New Journal of Physics
The open–access journal for physics

Transmutation of singularities in optical instruments

Tomáš Tyc[1,2] **and Ulf Leonhardt**[2,3]
[1] Institute of Theoretical Physics and Astrophysics, Masaryk University, Kotlářská 2, 61137 Brno, Czech Republic
[2] School of Physics and Astronomy, University of St Andrews, North Haugh, St Andrews KY16 9SS, UK
[3] Physics Department, National University of Singapore, 2 Science Drive 3, Singapore 117542
E-mail: tomtyc@physics.muni.cz

New Journal of Physics **10** (2008) 115038 (8pp)
Received 23 September 2008
Published 27 November 2008
Online at http://www.njp.org/
doi:10.1088/1367-2630/10/11/115038

Abstract. We propose a method for eliminating a class of singularities in optical media where the refractive index goes to zero or infinity at one or more isolated points. Employing transformation optics, we find a refractive index distribution equivalent to the original one that is nonsingular but shows a slight anisotropy. In this way, the original singularity is 'transmuted' into another, weaker type of singularity where the permittivity and permeability tensors are discontinuous at one point. The method is likely to find applications in designing and improving optical devices by making them easier to implement or to operate in a broad band of the spectrum.

Contents

1. Introduction 2
2. Singularities in refractive index 2
3. Heuristic argument 4
4. Singularity transmutation 5
5. Generalizing the method 6
6. Conclusion 7
Acknowledgments 8
References 8

1. Introduction

With the advent of metamaterials [1]–[4], new types of optical devices and instruments have been proposed and implemented. However, in some of them the required parameters of the optical medium can be unrealistic or very difficult to achieve. An example is the requirement that the refractive index goes to zero or to infinity at some points, which, for example, is the case for the Eaton lens [5]–[7] known from radar technology where the index tends to infinity. Another example is the proposal for conformal invisibility cloaking [8]–[10], which requires zeros in the index profile. As the zero or infinity of the refractive index can be satisfied only for a single frequency, such a singularity can be a serious obstacle in constructing the corresponding device or in operating it broadband.

As we show in this paper, a large class of such singularities of the refractive index can be eliminated, or rather 'transmuted' into less demanding singularities, using the methods of transformation optics developed in [8], [11]–[14] that have become a vibrant research area, see [15] for a review. By squeezing or expanding the neighborhood of the point where singularity occurs the refractive index can be made finite and nonzero while a slight optical anisotropy emerges. It is then certainly easier to design the corresponding anisotropic material than a material where the speed of light has to go to zero or infinity.

The paper is organized as follows. In section 2, we give a few examples of situations with refractive-index singularities and in sections 3 and 4, we present heuristic and rigorous arguments, respectively, of how to eliminate the singularity. In section 5, we show how our method works in general and we conclude in section 6.

2. Singularities in refractive index

In a number of theoretical proposals for optical devices optical singularities occur where the refractive index approaches zero or infinity at some point or points, which often prevents them from being practically implemented as devices. We give three examples here—the Eaton lens, an invisible sphere and the invisibility cloak based on optical conformal mapping.

The first example is the Eaton lens [5]–[7]: a device that returns the incident light rays back to their source, a perfect retroreflector, see figure 1. It has a spherically symmetric refractive-index distribution given by

$$n(r) = \sqrt{2a/r - 1} \quad \text{for} \quad r \leqslant a \quad \text{and} \quad n = 1 \quad \text{for} \quad r \geqslant a. \tag{1}$$

Clearly, the refractive index goes to infinity for $r \to 0$ and hence the speed of light goes to zero there. It can be easily shown that near $r = 0$ the refractive index behaves like $n \sim r^{-1/2}$.

The second example is an invisible sphere considered, among other things, in [16]. In this proposal, a spherically symmetrical refractive-index distribution $n(r)$ is employed within a sphere of radius a where $n(r)$ is given by the implicit equation[4]

$$\left(\frac{a}{nr} + \sqrt{\frac{a^2}{n^2r^2} - 1}\right)^2 = n \tag{2}$$

[4] This equation follows from formulae (11) and (2) of [16] by substituting $\nu = 1, n = \sqrt{1 - U/E}$ and replacing b_0 by a.

