# Transmuted Complementary Weibull Geometric Distribution

Ahmed Z. Afify
Department of Statistics, Mathematics and Insurance
Benha University, Egypt
ahmed.afify@fcom.bu.edu.eg

Zohdy M. Nofal
Department of Statistics, Mathematics and Insurance
Benha University, Egypt
dr.zohdynofal@fcom.bu.edu.eg

Nadeem Shafique Butt
Department of Statistics, COMSATS Institute of Information Technology
Lahore, Pakistan
nadeemshafique@ciitlahore.edu.pk

## Abstract

This paper provides a new generalization of the complementary Weibull geometric distribution introduced by Tojeiro et al. (2014), using the quadratic rank transmutation map studied by Shaw and Buckley (2007). The new distribution is referred to as transmuted complementary Weibull geometric distribution (TCWGD). The TCWG distribution includes as special cases **11** sub models such as the complementary Weibull geometric distribution (CWGD), complementary exponential geometric distribution (CEGD), Weibull distribution (WD), exponential distribution (ED) and **three** new submodels. Various structural properties of the new distribution including moments, quantiles, moment generating function and Rényi entropy of the subject distribution are derived. We proposed the method of maximum likelihood for estimating the model parameters. Two real data sets are used to compare the flexibility of the new model versus the complementary Weibull geometric distribution.

**Keywords**: Transmutation, complementary Weibull geometric, Reliability Function, Moment Generating Function, Rényi Entropy, Order Statistics, Maximum Likelihood Estimation.

## 1. Introduction

The Weibull distribution is of utmost interest to theory-orientated statisticians because of its great number of special features and to practitioners because of its ability to fit to data from various fields, ranging from life data to observations made in economics and business administration, meteorology, hydrology, maintenance, replacement, quality control, acceptance sampling, statistical process control, inventory control, geology, geography, astronomy, medicine, psychology, pharmacy, material science, engineering, physics, chemistry, biology, warranty and weather data, see, e.g., Rinne (2009). For more than half a century the Weibull distribution has attracted the attention of statisticians working on theory and methods as well as in various fields of applied statistics.

However, the Weibull distribution does not provide a reasonable parametric fit for some practical applications where the underlying hazard rates may be bathtub or unimodal shapes. Recently, some generalizations of the Weibull distribution are considered. Aryal and Tsokos (2011) presented a new generalization of Weibull distribution called the transmuted Weibull distribution. Khan and King (2013) introduced the transmuted modified Weibull distribution. Ashour and Eltehiwy (2013a, 2013b) studied the transmuted exponentiated modi.ed Weibull and transmuted exponentiated Lomax distributions. Ebraheim (2014) introduced exponentiated transmuted Weibull distribution.

An interesting idea of generalizing a distribution, known in the literature by transmution, is derived by using the quadratic rank transmutation map studied by Shaw and Buckley (2007). In this paper we propose a new distribution family by extending the complementary Weibull geometric (CWG), introduced by Tojeiro et al. (2014) by using the quadratic rank transmutation map.

Louzada et al. (2011) introduced the complementary exponential geometric distribution, which is complementary to the exponential geometric model proposed by Adamidis and Loukas (1998), based on a complementary risk problem (Basu and J., 1982) in presence of latent risks, in the sense that there is no information about which factor was responsible for the component failure but only the maximum lifetime value among all risks is observed. Louzada et al. (2013) introduced complementary exponentiated exponential geometric distribution which considered a generalization to the complementary exponential geometric distribution. Tojeiro et al. (2014) introduced the complementary Weibull geometric (CWG) as a complementary distribution to the Weibull geometric (WG) model proposed by Barreto-Souza et al. (2011).

The cumulative distribution function ($cdf$) of the complementary Weibull geometric distribution (CWGD) is given by

$$F_{CWG}(y,\alpha,\beta,\gamma)=\frac{\alpha\left(1-e^{-(\gamma y)^{\beta}}\right)}{\alpha+(1-\alpha)e^{-(\gamma y)^{\beta}}},y,\alpha,\beta,\gamma>0, \tag{1}$$

where $\gamma$ is a scale parameter and $\alpha$and $\beta$ are shape parameters. The corresponding probability density function ($pdf$) is given by

$$f_{CWG}(y,\alpha,\beta,\gamma)=\frac{\alpha\beta\gamma(\gamma y)^{\beta-1}e^{-(\gamma y)^{\beta}}}{\left(\alpha+(1-\alpha)e^{-(\gamma y)^{\beta}}\right)^{2}}. \tag{2}$$

The procedure of expanding a family of distributions for added flexibility or to construct covariate models is a well-known technique in the literature. In this article we present a new generalization of complementary Weibull geometric distribution called transmuted complementary Weibull geometric distribution. We derived the subject distribution using the quadratic rank transmutation map studied by Shaw and Buckley (2007). A random variable $X$ is said to have transmuted distribution if its cumulative distribution function ($cdf$) is given by

$$F(x)=(1+\lambda)G(x)-\lambda G^{2}(x),|\lambda|\leq 1.$$

where $G(x)$ is the ($cdf$) of the base distribution, which on differentiation yields,

$$f(x)=g(x)[1+\lambda-2\lambda G(x)],|\lambda|\leq 1.$$

where $f(x)$ and $g(x)$ are the corresponding $pdfs$ associated with $cdfs$ $F(x)$ and $G(x)$ respectively. For more information about the quadratic rank transmutation map is given in Shaw and Buckley (2007). Observe that at $\lambda = 0$, we have the base distribution.

In this paper we provide mathematical formulation of the transmuted complementary Weibull geometric distribution (TCWGD) and some of its properties. We will also provide possible area of applications. The rest of the paper is organized as follows. In Section 2 we demonstrate the subject distribution. In Section 3, we find the reliability function, hazard rate and cumulative hazard rate of the subject model. The statistical properties include quantile functions, random number generation, moments, moment generating functions and Rényi entropy are derived in Section 4. In section 5, the minimum, maximum and median order statistics models are discussed. We also demonstrate the joint density functions $f_{i:j;n}(x_i, x_j)$ of the transmuted complementary Weibull geometric distribution. In Section 6, we demonstrate the maximum likelihood estimates (MLEs) and the asymptotic confidence intervals of the unknown parameters. In section 7, the TCWG distribution is applied to a real data set to illustrate its usefulness. Finally, in Section 8, we provide some concluding remarks.

## 2. Transmuted Complementary Weibull Geometric Distribution (TCWGD)

The transmuted complementary Weibull geometric distribution (TCWGD) and its sub-models are presented in this section. A random variable $Y$ is said to have transmuted complementary Weibull geometric distribution with parameters $\alpha, \beta, \gamma$ and $\delta$ if its cumulative distribution function ($cdf$) is defined as

$$F_{TCWG}(y, \alpha, \beta, \gamma, \delta) = \frac{\alpha^2 + \left((\alpha\delta + \alpha - 2\alpha^2) - (\alpha\delta + \alpha - \alpha^2)e^{-(\gamma y)^\beta}\right)e^{-(\gamma y)^\beta}}{\left(\alpha + (1-\alpha)e^{-(\gamma y)^\beta}\right)^2},$$

$$y > 0, \alpha, \beta, \gamma > 0, |\delta| \le 1. \tag{3}$$

Using the series expansion

$$(1 - Z)^{-k} = \sum_{j=0}^{\infty} \frac{\Gamma(k+j)}{j!\,\Gamma(k)} Z^j, 0 < Z < 1, k > 0.$$

The $cdf$ of the transmuted complementary Weibull geometric in (3) can be expressed in the mixture form

$$F_{TCWG}(y, \alpha, \beta, \gamma, \delta) = \textstyle\sum_{j=0}^{\infty}(j+1)\left(1 - \frac{1}{\alpha}\right)^j e^{-j(\gamma y)^\beta}$$
$$+ \frac{h}{\alpha^2}\textstyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i(j+2)}{\Gamma(2-i)i!j!}\ell^j\left(1 - \frac{1}{\alpha}\right)^j e^{-(i+j+1)(\gamma y)^\beta}. \tag{4}$$

where $\ell = (\alpha\delta + \alpha - \alpha^2)/(\alpha\delta + \alpha - 2\alpha^2), h = \alpha\delta + \alpha - 2\alpha^2$, $\gamma$ is a scale parameter representing the characteristic life, $\alpha$ and $\beta$ are the shape parameters representing the different patterns of the transmuted complementary Weibull geometric distribution and $\delta$

is the transmuted parameter. The corresponding probability density function ($pdf$) of the transmuted complementary Weibull geometric distribution is given by

$$f_{TCWG}(y,\alpha,\beta,\gamma,\delta)=\frac{\alpha\beta\gamma(\gamma y)^{\beta-1}e^{-(\gamma y)^{\beta}}\left(\alpha(1-\delta)-(\alpha-\alpha\delta-\delta-1)e^{-(\gamma y)^{\beta}}\right)}{\left(\alpha+(1-\alpha)e^{-(\gamma y)^{\beta}}\right)^{3}}. \tag{5}$$

Using the series expansion the $pdf$ of the transmuted complementary Weibull geometric distribution in (5) can be expressed in the mixture form

$$f_{TCWG}(y,\alpha,\beta,\gamma,\delta)=\frac{\beta\gamma(1-\delta)}{2\alpha}(\gamma y)^{\beta-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i}\Gamma(j+3)}{\Gamma(2-i)i!j!}k^{i}\left(1-\frac{1}{\alpha}\right)^{j}e^{-(i+j+1)(\gamma y)^{\beta}}. \tag{6}$$

where $k=(\alpha-\alpha\delta-\delta-1)/\alpha(1-\delta),\delta\neq1.$

Figures 1 and 2 show some of the possible shapes of the $pdf$ and $cdf$ of TCWGD for different choices of the parameters $\alpha,\beta,\gamma$ and $\delta$ respectively.

