Journal of Statistics Applications & Probability
*An International Journal*



# Transmuted Log-Logistic Distribution

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Received: 14 Jul 2012, Revised: 26 Nov. 2012, Accepted: 29 Nov. 2012
Published online: 1 Mar. 2013

**Abstract:** A generalization of the log-logistic distribution so-called the transmuted log-logistic distribution is proposed and studied. Various structural properties including explicit expressions for the moments, quantiles, mean deviations of the new distribution are derived. The estimation of the model parameters is performed by maximum likelihood method. We hope that the new distribution proposed here will serve as an alternative model to the other models which are available in the literature for modeling positive real data in many areas.

**Keywords:** Log-logistic distribution, Hazard Rate Function, Reliability Function, Parameter Estimation, Order Statistics.

## 1 Introduction

The quality of the procedures used in a statistical analysis depends heavily on the assumed probability model or distributions. Because of this, considerable effort has been expended in the development of large classes of standard probability distributions along with revelent statistical methodologies. In fact, the statistics literature is filled with hundreds of continuous univariate distributions. However, in recent years, applications from the environmental, financial, biomedical sciences, engineering among others, have further shown that data sets following the classical distributions are more often the exception rather than the reality. Since there is a clear need for extended forms of these distributions a significant progress has been made toward the generalization of some well-known distributions and their successful application to problems in areas such as engineering, finance, economics and biomedical sciences, among others.

In this article we use transmutation map approach suggested by Shaw *et al.* [15] to define a new model which generalizes the log-logistic (LLog) model. We will call the generalized distribution as the transmuted log-logistic (TLLog) distribution. According to the Quadratic Rank Transmutation Map,(QRTM), approach the cumulative distribution function(cdf) satisfy the relationship

$$F_2(x) = (1+\lambda)F_1(x) - \lambda F_1(x)^2 \tag{1}$$

which on differentiation yields,

$$f_2(x) = f_1(x)\left[1+\lambda - 2\lambda F_1(x)\right] \tag{2}$$

where $f_1(x)$ and $f_2(x)$ are the corresponding probability density function(pdf) associated with $F_1(x)$ and $F_2(x)$ respectively and $-1 \leq \lambda \leq 1$. An extensive information about the quadratic rank transmutation map is given in Shaw *et al.*[15].

We will use the above formulation for a pair of distributions $F(x)$ and $G(x)$ where $G(x)$ is a submodel of $F(x)$. Therefore, a random variable X is said to have a transmuted probability distribution with cdf $F(x)$ if

$$F(x) = (1+\lambda)G(x) - \lambda G(x)^2, \qquad |\lambda| \leq 1 \tag{3}$$

where $G(x)$ is the cdf of the base distribution. Observe that at $\lambda = 0$ we have the distribution of the base random variable. Aryal *et al.* [ 2, 3] studied the transmuted extreme value distributions. The authors provided the mathematical

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characterization of transmuted Gumbel and transmuted Weibull distributions and their applications to analyze real data sets. In the present study we will provide mathematical formulation of the transmuted log-logistic(TLLog) distribution and some of its properties.

## 2 Transmuted Log-Logistic Distribution

The log-logistic distribution is a derivative of the very popular logistic distribution which was initially developed to model population growth by Verhulst [19]. Since the development of logistic growth curve there have been several contributions suggesting alternative functional forms for growth whilst retaining the sigmoid and asymptotic property of the Verhulst logistic curve. Several well known growth functions which extend the standard Verhulst equation are discussed in [18]. The use of the logistic distribution for economic and demographic purposes was very popular in the nineteenth century. The logistic distribution is also known by names such as growth function, autocatalytic curve and so on depending on its applications. The importance of the logistic distribution is already been included in many areas of human endeavor including biology, epidemiology, psychology, technology, energy and others.
A random variable X is said to have the log-logistic (LLog) distribution, also known as the Fisk distribution in economics, with parameters $\alpha$ and $\beta$ if its cdf is given by

$$G(x) = \frac{x^{\beta}}{\alpha^{\beta}+x^{\beta}}, \qquad x>0 \tag{4}$$

where, $\alpha > 0$ is a scale parameter and $\beta > 0$ is a shape parameter. Note that this distribution is unimodel if $\beta > 1$ and the mode is $\alpha$. The log-logistic distribution is widely used in practice and it is an alternative to the log-normal distribution since it presents a failure rate function that increases, reaches a peak after some finite period and then declines gradually. The properties of the log-logistic distribution make it an attractive alternative to the log-normal and Weibull distributions in the analysis of survival data [5]. Recent study by Dey *et al.* [6] helps to discriminate between the log-normal and log-logistic distributions. The log-logistic distribution has also been used in hydrology to model stream flow and precipitation [16], and [4], and for modeling flood frequency [1]. Additionally, it is used in economics as a simple model of the distribution of income [7]. This distribution is a survival model useful in reliability studies. Gupta *et al.*[9] made a study of log-logistic model in survival analysis. Ragab *et al.* [14] developed order statistics from the log-logistic distribution and their properties. Collet [5] suggested the log-logistic distribution for modeling the time following a heart transplantation. Kantam *et al.* [11] studied acceptance sampling based on life tests: log- logistic model.

