclusions

The problem of far-field multicast beamforming has been considered, under line-of-sight propagation conditions, for transmitters employing ULA arrays. Both QoS and MMF design criteria have been adopted and the Vandermonde structure of the channel vectors has been exploited to derive equivalent convex reformulations, which are amenable to exact and efficient solution using modern interior point methods. Therefore, it has been proved that, whereas the general multicast beamforming problem, presented in the previous chapter, is NP-hard, the important special case considered here is not. In addition, it has been shown that the natural (Lagrange bi-dual) SDR of the problem is tight in the case of Vandermonde channel vectors. The key tool behind these developments is spectral factorization and the representation of finite autocorrelation sequences via linear matrix inequalities. Departing from the idealized assumption of perfect CSI knowledge at the transmitter, a more realistic situation has also been considered, where the receiver directions are only known to lie in a certain interval. Robust SDP-based designs have been proposed for this case, under both QoS and MMF criteria, that yield exact solutions. The proposed designs have potential for practical use in a number of current and emerging wireless systems.

Chapter 3

Quality of Service Scheduling, Admission and Multicast Power Control

The joint scheduling, admission, and power control problem is considered under QoS constraints, and a general formulation that incorporates multicasting, cochannel or orthogonal transmission modalities, and access point selection. Several special cases are well-known to be NP-hard, yet important for QoS provisioning and bandwidth-efficient operation of existing and emerging cellular and overlay / underlay networks. Recognizing this, there have been several attempts to develop reasonable heuristics for joint scheduling and power control; e.g., [9, 27, 42]. A more disciplined approach has been followed in this contribution. The general problem is first concisely formulated as constrained optimization, whose objective combines the scheduling, admission, and power control components. The formulation also allows for multicasting. A GP approximation is then developed, which forms the core of a heuristic, yet well-motivated centralized algorithm that generates approximate solutions to the original NP-hard problem. Numerical results against an enumeration baseline and a state-of-art joint scheduling and power control algorithm [27] illustrate the merits of the approach.

3.1 Introduction

Consider a wireless network comprising K single-antenna receivers. In a general multicast scenario, there are $1 \leq G \leq K$ distinct data streams to be carried over the network. Each receiver is interested in a single stream, but in multicast mode several receivers may request the same stream. The two extreme values of G are of particular interest: when $G = 1$ all receivers are interested in the same content (i.e., the broadcast scenario), and when $G = K$ each receiver is interested in its own stream (e.g., the multiuser downlink scenario). Let $\mathcal{G}_i$ denote the index set of the receivers forming the ith multicast group, i.e., listening to the ith multicast stream. Under this setup, the multicast groups are mutually exclusive and partition the entire set of receivers, i.e., $\mathcal{G}_i \cap \mathcal{G}_j = \emptyset$, $j \neq i$ and $\cup_i \mathcal{G}_i = \mathcal{K} := \{1, \ldots, K\}$.

Regarding the transmissions, it is assumed that there are N degrees of freedom in the network. Specifically, each multicast stream can be transmitted in one or more (up to N) different dimensions. As dimensions we refer to access points located spatially apart but transmitting in the same channel, or to orthogonal dimensions, such as disjoint time slots or frequency bands. Let $p_i^{(n)}$ denote the transmission power of multicast stream $i \in \mathcal{G} := \{1, \ldots, G\}$ in dimension $n \in \mathcal{N} := \{1, \ldots, N\}$. The allowed or available transmission power is upper bounded by $P_i^{(n)}$, due to regulatory or power amplifier limitations. For every receiver $k \in \mathcal{G}_i$, let $\alpha_{i,k}^{(n)}$ and $\alpha_{j,k}^{(n)}$ denote the corresponding power gain of the direct and coupling links, respectively, in the nth dimension. The link gains include the effects of propagation loss, shadowing, and fading, as well as beamforming and coding gains if any. They are considered known to the design center.

Due to the broadcast nature of the wireless medium and the cochannel transmissions, there is interference which must be accounted for. The quality of communication that receiver $k \in \mathcal{G}_i$ experiences, when tuned in the nth dimension, is measured by the received SINR

$$\mathrm{SINR}_k^{(n)} = \frac{\alpha_{i,k}^{(n)} p_i^{(n)}}{\sum\limits_{j \neq i} \alpha_{j,k}^{(n)} p_j^{(n)} + \sum\limits_{m \neq n} \sum\limits_{i=1}^{G} \alpha_{i,k}^{(m)} p_i^{(m)} + \sigma_k^2} \tag{3.1}$$

where σ_k^2 denotes receiver's noise variance. Interference consists of two terms: the first accounts for the interfering multicasts received in the same dimension, and the second for all the transmissions from the other dimensions. Note that even the stream of interest is included in the second term, due to incoherent combining; for orthogonal scenarios, the inter-dimension (i.e., $\forall m \neq n$) coupling gains $\alpha_{i,k}^{(m)}$ are 0 and the second term vanishes. The multicast stream can be decoded successfully by the kth receiver when the received SINR is equal to or greater than a threshold γ_k, determined by application requirements, and the modulation and coding scheme. A receiver k is considered *assigned* in the nth dimension when $\text{SINR}_k^{(n)} \geq \gamma_k$, and *served* or *admitted* when its QoS requirement can be guaranteed, i.e., when it is assigned in any dimension.

3.2 Joint Problem Formulation

The QoS problem of interest is to find the optimum i) schedule, i.e., assignment of receivers in dimensions, (including a "void" assignment that accounts for admission control); and ii) transmission powers, in order to maximize the number of admitted receivers and minimize the total transmission power required to serve them. Let us introduce the auxiliary binary variables $\{s_k^{(n)} \in \{0,1\}\}_{k\in\mathcal{K}}^{n\in\mathcal{N}}$, one per degree of freedom and receiver. The role of the binary variable $s_k^{(n)}$ is to determine whether the kth receiver can be assigned in the nth dimension. Using these auxiliary variables, the joint QoS scheduling, admission, and multicast power control problem can be written as the constrained optimization

$$\min_{\{p_i^{(n)}\}_{i\in\mathcal{G}}^{n\in\mathcal{N}},\{s_k^{(n)}\}_{k\in\mathcal{K}}^{n\in\mathcal{N}}} \quad \epsilon\sum_{i=1}^{G}\sum_{n=1}^{N} p_i^{(n)} + (1-\epsilon)\prod_{k=1}^{K}\prod_{n=1}^{N}\frac{1}{1+s_k^{(n)}} \tag{3.2}$$

$$\begin{aligned}
\text{s. t. :} \quad & 0 \leq p_i^{(n)} \leq P_i^{(n)} && \forall i\in\mathcal{G} \;\; \forall n\in\mathcal{N}, && (3.3a)\\
& \text{SINR}_k^{(n)} \geq \gamma_k s_k^{(n)} && \forall k\in\mathcal{G}_i \;\; \forall i\in\mathcal{G} \;\; \forall n\in\mathcal{N}, && (3.3b)\\
& s_k^{(n)} \in \{0,1\} && \forall k\in\mathcal{K} \;\; \forall n\in\mathcal{N}, && (3.3c)\\
& \sum_{n=1}^{N} s_k^{(n)} \leq 1 && \forall k\in\mathcal{K}, && (3.3d)\\
& && && (3.3e)
\end{aligned}$$

where the left-hand term of (3.3b) is defined in (3.1).

As per (3.3b), N SINR constraints (one per available dimension) are *a priori* defined for every receiver. Focusing on the (k, n)th inequality of (3.3b), it is seen that the auxiliary variable $s_k^{(n)}$ multiplies the SINR threshold γ_k. The value of the binary variable determines whether the respective SINR constraint is accounted (or not) in the power control part of the joint problem; thus, $s_k^{(n)}$ can be interpreted as a switch that activates (or not) the inequality. Specifically, for $s_k^{(n)} = 1$, the constraint is enforced on the transmission powers, so that $\text{SINR}_k^{(n)}$ is requested to be at least equal to γ_k. On the contrary, for $s_k^{(n)} = 0$, the respective inequality (3.3b) is reduced to the trivial nonnegativity constraint for the transmission power $p_i^{(n)}$, which is already included in the left part of (3.3a); hence, no further constraint is actually imposed. Note that when no receiver of the ith multicast group is assigned in the nth dimension (i.e., $s_k^{(n)} = 0 \;\; \forall k \in \mathcal{G}_i$), then, the optimization problem (3.2)–(3.3) zeroes the respective transmission power (i.e., $p_i^{(n)} = 0$). This is because $p_i^{(n)}$ counts solely as interference in this case, since it appears only in the denominator of "active" SINR constraints (3.3b) and the left term of the objective function (3.2) minimizes the total transmission power. Hence, there is no need to explicitly account for the non-transmitted multicast streams in the denominator of $\text{SINR}_k^{(n)}$ (cf. (3.1)) in (3.3b).

The inequalities in (3.3d) constrain each receiver to be assigned in at most one dimension. Since QoS is already guaranteed to a receiver when just one out of N respective constraints (3.3b) is "active", multiple assignments solely increase the interference in the wireless network. In case there are extra degrees of freedom (i.e., other feasible assignments), the system would rather utilize them to serve more receivers or decrease the total transmission power. Note that by letting the sum in (3.3d) take values smaller than 1 (actually 0), admission control is included in the joint optimization problem (3.2)–(3.3). When there are not enough resources, the system is allowed to deny service to any receiver, say k, (setting $s_k^{(n)} = 0 \;\; \forall n \in \mathcal{N}$) in order to ensure service to the remaining ones. If the inequalities (3.3d) were replaced with equalities, the resulting joint problem would be a restriction of (3.2)–(3.3), since the set of feasible solutions would then be a subset of the original one. The restricted problem becomes infeasible when it is not possible to admit all receivers. On the contrary, inequalities (3.3d) imply the following result.