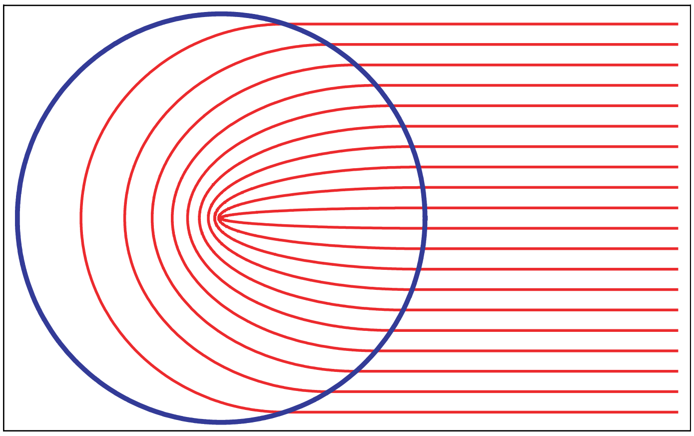

Figure 1. Eaton lens. The Eaton lens is a perfect retroreflector based on the index-profile (1). Light (shown in red) incident from the right impinges on the Eaton lens (blue) where it turns along an ellipse segment and is reflected back to the direction it came from. Since the index-profile (1) is radially symmetric the retroreflector works for light from all directions.

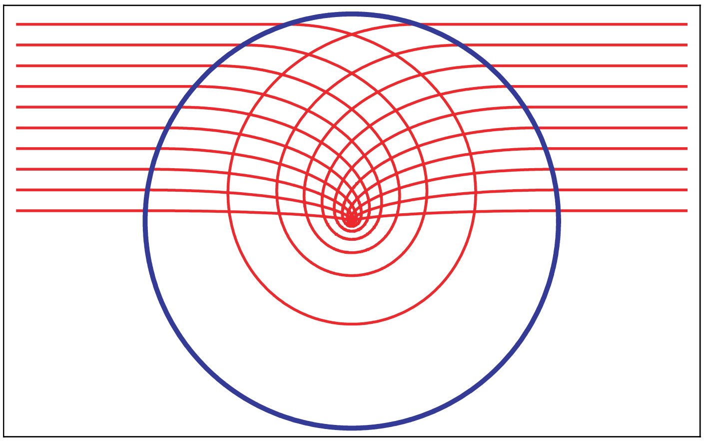

Figure 2. Invisible sphere. Incident light (red) performs loops in the sphere (blue) and leaves in the direction of incidence as if the sphere were not present. This device is based on the index-profile (2).

and $n = 1$ outside the sphere. Remarkably, the sphere is not visible within the regime of geometrical optics. The light rays that hit the sphere perform loops around its center and return to their original paths, see figure 2. The only potentially observable effect is a constant time delay of the rays. Although nothing can be hidden in the sphere, so it cannot be used as an invisibility cloak, it still represents an ultimate optical illusion, because the space appears to be empty while it is not. Using equation (2), it can be shown that n diverges like $n \sim r^{-2/3}$ near $r = 0$.

The third example is the neighborhood of a branch point in the proposal of an invisibility cloak based on conformal mapping [8]–[10]. In this proposal, transformation optics [15] is employed for constructing a cloaking device. The transformation from electromagnetic space (complex plane w) to physical space (complex plane z) is given by the formulae

$$w = z + \frac{a^2}{z}, \qquad z = \frac{1}{2}\left(w \pm \sqrt{w^2 - 4a^2}\right). \tag{3}$$

The refractive index in physical space is $n = |\,\mathrm{d}w/\,\mathrm{d}z|$, see [8]. It is easy to verify that the derivative vanishes at $z = \pm a$, at the images of the branch points $w = \pm 2a$. Hence, the refractive index goes to zero and the speed of light goes to infinity as one approaches either of these points $z = \pm a$. The Taylor expansion of n shows that $n \sim r$ near $z = a$ where $r = |z - a|$ denotes the distance from the point $z = a$, and the behavior of n near $z = -a$ is similar.