The pdf of a log-logistic distribution is given by

$$g(x) = \frac{\beta}{\alpha}\frac{(x/\alpha)^{\beta-1}}{[1+(x/\alpha)^{\beta}]^2}, \qquad x>0 \tag{5}$$

Now using (3) and (4) we have the cdf of a transmuted log-logistic (TLLog)distribution given by

$$F(x) = \frac{(1+\lambda)\alpha^{\beta}x^{\beta}+x^{2\beta}}{(\alpha^{\beta}+x^{\beta})^2} \tag{6}$$

Hence, the pdf of transmuted log-logistic(TLLog) distribution with parameters $\alpha, \beta$, and $\lambda$ is

$$f(x) = \frac{\beta\alpha^{\beta}x^{\beta-1}[(1+\lambda)(\alpha^{\beta}+x^{\beta})-2\lambda x^{\beta}]}{(\alpha^{\beta}+x^{\beta})^3} \qquad x>0. \tag{7}$$

Note that for $\lambda = 0$ we have the pdf of a log-logistic distribution. Also note that for $x = \alpha$ we have $f(x) = \frac{\beta}{4\alpha}$ which is clearly independent of $\lambda$. Figure 1 illustrates some of the possible shapes of the density function of transmuted log-logistic distribution for selected parameters.

## 3 Moments and Quantiles

In this section we shall present the moments and qunatiles for the transmuted log logistic distribution. The $k^{th}$ order moments, for $k < \beta$, of a transmuted log-logistic random variable $X$, is given by

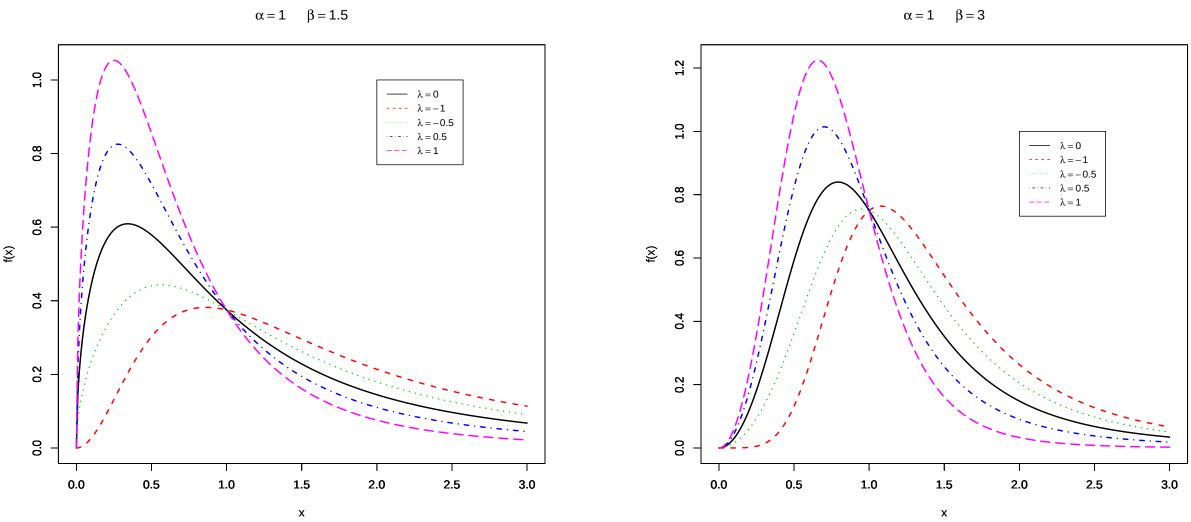


**Fig. 1** PDF of TLLog distribution

$$E(X^k) = (1+\lambda)\alpha^k B\left(1-\frac{k}{\beta}, 1+\frac{k}{\beta}\right) - 2\lambda\alpha^{k+\beta} B\left(1-\frac{k}{\beta}, 2+\frac{k}{\beta}\right) \tag{8}$$

where, $B(.,.)$ is the beta function defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$$

Using the functional relationships

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

and

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$$

we have the $k^{th}$ order moments givable by

$$E(X^k) = \alpha^k \frac{k\pi/\beta}{\sin\left(\frac{k\pi}{\beta}\right)}\left[1+\lambda-\lambda\alpha^\beta\left(1+\frac{k}{\beta}\right)\right] \tag{9}$$

In particular, the mean of the transmuted log-logistic distribution is given by

$$\mathrm{E}(X) = \frac{\pi\alpha/\beta}{\sin(\pi/\beta)}\left[1+\lambda-\lambda\alpha^\beta(1+1/\beta)\right]$$

Table 1 lists the first four ordinary moments for selected values of the parameter $\lambda$ of the transmuted log-logistic distribution for $\alpha = 1$ and $\beta = 10$.