Claim 10. *Optimization problem* (3.2)–(3.3) *is always feasible.*

Proof: A trivial feasible solution, i.e., satisfying the constraints (3.3), is $\{s_k^{(n)} = 0\}_{k\in\mathcal{K}}^{n\in\mathcal{N}}$ and any $\{p_i^{(n)}\}_{i\in\mathcal{G}}^{n\in\mathcal{N}}$ conforming to (3.3a). ■

However, this trivial solution is the worst possible from a QoS perspective, since none of the K receivers is served. As noted before, the primal design goal of the joint QoS problem of interest is to maximize the number of receivers served, which, by virtue of (3.3d), equals to the value of the auxiliary variables' total sum. For this reason, apart from the total power minimization, a second term has been included in the objective function (3.2), which is pertinent to the scheduling / admission part of the joint problem. This term is a function that favors the value 1 for the binary variables, by increasing the penalty when they take the value 0 instead. Specifically, focusing on any receiver, say k, the objective function term of the scheduling / admission problem is $\prod_{n=1}^{N} 1/(1+s_k^{(n)})$. The associated cost for assigning the receiver in any dimension, say n, is 2^{-1} ($s_k^{(n)} = 1$ and $s_k^{(m)} = 0 \;\; \forall m \neq n$) and 1 for not admitting it ($s_k^{(n)} = 0 \;\; \forall n \in \mathcal{N}$). From (3.2), the total assignment cost is the product of the individual ones; hence, it is equal to $2^{-\hat{K}}$, where $\hat{K} \in \{0, \mathcal{K}\}$ is the number of receivers served. When all receivers are served, the assignment cost is equal to 2^{-K} and doubles for every receiver that is not admitted. Observe that the assignment cost is discrete-valued with minimum step size of 2^{-K} that doubles for every step.

The overall cost of the joint QoS problem (3.2)–(3.3) is the weighted sum of the power and the assignment cost. The power is continuous and upper bounded by

$$\sum_{i=1}^{G}\sum_{n=1}^{N} p_i^{(n)} \overset{(3.3a)}{\leq} \sum_{i=1}^{G}\sum_{n=1}^{N} P_i^{(n)} \leq GNP, \tag{3.4}$$

where $P \geq \max_{i\in\mathcal{G}, n\in\mathcal{N}} P_i^{(n)}$. The weighting factor $\epsilon \in [0,1]$ is chosen so that service is denied to some receivers only if this event cannot be avoided. Adapting the ruler analogy argument in [30] to a logarithmic scale, ϵ is chosen so that the minimum penalty paid for denying service to a receiver is higher than the maximum potential gain from the total power minimization, i.e.,

$$(1-\epsilon)2^{-K} > \epsilon GNP \Leftrightarrow \epsilon < \frac{2^{-K}}{GNP + 2^{-K}}. \tag{3.5}$$

Claim 11. *For* $\epsilon < \frac{2^{-K}}{GNP+2^{-K}}$*, optimization problem* (3.2)–(3.3) *maximizes the number of admitted receivers and minimizes the total transmission power required to serve them.*

Proof: Let $\left\{\{\breve{p}_i^{(n)}\}_{i\in\mathcal{G}}^{n\in\mathcal{N}}, \{\breve{s}_k^{(n)}\}_{k\in\mathcal{K}}^{n\in\mathcal{N}}\right\}$ denote an optimal solution of problem (3.2)–(3.3). Assume the existence of another feasible solution $\left\{\{\tilde{p}_i^{(n)}\}_{i\in\mathcal{G}}^{n\in\mathcal{N}}, \{\tilde{s}_k^{(n)}\}_{k\in\mathcal{K}}^{n\in\mathcal{N}}\right\}$ which serves 1 receiver more than the optimal solution, i.e.,

$$\sum_{k=1}^{K}\sum_{n=1}^{N}\tilde{s}_k^{(n)} = \sum_{k=1}^{K}\sum_{n=1}^{N}\breve{s}_k^{(n)} + 1. \tag{3.6}$$

Then, it holds that

$$\prod_{k=1}^{K}\prod_{n=1}^{N}\frac{1}{1+\tilde{s}_k^{(n)}} = \frac{1}{2}\prod_{k=1}^{K}\prod_{n=1}^{N}\frac{1}{1+\breve{s}_k^{(n)}} \tag{3.7}$$

and

$$\prod_{k=1}^{K}\prod_{n=1}^{N}\frac{1}{1+\tilde{s}_k^{(n)}} \geq 2^{-K} \overset{(3.7)}{\Rightarrow} \prod_{k=1}^{K}\prod_{n=1}^{N}\frac{1}{1+\breve{s}_k^{(n)}} \geq 2^{-K+1}. \tag{3.8}$$

The cost of the feasible solution is

$$\begin{aligned}
&\epsilon\sum_{i=1}^{G}\sum_{n=1}^{N}\tilde{p}_i^{(n)} + (1-\epsilon)\prod_{k=1}^{K}\prod_{n=1}^{N}\frac{1}{1+\tilde{s}_k^{(n)}} \overset{(3.4),\ (3.7)}{\leq} \\
&\epsilon GNP + (1-\epsilon)\frac{1}{2}\prod_{k=1}^{K}\prod_{n=1}^{N}\frac{1}{1+\breve{s}_k^{(n)}} = \\
&\epsilon GNP + (1-\epsilon)\prod_{k=1}^{K}\prod_{n=1}^{N}\frac{1}{1+\breve{s}_k^{(n)}} - (1-\epsilon)\frac{1}{2}\prod_{k=1}^{K}\prod_{n=1}^{N}\frac{1}{1+\breve{s}_k^{(n)}} \overset{(3.8)}{\leq} \\
&(1-\epsilon)\prod_{k=1}^{K}\prod_{n=1}^{N}\frac{1}{1+\breve{s}_k^{(n)}} + \underbrace{\epsilon GNP - (1-\epsilon)2^{-1}2^{-K+1}}_{<0\ (3.5)} < \\
&(1-\epsilon)\prod_{k=1}^{K}\prod_{n=1}^{N}\frac{1}{1+\breve{s}_k^{(n)}} \leq \\
&\underbrace{\epsilon\sum_{i=1}^{G}\sum_{n=1}^{N}\breve{p}_i^{(n)}}_{>0\ (3.3a)} + (1-\epsilon)\prod_{k=1}^{K}\prod_{n=1}^{N}\frac{1}{1+\breve{s}_k^{(n)}}.
\end{aligned}$$

Hence, it is smaller than the minimum objective value, which contradicts optimality of $\left\{\{\breve{p}_i^{(n)}\}_{i\in\mathcal{G}}^{n\in\mathcal{N}}, \{\breve{s}_k^{(n)}\}_{k\in\mathcal{K}}^{n\in\mathcal{N}}\right\}$ for problem (3.2)–(3.3). Thus, solution of (3.2)–(3.3) serves the maximum number of receivers possible under the constraints in (3.3). Given the optimum schedule $\{\breve{s}_k^{(n)}\}_{k\in\mathcal{K}}^{n\in\mathcal{N}}$, the joint QoS problem (3.2)–(3.3) reduces to a *feasible* multicast power control problem, with respect to the variables $\{p_i^{(n)}\}_{i\in\mathcal{G}}^{n\in\mathcal{N}}$, which minimizes the total transmission power, while ensuring the QoS targets of the admitted receivers. ■

The joint QoS problem (3.2)–(3.3) is nonconvex, since the auxiliary variables are binary. The computationally intensive components of the problem are the scheduling and admission control. Due to its combinatorial nature, the optimization with respect to the variables $\{s_k^{(n)}\}_{k\in\mathcal{K}}^{n\in\mathcal{N}}$ has exponential complexity; it can be proven that problem (3.2)–(3.3) is NP-hard. Given an assignment, the joint problem (3.2)–(3.3) reduces to the multicast power control problem, which is an LP optimization, with respect to the variables $\{p_i^{(n)}\}_{i\in\mathcal{G}}^{n\in\mathcal{N}}$, see, e.g. [26, 42].

3.3 Relaxation to Geometric Programming

The binary constraints (3.3c) on the auxiliary variables are nonconvex. They can be equivalently written as

$$s_k^{(n)}(s_k^{(n)} - 1) = 0 \qquad \forall k \in \mathcal{K} \;\; \forall n \in \mathcal{N} \quad \text{or} \tag{3.9}$$

$$\begin{cases} s_k^{(n)}(s_k^{(n)} - 1) \leq 0 \\ s_k^{(n)}(s_k^{(n)} - 1) \geq 0 \end{cases} \qquad \forall k \in \mathcal{K} \;\; \forall n \in \mathcal{N}. \tag{3.10}$$

The quadratic inequalities on the upper branch of (3.10) are convex, whereas the others on the lower are not. Discarding the nonconvex ones, the auxiliary variables $\{s_k^{(n)}\}_{k\in\mathcal{K}}^{n\in\mathcal{N}}$ are merely allowed to take any value in the interval $[0, 1]$, which is obviously convex. An immediate consequence of this relaxation is that the auxiliary variables lose their strict functionality of taking "yes" or "no" decisions for the scheduling and admission control problems. Now, they merely measure the ratio of SINR target satisfaction in (3.3b). Hence, the motivation for introducing constraint (3.3d) does not hold anymore and it can be discarded as well.