In all of the previous examples, as well as others not explicitly stated here, the refractive index behaves like $n \sim r^p$, $p \in \mathbb{R}$ near the point $r = 0$ and the speed of light diverges or goes to zero at this point. We will refer to such a singularity of the material properties as 'material singularity'. As we will show in the following, if $p > -1$ (which includes all of the above examples), the material singularity of the refractive index can be efficiently removed by means of transformation optics.

3. Heuristic argument

Here, we give a heuristic derivation of the transformation that removes the material singularity using the properties of the refractive index. Then we proceed to a rigorous calculation of the transformed permittivity and permeability tensors in the next section.

Suppose we have a spherically symmetric refractive index distribution $n(r) = r^p$, $p \neq 0$ in three-dimensional space[5], and wish to remove its singularity at $r = 0$. Suppose for a moment that $p < 0$, which means that the light gets very slow near $r = 0$. If we could expand the neighborhood of this point in such a way that the expansion factor would increase as one approached $r = 0$, then we might be able to provide light with more space such that its speed could remain above some fixed positive value. This is indeed possible if we expand the space according to the rule $R = r^{p+1}$, where R is a new radial coordinate, and leave the spherical angles θ, ϕ intact. According to the general terminology of transformation optics, (R, θ, ϕ) will be referred to as physical space, whereas the original (r, θ, ϕ) space will be called electromagnetic space. If the light travels in the radial direction, then the expansion factor is $\eta_R = \mathrm{d}R/\,\mathrm{d}r = (p+1)\,r^p$, so the light moves η_R-times faster in physical space than in electromagnetic space. The refractive index is therefore η_R-times smaller in physical space and becomes $n_r = 1/(p+1)$, which means that the singularity has been removed. If the light travels in the angular direction instead (say in the direction of increasing ϕ), the expansion factor will be $\eta_\phi = R\sin\theta\,\mathrm{d}\phi/(r\sin\theta\,\mathrm{d}\phi) = r^p$, which leads to $n_\phi = 1$, and in a similar way we get $n_\theta = 1$.

We see that the singularity of the refractive index has been removed in both the radial and angular directions. The price to be paid is a slight optical anisotropy because the refractive indices in the radial and angular directions differ by the factor of $p + 1$. One also sees that the transformation works only if $p > -1$. In this way we can eliminate all singularities with $n \to 0$ and many singularities with $n \to \infty$.

[5] To avoid dimensional discrepancies in equations like $n = r^p$ and in order to make formulae as simple as possible, we will consider all distances dimensionless.

4. Singularity transmutation

Suppose again that we have a spherically symmetric refractive-index distribution $n(r) = r^p$. In order to remove the material singularity at $r = 0$, we apply the transformation $R = r^{p+1}$, where R is a new radial coordinate. To calculate the optical properties of the medium in physical space (R, θ, ϕ), we apply the method described in [14] and [15].

We use the connection between dielectric media and spatial geometries. We start with the covariant metric tensors of electromagnetic space $g'_{ij} = \mathrm{diag}\,(1, r^2, r^2\sin^2\theta)$ and of physical space $\gamma_{ij} = \mathrm{diag}\,(1, R^2, R^2\sin^2\theta)$ in spherical coordinates. The transformation matrix between the two spaces is denoted by $\Lambda^i_j = \partial(R, \theta, \phi)/\partial(r, \theta, \phi)$; we find $\Lambda^i_j = \mathrm{diag}\,(\mathrm{d}R/\,\mathrm{d}r, 1, 1)$. Equation (5.1) of [15] gives the recipe for calculating the contravariant components of the permittivity and permeability tensors $\varepsilon^{ij}, \mu^{ij}$ in physical space from their values ε', μ' in electromagnetic space that we assume to be scalar:

$$\varepsilon^{ij} = \varepsilon' \frac{\sqrt{g}}{\sqrt{\gamma}}\, g'^{kl} \Lambda^i_k \Lambda^j_l, \qquad \mu^{ij} = \mu' \frac{\sqrt{g}}{\sqrt{\gamma}}\, g'^{kl} \Lambda^i_k \Lambda^j_l. \tag{4}$$