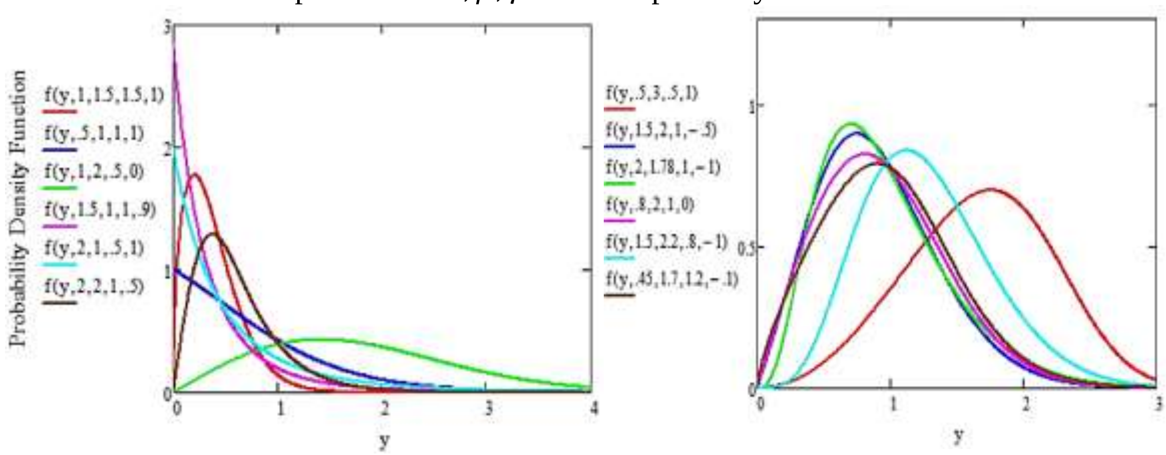


Figure 1: Probability Density Function of the TCWGD

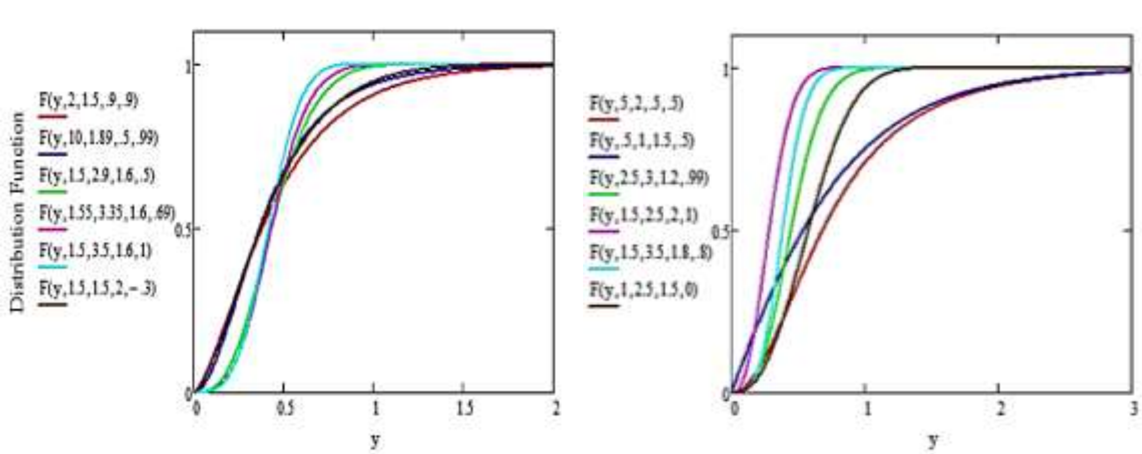


Figure 2: Distribution Function of the TCWGD

The transmuted complementary Weibull geometric distribution is very flexible model that approaches to different distributions when its parameters are changed. The subject distribution contains 11 sub models of well known and unknown probability distributions, as special cases, such as the transmuted complementary exponential geometric distribution (TCEGD), the transmuted complementary Rayleigh geometric distribution (TCRGD) and the complementary Rayleigh geometric distributio (CRGD). These three sub models are new distributions. The flexibility of the transmuted complementary Weibull geometric distribution is illustrated in the following .

**Corollary 1** *If $Y$ is a random variable with $pdf$ in (5), then we have the following special cases.*

1. When $\delta = 0$, we get the complementary Weibull geometric distribution, CWGD$(\alpha, \beta, \gamma, y)$.
2. When $\beta = 1$, we get the transmuted complementary exponential geometric distribution, TCEGD$(\alpha, \gamma, \delta, y)$. (New)
3. When $\beta = 2$, we get the transmuted complementary Rayleigh geometric distribution, TCRGD$(\alpha, \gamma, \delta, y)$. (New)
4. When $\beta = 1$ and $\delta = 0$, we get the complementary exponential geometric distribution, CEGD$(\alpha, \gamma, y)$.
5. When $\beta = 2$ and $\delta = 0$, we get the complementary Rayleigh geometric distribution, CRGD$(\alpha, \gamma, y)$. (New)
6. When $\alpha = 1$, we get the transmuted Weibull distribution, TWD$(\beta, \gamma, \delta, y)$.
7. When $\alpha = 1$ and $\delta = 0$, we get the Weibull geometric distribution, WD$(\beta, \gamma, y)$.
8. When $\alpha = \beta = 1$, we get the transmuted exponential distribution, TED$(\gamma, \delta, y)$.
9. When $\alpha = \beta = 1$, and $\delta = 0$, we get the exponential distribution, ED$(\gamma, y)$.
10. When $\alpha = 1$ and $\beta = 2$, we get the transmuted Rayleigh distribution, TRD$(\gamma, \delta, y)$.
11. When $\alpha = 1$, $\beta = 2$, and $\delta = 0$, we get the Rayleigh distribution, RD$(\gamma, y)$.

## 3. Reliability Analysis

The characteristics in reliability analysis which are the reliability function (RF), the hazard rate function (HF), the cumulative hazard rate function (CHF) for the TCWGD$(\alpha, \beta, \gamma, \delta, y)$ are introduced in this section.

### 3.1 Reliability Function

The reliability function $(RF)$ also known as the survival function, which is the probability of an item not failing prior to some time $t$, is defined by $R(y) = 1 - F(y)$. The reliability function of the transmuted complementary Weibull geometric distribution denoted by $R_{TCWG}(y, \alpha, \beta, \gamma, \delta)$, can be a useful characterization of life time data analysis. It can be defined as $R_{TCWG}(y, \alpha, \beta, \gamma, \delta) = 1 - F_{TCWG}(y, \alpha, \beta, \gamma, \delta)$,

$$R_{TCWG}(y, \alpha, \beta, \gamma, \delta) = \frac{\left(\alpha(1-\delta) - (\alpha - \alpha\delta - 1)e^{-(\gamma y)^{\beta}}\right)e^{-(\gamma y)^{\beta}}}{\left(\alpha + (1-\alpha)e^{-(\gamma y)^{\beta}}\right)^{2}}. \tag{7}$$

Using the series expansion the $RF$ of the transmuted complementary Weibull geometric distribution in (7) can be expressed in the mixture form as follows

$$R_{TCWG}(y,\alpha,\beta,\gamma,\delta)=\frac{1-\delta}{\alpha}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i}(j+1)}{\Gamma(2-i)i!}p^{i}\left(1-\frac{1}{\alpha}\right)^{j}e^{-(i+j+1)(\gamma y)^{\beta}}. \tag{8}$$

where $p=(\alpha-\alpha\delta-1)/\alpha(1-\delta),\delta\neq 1$.

Figure 3 illustrates the pattern of the transmuted complementary Weibull geometric distribution reliability function with different choices of parameters $\alpha,\beta,\gamma$ and $\delta$.