The relaxation of the feasible set for the auxiliary variables has the side effect that the right-hand term of the objective function (3.2) does not take discrete values anymore. Furthermore, due to the absence of (3.3d), it now tries to push *all* auxiliary variables as close as possible to 1, i.e., it tries to assign every receiver to all dimensions. Note that the denominator of the right-hand term of the objective function consists of all possible cross-products of the auxiliary variables. Since all variables are positive and smaller than or equal to 1, the smallest term in the denominator, dominating the cost, is the product of the highest order, i.e., the one involving all variables. Thus, the denominator may be well approximated by this term. The resulting objective function term has simpler form: it is a monomial. It upper bounds the original objective function term and has essentially the same functionality, i.e., it penalizes severely the values of the auxiliary variables close to 0.

After the aforementioned relaxations, the joint QoS problem (3.2)–(3.3) reads

$$\min_{\{p_i^{(n)}\}_{i\in\mathcal{G}}^{n\in\mathcal{N}},\{s_k^{(n)}\}_{k\in\mathcal{K}}^{n\in\mathcal{N}}} \quad \epsilon\sum_{i=1}^{G}\sum_{n=1}^{N} p_i^{(n)} + (1-\epsilon)\prod_{k=1}^{K}\prod_{n=1}^{N}\frac{1}{s_k^{(n)}} \tag{3.11}$$

$$\text{s. t.} \quad 0 \le p_i^{(n)} \le P_i^{(n)} \qquad \forall i \in \mathcal{G} \;\; \forall n \in \mathcal{N}, \tag{3.12a}$$

$$\sum_{j\neq i}\alpha_{j,k}^{(n)} p_j^{(n)} + \sum_{m\neq n}\sum_{i=1}^{G}\alpha_{i,k}^{(m)} p_i^{(m)} + \sigma_k^2 \le \alpha_{i,k}^{(n)} p_i^{(n)} (\gamma_k s_k^{(n)})^{-1} \qquad \forall k \in \mathcal{G}_i \;\; \forall i \in \mathcal{G} \;\; \forall n \in \mathcal{N}, \tag{3.12b}$$

$$0 \le s_k^{(n)} \le 1 \qquad \forall k \in \mathcal{K} \;\; \forall n \in \mathcal{N}. \tag{3.12c}$$

The relaxed problem (3.11)–(3.12) is a posynomial minimization subject to posynomial upper bound inequality constraints. Hence, it is a GP problem in *standard form* [5,7]. It can be readily transformed into *convex form*, by a logarithmic change of variables, and efficiently solved by means of interior point methods. In fact, there exist freely available MATLAB toolboxes, e.g., GGPLAB [31] and CVX [21], that accept as input GP problems in standard form.

3.4 Applications

The introduction, in Section 3.2, of the auxiliary variables $\{s_k^{(n)}\}_{k\in\mathcal{K}}^{n\in\mathcal{N}}$ has enabled us to approach a general class of cross-layer QoS problems *jointly* with the single-step constrained optimization (3.2)–(3.3), which is intractable, due to its combinatorial nature. The relaxation, in Section 3.3, of the binary constraints (3.3c) to the intervals (3.12c) has lead, on one hand, to the convex approximation (3.11)–(3.12) of the problem of interest. On the other hand, convexity is obtained at a considerable cost; the strict functionality of the auxiliary variables has been sacrificed in favor of the relaxation. However, the GP problem (3.11)–(3.12) is still useful; it can used as the core component of heuristic algorithms to find approximate solutions to the original joint QoS problem. In the following, two iterative algorithms are presented for respective problems, which take scheduling and admission decisions by pruning degrees of freedom in each iteration.

3.4.1 Access Point Assignment, Admission & Multicast Power Control

Consider a multicast scenario, where G streams are provided by N sources on the same channel. The sources can be either GN single-antenna access points (AP's) (each transmitting a single stream) or N multi-antenna AP's (each spatially multiplexing the G multicasts by means of beamforming). Let $\mathcal{G}_{i,n} \subseteq \mathcal{G}_i$ denote the set of receivers, interested in the ith multicast, which are *tentatively* tuned in the nth dimension. Using these sets, the GP problem (3.11)–(3.12) is equivalently rewritten as

$$\min_{\left\{\{p_i^{(n)}\},\{s_k^{(n)}\}_{k\in\mathcal{G}_{i,n}}\right\}_{i\in\mathcal{G}}^{n\in\mathcal{N}}} \quad \epsilon\sum_{i=1}^{G}\sum_{n=1}^{N} p_i^{(n)} + (1-\epsilon)\prod_{n=1}^{N}\prod_{i=1}^{G}\prod_{k\in\mathcal{G}_{i,n}} \frac{1}{s_k^{(n)}} \tag{3.13}$$

$$\text{s. t.} \quad 0 \leq p_i^{(n)} \leq P_i^{(n)} \qquad \forall i\in\mathcal{G} \;\; \forall n\in\mathcal{N}, \tag{3.14a}$$

$$\sum_{j\neq i}\alpha_{j,k}^{(n)}p_j^{(n)} + \sum_{m\neq n}\sum_{i=1}^{G}\alpha_{i,k}^{(m)}p_i^{(m)} + \sigma_k^2 \leq \alpha_{i,k}^{(n)}p_i^{(n)}(\gamma_k s_k^{(n)})^{-1} \qquad \forall k\in\mathcal{G}_{i,n} \;\; \forall i\in\mathcal{G} \;\; \forall n\in\mathcal{N}, \tag{3.14b}$$

$$0 \leq s_k^{(n)} \leq 1 \qquad \forall k\in\mathcal{G}_{i,n} \;\; \forall i\in\mathcal{G} \;\; \forall n\in\mathcal{N}. \tag{3.14c}$$

As noted in Section 3.2 for problem (3.2)–(3.3), when no receiver of the ith multicast group is assigned in the nth dimension, i.e., $\mathcal{G}_{i,n} = \emptyset$, the optimization (3.13)–(3.14) yields $p_i^{(n)} = 0$. Moreover, it is important to take into consideration a peculiarity of the QoS multicast power control problem. The power of each multicast stream is controlled by one variable, but many SINR constraints (one for each receiver of the respective multicast group) need to be fulfilled at once. Typically, for a feasible instance of the problem, optimization yields SINR values which are greater than the targeted thresholds, for all but one receivers per multicast group.

The following algorithm initially assumes that all receivers can be tuned to all AP's that transmit the multicast stream of interest. Then, it iteratively rejects possible AP assignments in order to decrease the interference level, until (if possible) a single assignment is preserved per receiver that satisfies the requested QoS target.

Algorithm 1

1. Initialization: $\mathcal{G}_{i,n} = \mathcal{G}_i \quad \forall n \in \mathcal{N} \;\; \forall i \in \mathcal{G}$.

2. Solve the GP problem (3.13)–(3.14), denote the solution $\left\{\{\breve{p}_i^{(n)}\}, \{\breve{s}_k^{(n)}\}_{k\in\mathcal{G}_{i,n}}\right\}_{i\in\mathcal{G}}^{n\in\mathcal{N}}$, and calculate the $\mathrm{SINR}_k^{(n)} \quad \forall n \in \mathcal{N} \;\; \forall k \in \mathcal{K}$.

3. **If** there exist receivers with multiple assignments, whose SINR target is reached by more than one AP,
   then find among them the receiver-AP pair $\{\tilde{k}, \tilde{n}\}$ with the largest SINR over-satisfaction, preserve assignment of receiver $\tilde{k}$ only to AP $\tilde{n}$ (when $\tilde{k} \in \mathcal{G}_{\tilde{i}}$ then $\mathcal{G}_{i,n} = \mathcal{G}_{i,n} - \tilde{k} \quad \forall n \neq \tilde{n} \;\; \forall i \neq \tilde{i}$), and go to 2,
   else go to 4.

4. **If** there exist receivers with multiple assignments, whose SINR target is reached by a single AP,
   then find among them the pair $\{\tilde{k}, \tilde{n}\}$ with the largest SINR, preserve assignment of receiver $\tilde{k}$ only to AP $\tilde{n}$, and go to 2,
   else go to 5.

5. **If** there exist receivers with multiple assignments, whose SINR target is not reached by any AP,
   then find among them the pair $\{\tilde{k}, \tilde{n}\}$ with the largest SINR, preserve assignment of receiver $\tilde{k}$ only to AP $\tilde{n}$, and go to 2,
   else go to 6.

6. **If** there exist receiver(s) whose SINR target is not reached,
   then find among them the pair $\{\tilde{k}, \tilde{n}\}$ with the smallest SINR, drop assignment of receiver $\tilde{k}$ to AP $\tilde{n}$ (when $\tilde{k} \in \mathcal{G}_{\tilde{i}}$ then $\mathcal{G}_{\tilde{i},\tilde{n}} = \mathcal{G}_{\tilde{i},\tilde{n}} - \tilde{k}$), and go to 2,
   else the sets $\{\mathcal{G}_{i,n}\}_{i\in\mathcal{G}}^{n\in\mathcal{N}}$ contain a feasible assignment $\{\{\breve{s}_k^{(n)} = 1\}_{k\in\mathcal{G}_{i,n}}\}_{i\in\mathcal{G}}^{n\in\mathcal{N}}$ and the solution $\{\breve{p}_i^{(n)}\}_{i\in\mathcal{G}}^{n\in\mathcal{N}}$ determines the minimum power levels required for service.