Here g'^{kl} is the contravariant form of the metric tensor (the inverse of g'_{ij}), γ is the determinant of the tensor γ_{ij} and $g = \det(g'_{ij})/\det^2(\Lambda^i_j)$. We adopt Einstein's summation convention over repeated indices. Note that the ϵ^{ij} and μ^{ij} are tensors with respect to the background geometry γ_{ij} although they are not tensors for the electromagnetic geometry g_{ij} [15]. Substituting all the relevant quantities into equation (4), we obtain

$$\varepsilon^{ij} = \varepsilon' \,\mathrm{diag}\left(\frac{r^2}{R^2}\frac{\mathrm{d}R}{\mathrm{d}r}, \frac{1}{R^2\,\mathrm{d}R/\,\mathrm{d}r}, \frac{1}{R^2\sin^2\theta\,\mathrm{d}R/\,\mathrm{d}r}\right) \tag{5}$$

and the analogous equation holds for μ^{ij}. However, it is the mixed (covariant–contravariant) components of the permittivity and permeability tensors that are of physical relevance [15], because they correspond to the tensors expressed in Cartesian coordinates and are directly related to the refractive indices. Lowering one index of ε^{ij} using the metric tensor γ_{ij}, we get

$$\varepsilon^i_j = \varepsilon' \,\mathrm{diag}\left(\frac{r^2}{R^2}\frac{\mathrm{d}R}{\mathrm{d}r}, \frac{\mathrm{d}r}{\mathrm{d}R}, \frac{\mathrm{d}r}{\mathrm{d}R}\right), \quad \mu^i_j = \mu' \,\mathrm{diag}\left(\frac{r^2}{R^2}\frac{\mathrm{d}R}{\mathrm{d}r}, \frac{\mathrm{d}r}{\mathrm{d}R}, \frac{\mathrm{d}r}{\mathrm{d}R}\right), \tag{6}$$

which is a general result. Assuming that the original refractive index in electromagnetic space was realized by an impedance-matched medium for which $\varepsilon' = \mu' = n(r) = r^p$ and using the transformation $R = r^{p+1}$, we obtain from equation (6) for the mixed tensors in physical space

$$\varepsilon^i_j = \mu^i_j = \mathrm{diag}\left(p+1, \frac{1}{p+1}, \frac{1}{p+1}\right). \tag{7}$$

We see that all the components of the tensors ε and μ are non-singular, in contrast to their original values in electromagnetic space. In this way, the infinity or zero of the speed of light at $r = 0$ has been removed. At the same time, equation (7) shows that the tensors ε and μ are anisotropic, which is due to the fact that the transformation from electromagnetic to physical space is not conformal. Clearly, the tensors are diagonal in the spherical coordinates and their angular eigenvalues coincide. This means that the optical medium is uniaxial with the optical axis oriented in the radial direction, which resembles the field lines of a monopole, see figure 3. At $R = 0$ the optical axes meet at a topological defect, a singularity similar to the one known from the problem of combing a hedgehog. We have thus arrived at a new type of singularity of