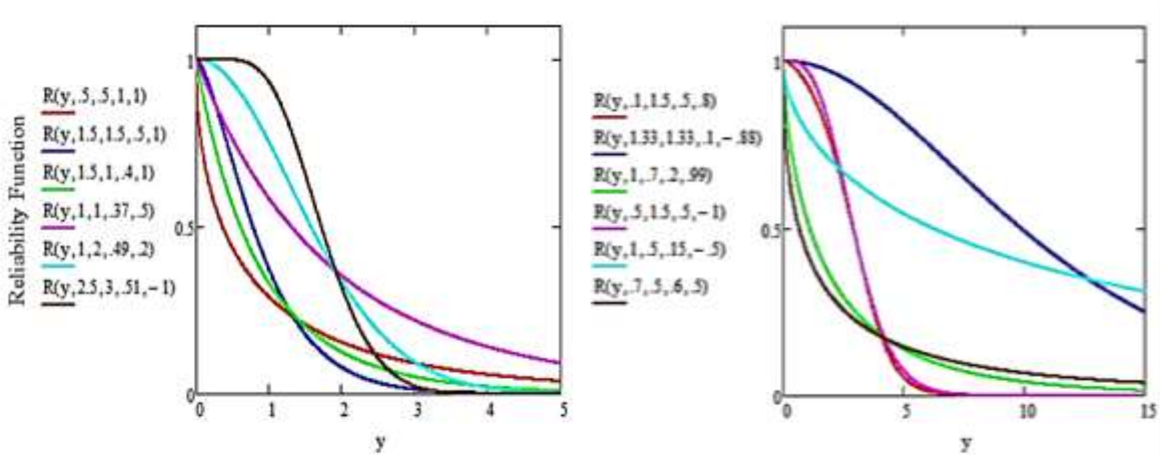


Figure 3: Reliability Function of the TCWGD

### 3.2 Hazard Rate Function

The other characteristic of interest of a random variable is the hazard rate function $(HF)$.The hazard function of the transmuted complementary Weibull geometric distribution also known as instantaneous failure rate denoted by $h_{TCWG}(y)$, is an important quantity characterizing life phenomenon. It can be loosely interpreted as the conditional probability of failure, given it has survived to the time $t$. The $HF$ of the transmuted complementary Weibull geometric distribution is defined by $h_{TCWG}(y,\alpha,\beta,\gamma,\delta)=f_{TCWG}(y,\alpha,\beta,\gamma,\delta)/R_{TCWG}(y,\alpha,\beta,\gamma,\delta)$,

$$h_{TCWG}(y,\alpha,\beta,\gamma,\delta)=\frac{\alpha\beta\gamma(\gamma y)^{\beta-1}e^{-(\gamma y)^{\beta}}\left(\alpha(1-\delta)+(1-\alpha+\alpha\delta+\delta)e^{-(\gamma y)^{\beta}}\right)}{\left(\alpha+(1-\alpha)e^{-(\gamma y)^{\beta}}\right)\left(\alpha(1-\delta)e^{-(\gamma y)^{\beta}}+(1-\alpha+\alpha\delta)e^{-2(\gamma y)^{\beta}}\right)}. \tag{9}$$

Using the series expansion, the $HF$ of the transmuted complementary Weibull geometric distribution in (9) can be expressed in the mixture form as follows

$$h_{TCWG}(y,\alpha,\beta,\gamma,\delta)=\beta\gamma(\gamma y)^{\beta-1}\frac{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i}\Gamma(j+3)}{\Gamma(2-i)i!j!}k^{i}\left(1-\frac{1}{\alpha}\right)^{j}e^{-(i+j+1)(\gamma y)^{\beta}}}{2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i}(j+1)}{\Gamma(2-i)i!}p^{i}\left(1-\frac{1}{\alpha}\right)^{j}e^{-(i+j+1)(\gamma y)^{\beta}}}. \tag{10}$$

where $p$ and $k$ are defined above. It is important to note that the units for $h_{TCWG}(y)$ is the probability of failure per unit of time, distance or cycles. These failure rates are defined with different choices of parameters.

Figure 4 illustrates some of the possible shapes of the hazard rate function of the transmuted complementary Weibull geometric distribution for different values of the parameters$\alpha, \beta, \gamma$ and $\delta$.

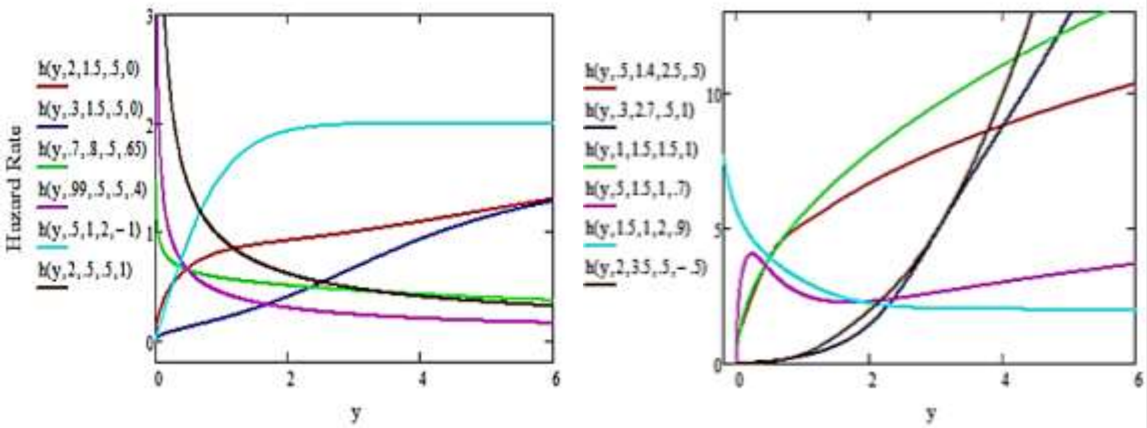


Figure 4: Hazard Rate of the TEFD

**Corollary 2** *The hazard rate function of the transmuted complementary Weibull geometric distribution TCWGD*$(\alpha, \beta, \gamma, \delta, y)$ *has the following special cases*

1. When $\delta = 0$, the failure rate is same as the CWGD$(\alpha, \beta, \gamma, y)$

$$h_{CWG}(y, \alpha, \beta, \gamma) = \frac{\alpha\beta\gamma(\gamma y)^{\beta-1}}{\alpha+(1-\alpha)e^{-(\gamma y)^{\beta}}}.$$

2. When $\beta = 1$, the failure rate is same as the TCEGD$(\alpha, \gamma, \delta, y)$.

$$h_{TCEG}(y, \alpha, \gamma, \delta) = \frac{\alpha\gamma e^{-(\gamma y)}\left(\alpha(1-\delta) + (1-\alpha+\alpha\delta+\delta)e^{-(\gamma y)}\right)}{\left(\alpha+(1-\alpha)e^{-(\gamma y)}\right)\left(\alpha(1-\delta)e^{-(\gamma y)} + (1-\alpha+\alpha\delta)e^{-2(\gamma y)}\right)}.$$

3. When $\beta = 1$ and $\delta = 0$, the failure rate is same as the CEGD$(\alpha, \gamma, y)$

$$h_{CEG}(y, \alpha, \gamma) = \frac{\alpha\gamma}{\alpha+(1-\alpha)e^{-(\gamma y)}}.$$

4. When $\beta = 2$, the failure rate is same as the TCRGD$(\alpha, \gamma, \delta, y)$. (New)

$$h_{TCRG}(y, \alpha, \gamma, \delta) = \frac{2\alpha\gamma^2 \mathrm{y} e^{-(\gamma y)^2}\left(\alpha(1-\delta) + (1-\alpha+\alpha\delta+\delta)e^{-(\gamma y)^2}\right)}{\left(\alpha+(1-\alpha)e^{-(\gamma y)^2}\right)\left(\alpha(1-\delta)e^{-(\gamma y)^2} + (1-\alpha+\alpha\delta)e^{-2(\gamma y)^2}\right)}.$$

5. When $\beta = 2$ and $\delta = 0$, the failure rate is same as the CRGD$(\alpha, \gamma, y)$. (New)

$$h_{CRG}(y, \alpha, \gamma) = \frac{2\alpha\gamma^2 \mathrm{y}}{\left(\alpha+(1-\alpha)e^{-(\gamma y)^2}\right)}.$$

6. When $\alpha = 1$, the failure rate is same as the TWD$(\beta, \gamma, \delta, y)$.

$$h_{TW}(y, \beta, \gamma, \delta) = \frac{\beta\gamma(\gamma y)^{\beta-1} e^{-(\gamma y)^\beta}\left((1-\delta) + 2\delta e^{-(\gamma y)^\beta}\right)}{\left((1-\delta) e^{-(\gamma y)^\beta} + \delta e^{-2(\gamma y)^\beta}\right)}.$$