Using these ordinary moments one can easily compute the variance, skewness and kurtosis of the transmuted log-logistic distribution for the selected values of the parameters.

The $q^{th}$ quantile $x_q$ of the transmuted log logistic distribution can be obtained from (6) as

$$x_q = \alpha\left[\frac{-(1+\lambda-2q)+\sqrt{(1+\lambda)^2-4\lambda q}}{2(1-q)}\right]^{1/\beta} \tag{10}$$

**Table 1** Moments of transmuted log-logistic distribution for selected values of the parameters

| | $\lambda=-1$ | $\lambda=-0.5$ | $\lambda=0$ | $\lambda=0.5$ | $\lambda=1$ |
|---|---|---|---|---|---|
| $k=1$ | 1.12 | 1.07 | 1.02 | 0.97 | 0.91 |
| $k=2$ | 1.28 | 1.18 | 1.07 | 0.96 | 0.86 |
| $k=3$ | 1.51 | 1.34 | 1.16 | 0.99 | 0.82 |
| $k=4$ | 1.85 | 1.59 | 1.32 | 1.06 | 0.79 |

Hence, the distribution median is

$$x_{0.5}=\alpha\left[-\lambda+\sqrt{1+\lambda^2}\right]^{1/\beta}.$$

In particular if $\lambda=0$ then the median of the resulting log-logistic distribution is simply the parameter $\alpha$. To illustrate the effect of the shape parameter $\lambda$ on skewness and kurtosis we consider measures based on quantiles. The shortcomings of the classical kurtosis measure are well known. There are many heavy-tailed distributions for which this measure is infinite, so it becomes uninformative. The Bowley's skewness [12] is one of the earliest skewness measures defined by the average of the quartiles minus the median, divided by half the interquartile range, given by

$$\mathscr{B}=\frac{Q_3+Q_1-2Q_2}{Q_3-Q_1}=\frac{Q(3/4)+Q(1/4)-2Q(2/4)}{Q(3/4)-Q(1/4)}$$

and the Moors kurtosis [13] is based on octiles and is given by

$$\mathscr{M}=\frac{(E_3-E_1)+(E_7-E_5)}{E_6-E_2}=\frac{Q(3/8)-Q(1/8)+Q(7/8)-Q(5/8)}{Q(6/8)-Q(2/8)}$$

For any distribution symmetrical to 0 the Moors kurtosis reduces to

$$\mathscr{M}=\frac{(E_7-E_5)}{E_6}$$

It is easy to calculate that for standard normal distribution $E_1=-E_7=-1.15, E_2=-E_6=-0.67$ and, $E_3=-E_5=-0.32$. Therefore, $\mathscr{M}=1.23$. Hence, the centered Moor's coefficient is given by:

$$\mathscr{M}=\frac{(E_7-E_5)+(E_3-E_1)}{E_6-E_2}-1.23.$$

Figure 2 displays the Bowley ($\mathscr{B}$) and Moors ($\mathscr{M}$) kurtosis as a function of the parameter $\lambda$ for $\alpha=1$ and $\beta=10$. It is evident that both measures depend on the parameter $\lambda$.

## 4 Mean Deviation

The amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and the median. These are known as the mean deviation about the mean and the mean deviation about the median respectively and are defined by

$$\delta_1(X)=\int_0^\infty |x-\mu|\,f(x)dx \tag{11}$$

and

$$\delta_2(X)=\int_0^\infty |x-M|\,f(x)dx, \tag{12}$$

where

$$\mu=\mathrm{E}(X)=\frac{\pi\alpha/\beta}{\sin(\pi/\beta)}\left[1+\lambda-\lambda\alpha^{\beta}(1+1/\beta)\right]$$

and

$$M=\mathrm{Median}(X)=\alpha\left[-\lambda+\sqrt{1+\lambda^2}\right]^{1/\beta}.$$

