TRANSPLANTATION, SUMMABILITY AND MULTIPLIERS FOR MULTIPLE LAGUERRE EXPANSIONS

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Abstract. This paper is concerned with Cesàro summability and Marcinkiewicz multipliers for the n-dimensional case of Laguerre expansions of a different kind. The results are obtained from the corresponding results for the n-dimensional Hermite expansions by appealing to a transplantation theorem which is also proved here.

1. Introduction. The aim of this paper is to study the mean summability of multiple Laguerre expansions. There are at least four types of Laguerre expansions on the half line studied in the literature. The fourth type studied by Markett [7] and the author [15] seems to behave better than the other types with respect to the Riesz or Cesàro summability. In this paper we consider the n-dimensional version of this fourth type of expansions. Before describing the results, some comparison between the Hermite and Laguerre expansions on the real line are in order.

In the case of the Hermite series there are only two possibilities: one can either consider the Hermite polynomials H_k forming an orthogonal system in $L^2(\boldsymbol{R}, e^{-x^2}dx)$ or consider the Hermite functions h_k forming an orthogonal system in $L^2(\boldsymbol{R}, dx)$. In the former case we expand f in $L^p(\boldsymbol{R}, e^{-x^2}dx)$ in terms of H_k. For this series it has been proved by Pollard [9] and Askey-Hirschman [1] that no Cesàro means of any order whatsoever will converge to the function in the norm for $p \neq 2$. On the other hand for expansions in terms of h_k even the partial sums $S_N f$ converge to f in the norm provided $4/3 < p < 4$. This fundamental result concerning the summability of Hermite series is due to Askey-Wainger [2]. Later several authors studied the Riesz and Cesàro means for the Hermite series. (See [12] and the references thereof). The author has proved in [12] that the critical index for the Riesz summability is $1/6$.

As we have mentioned in the beginning, there are four types of Laguerre expansions on the half line $\boldsymbol{R}_+ = (0, \infty)$. The Laguerre polynomials $L_k^\alpha(x)$ form an orthogonal system in $L^2(\boldsymbol{R}_+, x^\alpha e^{-x}dx)$. As in the case of Hermite polynomials the series in terms of L_k^α turns out to be nonconvergent for $p \neq 2$. Therefore, we define the following three families of Laguerre functions:

$$(1.1) \qquad \mathscr{L}_k^\alpha(t) = \left(\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)} \right)^{1/2} L_k^\alpha(t) e^{-t/2} t^{-\alpha/2} \,,$$

$$(1.2) \qquad \psi_k^\alpha(t) = \mathscr{L}_k^\alpha\left(\frac{1}{2}t^2\right)t^{-\alpha},$$

$$(1.3) \qquad \varphi_k^\alpha(t) = \mathscr{L}_k^\alpha(t^2)(2t)^{1/2}.$$

The functions $\mathscr{L}_k^\alpha$ and φ_k^α form orthonormal systems in $L^2(\mathbf{R}_+, dt)$, whereas the functions ψ_k^α form an orthonormal system in $L^2(\mathbf{R}_+, t^{2\alpha+1}dt)$. Consequently, we have three types of Laguerre expansions. Let us call them expansions of type II, type III and type IV, respectively.

For type III expansions with $\alpha \geq 0$ it has been proved by Görlich-Markett [4] that the critical index for the Cesàro summability is $(\alpha + 1/2)$. On the other hand the critical index for type II expansions is $1/2$ (see Markett [6]). When $\alpha = 0$ these two types coincide. Thus, we see that there is a basic difference between the Hermite and Laguerre expansions in that they have different critical indices of summability. This difference is well explained by the fact that the Laguerre expansions of type III with $\alpha = 0$ arise as a particular case of special Hermite expansions on C, see [18]. Thus Laguerre expansions of type III are to be treated as some two-dimensional expansions even though they are on the half line. Therefore, the critical index is $1/2$ which is only to be expected.