7. When $\alpha = 1$ and $\delta = 0$, the failure rate is same as the WD$(\beta, \gamma, y)$.

$$h_W(y, \beta, \gamma) = \beta\gamma(\gamma y)^{\beta-1}.$$

8. When $\alpha = \beta = 1$, the failure rate is same as the TED$(\gamma, \delta, y)$.

$$h_{TE}(y, \gamma, \delta) = \frac{\gamma e^{-(\gamma y)}\left((1-\delta) + 2\delta e^{-(\gamma y)}\right)}{\left((1-\delta) e^{-(\gamma y)} + \delta e^{-2(\gamma y)}\right)}.$$

9. When $\alpha = \beta = 1$, and $\delta = 0$, the failure rate is same as the ED$(\gamma, x)$.

$$h_E(y, \gamma) = \gamma.$$

10. When $\alpha = 1$ and $\beta = 2$, the failure rate is same as the TRD$(\gamma, \delta, y)$.

$$h_{TR}(y, \gamma, \delta) = \frac{2\gamma^2 \mathrm{y} e^{-(\gamma y)^2}\left((1-\delta) + 2\delta e^{-(\gamma y)^2}\right)}{\left((1-\delta) e^{-(\gamma y)^2} + \delta e^{-2(\gamma y)^2}\right)}.$$

11. When $\alpha = 1$, $\beta = 2$, and $\delta = 0$, the failure rate is same as the RD$(\gamma, y)$.

$$h_R(y, \gamma) = 2\gamma^2 \mathrm{y}.$$

## 3.3 Cumulative Hazard Rate Function

The Cumulative hazard function $(CHF)$ of the transmuted complementary Weibull geometric distribution, denoted by $H_{TCWG}(y, \alpha, \beta, \gamma, \delta)$, is defined as

$$H_{TCWG}(y, \alpha, \beta, \gamma, \delta) = \int_0^x h_{TCWG}(y, \alpha, \beta, \gamma, \delta)\, dx = -\ln R_{TCWG}(y, \alpha, \beta, \gamma, \delta),$$

$$H_{TCWG}(y, \alpha, \beta, \gamma, \delta) = \ln\left(\frac{\left(\alpha + (1-\alpha) e^{-(\gamma y)^\beta}\right)^2}{\alpha(1-\delta) e^{-(\gamma y)^\beta} + (1-\alpha+\alpha\delta) e^{-2(\gamma y)^\beta}}\right). \tag{11}$$

We can express the $CHF$ of the TCWGD using the series expansion as follows

$$H_{TCWG}(y, \alpha, \beta, \gamma, \delta) = -\ln\left(\frac{1-\delta}{\alpha} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^i (j+1)}{\Gamma(2-i) i!} p^i \left(1 - \frac{1}{\alpha}\right)^j e^{-(i+j+1)(\gamma y)^\beta}\right). \tag{12}$$

where $\ell$ is defined above. It is important to note that the units for $H_{TCWG}(y, \alpha, \beta, \gamma, \delta)$ is the cumulative probability of failure or death per unit of time, distance or cycles.

**Corollary 3** *The cumulative hazard rate function of the transmuted complementary Weibull geometric distribution TCWGD$(\alpha, \beta, \gamma, \delta, y)$ has the following special cases*

1. When $\delta = 0$, the cumulative failure rate is same as the CWGD$(\alpha, \beta, \gamma, y)$

$$H_{CWG}(y, \alpha, \beta, \gamma) = \ln\left(\frac{\left(\alpha+(1-\alpha)e^{-(\gamma y)^{\beta}}\right)^2}{\alpha e^{-(\gamma y)^{\beta}}+(1-\alpha)e^{-2(\gamma y)^{\beta}}}\right).$$

2. When $\beta = 1$, the cumulative failure rate is same as the TCEGD$(\alpha, \gamma, \delta, y)$.

$$H_{TCEG}(y, \alpha, \gamma, \delta) = \ln\left(\frac{\left(\alpha + (1-\alpha)e^{-(\gamma y)}\right)^2}{\alpha(1-\delta)e^{-(\gamma y)} + (1-\alpha+\alpha\delta)e^{-2(\gamma y)}}\right).$$

3. When $\beta = 1$ and $\delta = 0$, the cumulative failure rate is same as the CEGD$(\alpha, \gamma, y)$

$$H_{CEG}(y, \alpha, \gamma) = \gamma y + \ln(\alpha + (1-\alpha)e^{-\gamma y})$$

4. When $\beta = 2$, the cumulative failure rate is same as the TCRGD$(\alpha, \gamma, \delta, y)$. (New)

$$H_{TCRG}(y, \alpha, \gamma, \delta) = \ln\left(\frac{\left(\alpha+(1-\alpha)e^{-(\gamma y)^2}\right)^2}{\alpha(1-\delta)e^{-(\gamma y)^2}+(1-\alpha+\alpha\delta)e^{-2(\gamma y)^2}}\right).$$

5. When $\beta = 2$ and $\delta = 0$, the cumulative failure rate is same as the CRGD$(\alpha, \gamma, y)$. (New)

$$H_{CRG}(y, \alpha, \gamma) = (\gamma y)^2 + \ln\left(\alpha + (1-\alpha)e^{-(\gamma y)^2}\right).$$

6. When $\alpha = 1$, the cumulative failure rate is same as the TWD$(\beta, \gamma, \delta, y)$.

$$H_{TW}(y, \beta, \gamma, \delta) = -\ln\left((1-\delta)e^{-(\gamma y)^{\beta}} + \delta e^{-2(\gamma y)^{\beta}}\right).$$

7. When $\alpha = 1$ and $\delta = 0$, the cumulative failure rate is same as the WD$(\beta, \gamma, y)$.

$$H_W(y, \beta, \gamma) = \ln\left(e^{(\gamma y)^{\beta}}\right) = (\gamma y)^{\beta}.$$

8. When $\alpha = \beta = 1$, the cumulative failure rate is same as the TED$(\gamma, \delta, y)$.

$$H_{TE}(y, \gamma, \delta) = -\ln\left((1-\delta)e^{-(\gamma y)} + \delta e^{-2(\gamma y)}\right).$$

9. When $\alpha = \beta = 1$, and $\delta = 0$, the cumulative failure rate is same as the ED$(\gamma, x)$.

$$H_E(y, \gamma) = \ln\left(e^{(\gamma y)}\right) = \gamma y.$$

10. When $\alpha = 1$ and $\beta = 2$, the cumulative failure rate is same as the TRD$(\gamma, \delta, y)$.

$$H_{TR}(y, \gamma, \delta) = -\ln\left((1-\delta)e^{-(\gamma y)^2} + \delta e^{-2(\gamma y)^2}\right).$$

11. When $\alpha = 1$, $\beta = 2$, and $\delta = 0$, the cumulative failure rate is same as the RD$(\gamma, y)$.

$$H_R(y, \gamma) = (\gamma y)^2.$$

Figure 5. illustrates some of the possible shapes of the cumulative hazard rate of the transmuted complementary Weibull geometric distribution for different values of the parameters$\alpha, \beta, \gamma$ and $\delta$.

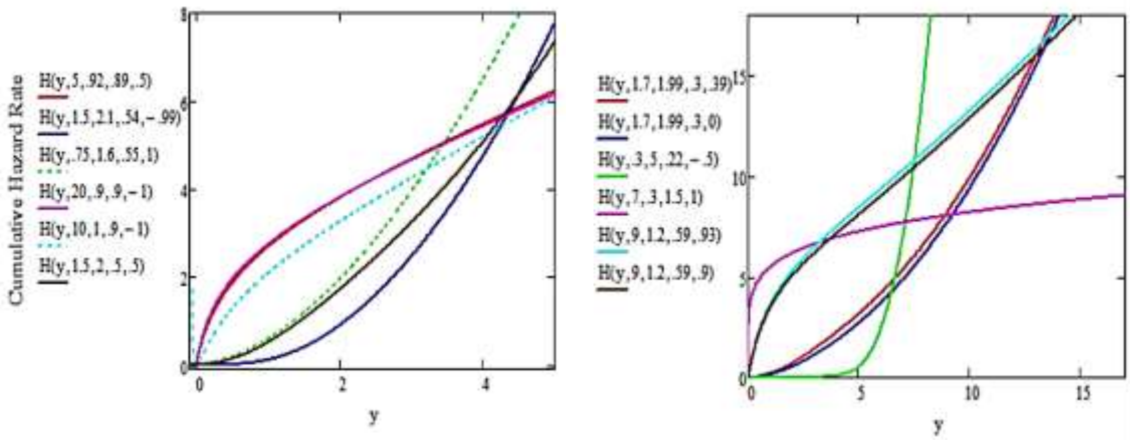


Figure 5: Cumulative Hazard Rate of the TCWGD

## 4. Statistical properties

The statistical properties of the transmuted complementary Weibull geometric distribution including quantile and random number generation, moments, moment generating function and Rényi entropy are discussed in this section.