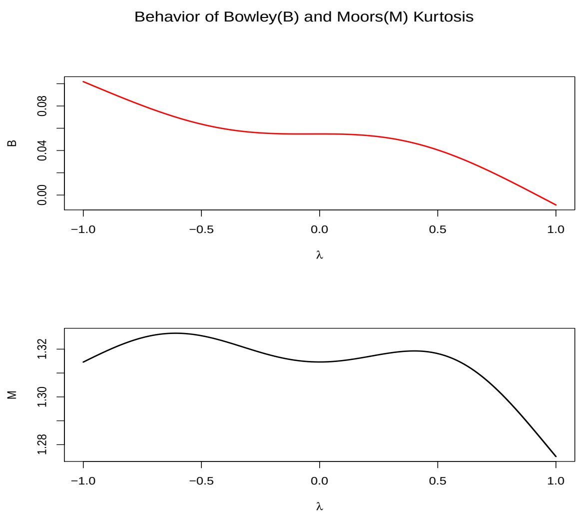


**Fig. 2** Behavior of Bowley(B) and Moors(M) kurtosis for TLLog distribution

The measures $\delta_1(X)$ and $\delta_2(X)$ can be expressed as $\delta_1(X) = 2\mu F(\mu) - 2J(\mu)$ and $\delta_2(X) = \mu - 2J(M)$ where $J(q) = \int_0^q xf(x)dx$. For a transmuted log-logistic distribution

$$J(q) = (1+\lambda)\alpha\beta \int_0^{q/\alpha} \frac{y^\beta}{(1+y^\beta)^2} dy - 2\lambda\alpha\beta \int_0^{q/\alpha} \frac{y^{2\beta}}{(1+y^\beta)^3} dy \tag{13}$$

One can easily compute these integrals numerically in software such as MAPLE[8], MATLAB [17], and R [10] and hence get the mean deviations about the mean and about the median as desired. From the mean deviations we can construct Lorenz and Bonferroni curves, which are used in several areas including economics, reliability, insurance and medicine and others.

Some numerical values of the mean deviation from mean and median for selected value of $\alpha = 1$ and $\beta = 10$ and different values of $\lambda$ are listed in the table 2 below.

**Table 2** Mean deviation from the mean and the median for selected values of the parameters

| | $\lambda = -1$ | $\lambda = -0.5$ | $\lambda = 0$ | $\lambda = 0.5$ | $\lambda = 1$ |
|---|---|---|---|---|---|
| $\delta_1$ | 0.135 | 0.145 | 0.145 | 0.132 | 0.099 |
| $\delta_2$ | 0.133 | 0.144 | 0.144 | 0.132 | 0.099 |

## 5 Random Number Generation and Parameter Estimation

Using the method of inversion we can generate random numbers from the transmuted log-logistic distribution as

$$\frac{(1+\lambda)\alpha^\beta x^\beta + x^{2\beta}}{(\alpha^\beta + x^\beta)^2} = u$$

where $u \sim U(0,1)$. After simple calculation this yields

$$x = \alpha \left[ \frac{-(1+\lambda-2u) + \sqrt{(1+\lambda)^2 - 4\lambda u}}{2(1-u)} \right]^{1/\beta}. \tag{14}$$

One can use equation (14) to generate random numbers when the parameters $\alpha,\beta$ and $\lambda$ are known. The maximum likelihood estimates, MLEs, of the parameters that are inherent within the transmuted log-logistic probability distribution is given by the following: Let $X_1,X_2,\cdots,X_n$ be a sample of size $n$ from a transmuted log logistic distribution. Then the likelihood function is given by

$$L=\frac{\beta^n\alpha^{n\beta}\left(\prod_{i=1}^n x_i\right)^{\beta-1}\prod_{i=1}^n\left[(1+\lambda)(\alpha^\beta+x_i^\beta)-2\lambda x_i^\beta\right]}{\prod_{i=1}^n\left(\alpha^\beta+x_i^\beta\right)^3} \tag{15}$$

Hence, the log-likelihood function $\mathscr{L}=\ln L$ becomes

$$\mathscr{L}=n\ln\beta+n\beta\ln\alpha+(\beta-1)\sum_{i=1}^n\ln(x_i)-3\sum_{i=1}^n\ln(\alpha^\beta+x_i^\beta)$$
$$+\sum_{i=1}^n\ln[(1+\lambda)(\alpha^\beta+x_i^\beta)-2\lambda x_i^\beta] \tag{16}$$