In [7] Markett initiated the study of the fourth type of expansions. He made the interesting observation that the L^p-norms of the partial sum operators S_N for type IV expansions grow slower than that for type III expansions. This indicated a smaller critical index and in [15] the author proved that it is $1/6$ for the type IV expansions. Thus the type IV expansions is similar in nature to the Hermite expansions. We expect, therefore, that the n-dimensional version of this Laguerre expansion will behave just like the n-dimensional Hermite expansions. This claim is proved to be correct in this paper.

We conclude this introduction with some words about the organisation of this paper. In the next section we set the notation and state the main theorems. We reduce everything to the Hermite case by appealing to a transplantation theorem to be proved in the following sections. Section 3 consists of the sketch of the proof of the transplantation theorem. Section 4 proves a multiplier theorem needed to prove the transplantation theorem. Section 5 gives the proof of the transplantation theorem.

Finally, we wish to thank Professor Yuichi Kanjin for sending us the preprint [5] which inspired this work.

2. Notation and the main results.

For $\alpha > -1$ the Laguerre polynomials of type α are defined by

$$(2.1) \qquad L_k^\alpha(t) = \frac{1}{k!}e^t t^{-\alpha}\frac{d^k}{dt^k}(e^{-t}t^{k+\alpha}), \quad t > 0, \; k = 0, 1, 2.$$

Then the Laguerre functions ψ_k^α, $\mathscr{L}_k^\alpha$ and φ_k^α are defined as in Section 1. The Hermite

polynomials and the Hermite functions are defined by the formulas

$$(2.2) \qquad H_k(t) = (-1)^k e^{t^2} \frac{d^k}{dt^k} (e^{-t^2}) ,$$

$$(2.3) \qquad h_k(t) = (2^k k! \sqrt{\pi})^{-1/2} H_k(t) e^{-t^2/2} .$$

In the n-dimensional case we define the Hermite functions $\Phi_m(x)$ on R^n by

$$(2.4) \qquad \Phi_m(x) = \prod_{j=1}^{n} h_{m_j}(x_j) .$$

Here $x = (x_1, \ldots, x_n) \in R^n$, and $m = (m_1, \ldots, m_n)$ is a multi-index. The functions Φ_m form a complete orthonormal system in $L^2(R^n)$ and one has the Hermite series

$$(2.5) \qquad f = \sum_m (f, \Phi_m)\Phi_m \quad \text{with} \quad (f, \Phi_m) = \int_{R^n} f(x)\Phi_m(x)dx .$$

In [12], [13] and [17] we have studied the Riesz summability of this series.

We now define the n-dimensional Laguerre functions. Let $R^n_+ = \{x \in R^n : x_j \geq 0\}$ and let $\alpha = (\alpha_1, \ldots, \alpha_n)$ where $\alpha_j > -1$ for all j. We define $\mathscr{L}^\alpha_m(x)$ and $\Phi^\alpha_m(x)$ by

$$(2.6) \qquad \mathscr{L}^\alpha_m(x) = \prod_{j=1}^{n} \mathscr{L}^{\alpha_j}_{m_j}(x_j) ,$$

$$(2.7) \qquad \Phi^\alpha_m(x) = \prod_{j=1}^{n} \varphi^{\alpha_j}_{m_j}(x_j) .$$

Associated to the families $\mathscr{L}^\alpha_m(x)$ and $\Phi^\alpha_m(x)$, we have the following two types of expansions:

$$(2.8) \qquad f = \sum_m (f, \mathscr{L}^\alpha_m)\mathscr{L}^\alpha_m ,$$

$$(2.9) \qquad f = \sum_m (f, \Phi^\alpha_m)\Phi^\alpha_m .$$

In (2.8) and (2.9), (f, g) stands for the inner product in the Hilbert space $L^2(R^n_+, dx)$. We then define the projections

$$(2.10) \qquad Q^\alpha_k f = \sum_{|m|=k} (f, \mathscr{L}^\alpha_m)\mathscr{L}^\alpha_m ,$$

$$(2.10') \qquad P^\alpha_k f = \sum_{|m|=k} (f, \Phi^\alpha_m)\Phi^\alpha_m .$$

In the same way we define

$$(2.11) \qquad P_k f = \sum_{|m|=k} (f, \Phi_m)\Phi_m$$

to be the projection for the Hermite series.

