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Transmuted Inverse Rayleigh Distribution: A Generalization of the Inverse Rayleigh Distribution.

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Abstract:

In this article, we generalize the Inverse Rayleigh distribution using the quadratic rank transmutation map studied by Shaw et al. (2007) to develop a transmuted inverse Rayleigh distribution. The properties of this distribution are derived and the estimation of the model parameters is performed by maximum likelihood method.

Keywords: Inverse Rayleigh Distribution, Transmutation Map, Hazard Rate Function, Reliability Function, Order Statistics, Parameter Estimation.

1. Introduction:

The quality of the procedures used in a statistical analysis depends heavily on the assumed probability model or distribution. Because of this, considerable effort has been expended in the development of large classes of standard probability distributions along with relevant statistical methodologies. However, there still remain many important problems where the real data does not follow any of the classical or standard probability models.

The inverse Rayleigh distribution has many applications in the area of reliability studies. Voda (1972) mentioned that the distribution of lifetimes of several types of experimental units can be approximated by the inverse Rayleigh distribution. In this article we use transmutation map approach suggested by Shaw et al. (2007) to define a new model which generalizes the Inverse Rayleigh model. We will call the generalized distribution as the transmuted inverse Rayleigh distribution. According to the quadratic rank transmutation map (QRTM), approach the cumulative distribution (cdf) satisfy the relationship

$$F_2(x)=(1+\lambda)F_1(x)-F_1^{\,2}(x)$$

which on differentiation yields

$$f_2(x)=f_1(x)\left[1+\lambda-2\lambda F_1(x)\right]$$

where $f_1(x)$ and $f_2(x)$ are the corresponding probability density function (pdf) associated with $F_1(x)$ and $F_2(x)$ respectively and $-1\le\lambda\le 1$.

We will use the above formulation for a pair of distributions F(x) and G(x) where G(x) is a sub model of F(x). Therefore, a random variable X is said to have transmuted probability distribution with cdf F(x) if

$$F(x)=(1+\lambda)G(x)-\lambda G^2(x) \quad ,|\lambda|\le 1 \tag{1}$$

where G(x) is the cdf of the base distribution. Observe that at $\lambda=0$, we have the distribution of the base random variable. Aryal and Tsokos (2009,2011) studied the transmuted extreme distributions. The authors provided the mathematical characterization of transmuted Gumbel and transmuted Weibull distributions and their applications to analyze real data sets. Faton Merovci (2013) studied the transmuted Rayleigh distribution, Ashouret et al (2013). studied the transmuted exponentiated Lomax distribution and discussed some properties of this family. In the present study we will provide mathematical formulation of the transmuted inverse Rayleigh distribution and some of its properties.

2. Transmuted Inverse Rayleigh Distribution

A random variable X is said to have a inverse Rayleigh distribution with parameter θ>0 if its pdf is given by

$$g(x,\theta)=\frac{2\theta}{x^3}\exp\left(-\frac{\theta}{x^2}\right), \qquad x>0,\theta>0 \tag{2}$$

and the corresponding cdf is

$$G(x,\theta)=\exp\left(-\frac{\theta}{x^2}\right), \qquad x>0,\theta>0 \tag{3}$$

Now using (2) and (3) we have the cdf of transmuted inverse Rayleigh distribution

$$F(x,\theta,\lambda)=\exp\left(-\frac{\theta}{x^2}\right)\left(1+\lambda-\lambda\exp\left(-\frac{\theta}{x^2}\right)\right) \tag{4}$$

Hence, the pdf of transmuted inverse Rayleigh distribution with parameters $\theta \, and \, \lambda$ is

$$f(x,\theta,\lambda)=\frac{2\theta}{x^3}\exp\left(-\frac{\theta}{x^2}\right)\left(1+\lambda-2\lambda\exp\left(-\frac{\theta}{x^2}\right)\right)$$

Note that the transmuted inverse Rayleigh distribution (TIR) is an extended model to analyze more complex data. The Rayleigh distribution is clearly a special case for $\lambda=0$. Figure 1 illustrates some of the possible shapes of the pdf of a transmuted inverse Rayleigh distribution for selected values of the parameters θ and λ.