Therefore, the components of the score vector are given by

$$\frac{\partial\mathscr{L}}{\partial\alpha}=\frac{n\beta}{\alpha}-3\beta\alpha^{\beta-1}\sum_{i=1}^n\left\{\alpha^\beta+x_i^\beta\right\}^{-1}+(1+\lambda)\beta\alpha^{\beta-1}\sum_{i=1}^n\left\{(1+\lambda)(\alpha^\beta+x_i^\beta)-2\lambda x_i^\beta\right\}^{-1}, \tag{17}$$

$$\frac{\partial\mathscr{L}}{\partial\beta}=\frac{n}{\beta}+\sum_{i=1}^n\ln(\alpha x_i)-3\sum_{i=1}^n\frac{\alpha^\beta\ln\alpha+x_i^\beta\ln(x_i)}{(\alpha^\beta+x_i^\beta)}+\sum_{i=1}^n\frac{[1+\lambda][\alpha^\beta\ln\alpha+x_i^\beta\ln(x_i)]-2\lambda x_i^\beta\ln(x_i)}{\left\{(1+\lambda)(\alpha^\beta+x_i^\beta)-2\lambda x_i^\beta\right\}}, \tag{18}$$

$$\frac{\partial\mathscr{L}}{\partial\lambda}=\sum_{i=1}^n\frac{(\alpha^\beta-x_i^\beta)}{\left\{(1+\lambda)(\alpha^\beta+x_i^\beta)-2\lambda x_i^\beta\right\}}. \tag{19}$$

The maximum likelihood estimator $\hat{\boldsymbol{\theta}}=\left(\hat{\alpha},\hat{\beta},\hat{\lambda}\right)'$ of $\boldsymbol{\theta}=(\alpha,\beta,\lambda)'$ is obtained by setting the score vector to zero and solving the nonlinear system of equations. It is usually more convenient to use nonlinear optimization algorithms such as quasi- Newton algorithm to numerically maximize the log-likelihood function given in (16).

We can compute the maximum values of the unrestricted and restricted log-likelihood functions to obtain likelihood ratio (LR) statistics for testing the sub-model of the new distribution. For example, we can use the LR statistic to check if the fit using the transmuted log-logistic distribution is statistically "superior" to a fit using the log-logistic distribution for a given data set, i.e. we can compare the first model against the second model by testing $H_0:\lambda=0$ versus $H_a:\lambda\neq 0$.

## 6 Reliability Analysis

The survival function, also known as the reliability function in engineering, is the characteristic of an explanatory variable that maps a set of events, usually associated with mortality or failure of some system onto time. It is the probability that the system will survive beyond a specified time.

The transmuted log-logistic distribution can be a useful model to characterize failure time of a given system because of the analytical structure. The reliability function $R(t)$, which is the probability of an item not failing prior to some time $t$, is defined by $R(t)=1-F(t)$. The reliability function of a transmuted log-logistic distribution is given by

$$R(t)=\frac{\alpha^{2\beta}+(1-\lambda)\alpha^\beta t^\beta}{(\alpha^\beta+t^\beta)^2}. \tag{20}$$

Note that at $t=\alpha$, $R(t)=\frac{2-\lambda}{4}$ which is a constant independent of the parameter $\beta$. Figure 3 illustrates the reliability behavior of a transmuted log-logistic distribution as the value of the parameter $\lambda$ varies from $-1$ to 1.

The other characteristic of interest of a random variable is the hazard rate function also known as instantaneous failure rate defined by

$$h(t)=\frac{f(t)}{1-F(t)}$$

which is an important quantity characterizing life phenomenon. It can be loosely interpreted as the conditional probability of failure, given it has survived to the time $t$. The hazard rate function for a transmuted log-logistic distribution is given by