### 4.1 Quantile and Median

The quantile $y_q$ of the transmuted complementary Weibull geometric distribution TCWGD$(\alpha,\beta,\gamma,\delta,y)$ is the real solution of the following equation $F(y_q)=q$, and is given by the following

$$y_q=\gamma^{-1}\left\{\ln\left(\frac{-k+\alpha\sqrt{1+\delta(\delta-4q+2)}}{2\alpha^2(1-q)}\right)\right\}^{1/\beta},0\leq q\geq 1, \tag{13}$$

where $k=2\alpha q(\alpha-1)-2\alpha^2+\alpha(1+\delta)$.

If we put $q=0.5$ in the above equation we can get the median of the TCWGD$(\alpha,\beta,\gamma,\delta,y)$. The quantile $y_q$ of the transmuted complementary Weibull geometric distribution. The median life of the subject distribution is the 50-th percentile. In practice, this is the life by which 50 percent of the units will be expected to have failed and so it is the life at which 50 percent of the units would be expected to still survive.

### 4.2 Random Number Generation

The random number $y$ of the TCWGD$(\alpha,\beta,\gamma,\delta,y)$ is defined by the following relation $F_{TCWG}(y,\alpha,\beta,\gamma,\delta)=\zeta$, where $\zeta\sim U(0,1)$, then

$$\frac{\alpha^2+\left((\alpha\delta+\alpha-2\alpha^2)-(\alpha\delta+\alpha-\alpha^2)e^{-(\gamma y)^\beta}\right)e^{-(\gamma y)^\beta}}{\left(\alpha+(1-\alpha)e^{-(\gamma y)^\beta}\right)^2}=\zeta.$$

Solving for $y$,we get

$$y=\gamma^{-1}\left\{\ln\left(\frac{2\alpha^2-2\alpha q(\alpha-1)-\alpha(1+\delta)+\alpha\sqrt{1+\delta(\delta-4q+2)}}{2\alpha^2(1-q)}\right)\right\}^{1/\beta}. \tag{14}$$

Using a random number uniformly distributed from zero to one, we have solved the above equation to obtain a random number in $y$.

### 4.3 Moments

The $r^{th}$ moment, denoted by $\mu'_r$,of the TCWGD$(\alpha,\beta,\gamma,\delta,y)$ is given by the following theorem.

**Theorem 1.** *If $Y$ is a continuous random variable has the* TCWGD$(\alpha,\beta,\gamma,\delta,y)$ *with* $|\delta|\leq 1$, *then the $r^{th}$ non-central moment of $Y$ is given as follows*

$$\mu'_r = E(Y^r) = \frac{(1-\delta)}{2\alpha\gamma^r}\Gamma\left(1+\frac{r}{\beta}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i\Gamma(j+3)}{\Gamma(2-i)i!j!}k^i\left(1-\frac{1}{\alpha}\right)^j(i+j+)^{-(1+r/\beta)}. \quad (15)$$

*Proof:*

By definition

$$\begin{aligned}\mu'_r &= \int_0^\infty y^r f_{TCWG}(x,\alpha,\beta,\gamma,\delta)dy \\ &= \frac{\beta\gamma^\beta(1-\delta)}{2\alpha}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i\Gamma(j+3)}{\Gamma(2-i)i!j!}k^i\left(1-\frac{1}{\alpha}\right)^j\int_0^\infty y^{r+\beta-1}e^{-(i+j+1)(\gamma y)^\beta}dy.\end{aligned} \quad (16)$$

To compute $\int_0^\infty y^{r+\beta-1}e^{-(i+j+1)(\gamma y)^\beta}dy$,let $t=(i+j+1)(\gamma y)^\beta$. Then $y=\frac{1}{\gamma}\left(\frac{t}{i+j+1}\right)^{1/\beta}$. Therefore

$$\int_0^\infty y^{r+\beta-1}e^{-(i+j+1)(\gamma y)^\beta}dy = \frac{(i+j+1)^{-(1+r/\beta)}}{\beta\gamma^{r+\beta}}\Gamma\left(1+\frac{r}{\beta}\right). \quad (17)$$

By substituting from Equation (17) into Equation (16), we obtain

$$\mu'_r = \frac{(1-\delta)}{2\alpha\gamma^r}\Gamma\left(1+\frac{r}{\beta}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i\Gamma(j+3)}{\Gamma(2-i)i!j!}k^i\left(1-\frac{1}{\alpha}\right)^j(i+j+1)^{-\left(1+\frac{r}{\beta}\right)}.$$

where $k=(\alpha-\alpha\delta-\delta-1)/\alpha(1-\delta), \delta\neq 1$. This completes the proof.

Based on the above Theorem (1) the coefficient of variation, coefficient of skewness and coefficient of kurtosis of the TCWGD$(\alpha,\beta,\gamma,\delta,y)$ distribution can be obtained according to the following relations

$$CV_{TCWG} = \sqrt{\frac{\mu'_2-\mu'_1}{\mu'_1}},$$

$$CS_{TCWG} = \frac{\mu'_3-3\mu'_2\mu'_1+2(\mu'_1)^3}{(\mu'_2-(\mu'_1)^2)^{3/2}},$$

and

$$CK_{TCWG} = \frac{\mu'_4-4\mu'_3\mu'_1+6\mu'_2(\mu'_1)^2-3(\mu'_1)^4}{(\mu'_2-(\mu'_1)^2)^2}.$$

**Corollary 4** *Using the relation between the central moments and non-centeral moments, we can obtain the $n^{th}$ central moment, denoted by $M_n$, of a TCWG random variable as follows*

$$M_n = E(Y-\mu)^n = \sum_{r=0}^{n} \binom{n}{r} (-\mu)^{n-r} E(Y^r),$$

where $E(Y^r)$ is the on-central moments of the TCWGD$(\alpha,\beta,\gamma,\delta,y)$. Therefore the $n^{th}$ central moments of the TCWGD$(\alpha,\beta,\gamma,\delta,y)$ is given by

$$M_n = \frac{(1-\delta)}{2\alpha} \sum_{r=0}^{n} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{i+n-r}\Gamma(j+3)}{\Gamma(2-i)i!j!} \binom{n}{r} (\mu)^{n-r} \gamma^{-r} k^i \times \left(1-\frac{1}{\alpha}\right)^j (i+j+1)^{-(1+r/\beta)} \Gamma\left(1+\frac{r}{\beta}\right). \tag{18}$$

where $k$ is mentioned above.

### 4.4 Moment Generating Function

The moment generating function $(mgf)$ of the transmuted complementary Weibull geometric distribution is given by the following theorem.

**Theorem 2.** *If $Y$ is a continuous random variable has the TCWGD$(\alpha,\beta,\gamma,\delta,y)$ with $|\delta| \le 1$, then the moment generating function $(mgf)$of $Y$,denoted by $M_Y(t) = E(e^{tY})$, is given as follows*

$$M_Y(t) = \frac{(1-\delta)}{2\alpha} \sum_{r=0}^{n} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{i}\Gamma(j+3)}{\Gamma(2-i)r!i!j!} k^i \left(\frac{t}{\gamma}\right)^r \left(1-\frac{1}{\alpha}\right)^j (i+j+1)^{-(1+r/\beta)} \Gamma\left(1+\frac{r}{\beta}\right). \tag{19}$$

*Proof:*

By definition

$$\begin{aligned} M_Y(t) &= \int_0^\infty e^{tx} f_{TCWG}(x,\alpha,\beta,\gamma,\delta) dy \\ &= \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^\infty y^r f_{TCWG}(x,\alpha,\beta,\gamma,\delta) dy \\ &= \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'. \end{aligned} \tag{20}$$

By substituting from Equation (15) into Equation (20), we obtain the following

$$M_Y(t) = \frac{(1-\delta)}{2\alpha} \sum_{r=0}^{n} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{i}\Gamma(j+3)}{\Gamma(2-i)r!i!j!} k^i \left(\frac{t}{\gamma}\right)^r \left(1-\frac{1}{\alpha}\right)^j (i+j+1)^{-(1+r/\beta)} \Gamma\left(1+\frac{r}{\beta}\right).$$

where $k$is mentioned above. Which completes the proof. the measure of central tendency, measure of dispersion, coefficient of variation, coefficient of skewness and coefficient of kurtosis of the TCWGD$(\alpha,\beta,\gamma,\delta,y)$ can be obtained according to the above relation in Theorem 2.

