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TRANSNORMAL HYPERSURFACES*

— Generalized constant width for Riemannian manifolds —

Dedicated to Professor Shigeo Sasaki on his 60th birthday

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Introduction. In the area of "visualized geometry", the theory of convex figures has many geometrical questions and theorems which make a strong appeal to the intuition and visual senses. Among them, those concerning curves of constant width are full of interest [9].

The theory of Riemannian transnormality has evolved from attempts to find a fruitful analogue in a general Riemannian setting of this familiar idea of constant width for curves. In [7] and [8], Robertson has achieved this in the case where the ambient space is Euclidean. Recently the study was extended by Bolton in [2] and [3] to the case in which the ambient space is a complete Riemannian manifold. Following them, we shall investigate some global properties of transnormal hypersurfaces of a complete Riemannian manifold, and it is the main purpose of this paper.

Let M be a complete hypersurface of a complete C^∞ Riemannian manifold W. M is called a *transnormal hypersurface* of W if each geodesic of W which cuts M orthogonally at some point cuts M orthogonally at all points of intersection. Of course, each curve of constant width in a Euclidean plane has this property, so it can be a model of a transnormal hypersurface. The *generating frame* $\Phi(x)$ at a point x of M is defined to be the set of those vectors at x which are orthogonal to M and which are mapped into M by the exponential map of W.

Robertson and Bolton prove that the generating frames at any two points of M are isometric and that the isometry group of each generating frame is transitive. From these facts, Robertson shows that a transnormal hypersurface in the ordinary Euclidean space is homeomorphic to one of the following: a Euclidean plane, a Euclidean cylinder and a sphere. In contrast to this, Bolton proves that if M is *not* a regular transnormal hypersurface of W, i.e. the underlying topology of M is not the one induced from W, then M is dense in W and is a leaf of a foliation of W.

* This paper was written while the author was at Tokyo Metropolitan University.

A typical example of this case is provided by a leaf of an irrational flow on a flat torus.

In this paper, we are mainly concerned with *regular* transnormal hypersurfaces of a complete Riemannian manifold and obtain several theorems (e.g., Theorems 4.1, 4.2 and 4.3) on their topological structures. The fundamental notion of our study is the *order* of a transnormal hypersurface (see § 3, for definition), which is originally introduced by Robertson when the ambient space is Euclidean. In fact, transnormal hypersurfaces of order either one or two are classified up to homeomorphism in a fairly general Riemannian setting. More precisely, we prove that under some condition on cut loci, a transnormal hypersurface of order one is homeomorphic to a Euclidean space, and a compact transnormal hypersurface of order two is homeomorphic to a Euclidean sphere. In addition, as for compact transnormal hypersurfaces of order greater than two, we determine their Euler characteristics under the same condition on cut loci as in the above cases of low order. On the other hand, the order of a transnormal hypersurface is shown to be either one or two if the ambient space is a simply connected complete Riemannian manifold of non-positive sectional curvature.

In § 1, we give the definition (in another version) and some examples of transnormal hypersurfaces.

§ 2 contains basic propositions on the generating frame of a transnormal hypersurface. Proofs are omitted mostly, because they are found in the indicated references.

§§ 3 and 4 are devoted to prove the above-mentioned theorems and their corollaries. These results emerge from a study of the distance function, which is defined in § 3, of a transnormal hypersurface.

Throughout this paper, unless otherwise stated, *manifolds are always assumed to be connected and* C^∞. Furthermore it should be remarked that *transnormal hypersurfaces as well as their ambient Riemannian manifolds are assumed, in their definition, to be complete*. Finally the author wishes to express his hearty thanks to Professors M. Obata and H. Omori for their encouragement and suggestions during the preparation of this paper.

1. Definition and examples of transnormal hypersurfaces. Let M be a complete Riemannian n-manifold isometrically *imbedded* into a complete Riemannian $(n+1)$-manifold W. We donote the normal bundle of M by $p\colon NM \to M$, and by N_x the fibre of NM over $x \in M$, i.e. $N_x = p^{-1}(x)$. N_x is identified with the subspace of the tangent space $T_x W$ of W at x in the obvious manner. The restriction of the exponential map exp:

$TW \to W$ of W to NM is written as $e: NM \to W$, i.e. $e = \exp | NM$, and called the *normal exponential map* of M.