Riesz means of order $\delta > -1$ for the Hermite series is defined by

$$(2.12) \qquad S_R^\delta f = \sum_{k=0}^\infty \left(1 - \frac{2k+n}{R}\right)_+^\delta P_k f,$$

where $a_+ = a$ for $a \geq 0$ and $a_+ = 0$ for $a < 0$. Riesz means for the series (2.8) denoted by $s_R^{\delta,\alpha} f$ is defined in a similar way. For the series (2.9) we define the Riesz means $S_R^{\delta,\alpha} f$ with a slight change, namely,

$$(2.13) \qquad S_R^{\delta,\alpha} f = \sum_{k=0}^\infty \left(1 - \frac{4k+n}{R}\right)_+^\delta P_k^\alpha f.$$

We have used $4k+n$ instead of $2k+n$ because it will enable us to deduce results from the Hermite expansions. This change does not affect the behaviour of the Riesz means in any way. We define the critical index for the Riesz summability to be the smallest δ_0 for which $\delta > \delta_0$ implies that the Riesz means of order δ are uniformly bounded on $L^1(R^n)$ or $L^1(R_+^n)$ as the case may be.

For the Hermite expansions the critical index is $1/6$ for $n = 1$ [12] and $(n-1)/2$ for $n \geq 2$ [13], [17]. On the other hand, for the series (2.8) with $\alpha_j \geq 0$ for all j, the critical index is $n - 1/2$ as proved in [20]. In the case of (2.9) with $n = 1$ and $\alpha \geq 1/2$ we have proved in [15] that the critical index is $1/6$. In the n-dimensional case of (2.9) we expect the critical index to be $(n-1)/2$. Our first theorem shows that this is indeed the case.

THEOREM 2.1.   *Assume that $\alpha_j \geq -1/2$ for all j.*

(i)   *When $n = 1$ the uniform estimates*

$$\|S_R^{\delta,\alpha} f\|_p \leq c \|f\|_p, \qquad f \in L^p(R_+)$$

hold for all $1 < p < \infty$ provided $\delta > 1/6$.

(ii)   *When $n \geq 2$, the uniform estimates hold for $1 < p < \infty$ provided $\delta > (n-1)/2$.*
In both cases $S_R^{\delta,\alpha} f$ converges to f in the norm for $1 < p < \infty$.

We also study Marcinkiewicz multiplier theorem for the Laguerre series (2.9). Given a bounded function λ on R_+ we consider the operator M_λ^α defined by

$$(2.14) \qquad M_\lambda^\alpha f = \sum_{k=0}^\infty \lambda(2k+n) P_k^\alpha f.$$

Since λ is a bounded function, M_λ^α is evidently a bounded operator on $L^2(R_+^n)$. But they are bounded on $L^p(R_+^n)$ for $p \neq 2$ only under some conditions on λ. Such a condition is given in the next theorem.

THEOREM 2.2.   *Assume that $\alpha_j \geq -1/2$ as before. Let λ be a C^k-function for some $k > n/2$ and let λ satisfy the estimates*

(2.15)
$$\sup_{t>0} | t^j \lambda^{(j)}(t) | \le C_j$$

for $j = 0, 1, 2, \ldots, k$. Then the operator M_λ^α is bounded on $L^p(\mathbf{R}_+^n)$ for $1 < p < \infty$.

Both theorems will be deduced from the corresponding results for the Hermite expansions which are already proved elsewhere. The deduction is made possible by a transplantation theorem which we are going to describe now. For each pair of n-tuples α and β with $\alpha_j, \beta_j > -1$ we define an operator W_α^β formally by

(2.16)
$$W_\alpha^\beta f = \sum_m (f, \Phi_m^\beta) \Phi_m^\alpha .$$

For this operator W_α^β we prove the following theorem which is in fact the main result of this paper.