Note that in each iteration, the number of variables and constraints (hence, the complexity) of the core GP problem decreases. The overall worst-case complexity of the algorithm is $O((KN)^{4.5})$.

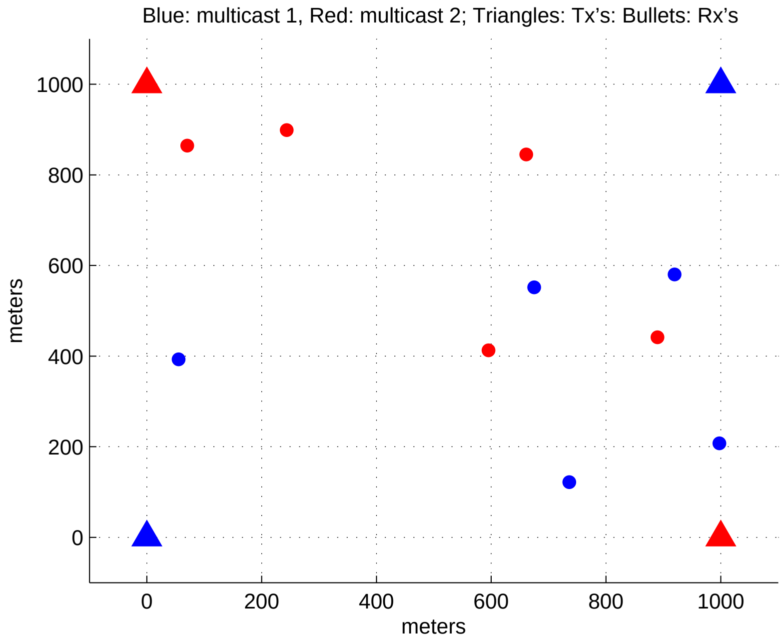

Figure 3.1: Wireless network instance with fixed transmitters' locations

The performance of the algorithm is evaluated on a 1km×1km network topology, an instance of which is depicted in Fig. 3.1. 2 multicast streams are transmitted, each by 2 dedicated transmitters located on the anti-diagonal vertices of the square. There are K receivers requesting service, randomly distributed within the network, forming 2 equally-sized multicast groups. Receivers are assumed stationary and the direct link gains account only the propagation loss with an attenuation exponent of 4, whereas the coupling gains are further reduced by a factor equal to 8. This setup can refer to a CDMA network of 4 different single-antenna AP's, each transmitting a single stream, where the coupling reduction is due to spreading. Alternatively, the antenna pairs on the left and right (or the upper and lower) side of the square may be viewed as separate AP's, each spatially multiplexing the 2 multicasts. Then, the difference in the values of the direct and coupling link gains of each receiver-AP pair is due to beamforming. All receivers ask for a SINR threshold of 10 dB and have a noise power level of 10^{-13} W. The transmission power per stream is upper bounded by 10 W.

Simulations were performed for $K = \{4, 6, 8, 10\}$ and 50 network instances. The network size has been kept small, to enable comparison with a brute-force approach, i.e., enumeration over all possible AP assignments, including the "void" one for admission control, and solution of QoS multicast power control. The toolboxes GGPLAB [31] and CVX [21] were used to solve GP and LP optimizations, respectively. The average number of receivers admitted, versus the number of receivers requesting service, is plotted in Fig. 3.2. Focusing on the case of $K = 10$, Fig. 3.3 depicts the number of admitted receivers versus the network instance. It is observed that on 60% of the occasions the algorithm serves the maximum possible number of receivers, and on nearly all others it serves just 1 receiver less. The total transmission power, averaged over the 30 network instances that the admission control problem is solved optimally, is reported in Fig. 3.4. For $K = 10$, the total power for these network instances is shown in Fig. 3.5. It can be seen that in 70% of those, the algorithm solves optimally the joint QoS problem.