DEFINITION 1.1. For each $x \in M$ there exists, up to parameterization, a unique geodesic τ_x of W which cuts M orthogonally at x. M is called a *transnormal hypersurface* (T. N. H.) if it satisfies the condition
(T–I) For each pair $x, y \in M$, the relation $\tau_x \ni y$ implies that $\tau_x = \tau_y$.

REMARK. As is easily seen, (T–I) is equivalent to
(T–II) For all $y \in M$ and $\xi \in N_x$ such that $e(\xi) = y$, it holds

$$d(e \mid N_x)_\xi(N_x) = N_y ,$$

where $d(e \mid N_x)_\xi$ denotes the differential of $e \mid N_x$ at ξ.

We will give now some

EXAMPLES. (cf. [2], [3] and [7])

(i) Every hypersurface of constant width in a Euclidean $(n+1)$-space $\boldsymbol{E}^{n+1}$ is the source of T. N. H.'s.

(ii) Let $S^n \subset S^{n+1}$ be the standard imbedding of a Euclidean n-sphere S^n as a great or small sphere of S^{n+1}. Then S^n is a T. N. H. of S^{n+1}.

(iii) The standard imbedding $P_n(\boldsymbol{R}) \subset P_{n+1}(\boldsymbol{R})$ of a real projective n-space $P_n(\boldsymbol{R})$ induced from the imbedding $S^n \subset S^{n+1}$ in (ii) yields a T. N. H. of $P_{n+1}(\boldsymbol{R})$.

(iv) Suppose M_1 and M_2 are T. N. H.'s of W_1 and W_2 respectively. Then the product manifold $M_1 \times M_2$ is a T. N. H. of $W_1 \times W_2$. A particular example of this type is the standard imbedding of $T^2 = S^1 \times S^1$ into $\boldsymbol{E}^4 = \boldsymbol{E}^2 \times \boldsymbol{E}^2$. It should be remarked that any imbedding of T^2 into $\boldsymbol{E}^3$ is not transnormal with respect to the induced Riemannian metric on T^2, i.e. T^2 can never be a T. N. H. of $\boldsymbol{E}^3$.

Here we make an elementary observation in connection with regular transnormal hypersurfaces.

PROPOSITION 1.1. *Let M be a regular complete hypersurface of a complete Riemannian manifold W. Then,*

(i) *M is closed in W, and*

(ii) *the normal exponential map $e: NM \to W$ is surjective.*

PROOF. (i) Since M is regula, the topology of M is the relative one. Therefore any sequence of points which is a Cauchy sequence in W is also a Cauchy one in M. Thus (i) follows from the completeness of M. (ii) is a direct consequence of (i) because W is complete. q.e.d.

2. Generating frames. Transnormality together with the well-known "Transversality Theorem" [1] leads us to the following

PROPOSITION 2.1 (Bolton [2]). *Let M^n be a T. N. H. of W^{n+1}. Then the following hold.*

(i) *The normal exponential map e is transversal to M.*

(ii) *$E = e^{-1}(M)$ is an n-submanifold of NM such that $e: E \to M$ is C^∞, and its tangent bundle TE coincides with $(de)^{-1}TM$.*

(iii) *$p: E \to M$ is a local diffeomorphism.*

(iv) *$e: E \to M$ is a local diffeomorphism.*

In fact, (i), (iii) and (iv) are obtained from the property (T-I) or (T-II), while (ii) is an implication of the transversality theorem.

Let N_tM be the normal sphere bundle of M with radius $t \in \boldsymbol{R}$, i.e.

$$N_tM = \{\xi \in NM; \|\xi\| = t\} ,$$

where $\| \ \|$ denotes the norm induced from the scalar product on each fibre. As usual M is identified with the zero cross section of NM, i.e. $M = N_0M$.

Now we can state the following theorem which tells the relation between Riemannian transnormality and the concept of generalized constant width for Riemannian manifolds.

THEOREM 2.2 (Bolton [2]). *Let M be a T. N. H. of W. Then the following hold.*

(i) *Let E_0 be a connected component of E. Then E_0 coincides with a component of N_tM for some $t \in \boldsymbol{R}$, and hence $p: E_0 \to M$ is a covering map which is at most two fold.*

(ii) *$E_x(= E \cap N_x)$ is isometric to E_y for all $x, y \in M$, where E_x is considered as a metric space with the distance induced from N_x.*

PROOF. (i) The Gauss lemma shows that E_0 is an open submanifold of N_tM for some $t \in \boldsymbol{R}$. Then, from the completeness of M, we get our first assertion immediately. For the latter half, remind that the codimension of M is one. (ii) is easily proved by (i).