### 4.5 Rényi Entropy

Entropy refers to the amount of uncertainty associated with a random variable. The Rényi entropy has numerous applications in information theoretic learning, statistics (e.g. classification, distribution identification problems, and statistical inference), computer science (e.g. average case analysis for random databases, pattern recognition, and image matching) and econometrics, see Källberg et al. (2014). The Rényi entropy of a random variable $Y$ represents a measure of variation of the uncertainty. The Rényi entropy is defined by

$$I_\theta(Y)=\frac{1}{1-\theta}\log\int_{-\infty}^{\infty} f^\theta(y)dy, \theta>0 \text{ and } \theta\neq 1.$$

Therefore, the Rényi entropy of a random variable$Y$which follows the TCWGD$(\alpha,\beta,\gamma,\delta,y)$ is given by

$$I_\theta(Y)=\frac{1}{1-\theta}\log\left(\frac{\beta(1-\delta)}{\alpha}\right)^\theta \gamma^{\theta\beta}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i\Gamma(\theta+1)\Gamma(3\theta+j)}{\Gamma(\theta-i+1)\Gamma(3\theta)i!\,j!}k^i\left(1-\frac{1}{\alpha}\right)^j$$
$$\times\int_0^\infty y^{\theta(\beta-1)}e^{-(\theta+i+j)(\gamma y)^\beta}dy.$$

But

$$\int_0^\infty y^{\theta(\beta-1)}e^{-(\theta+i+j)(\gamma y)^\beta}dy=\frac{1}{\beta}\gamma^{\theta(1-\beta)-1}(\theta+i+j)^{(\theta(1-\beta)-1)/\beta}\Gamma\left(\frac{\theta(\beta-1)+1}{\beta}\right),$$

and then

$$I_\theta(X)=\frac{1}{1-\theta}\log\left\{\begin{matrix}(\alpha)^{-\theta}(1-\delta)^\theta(\beta\gamma)^{\theta-1}\Gamma\left(\frac{\theta(\beta-1)+1}{\beta}\right)\\ \times\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i\Gamma(\theta+1)\Gamma(3\theta+j)}{\Gamma(\theta-i+1)\Gamma(3\theta)i!j!}k^i\left(1-\frac{1}{\alpha}\right)^j(\theta+j+i)^\zeta\end{matrix}\right\}, \theta> 0 \text{and} \theta\neq 1, \tag{21}$$

where $k$ is mentioned above, $\zeta=(\theta(1-\beta)-1)/\beta$.

## 5. Order Statistics

The order statistics and their moments have great importance in many statistical problems and they have many applications in reliability analysis and life testing. The order statistics arise in the study of reliability of a system. The order statistics can represent the lifetimes of units or components of a reliability system. Let $Y_1,Y_2,\dots,Y_n$ be a random sample of size $n$ from the TCWGD$(\alpha,\beta,\gamma,\delta,y)$ with cumulative distribution function $(cdf)$, and the corresponding probability density function $(pdf)$, as in (3) and (5), respectively. Let $Y_{(1)},Y_{(2)},\dots,Y_{(n)}$ be the corresponding order statistics. Then the $pdf$ of $Y_{(r:n)}, 1\leq r\leq n$,denoted by $f_{r:n}(y)$,is given by

$$f_{r:n}(y) = C_{r:n} f_{TCWG}(y,\alpha,\beta,\gamma,\delta)[F_{TCWG}(y,\alpha,\beta,\gamma,\delta)]^{r-1}[R_{TCWG}(y,\alpha,\beta,\gamma,\delta)]^{n-r},$$

$$f_{r:n}(y) = C_{r:n}\frac{\alpha\beta\gamma(\gamma y)^{\beta-1}e^{-(\gamma y)^\beta}\left(\ell_1-\ell_2 e^{-(\gamma y)^\beta}\right)}{\left(\alpha+(1-\alpha)e^{-(\gamma y)^\beta}\right)^3}\times\frac{\left(\alpha^2+\ell_3 e^{-(\gamma y)^\beta}-\ell_4 e^{-2(\gamma y)^\beta}\right)^{r-1}}{\left(\alpha+(1-\alpha)e^{-(\gamma y)^\beta}\right)^{2(r-1)}}$$
$$\times\frac{\left(\ell_1 e^{-(\gamma y)^\beta}-\ell_5 e^{-2(\gamma y)^\beta}\right)^{n-r}}{\left(\alpha+(1-\alpha)e^{-(\gamma y)^\beta}\right)^{2(n-r)}}.$$

Or equivalently

$$f_{r:n}(x) = \frac{\left\{\begin{matrix} C_{r:n}.\alpha\beta\gamma(\gamma y)^{\beta-1}e^{-(\gamma y)^\beta}\left(\ell_1-\ell_2 e^{-(\gamma y)^\beta}\right)\left(\alpha^2+\ell_3 e^{-(\gamma y)^\beta}-\ell_4 e^{-2(\gamma y)^\beta}\right)^{r-1} \\ \times\left(\ell_1 e^{-(\gamma y)^\beta}-\ell_5 e^{-2(\gamma y)^\beta}\right)^{n-r}\end{matrix}\right\}}{\left(\alpha+(1-\alpha)e^{-(\gamma y)^\beta}\right)^{2n+1}}. \quad (22)$$

The joint $pdf$ of $Y_{(r:n)}$ and $Y_{(j:n)}, 1 \le r \le j \le n,$ is given by

$$f_{r:j:n}(y_r,y_j) = C_{r:j:n}\frac{\alpha\beta\gamma(\gamma y_r)^{\beta-1}e^{-(\gamma y_r)^\beta}\left(\ell_1-\ell_2 e^{-(\gamma y_r)^\beta}\right)}{\left(\alpha+(1-\alpha)e^{-(\gamma y_r)^\beta}\right)^3}\times\frac{\left(\alpha^2+\ell_3 e^{-(\gamma y_r)^\beta}-\ell_4 e^{-2(\gamma y_r)^\beta}\right)^{r-1}}{\left(\alpha+(1-\alpha)e^{-(\gamma y_r)^\beta}\right)^{2(r-1)}}$$
$$\times\frac{\alpha\beta\gamma(\gamma y_j)^{\beta-1}e^{-(\gamma y_j)^\beta}\left(\ell_1-\ell_2 e^{-(\gamma y_j)^\beta}\right)}{\left(\alpha+(1-\alpha)e^{-(\gamma y_j)^\beta}\right)^3}\times\frac{\left(\ell_1 e^{-(\gamma y_j)^\beta}-\ell_5 e^{-2(\gamma y_j)^\beta}\right)^{n-j}}{\left(\alpha+(1-\alpha)e^{-(\gamma y_j)^\beta}\right)^{2(n-j)}}$$
$$\times\left\{\frac{\alpha^2+\ell_3 e^{-(\gamma y_j)^\beta}-\ell_4 e^{-2(\gamma y_j)^\beta}}{\left(\alpha+(1-\alpha)e^{-(\gamma y_j)^\beta}\right)^2}-\frac{\alpha^2+\ell_3 e^{-(\gamma y_r)^\beta}-\ell_4 e^{-2(\gamma y_r)^\beta}}{\left(\alpha+(1-\alpha)e^{-(\gamma y_r)^\beta}\right)^2}\right\}^{j-r-1}, \quad (23)$$

where $C_{r:n} = \frac{n!}{(r-1)!(n-r)!}, C_{r:j:n} = \frac{n!}{(r-1)!(j-r-1)!(n-j)!}, \ell_1 = \alpha(1-\delta),$
$\ell_2 = \alpha - \alpha\delta - \delta - 1, \ell_3 = \alpha + \alpha\delta - 2\alpha^2, \ell_4 = \alpha + \alpha\delta - \alpha^2$ and $\ell_5 = \alpha - \alpha\delta - 1.$

## 5.1 Distribution of Minimum and Maximum

Let $Y_1, Y_2, \ldots, Y_n$ be $n$ independently identically distributed order random variables from the transmuted complementary Weibull geometric distribution. Here we define $X_{(1)} = Min(Y_1, Y_2, \ldots, Y_n)$ and $Y_{(n)} = Max(Y_1, Y_2, \ldots, Y_n)$ ordered random variables. therefore, the $pdf$ of the smallest order statistic $Y_{(1)}$, the $pdf$ of the largest order statistic $Y_{(n)}$ and the $pdf$ of the median order statistic $Y_{(m+1)}$ are given by the following

$$f_{1:n}(y) = \frac{n\alpha\beta\gamma(\gamma y)^{\beta-1}e^{-(\gamma y)^\beta}\left(\ell_1-\ell_2 e^{-(\gamma y)^\beta}\right)\left(\ell_1 e^{-(\gamma y)^\beta}-\ell_5 e^{-2(\gamma y)^\beta}\right)^{n-1}}{\left(\alpha+(1-\alpha)e^{-(\gamma y)^\beta}\right)^{2n+1}}, \quad (24)$$

$$f_{n:n}(y) = \frac{n\alpha\beta\gamma(\gamma y)^{\beta-1}e^{-(\gamma y)^\beta}\left(\ell_1-\ell_2 e^{-(\gamma y)^\beta}\right)\left(\alpha^2+\ell_3 e^{-(\gamma y)^\beta}-\ell_4 e^{-2(\gamma y)^\beta}\right)^{n-1}}{\left(\alpha+(1-\alpha)e^{-(\gamma y)^\beta}\right)^{2n+1}}, \quad (25)$$

and

$$f_{m+1:n}(x) = \frac{(2m+1)!}{m!\,m!} f(x)[F(x)]^m [1-F(x)]^m,$$

$$f_{m+1:n}(y) = \frac{\left\{\begin{array}{c}\frac{(2m+1)!}{m!m!}\alpha\beta\gamma(\gamma y)^{\beta-1}e^{-(\gamma y)^\beta}\left(\ell_1-\ell_2 e^{-(\gamma y)^\beta}\right)\left(\alpha^2+\ell_3 e^{-(\gamma y)^\beta}-\ell_4 e^{-2(\gamma y)^\beta}\right)^m \\ \times\left(\ell_1 e^{-(\gamma y)^\beta}-\ell_5 e^{-2(\gamma y)^\beta}\right)^{\mathrm{m}}\end{array}\right\}}{\left(\alpha+(1-\alpha)e^{-(\gamma y)^\beta}\right)^{4m+3}}. \qquad (26)$$