THEOREM 2.3.   Assume that $\alpha_j, \beta_j \ge -1/2$ for all j. Then for $1 < p < \infty$ the operator W_α^β is bounded on $L^p(\mathbf{R}_+^n)$.

The operator W_α^β is called a transplantation operator for the following reason. It is easy to check that

(2.17)
$$W_\beta^\alpha M_\lambda^\alpha W_\alpha^\beta f = M_\lambda^\beta f .$$

In view of this relation and Theorem 2.3 it follows that M_λ^β is bounded on $L^p(\mathbf{R}_+^n)$ whenever M_λ^α is bounded. Therefore, we can transplant any norm inequality for expansions in terms of Φ_m^α from the same for expansions in terms of Φ_m^β and vice versa. Hence in order to prove Theorems 2.1 and 2.2 it is enough to prove them in the particular case $\alpha = (-1/2, -1/2, \ldots, -1/2)$. The rest of this section is devoted to showing how the particular case $\alpha = (-1/2, \ldots, -1/2)$ of the two theorems follow from the corresponding results for the Hermite expansions.

The relation between the Hermite and Laguerre polynomials is given by (see [11])

(2.18)
$$H_{2k}(t) = (-1)^k 2^{2k} k! L_k^{-1/2}(t^2) ,$$

(2.19)
$$H_{2k+1}(t) = (-1)^k 2^{2k+1} k! t L_k^{1/2}(t^2) .$$

From this it follows that h_{2k} is even and h_{2k+1} is odd. From the definition of $\varphi_k^{1/2}$ and $\varphi_k^{-1/2}$ we also observe that

(2.20)
$$h_{2k}(t) = (-1)^k \frac{1}{\sqrt{2}} \varphi_k^{-1/2}(t) ,$$

(2.21)
$$h_{2k+1}(t) = (-1)^k \frac{1}{\sqrt{2}} \varphi_k^{1/2}(t) .$$

Therefore, if f is a function on $\mathbf{R}^n$ which is even in each variable separately, then

(2.22)
$$P_{2k+1}f = \sum_{|m|=2k+1} (f, \Phi_m)\Phi_m = 0$$

and

(2.23) $$P_{2k}f = \sum_{|m|=2k} (f, \Phi_m)\Phi_m = \sum_{|m|=k} (f, \Phi_{2m})\Phi_{2m} = \sum_{|m|=k} (f, \Phi_m^{(-1/2)})\Phi_m^{(-1/2)} = P_k^{(-1/2)}f,$$

where $(-1/2) = (-1/2, \ldots, -1/2)$.

Therefore, if M_λ is the multiplier operator

(2.24)
$$M_\lambda f = \sum_{k=0}^{\infty} \lambda(2k+n)P_k f$$

for the Hermite series, then for f even in each variable we get

(2.25)
$$M_\lambda f = \sum_{k=0}^{\infty} \lambda(4k+n)P_k^{(-1/2)}f.$$

Hence Theorem 2.2 for $\alpha = (-1/2)$ follows from the multiplier theorem (see [14]) for the Hermite expansions. Since the Riesz means are also multiplier transforms, Theorem 2.1 for $\alpha = (-1/2)$ also follows from the results in [12], [13] for the Hermite expansions.

In the next section we sketch a proof of the transplantation Theorem 2.3.

3. **The transplantation theorem.** In order to establish the transplantation theorem it is enough to consider the one-dimensional case. So, let α and β be real numbers both greater than -1. Our operator W_α^β is related to the operator T_α^β studied by Kanjin [5]. In fact, as we will see presently, our theorem is equivalent to a weighted norm inequality for T_α^β. Recall from [5] that the operator T_α^β is defined by

(3.1)
$$T_\alpha^\beta f = \sum_{k=0}^{\infty} (f, \mathscr{L}_k^\beta)\mathscr{L}_k^\alpha.$$

For this operator Kanjin has proved the following result:

THEOREM 3.1 (Kanjin). *Let $v = \min\{\alpha, \beta\}$. If $v \geq 0$, then T_α^β is bounded on $L^p(\mathbf{R}_+)$ for $1 < p < \infty$. If $-1 < v < 0$, it is bounded on $L^p(\mathbf{R}_+)$ for p lying in the interval $(1 + v/2)^{-1} < p < -2/v$.*