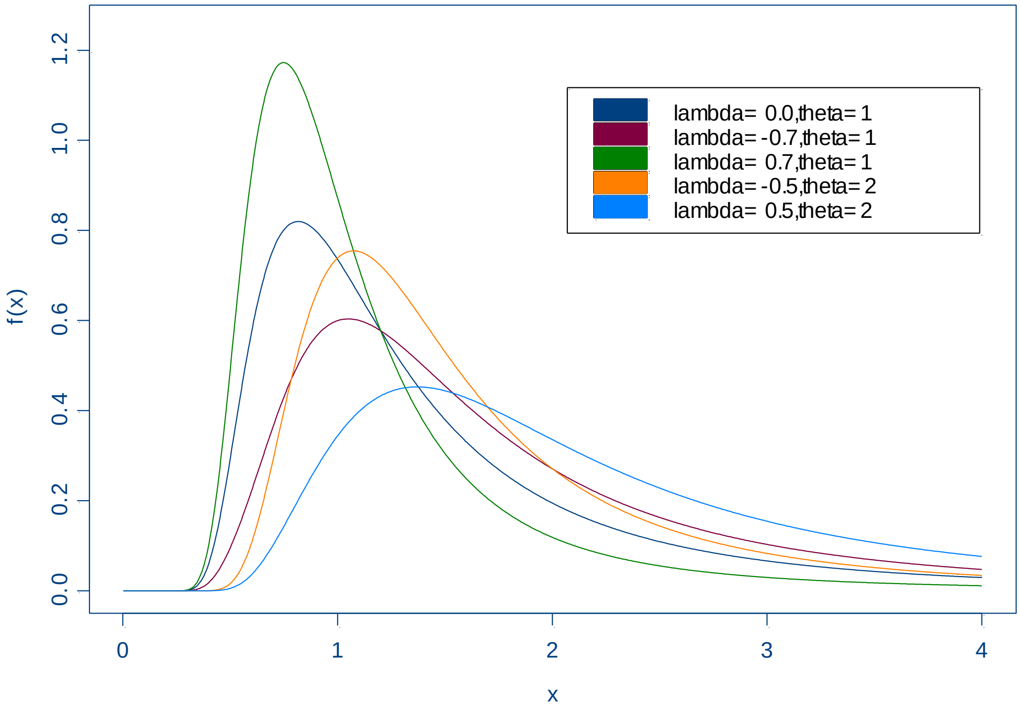

Figure 1:The pdfs of various transmuted inverse Rayleigh distributions

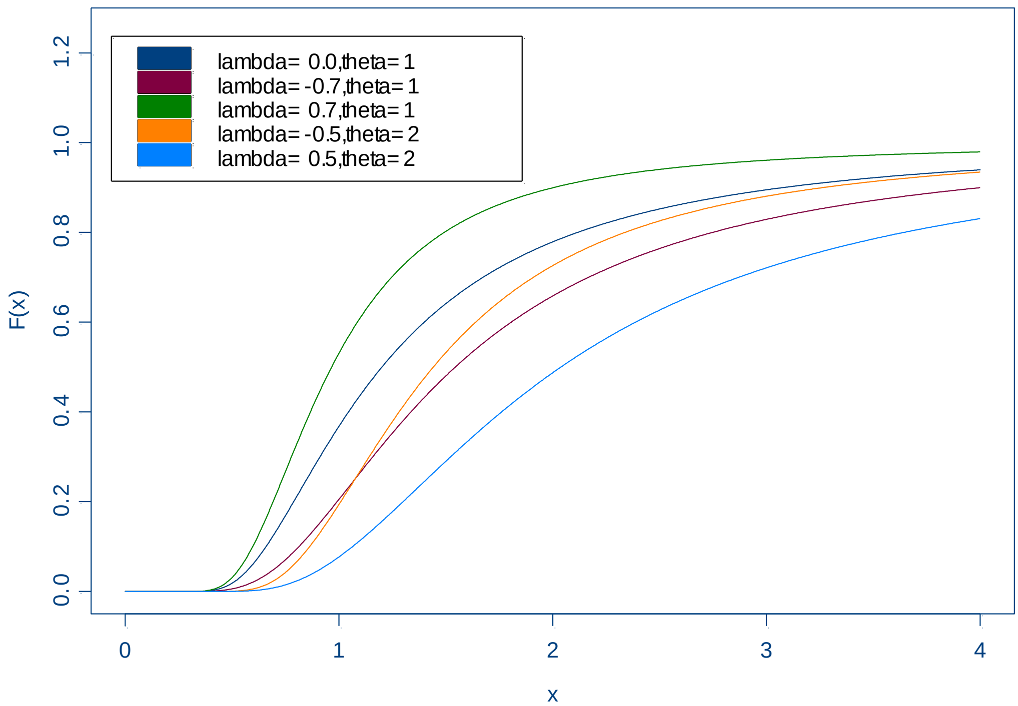

Figure 2:The cdfs of various transmuted inverse Rayleigh distributions

3. Statistical Properties

This section is devoted to study the statistical properties of the (TIR) distribution specially moments, quantile function, median, moment generating function.