## 5.2 Minimum and Maximum Joint Order Statistic

The joint probability density function of $r^{th}$ order statistic and $j^{th}$ order statistic is given in (23), then the minimum and maximum joint probability density of the TCWGD$(\alpha,\beta,\gamma,\delta,y)$, denoted by $f_{1:n:n}(y_1,y_n)$, can be obtained from Equation (23) by substituting $r=1$ and $j=n$ as follows

$$f_{1:n:n}(y_1,y_n) = C_{1:n:n}\frac{(\alpha\beta\gamma)^2(\gamma y_1)^{\beta-1}e^{-(\gamma y_1)^\beta}\left(\ell_1-\ell_2 e^{-(\gamma y_1)^\beta}\right)}{\left(\alpha+(1-\alpha)e^{-(\gamma y_1)^\beta}\right)^3}\cdot\frac{(\gamma y_n)^{\beta-1}e^{-(\gamma y_n)^\beta}\left(\ell_1-\ell_2 e^{-(\gamma y_n)^\beta}\right)}{\left(\alpha+(1-\alpha)e^{-(\gamma y_n)^\beta}\right)^3}$$
$$\times\left(\frac{\alpha^2+\ell_3 e^{-(\gamma y_n)^\beta}-\ell_4 e^{-2(\gamma y_n)^\beta}}{\left(\alpha+(1-\alpha)e^{-(\gamma y_n)^\beta}\right)^2}-\frac{\alpha^2+\ell_3 e^{-(\gamma y_1)^\beta}-\ell_4 e^{-2(\gamma y_1)^\beta}}{\left(\alpha+(1-\alpha)e^{-(\gamma y_1)^\beta}\right)^2}\right)^{n-2}. \qquad (27)$$

where$C_{1:n:n} = \frac{n!}{(n-2)!}$, $\ell_1 = \alpha(1-\delta)$, $\ell_2 = \alpha-\alpha\delta-\delta-1$, $\ell_3 = \alpha+\alpha\delta-2\alpha^2$ and

$$\ell_4 = \alpha+\alpha\delta-\alpha^2.$$

## 6. Maximum Likelihood Estimation

The maximum likelihood estimators (MLEs) for the parameters of the transmuted extended Fréchet distribution TCWGD$(\alpha,\beta,\gamma,\delta,y)$ is discussed in this section. Consider the random sample $x_1,x_2,\ldots,x_n$ of size $n$ from TCWGD$(\alpha,\beta,\gamma,\delta,y)$ with probability density function in (5), then the likelihood function can be expressed as follows

$$L(x_1,x_2,\ldots,x_n,\alpha,\beta,\gamma,\delta) = \prod_{i=1}^{n} f_{TCWGD}(x_i,\alpha,\beta,\gamma,\delta),$$

$$L = \frac{(\alpha\beta\gamma)^n\prod_{i=1}^{n}(\gamma y_i)^{\beta-1}e^{-(\gamma y_i)^\beta}\left(\alpha(1-\delta)-(\alpha-\alpha\delta-\delta-1)e^{-(\gamma y_i)^\beta}\right)}{\prod_{i=1}^{n}\left(\alpha+(1-\alpha)e^{-(\gamma y_i)^\beta}\right)^3}. \qquad (28)$$

Then, the log-likelihood function $Ł = \ln L$ becomes:

$$\begin{aligned} Ł = {} & n(\ln\alpha+\ln\beta+\ln\gamma)+(\beta-1)\sum_{i=1}^{n}\ln(\gamma y_i) \\ & -\sum_{i=1}^{n}(\gamma y_i)^\beta - 3\sum_{i=1}^{n}\ln\left(\alpha+(1-\alpha)e^{-(\gamma y_i)^\beta}\right) \\ & +\sum_{i=1}^{n}\ln\left(\alpha(1-\delta)-(\alpha-\alpha\delta-\delta-1)e^{-(\gamma y_i)^\beta}\right). \end{aligned} \qquad (29)$$

Differentiating Equation (29) with respect to $\alpha, \beta, \gamma$ and $\delta$ then equating it to zero, we obtain the MLEs of $\alpha, \beta, \gamma$ and $\delta$ as follows

$$\frac{\partial Ł}{\partial \alpha} = \frac{n}{\alpha} - 3\sum_{i=1}^{n} \frac{1-e^{-(\gamma y_i)^{\beta}}}{\left(\alpha+(1-\alpha)e^{-(\gamma y_i)^{\beta}}\right)} + \sum_{i=1}^{n} \frac{(1-\delta)\left(1-e^{-(\gamma y_i)^{\beta}}\right)}{\alpha(1-\delta)-(\alpha-\alpha\delta-\delta-1)e^{-(\gamma y_i)^{\beta}}} = 0, \quad (30)$$

$$\begin{aligned}\frac{\partial Ł}{\partial \beta} &= \frac{n}{\beta} + \sum_{i=1}^{n} \left(1-(\gamma y_i)^{\beta}\right)\ln(\gamma y_i) + \sum_{i=1}^{n} (\gamma y_i)^{\beta}\ln(\gamma y_i)e^{-(\gamma y_i)^{\beta}} \\ &\times \left(\frac{3(1-\alpha)}{\alpha+(1-\alpha)e^{-(\gamma y_i)^{\beta}}} + \frac{\alpha-\alpha\delta-\delta-1}{\alpha(1-\delta)-(\alpha-\alpha\delta-\delta-1)e^{-(\gamma y_i)^{\beta}}}\right) = 0,\end{aligned} \quad (31)$$

$$\begin{aligned}\frac{\partial Ł}{\partial \gamma} &= \frac{n\beta}{\gamma} - \frac{\beta}{\gamma}\sum_{i=1}^{n} (\gamma y_i)^{\beta} + \frac{\beta}{\gamma}\sum_{i=1}^{n} (\gamma y_i)^{\beta} e^{-(\gamma y_i)^{\beta}} \\ &\times \left(\frac{3(1-\alpha)}{\alpha+(1-\alpha)e^{-(\gamma y_i)^{\beta}}} + \frac{\alpha-\alpha\delta-\delta-1}{\alpha(1-\delta)-(\alpha-\alpha\delta-\delta-1)e^{-(\gamma y_i)^{\beta}}}\right) = 0,\end{aligned} \quad (32)$$

and

$$\frac{\partial Ł}{\partial \delta} = \sum_{i=1}^{n} \frac{(1+\alpha)e^{-(\gamma y_i)^{\beta}}-\alpha}{\alpha(1-\delta)-(\alpha-\alpha\delta-\delta-1)e^{-(\gamma y_i)^{\beta}}} = 0. \quad (33)$$

We can find the estimates of the unknown parameters by maximum likelihood method by setting these above nonlinear system of Equations (30) - (33) to zero and solve them simultaneously. These solutions will yield the ML estimators $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ and $\hat{\delta}$. For the four parameters transmuted complementary Weibull geometric distribution TCWGD$(\alpha, \beta, \gamma, \delta, y)$ $pdf$ all the second order derivatives exist. Thus we have the inverse dispersion matrix is given by

$$\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\\ \hat{\gamma}\\ \hat{\delta}\end{pmatrix} \sim N\left[\begin{pmatrix}\alpha\\ \beta\\ \gamma\\ \delta\end{pmatrix}, \begin{pmatrix}\hat{V}_{11} & \hat{V}_{12} & \hat{V}_{13} & \hat{V}_{14}\\ \hat{V}_{21} & \hat{V}_{22} & \hat{V}_{23} & \hat{V}_{24}\\ \hat{V}_{31} & \hat{V}_{32} & \hat{V}_{33} & \hat{V}_{34}\\ \hat{V}_{41} & \hat{V}_{42} & \hat{V}_{43} & \hat{V}_{44}\end{pmatrix}\right],$$

$$V^{-1} = -E\begin{pmatrix}V_{11} & V_{12} & V_{13} & V_{14}\\ V_{21} & V_{22} & V_{23} & V_{24}\\ V_{31} & V_{32} & V_{33} & V_{34}\\ V_{41} & V_{42} & V_{43} & V_{44}\end{pmatrix}. \quad (34)$$