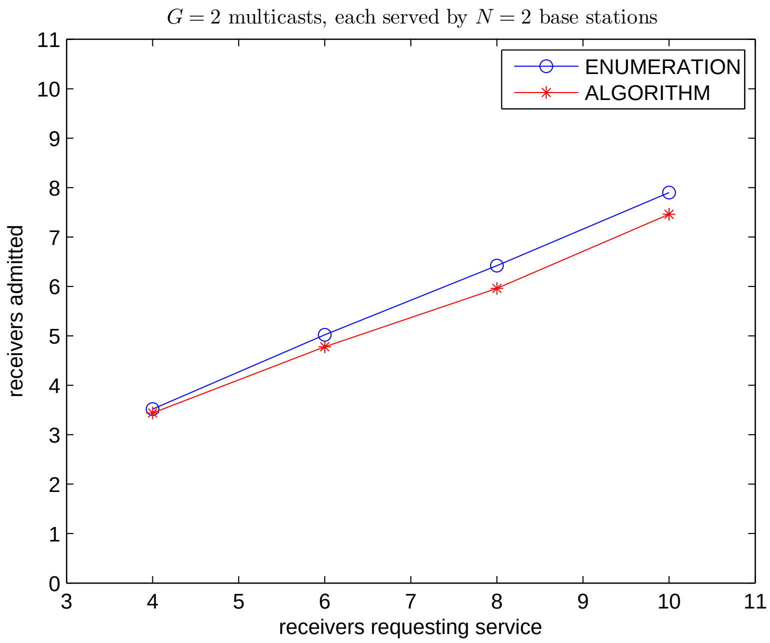

Figure 3.2: Receivers admitted on average

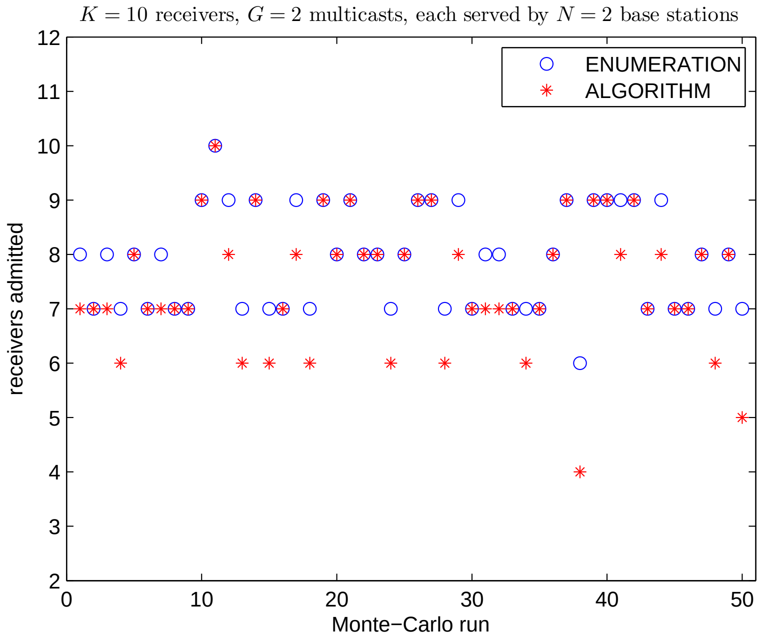

Figure 3.3: Receivers admitted

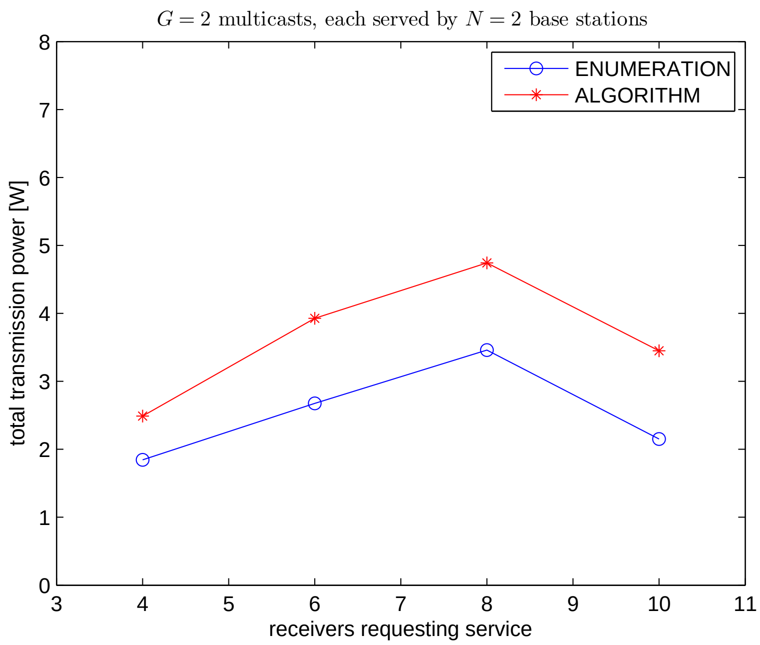

Figure 3.4: Total transmission power on average

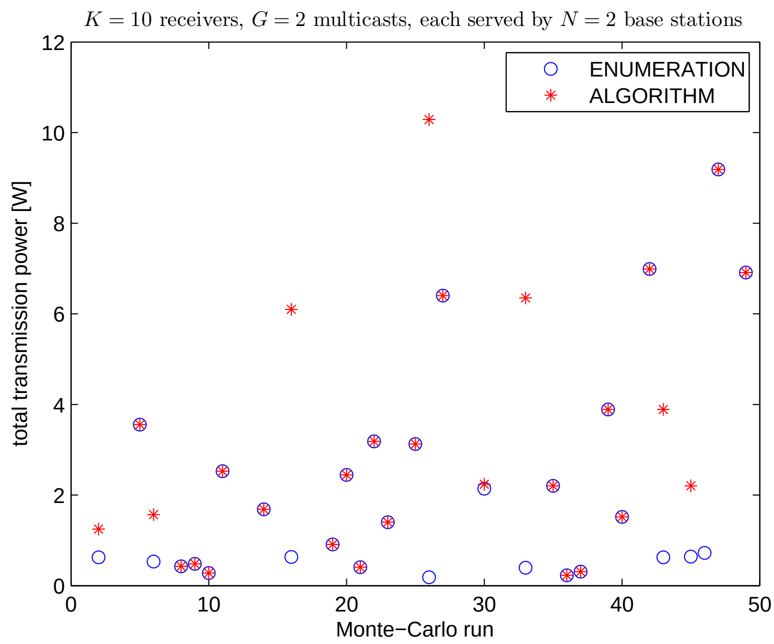

Figure 3.5: Total transmission power

3.4.2 Scheduling, Admission and Power Control

Consider a unicast scenario, where K active links of a wireless network carry different data streams. The transmissions can take place in any of N available non-overlapping time slots. Let $\mathcal{K}_n \subseteq \mathcal{K}$ denote the set of links *tentatively* scheduled in time slot $n \in \mathcal{N}$. Under this setup, the GP problem (3.11)–(3.12) is rewritten as

$$\min_{\{p_k^{(n)},\, s_k^{(n)}\}_{k\in\mathcal{K}_n}^{n\in\mathcal{N}}} \quad \epsilon \sum_{n=1}^{N} \sum_{k\in\mathcal{K}_n} p_k^{(n)} + (1-\epsilon) \prod_{n=1}^{N} \prod_{k\in\mathcal{K}_n} \frac{1}{s_k^{(n)}} \tag{3.15}$$

$$\text{s. t.} \quad 0 \le p_k^{(n)} \le P \qquad \forall n \in \mathcal{N} \;\; \forall k \in \mathcal{K}_n, \tag{3.16a}$$

$$\sum_{\ell\neq k} \gamma_k \alpha_{\ell,k}^{(n)} p_\ell^{(n)} s_k^{(n)} + \gamma_k \sigma_k^2 s_k^{(n)} \le \alpha_{k,k}^{(n)} p_k^{(n)} \qquad \forall n \in \mathcal{N} \;\; \forall k \in \mathcal{K}_n, \tag{3.16b}$$

$$0 \le s_k^{(n)} \le 1 \qquad \forall n \in \mathcal{N} \;\; \forall k \in \mathcal{K}_n, \tag{3.16c}$$

where the middle term of left-hand side in (3.12b) representing the inter-dimension interference disappears in (3.16b). The following algorithm initially assigns all links in all slots and then iteratively balances the interference level, preserving at most 1 slot assignment per link.

Algorithm 2

1. Initialization: $\mathcal{K}_n = \mathcal{K} \quad \forall n \in \mathcal{N}$.

2. Solve the GP problem (3.15)–(3.16) and denote the solution $\{\check{p}_k^{(n)}, \check{s}_k^{(n)}\}_{k\in\mathcal{K}_n}^{n\in\mathcal{N}}$.

3. **If** there exist links whose SINR target is reached in more than 1 slot,
   then find among them the link-slot pair $\{\tilde{k}, \tilde{n}\}$ requiring the minimum transmission power for service, schedule link $\tilde{k}$ only in slot $\tilde{n}$ ($\mathcal{K}_n = \mathcal{K}_n - \tilde{k} \quad \forall n \neq \tilde{n}$), and go to 2,
   else go to 4.

4. **If** there exist links with multiple assignments whose SINR target is reached in a single slot,
   then find among them the pair $\{\tilde{k}, \tilde{n}\}$ requiring the minimum transmission power for service, schedule link $\tilde{k}$ only in slot $\tilde{n}$, and go to 2,
   else go to 5.

5. **If** there exist links with multiple assignments, whose SINR target is not reached in any slot,
   then find among them the pair $\{\tilde{k}, \tilde{n}\}$ yielding the minimum SINR, remove link $\tilde{k}$ from slot $\tilde{n}$ ($\mathcal{K}_{\tilde{n}} = \mathcal{K}_{\tilde{n}} - \tilde{k}$), and go to 2,
   else go to 6.

6. **If** there exist links whose SINR target is not reached,
   then find among them the pair $\{\tilde{k}, \tilde{n}\}$ yielding the minimum SINR, remove link $\tilde{k}$ from slot $\tilde{n}$, and go back to step 2,
   else the sets $\{\mathcal{K}_n\}_{n\in\mathcal{N}}$ contain a feasible schedule $\{\check{s}_k^{(n)} = 1\}_{k\in\mathcal{K}_n}^{n\in\mathcal{N}}$ and the solution $\{\check{p}_k^{(n)}\}_{k\in\mathcal{K}_n}^{n\in\mathcal{N}}$ determines the minimum power levels required for service.

Steps 3–5 and 6 are pertinent to the scheduling and admission part, respectively, of the joint QoS problem. Note that in each iteration, the complexity of the GP problem (3.15)–(3.16) decreases, because the number of variables involved decreases (from $2KN$ to $2\hat{K}$).

The performance of the Algorithm is evaluated on a $100m \times 100m$ network topology. The transmitting and receiving nodes are randomly located within this square. A sample topology instance with $K = 12$ links is shown in Fig. 3.6. The nodes are assumed stationary for the duration of the $N = 2$ time slots and the direct link gains account only the propagation loss with an attenuation exponent of 3 ($\alpha_{k,k}^{(n)} = d_{k,k}^{-3} \quad \forall n \in \mathcal{N}$, where $d_{k,k}$ is the distance among the transmitter and the receiver of the kth link). Spreading with factor 64 is used to decrease the interference level and the corresponding spreading process gain is included in the coupling link gains along with the propagation loss ($\alpha_{\ell,k}^{(n)} = d_{\ell,k}^{-3}/64 \quad \forall n \in \mathcal{N}$). The noise power is the same for all receivers ($\sigma_k^2 = \sigma^2 \quad \forall k \in \mathcal{K}$), and the transmission powers are normalized with it and upper bounded by $P/\sigma^2 = 10^8$. A common SINR threshold of 10 dB is requested by all links ($\gamma_k = 10, \ \forall k \in \mathcal{K}$). Fig. 3.7 reports, as a histogram, the number of links served when Algorithm 1 is executed for 100 different network instances. As a baseline, we use Algorithm B, developed in [27] for a somewhat more general context, but represents the state-of-art in our present context. Admission control is not included in [27]; iterations stop and the problem is declared infeasible when a link cannot be scheduled. We use a slightly modified version here, which continues the iterations until the maximum possible number of links is scheduled.