We now bring out the relation between W_α^β and T_α^β. Since $\varphi_k^\alpha(t) = \mathscr{L}_k^\alpha(t^2)(2t)^{1/2}$ it is easy to see that

(3.2)
$$(f, \varphi_k^\alpha) = \frac{1}{\sqrt{2}}(g, \mathscr{L}_k^\alpha),$$

where $g(t) = f(\sqrt{t})t^{-1/4}$. Therefore, one has

(3.3)
$$W_\alpha^\beta f(\sqrt{t})t^{-1/4} = T_\alpha^\beta g(t).$$

In view (3.3) it is clear that the boundedness of W_α^β is equivalent to

(3.4)
$$\int_0^\infty |T_\alpha^\beta g(t)|^p t^{p/4-1/2} dt \le C \int_0^\infty |g(t)|^p t^{p/4-1/2} dt .$$

Hence, Theorem 2.3 follows once we prove:

THEOREM 3.2. *Assume that $\alpha, \beta \ge -1/2$. Then for $1 < p < \infty$ the inequality (3.4) is valid.*

In proving Theorem 3.2 we closely follow the proof of Theorem 3.1 given in [5]. Here we give a sketch of the proof. Let us define

(3.5)
$$\|f\|_{p,w}^p = \int_0^\infty |f(x)|^p x^{p/4-1/2} dx .$$

We claim that Theorem 3.2 follows from the next proposition. A function $M(\theta)$ defined on R is said to be admissible if it satisfies the growth condition

(3.6)
$$\sup_{\theta \in R} e^{-a|\theta|} \log M(\theta) < \infty \qquad \text{with} \quad a < \pi .$$

PROPOSITION 3.1. (i) *For $\alpha = 0, 1, 2, \ldots$ one has*

$$\|T_\alpha^{\alpha+k+i\theta} f\|_{p,w} \le M(\theta)\|f\|_{p,w}$$

for $1 < p < \infty$ and $k = 0, 2$ where $M(\theta)$ is admissible.
 (ii) *If $\alpha \ge -1/2$, then for $1 < p < \infty$*

$$\|T_\alpha^{\alpha+2} f\|_{p,w} \le C\|f\|_{p,w} .$$

We now briefly indicate how Theorem 3.2 follows from this proposition. Since $(T_\alpha^\beta f, g) = (f, T_\beta^\alpha g)$ it is enough to consider $-1/2 \le \alpha < \beta$. Choose an integer N so that $2N \le \beta < 2(N+1)$. Then it follows that if $-1/2 \le \alpha < 0$

(3.7)
$$T_\alpha^\beta = T_\alpha^{\alpha+2} \circ T_{\alpha+2}^0 \circ T_0^2 \circ \cdots \circ T_{2N}^\beta .$$

(If $\alpha \ge 0$ we can get a similar expression). Therefore it is enough to prove the following two things:
 (a) $\|T_\alpha^{\alpha+2} f\|_{p,w} \le C\|f\|_{p,w}$, $\alpha \ge -1/2$,
 (b) $\|T_n^\beta f\|_{p,w} \le C\|f\|_{p,w}$, $n \le \beta \le (n+2)$.
Since (a) is simply Part (ii) of the proposition we only need to prove (b).

To prove (b) we use analytic interpolation. We consider the analytic family $T_\alpha^{\alpha+2z}$ of operators where $0 \le \text{Re } z \le 1$. T_α^{2+2z} is an admissible family of operators in our situation (cf. [5]). Since $T_n^{n+i\theta}$ and $T_n^{n+2+i\theta}$ are bounded on L^p it follows that T_n^β with $n \le \beta \le n+2$ satisfies $\|T_n^\beta f\|_{p,w} \le C\|f\|_{p,w}$. Hence (b) is proved. Therefore, it remains to

establish the proposition.