From this theorem, we can see that E_x regareded as a subset of $\boldsymbol{E}^1$ is independent of the choice of a reference point $x \in M$. So the following definition is well-defined.

DEFINITION 2.1. An isometric copy of E_x in $\boldsymbol{E}^1$ is called the *generating frame* $\Phi(M)$ of M.

The most important property of the generating frame $\Phi(M)$ is contained in the following

THEOREM 2.3 (Robertson [7], Bolton [2], [3]). *There is an isometry group of E^1 which acts transitively on $\Phi(M)$ and leaves $\Phi(M)$ invariant.*

PROOF. Choose and fix a point $x \in M$ arbitrarily. It suffices to show that for each $\xi \in E_x$ there exists an isometry of N_x which sends x to ξ and maps E_x onto E_x.

Let $y = e(\xi)$, and join x with y by a C^∞-geodesic $\lambda: [0, 1] \to M$. For each $\zeta \in N_x$, we know the existence of the unique lift λ_ζ of λ to $N_{||\zeta||}M$, $\lambda_\zeta: [0, 1] \to N_{||\zeta||}M$, so that $\lambda_\zeta(0) = \zeta$ and $p \circ \lambda_\zeta = \lambda$. Then the mapping $\lambda^*: N_x \to N_y$ defined by $\lambda^*(\zeta) = \lambda_\zeta(1)$ is a linear isometry, as is the mapping $\eta: N_x \to N_y$ obtained by the parallel translation from x to y along the geodesic τ_x. Here $\lambda^* = \pm\eta$, since the dimension of each fibre of NM is one.

If $\lambda^* = \eta$ is the case, the mapping $N_x \to N_x$ defined by $\zeta \to \zeta + \xi$ is a required isometry. If $\lambda^* = -\eta$ is the case, so is the mapping $N_x \to N_x: \zeta \to \xi - \zeta$. In fact, one can easily check that both the above mappings send E_x onto E_x. For example, in case $\lambda^* = \eta$, we observe for each $\zeta \in E_x$

$$\exp_x(\zeta + \xi) = \exp_{e(\xi)} \eta(\zeta) = \exp_y \lambda^*(\zeta) \in M,$$

thus $\zeta + \xi \in E_x$. q.e.d.

3. Order of transnormality and distance functions. Let M be a T. N. H. of W. We now define an equivalence relation $\sim$ on M by writing $x \sim y$ to mean $y \in \tau_x$. Indeed, $\sim$ is an equivalence relation since (T–I) holds for M.

Take the quotient space $\hat{M} = M/\sim$ of M with respect to the relation $\sim$ and endow $\hat{M}$ with the quotient topology.

DEFINITION 3.1. We call M a *transnormal hypersurface of order* r, or briefly an *r-transnormal hypersurface*, if the natural projection $\psi: M \to \hat{M}$ is an r-fold (topological) covering map.

REMARK. In general, the projection ψ is not always a covering map. To take an illustration, suppose W is the Klein bottle constructed from the produdt $[0, 1] \times [0, 1]$ of unit intervals by identifying $(t, 0)$ with $(t, 1)$ and $(0, s)$ with $(1, 1 - s)$ respectively. Let M be a hypersurface resulting from $\{0\} \times [0, 1]$. Then M is a T. N. H. of W, but the projection ψ is not a covering map.

However the following proposition shows that the covering condition in Definition 3.1 is automatically satisfied in a fairly general family of Riemannian manifolds.

PROPOSITION 3.1. *Let M be a T. N. H. of a simply connected complete*

Riemannian manifold W of constant sectional curvature. Then the projection $\psi: M \to \hat{M}$ *is a covering map.*

PROOF. For each $\alpha \in \hat{M}$, we shall find a neighborhood $V(\alpha)$ of α in $\hat{M}$ which is evenly covered by M via ψ. Choose a point $x \in \psi^{-1}(\alpha)$, and let $U(x)$ be an arcwise connected open neighborhood of x in M such that $p^{-1}(U(x))$ is trivial. Put $V(\alpha) = \psi(U(x))$. Then $\psi^{-1}(V(\alpha)) = e(p^{-1}(U(x)) \cap E)$.