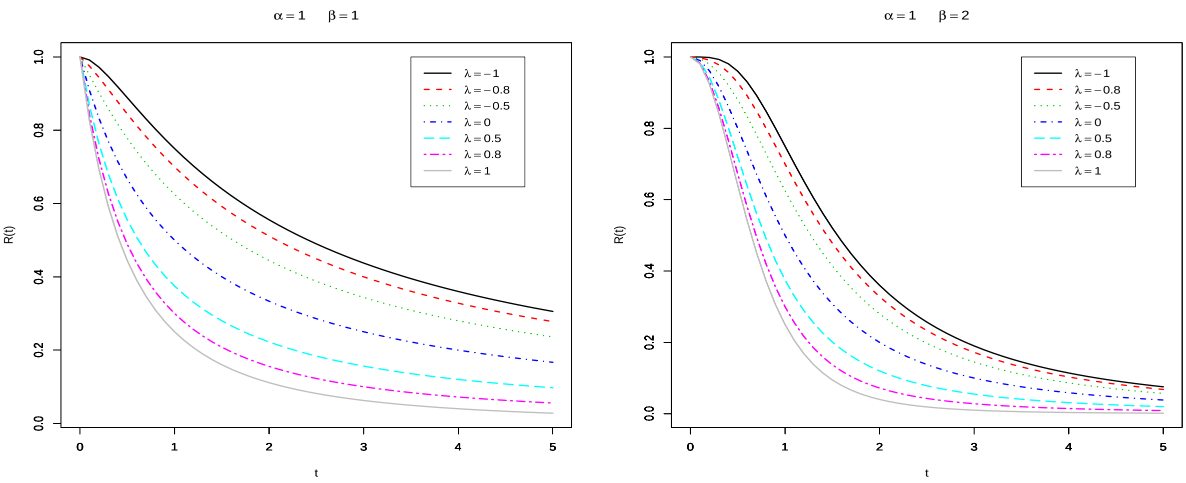


**Fig. 3** Reliability function of TLLog distribution

$$h(t) = \frac{\beta t^{\beta-1}\left[(1+\lambda)\alpha^{\beta}+(1-\lambda)t^{\beta}\right]}{(\alpha^{\beta}+t^{\beta})[\alpha^{\beta}+(1-\lambda)t^{\beta}]}. \tag{21}$$

It is important to note that the units for $h(t)$ is the probability of failure per unit of time, distance or cycles. Note that at $t = \alpha$ we have $h(t) = \frac{\beta}{\alpha(2-\lambda)}$.

Figure 4 illustrates the behavior of the hazard rate function of a transmuted log-logistic distribution for selected values of the parameters.

Observing the behavior of the hazard rate function it is worth noting that the transmuted log-logistic distribution will have more applicability than the log-logistic distribution and some of its generalizations.

Many generalized probability models have been proposed in reliability literature through the fundamental relationship between the reliability function $R(t)$ and its cumulative hazard function(CHF) $H(t)$ given by $H(t) = -\ln R(t)$. The CHF describes how the risk of a particular outcome changes with time.

The cumulative hazard rate function of a transmuted log-logistic distribution is given by

$$\begin{aligned} H(t) &= \int_0^t h(x)dx \\ &= 2\ln(\alpha^{\beta}+t^{\beta}) - \beta\ln\alpha - \ln\left(\alpha^{\beta}+(1-\lambda)t^{\beta}\right) \\ &= \ln\left(\frac{\left\{1+(t/\alpha)^{\beta}\right\}^2}{1+(1-\lambda)(t/\alpha)^{\beta}}\right) \end{aligned}$$

Observe that:

i) $H(t)$ is nondecreasing for all $t \geq 0$,

ii) $H(0) = 0$,

iii) $\lim_{t\to\infty} H(t) = \infty$.

It is important to note that the units for $H(t)$ are the cumulative probability of failure per unit of time, distance or cycles.

## 7 Order Statistics

Order statistics make their appearance in many areas of statistical theory and practice. We know that if $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ denotes the order statistics of a random sample $X_1, X_2, \cdots, X_n$ from a continuous population with cdf $F_X(x)$ and pdf $f_X(x)$