In order to prove the proposition it is necessary to study the L^p-mapping properties of a multiplier operator m_λ^α which is interesting in itself. This operator m_λ^α is defined by setting

$$m_\lambda^\alpha f = \sum_{k=0}^\infty \lambda(2k+1)(f, \mathscr{L}_k^\alpha)\mathscr{L}_k^\alpha,$$

where λ is a bounded function. We first establish the following theorem

THEOREM 3.3.   *Assume that $\alpha = n-1$, where $n = 1, 2, \ldots$. Let λ satisfy the estimates*

(3.9)
$$\sup_{t>0} |t^j \lambda^{(j)}(t)| \le C_j$$

for $j = 0, 1, \ldots, (n+2)$. Then for $1 < p < \infty$ we have

(3.10)
$$\|m_\lambda^\alpha f\|_{p,w} \le C \|f\|_{p,w}.$$

To apply this multiplier theorem to proof of the proposition we consider the function λ defined by

(3.11)
$$\lambda(2t) = \left(\frac{\Gamma\left(t + \alpha + \dfrac{1}{2} + i\theta\right)}{\Gamma\left(t + \alpha + \dfrac{1}{2}\right)} \right)^{1/2}.$$

Then as in [5] we can prove that this function λ satisfies the conditions (3.9) for any $\alpha = n-1$. Take $\varphi(k) = \lambda(2k+1)^{-1}$ and define

(3.12)
$$T_{\alpha,\varphi}^\beta f = \sum_{k=0}^\infty \varphi(k)(f, \mathscr{L}_k^\beta)\mathscr{L}_k^\alpha.$$

Then it is clear that

(3.13)
$$m_\lambda^\alpha(T_{\alpha,\varphi}^\beta f) = T_\alpha^\beta f.$$

Hence in view of the multiplier theorem, Proposition 3.1 would follow immediately once we have the following estimates.

PROPOSITION 3.2.   (i)   *When $\alpha \ge 0$ one has*

(3.14)
$$\|T_{\alpha,\varphi}^{\alpha+k+i\theta} f\|_{p,w} \le M(\theta) \|f\|_{p,w}$$

for $1 < p < \infty$, $k = 0, 2$ where $M(\theta)$ is admissible,
   (ii)   *When $\alpha \ge -1/2$ one has for $1 < p < \infty$*

(3.15)
$$\|T_{\alpha,\varphi}^{\alpha+2+i\theta} f\|_{p,w} \le M(\theta) \|f\|_{p,w}.$$

Thus the crux of the matter lies in proving the multiplier theorem and the above

proposition. First we will take up the multiplier theorem.

4. The multiplier theorem. In this section our aim is to establish the multiplier theorem for the Laguerre expansions. To this end we first study multipliers for the special Hermite expansions on C^n. By the term special Hermite expansion we mean an expansion of the form

$$(4.1) \qquad f = (2\pi)^{-n} \sum_{k=0}^{\infty} f \times \varphi_k .$$

Here f is a function on C^n, φ_k is the function

$$(4.2) \qquad \varphi_k(z) = L_k^{n-1}\left(\frac{1}{2}|z|^2\right)e^{-|z|^2/4}$$

and $f \times g$ stands for the twisted convolution

$$(4.3) \qquad (f \times g)(z) = \int_{C^n} f(z-w)g(w)e^{i\,\mathrm{Im}(z,\bar{w})/2}dw .$$

For facts about the twisted convolution and special Hermite expansions we refer to [16] and [18] and the references thereof.

For the special Hermite expansion (4.1) we define a multiplier transform T_λ by the prescription

$$(4.4) \qquad T_\lambda f(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} \lambda(2k+n)(f \times \varphi_k)(z) .$$

For this operator we establish the following.

THEOREM 4.1. *Assume that the function λ satisfies the conditions of Theorem 3.3 (with $\alpha = n - 1$). Then for $1 < p < \infty$,*

$$(4.5) \qquad \int_{C^n} |T_\lambda F(z)|^p |z|^{(n-1/2)p-(2n-1)}dz \leq C \int_{C^n} |F(z)|^p |z|^{(n-1/2)p-(2n-1)}dz$$

whenever $F \in L^p(C^n)$ is radial.