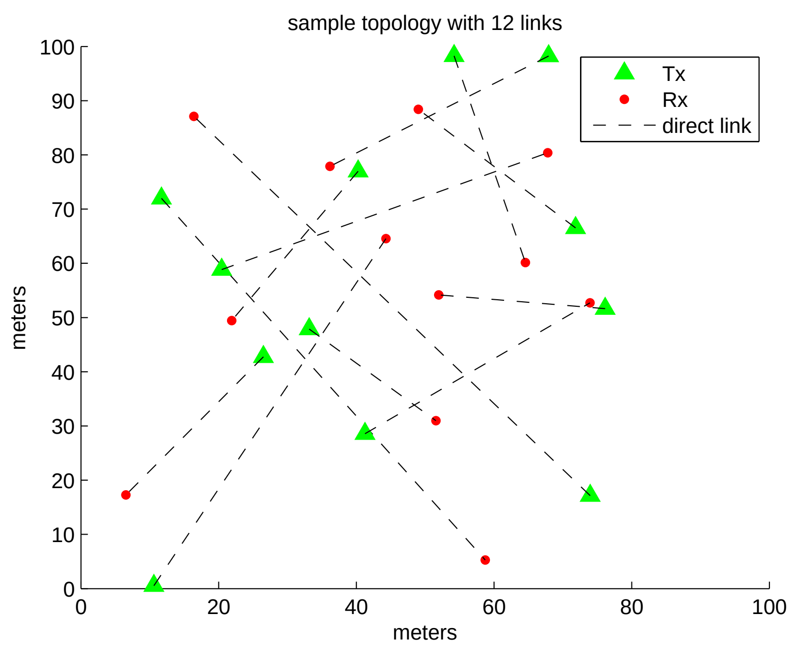

Figure 3.6: Wireless network instance

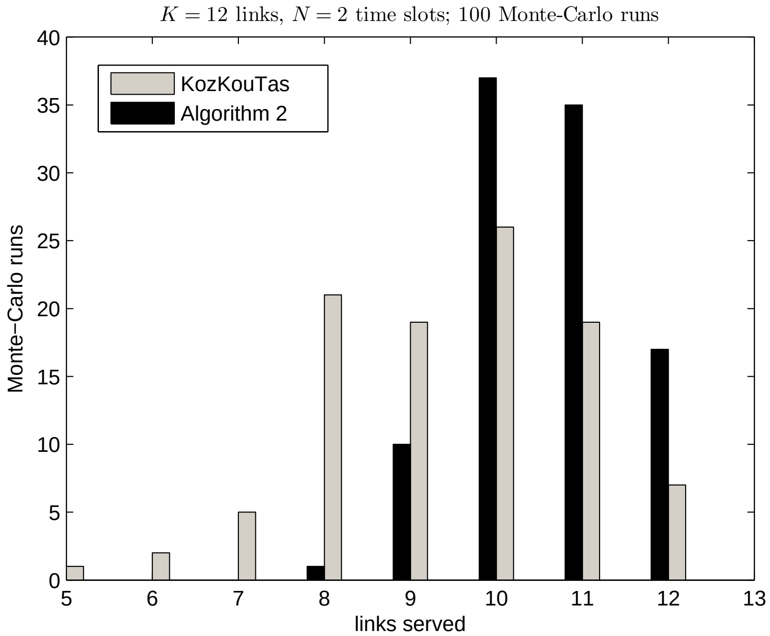

Figure 3.7: Links served

3.5 Conclusions

A disciplined convex approximation approach was developed for a wide class of cross-layer network optimization problems. The formulation is general enough to account for interfering cochannel transmissions, unicast, broadcast, or multicast modalities, admission control, and base station selection. The nonconvex and NP-hard joint optimization problem is approximated by a suitable geometric program, which is relatively easy to solve. This approximation forms the core of an iterative algorithm that generates approximate solutions to the original problem. While heuristic, this algorithm has a solid footing, and this shows in numerical experiments.

Chapter 4

Maximum Throughput Power Control

The throughput maximization problem has been thoroughly studied in the context of wired networks (cf. the celebrated max-flow min-cut theorem [12]). One is given a directed multi-hop network between a source and a destination, with fixed edge capacities. Each node may split its aggregate incoming flow to multiple outgoing flows, and the objective is to select flows to maximize the end-to-end flow from source to destination. This is an LP problem, which can be efficiently solved using a variety of centralized and distributed algorithms, including custom solutions using the max-flow min-cut theorem and related insights (see, e.g., [3]). The situation is very different in wireless networks, due to fading and mutual interference. Edge capacities are randomly time-varying and depend on interfering node transmission modalities (e.g., transmission powers) as well. In the following, we consider the case when time-variation due to fading is slow. Edge capacities are then a function of transmission powers, and power control can be used to obtain a favorable 'topology' from the viewpoint of maximizing flow. This suggests a joint max-flow power control problem that is basic, yet has not been considered in the cross-layer network optimization literature. Alternatively, power control may be coupled with dynamic routing by means of differential queue lengths information. Both approaches are sketched and convex approximations, in the high SINR regime, are provided for these cross-layer power control problems.

4.1 System Model

Consider a wireless ad-hoc network consisting of N nodes, located at fixed positions. The topology of the network is represented by a directed graph $(\mathcal{N}, \mathcal{L})$, where $\mathcal{N} := \{1, \ldots, N\}$ and $\mathcal{L} := \{1, \ldots, L\}$ denote the set of nodes and links, respectively. Each link $\ell \in \mathcal{L}$ corresponds to an ordered pair of nodes (i, j), where $i, j \in \mathcal{N}$ and $i \neq j$. Let $\mathrm{Tx}(\ell)$ and $\mathrm{Rx}(\ell)$ denote the transmitter and the receiver of link ℓ, i.e., when $\ell = (i, j)$, then $\mathrm{Tx}(\ell) = i$ and $\mathrm{Rx}(\ell) = j$. Let p_ℓ denote the power transmitted on link ℓ and $G_{\ell k}$ the path loss among the transmitter and the receiver of link ℓ and k, respectively. Every node except the destination can potentially transmit to any other node except the source, including transmission to more than one nodes simultaneously. All transmissions are concurrent and cochannel; hence, communication is interference-limited. The nodes are assumed capable of receiving and transmitting at the same time. In such a scenario, a common, often implicit, assumption that is adopted here too is that the transmissions of a node are not accounted as interference at its receiver. The reason is that since self-transmissions are known, they can therefore be canceled, at least in theory. Cancelation will not be perfect in practice, but so-called *near-end crosstalk* (NEXT) effects can be mitigated. Then, for each link $\ell \in \mathcal{L}$, the set of potentially interfering links can be defined as $\mathcal{I}(\ell) := \mathcal{L} - \{\ell\} - \{m | \mathrm{Tx}(m) = \mathrm{Rx}(\ell)\}$.

Under this setup, the SINR experienced at the receiver of link ℓ is equal to

$$\gamma_\ell = \frac{G_{\ell\ell} p_\ell}{\frac{1}{G} \sum\limits_{k \in \mathcal{I}(\ell)} G_{k\ell} p_k + V_\ell}, \tag{4.1}$$

where V_ℓ is the additive noise power and G accounts for potential gain due to physical layer processing, such as beamforming, spreading, or coding for known interference. In the following, for notational convenience, G is absorbed by the coupling factors $G_{k\ell},\ k \neq \ell$.

The data flow traversing link $\ell \in \mathcal{L}$ is denoted f_ℓ and it is upper bounded, according to information theory first principles, by the maximum achievable rate

$$c_\ell = \log(1 + \gamma_\ell), \tag{4.2}$$

i.e., each link is viewed as a single-user Gaussian channel with Shannon capacity. It is well known that capacity may be achieved by Gaussian signalling and infinite-length coding; however, for practical modulation schemes and in particular coding with limited block size, the actual maximum achievable rate is (sometimes significantly) smaller. This loss can be modeled introducing in (4.2) a constant factor, say $0 < c < 1$, that scales the SINR. Note that this modification does not alter the mathematical properties of (4.2).

Inserting (4.1) into (4.2), the capacity of link ℓ is written as

$$\begin{aligned} c_\ell &= \log\left(1 + \frac{G_{\ell\ell}p_\ell}{\sum_{k\in\mathcal{I}(\ell)} G_{k\ell}p_k + V_\ell}\right) = \log\left(\frac{\sum_{k\in\mathcal{I}(\ell)} G_{k\ell}p_k + V_\ell + G_{\ell\ell}p_\ell}{\sum_{k\in\mathcal{I}(\ell)} G_{k\ell}p_k + V_\ell}\right) \\ &= \log\left(\sum_{k\in\{\ell,\mathcal{I}(\ell)\}} G_{k\ell}p_k + V_\ell\right) - \log\left(\sum_{k\in\mathcal{I}(\ell)} G_{k\ell}p_k + V_\ell\right). \end{aligned} \tag{4.3}$$

It is seen that, c_ℓ is a function of all the powers transmitted in the network. In contrast, the capacity of a link in an (interference-free) wired network is determined solely by the average power available at the transmitter of that link.

The curvature of function (4.3) is not constant throughout its support, because the logarithm of an affine expression is concave, but the difference of concave functions is, in general, neither convex nor concave [20]. Hence, problems involving optimization of $\{c_\ell\}_{\ell\in\mathcal{L}}$ with respect to $\{p_\ell\}_{\ell\in\mathcal{L}}$ are in general difficult. However, for high SINR, i.e., $\gamma_\ell \gg 1$, (4.2) is approximated by

$$c_\ell \approx \log(\gamma_\ell) \overset{(4.1)}{=} \log(G_{\ell\ell}p_\ell) - \log\left(\sum_{k\in\mathcal{I}(\ell)} G_{k\ell}p_k + V_\ell\right). \tag{4.4}$$

Expression (4.4) can be reformulated to a *concave* function, by means of a logarithmic change of variables [6]. Specifically, defining the auxiliary variables

$$\tilde{p}_\ell := \log p_\ell \qquad \forall \ell \in \mathcal{L}, \tag{4.5}$$

expression (4.4) is *equivalently* rewritten as

$$c_\ell \approx \tilde{G}_{\ell\ell} + \tilde{p}_\ell - \log\left(\sum_{k\in\mathcal{I}(\ell)} e^{\tilde{G}_{k\ell}+\tilde{p}_k} + e^{\tilde{V}_\ell}\right) := \tilde{c}_\ell, \tag{4.6}$$

where the factors $\tilde{G}_{k_\ell} := \log(G_{k_\ell})$ and $\tilde{V}_\ell := \log(V_\ell)$ have been defined merely to compact the writing. The logarithm of a sum of exponentials is convex [7], the negative sign reverts the curvature, and addition with an affine expression preserves curvature; hence, (4.6) is a concave function of $\{\tilde{p}_\ell\}_{\ell\in\mathcal{L}}$.