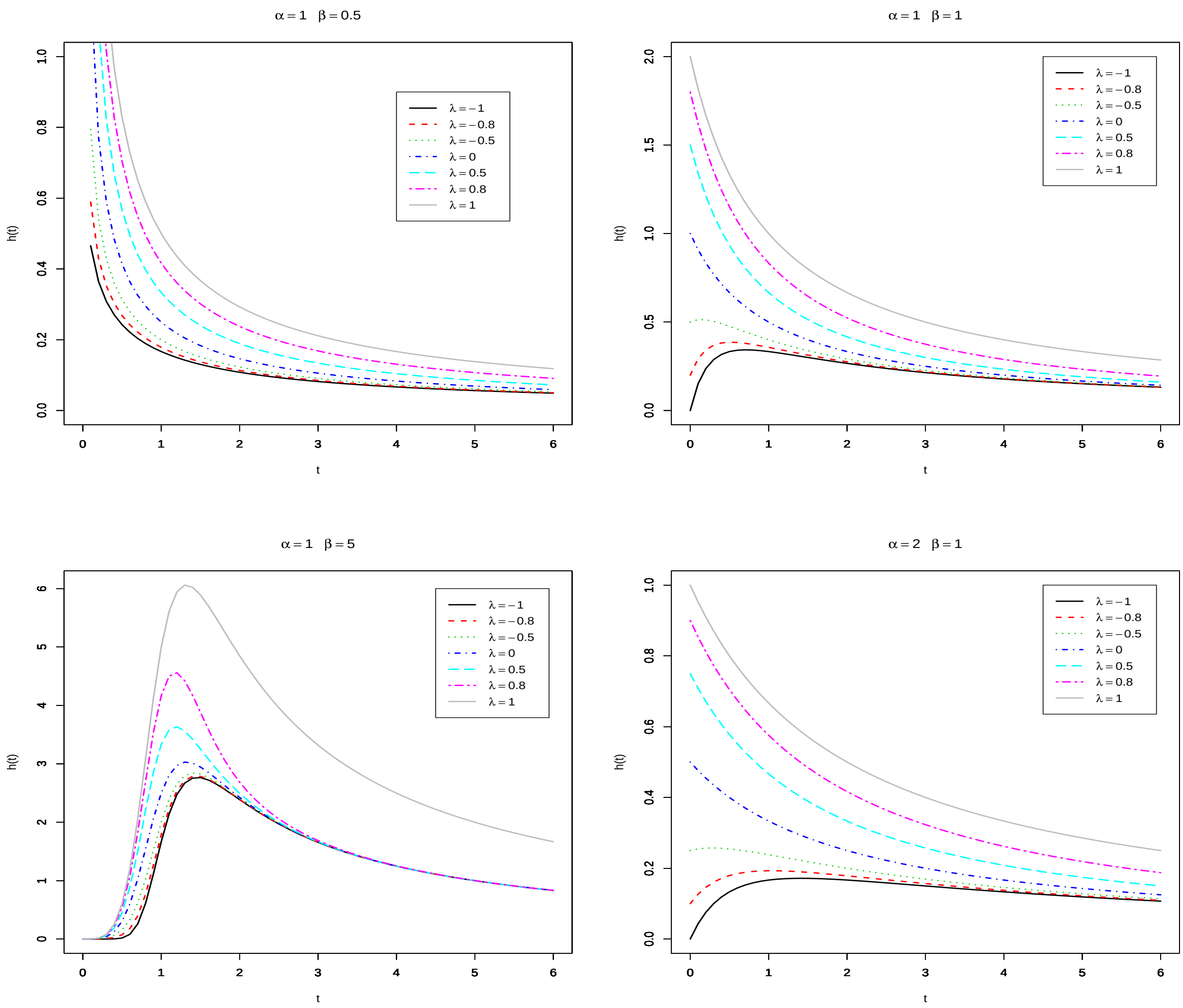


**Fig. 4** Hazard rate function of TLLog distribution

then the pdf of $X_{(j)}$ is given by

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!} f_X(x)[F_X(x)]^{j-1}[1-F_X(x)]^{n-j}$$

for $j = 1, 2, \cdots, n$.

We have from (4) and (5) the pdf of the $j^{th}$ order log-logistic random variable $X_{(j)}$ given by

$$g_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!} \frac{\beta\alpha^{(n+1-j)\beta} x^{\beta j-1}}{(\alpha^\beta + x^\beta)^{n+1}}$$

Therefore, the pdf of the $n^{th}$ order log-logistic statistic $X_{(n)}$ is given by

$$g_{X_{(n)}}(x) = \frac{n\beta\alpha^\beta x^{n\beta-1}}{(\alpha^\beta + x^\beta)^{n+1}} \tag{22}$$

and the pdf of the $1^{st}$ order log-logistic statistic $X_{(1)}$ is given by

$$g_{X_{(1)}}(x) = \frac{n\beta\alpha^{n\beta}x^{\beta-1}}{(\alpha^{\beta}+x^{\beta})^{n+1}} \tag{23}$$

In particular, we can express the following recursive relationship between the pdf of the $k^{th}$ order and $(k+1)^{th}$ order log-logistic statistic

$$g_{X_{(k+1)}}(x) = \left(\frac{n-k}{k}\right)\left(\frac{x}{\alpha}\right)^{\beta} g_{X_{(k)}}(x)$$

Now we provide the distribution of the order statistics for transmuted log-logistic random variable. The pdf of the $j^{th}$ order statistic for transmuted log-logistic distribution is given by

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!}\frac{\beta\alpha^{\beta}x^{\beta-1}}{(\alpha^{\beta}+x^{\beta})^{2n+1}}\left\{(1+\lambda)(\alpha^{\beta}+x^{\beta})-2\lambda x^{\beta}\right\} \\ \times\left\{(1+\lambda)\alpha^{\beta}x^{\beta}+x^{2\beta}\right\}^{j-1}\left\{(1-\lambda)\alpha^{\beta}x^{\beta}+\alpha^{2\beta}\right\}^{n-j}$$

Therefore, the pdf of the largest order statistic $X_{(n)}$ is given by

$$f_{X_{(n)}}(x) = \frac{n\beta\alpha^{\beta}x^{\beta-1}}{(\alpha^{\beta}+x^{\beta})^{2n+1}}\left\{(1+\lambda)(\alpha^{\beta}+x^{\beta})-2\lambda x^{\beta}\right\}\left\{(1+\lambda)\alpha^{\beta}x^{\beta}+x^{2\beta}\right\}^{n-1}$$

and the pdf of the smallest order statistic $X_{(1)}$ is given by

$$f_{X_{(1)}}(x) = \frac{n\beta\alpha^{\beta}x^{\beta-1}}{(\alpha^{\beta}+x^{\beta})^{2n+1}}\left\{(1+\lambda)(\alpha^{\beta}+x^{\beta})-2\lambda x^{\beta}\right\}\left\{(1-\lambda)\alpha^{\beta}x^{\beta}+\alpha^{2\beta}\right\}^{n-1}$$

Note that $\lambda = 0$ yields the order statistics of the two parameter log-logistic distribution.