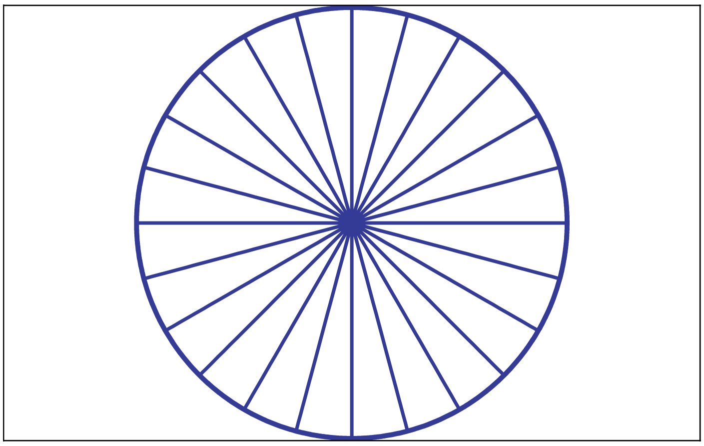

Figure 3. The optical axes of the transmuted medium resemble field lines of a monopole that emerge radially from a single point.

the medium that may be called a 'geometrical singularity'. Unlike the material singularity, the geometrical singularity is not characterized by any diverging properties of the material (because the speed of light is finite and nonzero and the anisotropy is slight and constant), but rather by the fact that the orientation of the optical axis changes arbitrarily quickly near the point $R = 0$. As the direction of the optical axis is not defined at $R = 0$, the tensors ε and μ are discontinuous at this point.

Our method thus describes a transformation of one type of singularity to another, a 'transmutation of singularities'. Certainly, constructing a material with a geometrical singularity is significantly less demanding in practice than creating the material singularity where light has to stop or propagate infinitely quickly. Moreover, a medium with a geometrical singularity can be made broadband in contrast to the material singularity, because one does not need to rely on resonance effects. However, as our result (7) suggests, it may be necessary to use materials with one ε component smaller than unity—anomalous dispersion where the phase velocity of light exceeds the speed of light in vacuum, which is possible in principle, but restricted to certain frequency bands for a given material. As an easy alternative, one could put the device into a host dielectric with sufficiently high refractive index, because only the relative variation of ε matters. The only real practical limitation in constructing the transmuted medium will occur in a very small vicinity of the point $R = 0$: if the features of the metamaterial to be used have a typical dimension d, then, clearly, the proposed properties of the transmuted medium cannot be achieved within a sphere with radius of about d. However, this is not a major problem as d is typically of the order of the wavelength, which is by several orders of magnitude smaller than the size of a typical device. Our method can therefore enable construction of devices that would otherwise be very difficult to make, by relaxing the strong requirements placed on the material.

5. Generalizing the method

In the previous section, we showed how the singularity $n = r^p$ of the refractive index at $r = 0$ can be transmuted by the transformation $R = r^{p+1}$. Moreover, our method can be applied also to situations where $n(r)$ has a more general form but still behaves like r^p for $r \to 0$. The singularity will be transmuted if we apply any suitable transformation in which R behaves like r^{p+1} for

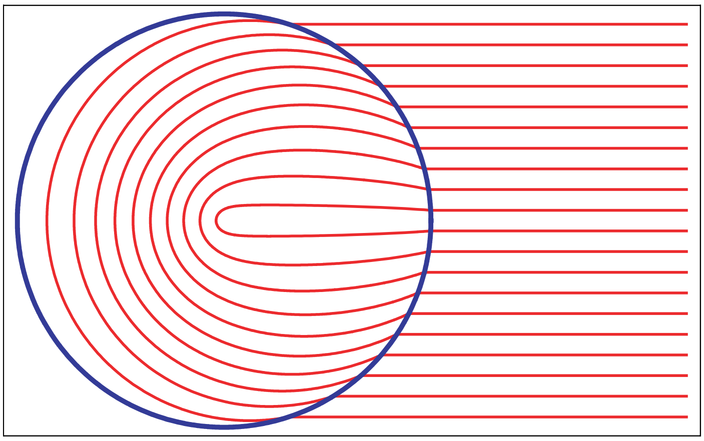

Figure 4. Transformed Eaton lens. Equally spaced incident light rays are nearly equally spaced around the origin of the transformed Eaton lens, which illustrates that the refractive-index profile is not singular anymore.