3.1 Moments: In this subsection we derive the rth moment for the (TIR) distribution.

Theorem 1 The rth moment $E\left(X^r\right)$ of a transmuted inverse Rayleigh distributed random variable X is given as

$$E\left(X^r\right)=\theta^{\frac{r}{2}}\,\Gamma\left(1-\frac{r}{2}\right)\left(1+\lambda-2^{\frac{r}{2}}\lambda\right)$$

Especially we have

$$E(X)=\sqrt{\theta\pi}\left(1+\lambda-\sqrt{2}\,\lambda\right)$$

$$\operatorname{var}(X)=\theta(1-\lambda)-\theta\pi\left(1+\lambda-\sqrt{2}\,\lambda\right)^2$$

Proof.

$$E(X^r)=\int_0^\infty x^r f(x,\theta,\lambda)\;dx$$

$$=2\theta\int_0^\infty x^{r-3}\exp\left(-\frac{\theta}{x^2}\right)\left(1+\lambda-2\lambda\exp\left(-\frac{\theta}{x^2}\right)\right)\,dx$$

$$=2\theta(1+\lambda)\int_0^\infty x^{r-3}\exp\left(-\frac{\theta}{x^2}\right)\,dx-4\theta\lambda\int_0^\infty x^{r-3}\exp\left(-\frac{2\theta}{x^2}\right)\,dx$$

$$=\theta(1+\lambda)\,\frac{\Gamma\left(1-\frac{r}{2}\right)}{\theta^{1-\frac{r}{2}}}-2\theta\lambda\frac{\Gamma\left(1-\frac{r}{2}\right)}{(2\theta)^{1-\frac{r}{2}}}$$

$$=\theta^{\frac{r}{2}}\,\Gamma\left(1-\frac{r}{2}\right)\left(1+\lambda-2^{\frac{r}{2}}\lambda\right) \tag{6}$$

3.2 Moment Generating Function: In this subsection we derive the moment generating function for the (TIR) distribution.

Theorem 2. Let X have a transmuted inverse Rayleigh distribution. Then the moment generating function of X is given by

$$M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\theta^{\frac{r}{2}}\,\Gamma\left(1-\frac{r}{2}\right)\left(1+\lambda-2^{\frac{r}{2}}\lambda\right)$$

Proof.

$$M_X(t) = E(e^{tx}) = \int_0^\infty \exp(tx)\, f(x,\theta,\lambda)\, dx$$

$$= \int_0^\infty \left(1 + tx + \frac{(tx)^2}{2!} + \ldots\right) f(x,\theta,\lambda)\, dx$$

$$= \int_0^\infty \sum_{r=0}^\infty \frac{t^r}{r!} x^r\ f(x,\theta,\lambda)\, dx$$

$$= \sum_{r=0}^\infty \frac{t^r}{r!} E(X^r)$$

$$= \sum_{r=0}^\infty \frac{t^r}{r!} \theta^{\frac{r}{2}} \Gamma\left(1 - \frac{r}{2}\right)\left(1 + \lambda - 2^{\frac{r}{2}} \lambda\right)$$

3.3 Quantiles and Median:

The quantile x_q of the transmuted inverse Rayleigh distribution is real solution of the following equation

$$x_q = \left[- \frac{\theta}{\ln\left(\frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda} \right)} \right]^{\frac{1}{2}}$$

In particular, the median of the distribution is

$$x_{0.5} = \left[- \frac{\theta}{\ln\left(\frac{(1+\lambda) - \sqrt{(1+\lambda^2}}{2\lambda} \right)} \right]^{\frac{1}{2}}$$

4. Random Number Generation and Parameter Estimation

Using the method of inversion we can generate random numbers from the transmuted inversion Rayleigh distribution as

$$\exp\left(-\frac{\theta}{x^2}\right)\left(1 + \lambda - \lambda \exp\left(-\frac{\theta}{x^2}\right)\right) = u$$

where u ~ U (0, 1). After simplification this yields

$$x = \left[-\frac{\theta}{\ln\left(\frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda u}}{2\lambda} \right)} \right]^{\frac{1}{2}} \tag{7}$$

One can use equation (7) to generate random numbers when the parameters θ and λ are known. The maximum likelihood estimates (MLE's) of the parameters that are inherent within the transmuted inverse Rayleigh distribution function is given by the following:

Let $x_1, x_2, \ldots, x_n$ be a sample of size n from a transmuted inverse Rayleigh distribution. Then the likelihood function is given by

$$L = \frac{(2\theta)^n}{\prod_{i=1}^{n} x_i^{\,3}} \exp\left(-\sum_{i=1}^{n} \frac{\theta}{x_i^{\,2}} \right) \prod_{i=1}^{n} \left(1 + \lambda - 2\lambda \exp\left(-\frac{\theta}{x_i^{\,2}} \right) \right)$$

The log likelihood function is given by

$$\ln L = n \ln 2 + n \ln\theta - \sum_{i=1}^{n} \ln x_i^{\,3} - \sum_{i=1}^{n} \frac{\theta}{x_i^{\,2}} + \sum_{i=1}^{n} \ln\left(1 + \lambda - 2\lambda \exp\left(-\frac{\theta}{x_i^{\,2}} \right) \right) \tag{8}$$

Therefore MLE's of θ and λ which maximizes (8) must satisfy the following normal equations

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} \frac{1}{x_i^{\,2}} + \sum_{i=1}^{n} \frac{\frac{2\lambda}{x_i^{\,2}} \exp\left(-\frac{\theta}{x_i^{\,2}} \right)}{\left(1 + \lambda - 2\lambda \exp\left(-\frac{\theta}{x_i^{\,2}} \right) \right)}$$

$$\frac{\partial \ln L}{\partial \lambda} = \sum_{i=1}^{n} \frac{\left(1 - 2\exp\left(-\frac{\theta}{x_i^{\,2}} \right) \right)}{\left(1 + \lambda - 2\lambda \exp\left(-\frac{\theta}{x_i^{\,2}} \right) \right)} = 0$$

The MLE $\hat{\eta} = (\hat{\theta}, \hat{\lambda})$ of $\eta = (\theta, \lambda)$ is obtained by solving this nonlinear system of equations. It is usually more convenient to use nonlinear optimization algorithms such as quasi-Newton algorithm to numerically maximize the log likelihood function given in (8). Applying the usual large sample approximation, the MLE $\hat{\eta}$ can be treated as being approximately bivariate normal with variance-covariance matrix equal to the inverse of the expected information matrix, i.e.

$$\sqrt{n}\,(\hat{\eta}-\eta) \rightarrow N\left(0, n\,I^{-1}(\eta)\right)$$

where $I^{-1}(\eta)$ is the limiting variance-covariance matrix of $\hat{\eta}$. The elements of the 2×2 matrix $I(\eta)$ can be estimated by $I_{ij}(\hat{\eta}) = -\ln L_{\eta_i \eta_j}\Big|_{\eta=\hat{\eta}}, i, j \in \{1,2\}$.

Approximate two sided 100(1-α) % confidence intervals for θ and λ are, respectively given by

$$\hat{\theta} \pm z_{\alpha/2}\sqrt{I_{11}^{-1}(\hat{\eta})} \quad \text{and} \quad \hat{\lambda} \pm z_{\alpha/2}\sqrt{I_{22}^{-1}(\hat{\eta})}$$

where z_α is the upper αth quantile of the standard normal distribution. Using R we can easily compute the Hessian matrix and its inverse and hence the standard errors and asymptotic confidence intervals.

5. Reliability Analysis

The reliability function $R(t)$, which is the probability of an item not failing prior to some time t, is defined by $R(t) = 1 - F(t)$. The reliability function of a transmuted inverse Rayleigh distribution is given by

$$R(t,\theta,\lambda) = 1 - \exp\left(-\frac{\theta}{t^2}\right)\left(1 + \lambda - \lambda\exp\left(-\frac{\theta}{t^2}\right)\right) \tag{9}$$

The other characteristic of interest of a random variable is the hazard function defined by

$$h(t) = \frac{f(t)}{1 - F(t)}.$$

which is an important quantity characterizing life phenomenon. The hazard rate function for a transmuted inverse Rayleigh random variable is given

$$h(t,\theta,\lambda) = \frac{\frac{2\theta}{t^3}\exp\left(-\frac{\theta}{t^2}\right)\left(1 + \lambda - 2\lambda\exp\left(-\frac{\theta}{t^2}\right)\right)}{1 - \exp\left(-\frac{\theta}{t^2}\right)\left(1 + \lambda - \lambda\exp\left(-\frac{\theta}{t^2}\right)\right)} \tag{10}$$

6. Order Statistics

Order statistics make their appearance in many statistical theory and practice. We know that if $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denotes the order statistics of a random sample $X_1, X_2, \ldots, X_n$ from a continuous population with cdf $F_X(x)$ and pdf $f_X(x)$, then the pdf of rth order statistics $X_{(r)}$ is given by

$$f_{X\,(r)}(x) = \frac{n!}{(r-1)!(n-r)!} f_X(x)\left(F_X(x)\right)^{r-1}\left(1 - F_X(x)\right)^{n-r}$$