4.2 Max-Flow Power Control

Consider a single unicast session[1], where traffic inserted at node 1 (the source) is relayed by the intermediate nodes of the network and eventually reaches node N (the destination). The problem of interest is to jointly adjust the transmission powers and determine feasible flows for all cochannel links, with the goal of maximizing session's throughput. The joint max-flow power control problem can be formulated as

$$\max_{\{f_\ell \geq 0,\ p_\ell \geq 0\}_{\ell \in \mathcal{L}}} \sum_{\ell : \mathrm{Tx}(\ell)=1} f_\ell \tag{4.7}$$

$$\text{s.t. :} \quad \sum_{\ell : \mathrm{Tx}(\ell)=i} p_\ell \leq P_i \qquad \forall i \in \mathcal{N} - \{N\}, \tag{4.8a}$$

$$\sum_{\ell : \mathrm{Rx}(\ell)=i} f_\ell = \sum_{\ell : \mathrm{Tx}(\ell)=i} f_\ell \qquad \forall i \in \mathcal{N} - \{1, N\}, \tag{4.8b}$$

$$f_\ell \leq c_\ell \qquad \forall \ell \in \mathcal{L}, \tag{4.8c}$$

where it assumed that the source (destination) can only transmit (receive). The flow conservation constraints (4.8b) ensure that nothing is lost in the network, thus the total flow emanating from the source reaches the destination. The objective function (4.7) and flow conservation constraints (4.8b) come directly from the classical max-flow problem for wired networks. The total power at which node i can transmit, due to regulation or physical limitations, is upper bounded by P_i (4.8a). The difficulty of formulation (4.7)–(4.8) is due to the flow bounds (4.8c); obviously, the remainder is a LP problem. Note that inequalities (4.8c) are convex if and only if their right-hand side is a concave function of the optimization variables [20]. However, the link capacities $\{c_\ell\}_{\ell \in \mathcal{L}}$ of the wireless network are functions of $\{p_\ell\}_{\ell \in \mathcal{L}}$, that do not have fixed curvature throughout their support, cf. (4.3). Nonconvexity is a serious difficulty that is sometimes mistakenly interpreted as intractability. Yet appropriate reformulation often reveals hidden convexity, thereby allowing exact and efficient solution.

In the high SINR regime, the link capacities can be closely approximated by $\{\tilde{c}_\ell\}_{\ell \in \mathcal{L}}$, defined in (4.6), which are concave functions of the logarithmic powers $\{\tilde{p}_\ell\}_{\ell \in \mathcal{L}}$. Then, problem (4.7)–(4.8) is written as

[1] A single session is considered for simplicity of exposition; generalization to multicommodity networks is straightforward.

$$\max_{\{f_\ell \geq 0,\ \tilde{p}_\ell \in \mathbb{R}\}_{\ell \in \mathcal{L}}} \sum_{\ell : \mathrm{Tx}(\ell)=1} f_\ell \tag{4.9}$$

$$\text{s.t.}: \quad \log\left(\sum_{\ell : \mathrm{Tx}(\ell)=i} e^{\tilde{p}_\ell}\right) \leq \log(P_i) \qquad \forall i \in \mathcal{N} - \{N\}, \tag{4.10a}$$

$$\sum_{\ell : \mathrm{Rx}(\ell)=i} f_\ell = \sum_{\ell : \mathrm{Tx}(\ell)=i} f_\ell \qquad \forall i \in \mathcal{N} - \{1, N\}, \tag{4.10b}$$

$$f_\ell \leq \tilde{G}_{\ell\ell} + \tilde{p}_\ell - \log\left(\sum_{k \in \mathcal{I}(\ell)} e^{\tilde{G}_{k\ell} + \tilde{p}_k} + e^{\tilde{V}_\ell}\right) \qquad \forall \ell \in \mathcal{L}, \tag{4.10c}$$

where inequalities (4.10c) are now convex. The side effect of the logarithmic change of variables (4.5) is the reformulation of linear inequalities (4.8a) to (4.10a). Luckily, (4.10a) are convex, since, as noted before, the logarithm of a sum of exponentials is a convex function. In fact, the sum of exponentials, resulting from the direct application of (4.5) into (4.8a), is already convex; the logarithm is taken merely to increase the accuracy of the numerical solution in practice. It can be easily seen that problem formulation (4.9)–(4.10) conforms to the disciplined convex programming (DCP) ruleset [20]; hence, not only it is convex by definition, but it can be readily solved efficiently, by means of interior point methods, when inserted in this form to the software package CVX [21].

4.3 Back-Pressure Power Control

Consider a system slotted with unit time slots, indexed by $t \in \mathbb{N}_+$. A *deterministic* amount of traffic, say X, is generated at the source in every slot. Contrary to the setup of Section 4.2, the nodes are assumed to have buffering capabilities; let $W_i(t)$ denote the traffic queue length of node i at the end of slot t. Then, the differential backlog [37] of link $\ell = (i, j)$, at the end of slot t, is defined as

$$D_\ell(t) := \begin{cases} \max\{0, W_i(t) - W_j(t)\}, & j \neq N \\ W_i(t), & j = N. \end{cases} \tag{4.11}$$

Traffic flows through the links in each slot, based on the link capacities resulting from the power selections at the particular slot. The powers for the next slot, $t+1$, can be determined by solving the optimization problem [17]

$$\max_{\{p_\ell \geq 0\}_{\ell \in \mathcal{L}}} \sum_{\ell=1}^{L} D_\ell(t) c_\ell \tag{4.12}$$

$$\text{s.t. :} \quad \sum_{\ell : \mathrm{Tx}(\ell)=i} p_\ell \leq P_i, \qquad \forall i \in \mathcal{N}. \tag{4.13}$$

The objective function (4.12) that is sought to be maximized is a weighted sum of the capacities of all network links, where the differential backlogs serve as weighting factors. The optimization problem (4.12)–(4.13) aims to maximize the throughput of the network by favoring the links whose receiver is less congested than the transmitter. Note that, due to the upper branch of (4.11), the links destined to more congested nodes have differential backlog 0 and consequently they are not included in the objective function (4.12). The transmission powers corresponding to such links are only accounted as interference and the optimization pushes them to 0 in order to increase the capacities of all other links. In fact, this is a *scheduling* decision that could have been taken beforehand, i.e., when $D_\ell(t) = 0$, set $p_\ell(t+1) = c_\ell(t+1) = 0$; however, problem formulation (4.12)–(4.13) yields the same result without requiring any explicit information (on the definition of the SINR). Inequalities (4.13) upper bound the total transmission power of each node.

In the high SINR regime, approximation (4.6) is valid and problem (4.12)–(4.13) can be formulated, with respect to the auxiliary variables defined in (4.5), as

$$\max_{\{\tilde{p}_\ell \in \mathbb{R}\}_{\ell \in \mathcal{L}}} \sum_{\ell=1}^{L} D_\ell(t) \tilde{c}_\ell \tag{4.14}$$

$$\text{s.t. :} \qquad \log\left(\sum_{\ell : \mathrm{Tx}(\ell)=i} e^{\tilde{p}_\ell} \right) \leq \log(P_i) \qquad \forall i \in \mathcal{N}. \tag{4.15}$$

The objective function (4.14) is concave, since $\{\tilde{c}_\ell\}_{\ell \in \mathcal{L}}$ are concave functions of $\{\tilde{p}_\ell\}_{\ell \in \mathcal{L}}$, cf. (4.6), and $\{D_\ell(t)\}_{\ell \in \mathcal{L}}$ are nonnegative, cf. (4.11). Moreover, inequalities (4.15) are convex, as noted before for (4.10a). Hence, optimization problem (4.14)–(4.15) is *convex*, since a concave function is maximized over a convex feasible set. The importance of this result is that the global optimum solution can be found with polynomial worst-case complexity, by means of modern interior point methods. The logarithmic change of variables (4.5) rings the bell of GP problems. Indeed, formulation (4.14)–(4.15) may be seen to belong to the wide class of *Extended GP* problems [7].

4.4 Simulation Results

A simulation experiment has been performed on a random network topology. $N = 6$ nodes are uniformly placed on a 100×100 square, the network has $L = 21$ directed links, the link gains equal to the path losses with exponential factor 4, and the processing gain is equal to 64. Each problem instance is solved using the CVX toolbox for MATLAB [21], in approximately 5 seconds on a typical modern workstation. Complexity is manageable when the number of variables is in the order of a few tens. However, for larger networks, e.g., $N > 10$, the number of variables exceeds 100, assuming fully connected topologies, and optimization is cumbersome. In such scenarios, receiver neighborhoods have to be defined for each node, in order to reduce the number of variables and ensure that the SINR of each link is high, so that the approximation (4.4) is valid.

Simulation results are displayed for three different amounts of input traffic per slot. Specifically, the three figure pairs correspond to traffic 9, 10.7, and 12, respectively. The first plot of each pair shows the evolution of the relays' backlogs with time. The second one depicts the power values resulting from optimization (4.14)–(4.15) for some typical links. It is seen that for the first two values of traffic the queues are stabilized after a few tens of iterations; actually, 10.7 is approximately the largest traffic for which this is true. As seen by the third simulation, higher traffic cannot reach destination and all backlogs grow to infinity with time. It is interesting that for stable setups, when the transitional phase is over, the optimized link powers are periodic and have a small number of feasible states.