$r \to 0$. If the original refractive index distribution $n(r)$ is spherically symmetric, it is even possible to make the angular eigenvalues of ε, μ constant by defining $R(r) = (1/n_0) \int_0^r n(r')\,\mathrm{d}r'$ where n_0 is a constant. It can be easily verified from equation (6) that in this case $\varepsilon_\theta^\theta = \varepsilon_\phi^\phi = n_0$ and $\varepsilon_r^r = (nr/R)^2/n_0$, which is non-singular and nonzero if $n \sim r^p$ and $R \sim r^{p+1}$. When we apply this transformation to the Eaton lens, we obtain

$$R = \frac{4a}{\pi + 2}\left[\arcsin\sqrt{\frac{r}{2a}} + \sqrt{\frac{r}{2a}\left(1 - \frac{r}{2a}\right)}\,\right], \tag{8}$$

for $r \leqslant a$ and we set $R = r$ for $r \geqslant a$. The multiplicative constant in equation (8) was chosen such that the function $R(r)$ is continuous at $r = a$. Figure 4 shows the light ray trajectories in the transformed Eaton lens.

Our method is rather flexible in treating singularities of the refractive index. The radial transformation may be applied just locally in the neighborhood of the singularity, or even several of such transformations can be applied at the same time, to transmute multiple singularities, which would be the case of the third example of section 2 where two singularities have to be treated. Understandably, our method works only for isolated singularities and cannot be used to transmute zero or infinite refractive indices on a line or surface. For example, the zero refractive index at the inner lining of a perfect cloaking device [13] cannot be transformed away.

6. Conclusion

In conclusion, we have proposed a method for eliminating material singularities where the refractive index goes to zero or infinity as $n \sim r^p$ for $r \to 0$ with $p > -1$. The material singularity is transmuted into a geometrical singularity where the permittivity and permeability tensors are not singular but are anisotropic, and the anisotropy forms a topological defect. One can never completely remove a singularity or a zero of the refractive index in an isotropic medium, but one can transmute it into a monopole in an anisotropic material, a much less harmful and practically realizable singularity[6]. Our method provides a useful

[6] Singularities are like curses in myths and prophecies: they cannot be taken back, but they can be altered.

tool for implementing some proposed optical devices or for letting them operate in a broad band of the spectrum, and it may also lead to the design and construction of new optical instruments.

Acknowledgments

We are greatly indebted to N V Korolkova for her generous support. TT acknowledges the grants MSM0021622409, MSM0021622419 and COMPAS; UL is supported by a Royal Society Wolfson Research Merit Award.

References

[1] Milton G W 2002 *The Theory of Composites* (Cambridge: Cambridge University Press)
[2] Smith D R, Pendry J B and Wiltshire M C K 2004 *Science* **305** 788
[3] Soukoulis C M, Linden S and Wegener M 2007 *Science* **315** 47
[4] Sarychev A K and Shalaev V M 2007 *Electrodynamics of Metamaterials* (Singapore: World Scientific)
[5] Hannay J H and Haeusser T M 1993 *J. Mod. Opt.* **40** 1437
[6] Eaton J E 1952 *Trans. IRE Antennas Propag.* **4** 66
[7] Kerker M 1969 *The Scattering of Light* (New York: Academic)
[8] Leonhardt U 2006 *Science* **312** 1777
[9] Leonhardt U 2006 *New J. Phys.* **8** 118
[10] Ochiai T, Leonhardt U and Nacher J C 2008 *J. Math. Phys.* **49** 032903
[11] Dolin L S 1961 *Isv. Vusov* **4** 964
[12] Greenleaf A, Lassas M and Uhlmann G 2003 *Math. Res. Lett.* **10** 1
[13] Pendry J B, Schurig D and Smith D R 2006 *Science* **312** 1780
[14] Leonhardt U and Philbin T G 2006 *New J. Phys.* **8** 247
[15] Leonhardt U and Philbin T G 2008 *Prog. Opt.* to appear (arXiv:0805.4778)
[16] Hendi A, Henn J and Leonhardt U 2006 *Phys. Rev. Lett.* **97** 073902