For r = 1, 2, . . . ,n.

we have from (2) and (3) the pdf of the rth order inverse Rayleigh random variable $X_{(r)}$ is given by

$$g_{X(r)}(x) = \frac{n!}{(r-1)!(n-r)!}\frac{2\theta}{x^3}\left(\exp\left(-\frac{\theta}{x^2}\right)\right)^r\left(1-\exp\left(-\frac{\theta}{x^2}\right)\right)^{n-r}$$

Therefore, the pdf of the nth order inverse Rayleigh statistic $X_{(n)}$ is given by

$$g_{X\,(n)}(x) = \frac{2n\theta}{x^3}\left(\exp\left(-\frac{\theta}{x^2}\right)\right)^n \tag{11}$$

and the pdf of the first order inverse Rayleigh statistic $X_{(1)}$ is given by

$$g_{X\,(1)}(x) = \frac{2n\theta}{x^3}\exp\left(-\frac{\theta}{x^2}\right)\left(1-\exp\left(-\frac{\theta}{x^2}\right)\right)^{n-1} \tag{12}$$

Note that in particular case of n=2, (11) yields

$$g_{X\,(2)}(x) = \frac{4\theta}{x^3}\left(\exp\left(-\frac{\theta}{x^2}\right)\right)^2 \tag{13}$$

and (12) yields

$$g_{X\,(1)}(x) = \frac{4\theta}{x^3}\exp\left(-\frac{\theta}{x^2}\right)\left(1-\exp\left(-\frac{\theta}{x^2}\right)\right) \tag{14}$$

Observe that (13) and (14) are special cases of (4) for $\lambda = -1 \, and \, \lambda = 1$ respectively. It has been observed that a transmuted inverse Rayleigh distribution with $\lambda = 1$ is the distribution of $\min(X_1, X_2)$ and a transmuted inverse Rayleigh distribution with $\lambda = -1$ is the $\max(X_1, X_2)$. Where X_1 and X_2 are independent and identically distributed inverse Rayleigh random variables. Now we provide the distribution of the order statistics for a transmuted inverse Rayleigh random variable. The pdf of the rth order statistic for a transmuted inverse Rayleigh distribution is given by

$$f_{X\,(r)}(x) = \frac{n!}{(r-1)(n-r)}\frac{2\theta}{x^3}\exp\left(-\frac{\theta}{x^2}\right)\left(1+\lambda-2\lambda\exp\left(-\frac{\theta}{x^2}\right)\right)$$
$$\left[\exp\left(-\frac{\theta}{x^2}\right)\left(1+\lambda-\lambda\exp\left(-\frac{\theta}{x^2}\right)\right)\right]^{r-1}\left[1-\exp\left(-\frac{\theta}{x^2}\right)\left(1+\lambda-\lambda\exp\left(-\frac{\theta}{x^2}\right)\right)\right]^{n-r}$$

Therefore, the pdf of the largest order statistic $X_{(n)}$ is given by

$$f_{X\,(n)}(r) = \frac{2n\theta}{x^3}\exp\left(-\frac{\theta}{x^2}\right)\left(1+\lambda-2\lambda\exp\left(-\frac{\theta}{x^2}\right)\right)\left[\exp\left(-\frac{\theta}{x^2}\right)\left(1+\lambda-\lambda\exp\left(-\frac{\theta}{x^2}\right)\right)\right]^{n-1}$$

and the pdf of the smallest order statistic $X_{(1)}$ is given by

$$f_{X\,(1)}(x)=\frac{2n\theta}{x^{3}}\exp\left(-\frac{\theta}{x^{2}}\right)\left(1+\lambda-2\lambda\exp\left(-\frac{\theta}{x^{2}}\right)\right)\left[1-\exp\left(-\frac{\theta}{x^{2}}\right)\left(1+\lambda-\lambda\exp\left(-\frac{\theta}{x^{2}}\right)\right)\right]^{n-1}$$

Note that $\lambda = 0$ yields the order statistics of the inverse Rayleigh distribution.

Conclusion

In the present study we have introduced a new generalization of the inverse Rayleigh distribution called the transmuted inverse Rayleigh distribution. The subject distribution is generated by using the quadratic rank transmutation map and taking the inverse Rayleigh distribution as the base distribution. Some mathematical properties along with estimation issues are discussed. The hazard rate function and reliability behavior of transmuted inverse Rayleigh distribution shows that subject distribution can be used to model reliability data.

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