Here remark that for each pair $x, y \in M$, there exists an isometry ϕ of W which satisfies the following conditions: $\phi(x) = y$, $\phi(\tau_x) = \tau_y$ and $\phi(e(E_x)) = e(E_y)$, since the ambient space W is a simply connected complete Riemannian manifold of constant curvature and Theorem 2.2 (ii) holds. From this it is observed without difficulty that on each component of $\psi^{-1}(V(\alpha))$, ψ is a bijection onto $V(\alpha)$. Thus, by the definition of the topology on $\hat{M}$, ψ maps each component of $\psi^{-1}(V(\alpha))$ homeomorphically onto $V(\alpha)$. This completes the proof. q.e.d.

This proposition also asserts that in a simply connected complete Riemannian manifold of constant sectional curvature, each geodesic which cuts a transnormal hypersurface orthogonally at some point (and then orthogonally at all points of intersection) intersects the hypersurface the same number of times.

We use elementary parts of the Morse theory to study the topological structure of an r-transnormal hypersurface M of W. Choose and fix a point $p \in M$. Let $C(p)$ be the cut locus of p in W, and put $M' = M - C(p)$. (For the definition of $C(p)$, see [4]).

DEFINITION 3.2. By the *distance function* Λ_p of M, we mean the real valued function $\Lambda_p: M' \to \boldsymbol{R}$ defined by

$$\Lambda_p(x) = d(p, x)^2 , \qquad x \in M' ,$$

where $d(,)$ denotes the distance in W.

To simplify our discussion, we impose the following assumption on M:

[Condition A] There is a point $p \in M$ such that $C(p) \cap M = \emptyset$, i.e. $M = M'$.

This assumption places no restriction if W is a simply connected complete Riemannian manifold of non-positive sectional curvature, in particular if W is either a Euclidean space $\boldsymbol{E}^{n+1}$ or a hyperbolic space $\boldsymbol{H}^{n+1}$, since $C(p) = \emptyset$ for all $p \in W$.

PROPOSITION 3.2 (Fundamental properties of Λ_p). *Let M be a T. N. H. of W. Then, under the assumption of Condition A,*

(i) *Λ_p is a C^∞-function on M,*

(ii) *$e(E_p)(=\tau_p \cap M)$ coincides with the set of critical points of Λ_p, and*

(iii) *Λ_p is non-degenerate, i.e. Λ_p: $M \to \boldsymbol{R}$ is a Morse funtion.*

PROOF. (i) This is well-known [4].

(ii) By Condition A, the minimizing geodesic $\tau(p, x)$ of W joining p with x is unique for each $x \in M$. Note that $\Lambda_p(x)$ is nothing but the square of the length of $\tau(p, x)$. Let $x \in M$ be a critical point of Λ_p, i.e. $d\Lambda_p(x) = 0$. Then from the formula for the first variation of Λ_p, we observe that $\tau(p, x)$ is perpendicular to M at x [4]. Since M is a T. N. H., $\tau(p, x)$ is perpendicular to M at p as well. Thus $x \in e(E_p)(=\tau_p \cap M)$.

Conversely, if $x \in e(E_p)$ then $d\Lambda_p(x) = 0$ as is easily seen.

(iii) By the formula for the second variation of Λ_p, Λ_p is degenerate at its critical point $x \in M$ if and only if p is a focal point of x [4]. However, Proposition 2.1 (iv) implies that there is no focal point of x on M. Hence Λ_p is non-degenerate. q.e.d.

4. Transnormal hypersurfaces of order r.

First, we prove the following

THEOREM 4.1. *Let M^n be a T. N. H. of W^{n+1}, and suppose M^n satisfies Condition A. Then the following hold.*

(i) *If M^n is 1-transnormal, then M^n is homeomorphic to a Euclidean n-space $\boldsymbol{E}^n$.*

(ii) *If M^n is compact and 2-transnormal, then M^n is homeomorphic to a Euclidean n-sphere S^n.*

PROOF. (i) By Proposition 3.2 (ii), Λ_p has only one critical point p which is a non-degenerate minimum of Λ_p. Thus M^n is homeomorphic to an open n-cell by one of the fundamental theorem of Morse [5]. So M^n is homeomorphic to $\boldsymbol{E}^n$.

(ii) M^n is compact and has exactly two non-degenerate critical point of Λ_p due to Proposition 3.2 (ii). From this, a well-known theorem of Reeb [5] implies that M^n is homeomorphic to S^n. q.e.d.