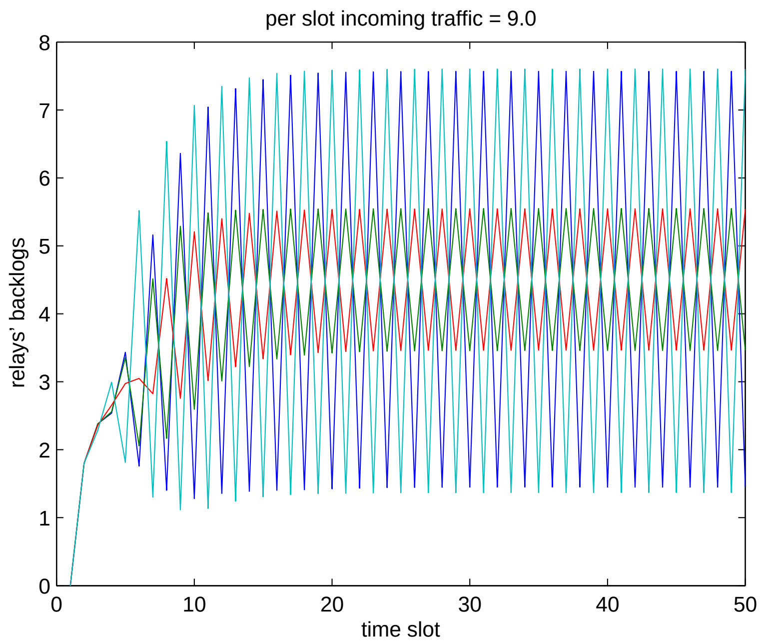

Figure 4.1: Relays' Backlogs; underflow

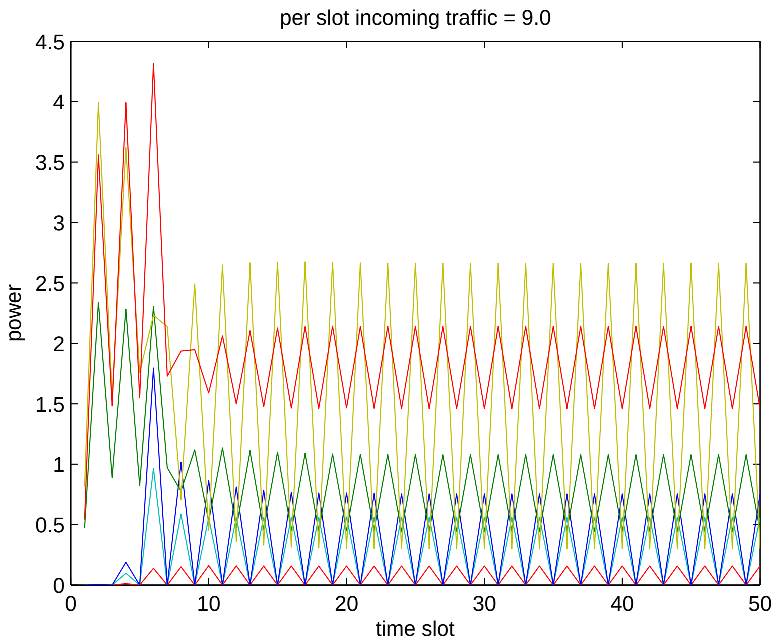

Figure 4.2: Typical resulting link powers; underflow

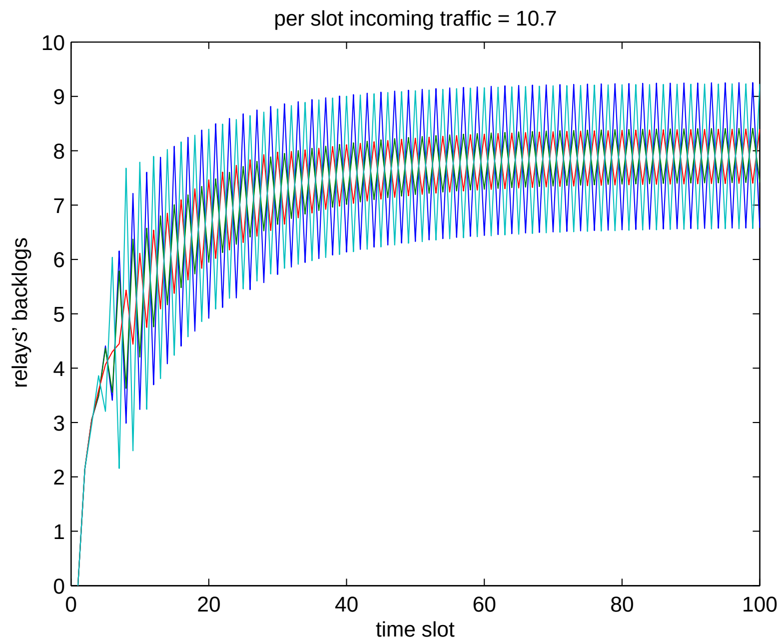

Figure 4.3: Relays' Backlogs; maxflow

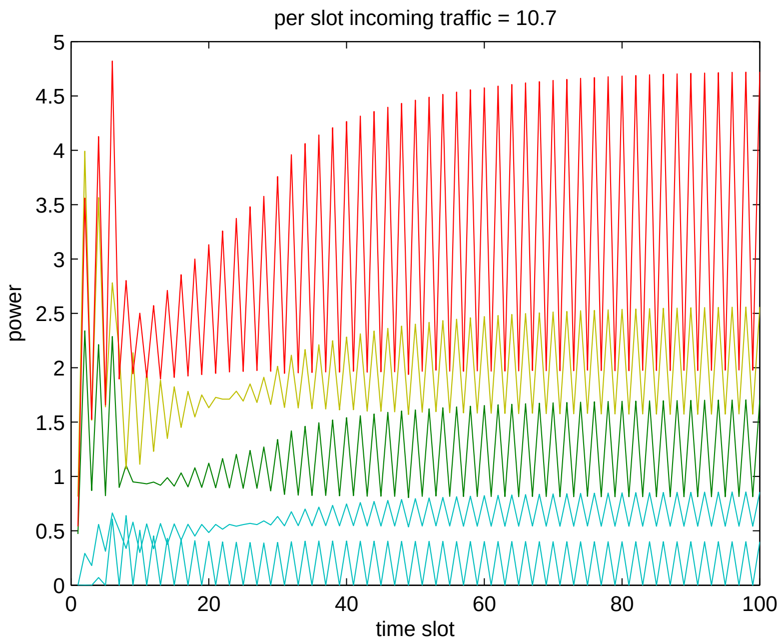

Figure 4.4: Typical resulting link powers; maxflow

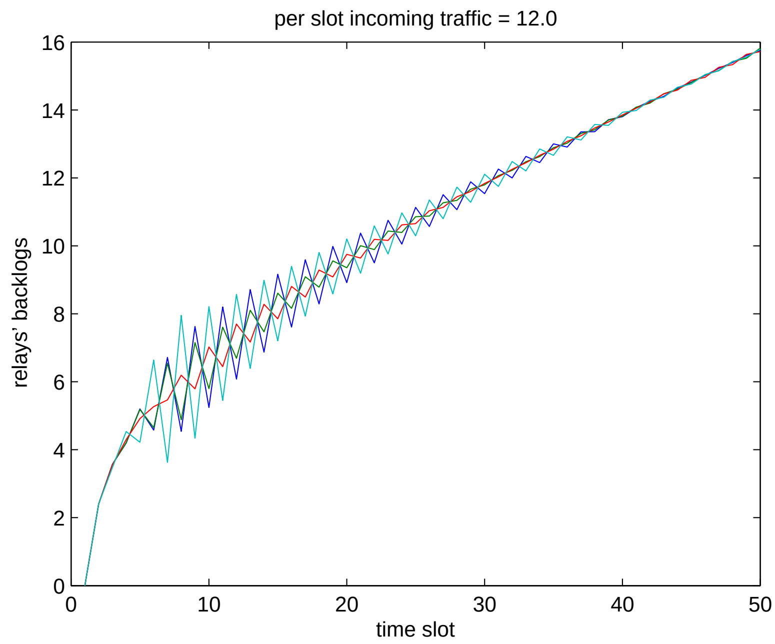

Figure 4.5: Relays' Backlogs; overflow

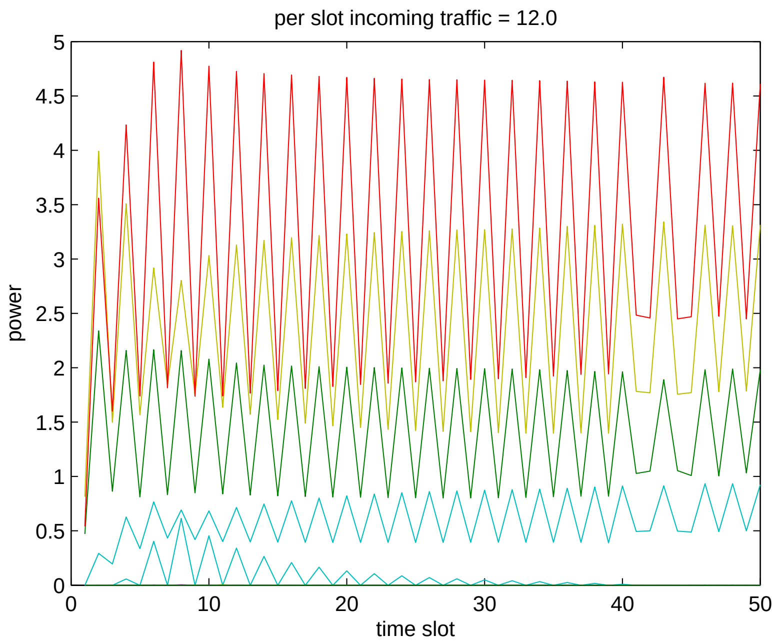

Figure 4.6: Typical resulting link powers; overflow

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