We can express the the pdf of the $(k+1)^{th}$ order transmuted log-logistic statistic in terms of the $(k)^{th}$ order transmuted log-logistic statistic using the following relationship

$$f_{X_{(k+1)}}(x) = \left(\frac{n-k}{k}\right)\left(\frac{x}{\alpha}\right)^{\beta}\left(\frac{(1+\lambda)\alpha^{\beta}+x^{\beta}}{(1-\lambda)x^{\beta}+\alpha^{\beta}}\right) f_{X_{(k)}}(x).$$

## 8 Concluding Remarks

In the present study, we have introduced a new generalization of the log-logistic distribution called the transmuted log-logistic distribution. The subject distribution is generated by using the quadratic rank transmutation map and taking the 2-parameter log-logistic distribution as the base distribution. Some mathematical properties along with estimation issues are addressed. The hazard rate function and reliability behavior of the transmuted log-logistic distribution shows that the subject distribution can be used to model reliability data. We expect that this study will serve as a reference and help to advance future research in the subject area.

## Acknowledgements

The author would like to thank the anonymous referee for carefully reading and commenting the paper.

## References

[1] M. I. Ahmad, C. D. Sinclair, and A. Werrity. Log-Logistic Flood Frequency Analysis. *Journal of Hydrology*, 98, 205-224, (1988).
[2] G. R. Aryal and C. P. Tsokos. On the transmuted extreme value distribution with application.*Nonlinear Analysis: Theory, Methods and Applications,* 71, 1401-1407, (2009).
[3] G. R. Aryal and C. P. Tsokos. Transmuted weibull distribution: A generalization of the weibull probability distribution.*European Journal of Pure and Applied Mathematics,* 4(2), 89-102, (2011).
[4] F. Ashkar and S. Mahdi. Fitting the log-logistic distribution by generalized moments. *Journal of Hydrology,* 328, 694-703, (2006).
[5] D. Collet. Modelling Survival data in medical research.*Chapman and Hall: London,* (2003).

[6] A. K. Dey and D. Kundu. Discriminating between the log-normal and log-logistic distributions.*Communications in Statistics-Theory and Methods,* 39, 280-292, (2010).
[7] P. R. Fisk. The Graduation of Income Distributions.*Econometrica,* 29(2), 171-185, (1961).
[8] F. Garvan. The Maple Book, 6th ed. *Champman and Hall/CRC, London,* (2002).
[9] R. C. Gupta, O. Akman, and S. Lvin. A study of log-logistic model in survival analysis. *Biometrical Journal,* 4(41), 431-443, (1999).
[10] R. Ihaka and R. Gentleman. R: A language for data analysisand graphics.*Journal of Computational and Graphical Statistics,* 5, 299-314, (1996).
[11] R. R. Kantam, K. Rosaiah, and G. Srinivasa Rao. Acceptance sampling based on life tests: log-logistic model . *Journal of Applied Statistics,* 28(1), 121-128, (2001).
[12] J. F. Kenney and E. S. Keeping. Mathematics of Statistics.*Princeton, NJ.* (1962).
[13] J. A. Moors. A quantile alternative for kurtosis.*Journal of the Royal Statistical Society, D,* 37, 25-32, (1998).
[14] A. Ragab and J. Green. On order statistics from the log-logistic distribution and their properties. *Communications in Statistics-Theory and Methods,* 3(21), 2713-2724, (1984).
[15] W. Shaw and I. Buckley. The alchemy of probability distributions: beyond Gram- Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map.*Research report,* (2007).
[16] M. M. Shoukri, I. M. Mian, and D. S. Tracy. Sampling Properties of Estimators of the Log-Logistic Distribution with Application to Canadian Precipitation Data.*The Canadian Journal of Statistics,* 16(3), 223-236, (1988).
[17] K. Sigmon and T.A. Davis. MATLAB Primer, 6th ed. *Champman and Hall/CRC, London,* (2002).
[18] A. Tsoularis. Analysis of Logistic Growth Models.*Res. Lett. Inf. Math. Sci.*, 2, 23-46, (2001).
[19] P. F. Verhulst. Notice sur la loi que la population suit dans son accroissement.*Corr. Math. Physics,* 10, 113-121, (1838).