REMARK. 1°. Without the assumption of Condition A, Theorem 4.1 does not hold. In fact, let W^2 be a Euclidean cylinder $W^2 = S^1 \times \boldsymbol{E}^1$ and M^1 a T. N. H. $M^1 = S^1 \times \{0\}$. Then M^1 is 1-transnormal and homeomorphic to S^1.

2°. If $\boldsymbol{E}^n$ is imbedded into W^{n+1} as a T.N.H. of order $r(<+\infty)$, then $r=1$. For $\boldsymbol{E}^n$ must be a covering manifold of finite order r.

3°. In a forthcoming paper [6], we shall study more differential geometric structures of a compact 2-transnormal hypersurface in a space of constant curvature.

1- and 2-transnormal hypersurfaces cover a rather wide class of transnormal hypersurfaces by the following

THEOREM 4.2. *Let M^n be an $r(<+\infty)$-transnormal hypersurface of W^{n+1}. Suppose W^{n+1} is a simply connected complete Riemannian manifold of non-positive sectional curvature. Then r is either one or two.*

PROOF. Choose a point $p \in M$ arbitrarily. Owing to the assumpiton on the curvature of W^{n+1}, the normal exponential map e is a diffeomorphism from N_p onto τ_p [4]. This together with r-transnormality of M implies that r is the number of elements in the generating frame $\Phi(M)$. However, by Theorem 2.3, an isometry group of E^1 acts transitively on $\Phi(M)$ and leaves $\Phi(M)$ invariant. Thus r must be either one or two. This completes the proof. q.e.d.

Now, following an idea of Robertson [8] in the case of a Euclidean space, we determine the Euler characteristic of a compact r-transnormal hypersurface.

THEOREM 4.3. *Let M be a compact $r(<+\infty)$-transnormal hypersurface of W, and suppose M satisfies Condition A. Then the Euler characteristic $\chi(M)$ of M is either zero or r.*

PROOF. Let r_i be the number of critical points of Λ_p with index i. Then from the Morse inequality [5],

$$\chi(M) = \sum_{i=0}^{n} (-1)^i r_i .$$

On the other hand, r-transnormality of M yields

$$r = \sum_{i=0}^{n} r_i .$$

Since the projection $\psi: M \to \hat{M}$ is an r-fold covering map,

$$\chi(M) = r \cdot \chi(\hat{M}) .$$

Also $r_0 > 0$ and $r_n > 0$, so that

$$-r < \chi(M) \leqq r ,$$

and hence

$$-1 < \chi(\hat{M}) \leqq 1 .$$

Thus either $\chi(\hat{M}) = 0$ or $\chi(\hat{M}) = 1$. Consequently,

$$\chi(M) = 0 \quad \text{or} \quad \chi(M) = r . \qquad \text{q.e.d.}$$

REMARK. Of course, if such M is odd dimensional, $\chi(M)$ is zero.

Theorem 4.3 has a number of interesting corollaries. For instance,

PROPOSITION 4.4. *If S^{2n} is imbedded into W^{n+1} as a T. N. H. of order $r(< +\infty)$ which satisfies Condition A, then $r = 2$.*

For other corollaries in the case $W^{n+1} = E^{n+1}$, see Robertson [8].

REFERENCES

[1] R. ABRAHAM and J. ROBBIN, Transversal mappings and flows, Benjamin, New York, 1967.
[2] J. BOLTON, Riemannian transnormality, Thesis, University of Liverpool, 1970.
[3] J. BOLTON, Transnormal hypersurfaces, Proc. Camb. Phil. Soc., 74 (1973), 43-48.
[4] S. KOBAYASHI and K. NOMIZU, Foundations of differential geometry, vol. I, vol. II, Interscience, New Nork, 1963, 1969.
[5] J. MILNOR, Morse theory, Ann. of Math. Studies, No. 51, Princeton University Press, 1963.
[6] S. NISHIKAWA, Compact two-transnormal hypersurfaces in a space of constant curvature, to appear.
[7] S. A. ROBERTSON, Generalized constant width for manifolds, Michigan Math. J., 11 (1964), 97-105.
[8] S. A. ROBERTSON, On transnormal manifolds, Topology, 6 (1967), 117-123.
[9] I. M. YAGLOM and V. G. BOLTYANSKIĬ, Convex figures, translation by P. J. Kelly and L. F. Walton, Holt, Reinhalt and Winston, New York, 1961.

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