Equation (34) is the variance covariance matrix of the TCWGD$(\alpha, \beta, \gamma, \delta, y)$, where

$$V_{11} = \frac{\partial^2 Ł}{\partial \alpha^2} \qquad V_{12} = \frac{\partial^2 Ł}{\partial \alpha\, \partial \beta} \qquad V_{13} = \frac{\partial^2 Ł}{\partial \alpha\, \partial \gamma} \qquad V_{14} = \frac{\partial^2 Ł}{\partial \alpha\, \partial \delta}$$

$$V_{22} = \frac{\partial^2 Ł}{\partial \beta^2} \qquad V_{23} = \frac{\partial^2 Ł}{\partial \beta\, \partial \gamma} \qquad V_{24} = \frac{\partial^2 Ł}{\partial \beta\, \partial \delta}$$

$$V_{33} = \frac{\partial^2 Ł}{\partial \gamma^2} \qquad V_{34} = \frac{\partial^2 Ł}{\partial \gamma\, \partial \delta}$$

$$V_{44} = \frac{\partial^2 Ł}{\partial \delta^2}$$

For interval estimation of the model parameters, we require the 4×4 observed information matrix. Under standard regularity conditions, the multivariate normal N_4 (0,V ̂_ij ) distribution can be used to construct approximate confidence intervals for the model parameters. Here, V ̂_ij is the total observed information matrix. Therefore, Approximate $100(1-\phi)\%$ confidence intervals for $\alpha, \beta, \gamma$ and $\delta$ can be determined as:

$$\hat{\alpha} \pm Z_{\frac{\phi}{2}}\sqrt{\hat{V}_{11}},\ \hat{\beta} \pm Z_{\frac{\phi}{2}}\sqrt{\hat{V}_{22}},\ \hat{\gamma} \pm Z_{\frac{\phi}{2}}\sqrt{\hat{V}_{33}} \text{ and } \hat{\delta} \pm Z_{\frac{\phi}{2}}\sqrt{\hat{V}_{44}},$$

where $Z_{\frac{\phi}{2}}$ is the upper $\phi$th percentile of the standard normal distribution.

## 7. Application

Now, we present an application of the proposed TCWG distribution (and their sub models, TCWG, TCRG, TCEG, CWG, and W) in two real data sets to illustrate its potentiality. The first real data set given below represents survival in months of 20 acute myeloid leukemia patients.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 2.226 | 2.113 | 3.631 | 2.473 | 2.720 | 2.050 | 2.061 | 3.915 | 0.871 | 1.548 |
| 2.746 | 1.972 | 2.265 | 1.200 | 2.967 | 2.808 | 1.079 | 2.353 | 0.726 | 1.958 |

The second real data set from Nichols and Padgett (2006) consisting of 100 observations on breaking stress of carbon fibres (in Gba). The data are as follows:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 3.70 | 2.74 | 2.73 | 2.50 | 3.60 | 3.11 | 3.27 | 2.87 | 1.47 | 4.42 |
| 3.11 | 2.41 | 3.19 | 3.22 | 1.69 | 3.28 | 3.09 | 1.87 | 3.15 | 4.90 |
| 3.75 | 2.43 | 2.95 | 2.97 | 3.39 | 2.96 | 2.53 | 2.93 | 3.22 | 2.67 |
| 2.38 | 3.39 | 2.81 | 4.20 | 3.33 | 2.55 | 3.31 | 3.31 | 2.85 | 2.56 |
| 1.57 | 3.65 | 3.56 | 3.15 | 2.35 | 2.55 | 2.59 | 2.81 | 2.77 | 3.19 |
| 2.17 | 2.83 | 1.92 | 1.41 | 3.68 | 2.97 | 1.36 | 0.98 | 2.76 | 4.91 |
| 1.25 | 3.68 | 1.84 | 1.59 | 0.81 | 5.56 | 1.73 | 1.59 | 2.00 | 2.82 |
| 1.89 | 1.22 | 1.12 | 1.71 | 2.17 | 1.17 | 5.08 | 2.48 | 1.18 | 2.05 |
| 3.51 | 2.17 | 1.69 | 4.38 | 1.84 | 0.39 | 3.68 | 2.48 | 0.85 | 1.61 |
| 2.79 | 4.70 | 2.03 | 1.80 | 1.57 | 1.08 | 2.03 | 1.61 | 2.12 | 2.88 |

In the following, we shall compare the proposed TCWG distribution with their sub-models, TCRG, TCEG, CWG and W distributions. We shall apply formal goodness-of-fit tests to verify which distribution fits better the real data sets. Here, we consider the Anderson–Darling ($A^*$) and Cramér-von Mises ($W^*$) statistics (full details can be found in Chen and Balakrishnan, 1995). In general, the distribution which has the smaller values of these statistics is the better the fit to the data. Table 1 lists the MLEs of the model parameters for TCWG, TCRG, TCEG, CWG, and W models, the corresponding standard errors (given in parentheses) and the statistics $A^*$ and $W^*$.

**Table 1: MLEs (standard errors in parentheses) for TCWG, TCRG, TCEG, CWG, and W models and the statistics $W^*$ and $A^*$ (first data set)**

| Model | Estimate | | | | Statistics | |
|---|---|---|---|---|---|---|
| | α | β | γ | δ | $W^*$ | $A^*$ |
| **TCWG** | 0.03589 (0.124) | 1.2857 (1.089) | 0.92613 (1.496) | 1.03523 (0.441) | **0.04537** | **0.28142** |
| **TCEG** | 0.01104 (0.047) | --- | 2.07469 (1.896) | 0.00001 (0.711) | **0.04147** | **0.26300** |
| **TCRG** | 0.14982 (0.055) | --- | 0.59094 (0.0.1299) | 0.44464 (0.094) | **0.04233** | **0.26854** |
| **CWG** | 0.57925 (1.6104) | 2.59402 (1.7188) | 0.45934 (0.2976) | --- | 0.05073 | 0.29914 |
| **W** | --- | 2.92369 (0.513) | 1.2857 (1.089) | --- | 0.05647 | 0.31946 |

These results show that the TCRG, TCWG and TCEG distributions give better fit than the CWG and W distributions and the TCEG distribution has the lowest $A^*$and $W^*$ values among all the fitted models, and so it could be chosen as the best model. Additionally, it is evident that the W distribution presents the worst fit according to the data

**Table 2: MLEs (standard errors in parentheses) for TCWG, TCRG, TCEG, CWG, and W models and the statistics $W^*$ and $A^*$ (second data set)**

| Model | Estimate | | | | Statistics | |
|---|---|---|---|---|---|---|
| | α | β | γ | δ | $W^*$ | $A^*$ |
| **TCWG** | 0.06872 (0.0021) | 1.44978 (0.097) | 0.76941 (0.0298) | 0.0002 (0.013) | **0.0555** | **0.41704** |
| **TCEG** | 0.013212 (0.00591) | --- | 1.675781 (0.15341) | 0.00002 (0.064) | **0.06721** | **0.43503** |
| **TCRG** | 0.17714 (0.0016) | --- | 0.46334 (0.018) | 0.56717 (0.0096) | **0.04929** | **0.4076** |
| **CWG** | 1.44381 (1.6401) | 3.00937 (0.701) | 0.31484 (0.0722) | --- | 0.07035 | 0.42941 |
| **W** | --- | 2.79286 (0.214) | 0.33971 (0.013) | --- | 0.06106 | 0.42092 |

Similarly, the results given in Table 2 illustrate that the TCRG, TCWG and TCEG distributions give better fit than the CWG and W distributions and the TCRG has the lowest $A^*$and $W^*$ values among all the fitted models, so the TCRG is the best model to fit this real data set.

## 8. Conclusions

In this paper, a new lifetime (TCWG) distribution is provided and discussed. The TCWG distribution extends the CWG distribution proposed by Tojeiro et.al., (2014). We provide a mathematical treatment of the new distribution, including expansions for its density and cumulative distribution functions, We derive Various structural properties of the new distribution including explicit expressions for the density function, moments, generating and quantile functions. The estimation of parameters is approached by the method of maximum likelihood. Finally, we fit the TCWG model to two real data sets to show its flexibility and potentially as a lifetime distribution. The new model has 11 well known and unknown probability distributions as special cases, 3 of them are new models. We hope that this new distribution may attract wider applications in the lifetime literature. Finally according to the results in tables 1, and 2 it is obvisely that the TCEG and TCRG distributions (as two new sub models of our proposed model) have the lowest $A^{*}$and $W^{*}$ values among all the fitted models, respectively. So they could be chosen as the best models.

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