We claim that Theorem 3.3 follows from the above theorem. To prove the claim we use the fact that when F is radial

$$(4.6) \qquad (F \times \varphi_k)(z) = \frac{k!(n-1)!}{(k+n-1)!}\left(\int_{C^n} F(z)\varphi_k(z)dz\right)\varphi_k(z)$$

which shows that $F \times \varphi_k$ is a radial function. With $r = |z|$ we have

$$(4.7) \quad (F \times \varphi_k)(z) = C_n \frac{k!(n-1)!}{(k+n-1)!}\left(\int_0^\infty F(r)L_k^{n-1}\left(\frac{1}{2}r^2\right)e^{-r^2/4}r^{2n-1}dr\right)L_k^{n-1}\left(\frac{1}{2}r^2\right)e^{-r^2/4}$$

which can be written as

(4.8) $(F \times \varphi_k)(z) = C'_n(f, \mathscr{L}_k^{n-1})\mathscr{L}_k^{n-1}(r^2/2)r^{1-n}$,

where f and F are related by

(4.9) $f(r^2/2) = F(r)r^{n-1}$.

Thus $T_\lambda F$ becomes

(4.10) $T_\lambda F(z) = C'_n \sum_{k=0}^{\infty} \lambda(2k+n)(f, \mathscr{L}_k^{n-1})\mathscr{L}_k^{n-1}(r^2/2)r^{1-n} = C''_n m_\lambda^{n-1} f(r^2/2)r^{1-n}$.

Therefore, Theorem 4.1 gives

(4.11) $\int_0^\infty \left| m_\lambda^{n-1} f\left(\frac{1}{2}r^2\right)\right|^p r^{(1-n)p+(n-1/2)p} dr \leq C \int_0^\infty \left| f\left(\frac{1}{2}r^2\right)\right|^p r^{(1-n)p+(n-1/2)p} dr$

which after a change of variables becomes

(4.12) $\int_0^\infty |m_\lambda^{n-1} f(r)|^p r^{p/4-1/2} dr \leq C \int_0^\infty |f(r)|^p r^{p/4-1/2} dr$.

Hence we obtain Theorem 3.3.

In the paper [16] on Weyl multipliers we have established the inequality

(4.13) $\int_{C^n} |T_\lambda F(z)|^p dz \leq C \int_{C^n} |F(z)|^p dz$

for all functions F in $L^p(C^n)$, $1 < p < \infty$. Let us briefly recall how this was proved. We consider the semigroup T^t defined by

(4.14) $T^t f = (2\pi)^{-n} \sum_{k=0}^{\infty} e^{-(2k+n)t} f \times \varphi_k$

and define Littlewood-Paley-Stein g functions

(4.15) $(g(f, z))^2 = \int_0^\infty |\partial_t T^t f(z)|^2 t dt$,

(4.16) $(g_k^*(f, z))^2 = \int_0^\infty \int_{C^n} t^{1-n}(1 + t^{-1}|z - w|^2)^{-k} |\partial_t T^t f(w)|^2 dt dw$,

where k is a nonnegative integer. For these functions we established the following bounds:

(4.17) $C_1\|f\|_p \leq \|g(f)\|_p \leq C_2\|f\|_p$, $1 < p < \infty$,

(4.18) $\|g_k^*(f)\|_p \leq C\|f\|_p$, $p > 2$ and $k > n$.

Then under the hypotheses on λ we established

(4.19) $$g(T_\lambda f, z) \le C g_k^*(f, z) ,$$

where $k = n + 1$. In view of (4.17) and (4.18) we obtain (4.13) for $p > 2$ from (4.19). Then a duality argument proves (4.13) for $p < 2$.

Since (4.19) is true, in order to prove Theorem 4.1 we only need to establish the following proposition. By slightly abusing the notation we denote by $\|F\|_{p, w}$ the norm

$$\|F\|_{p, w} = \int_{C^n} |F(z)|^p |z|^{(n-1/2)p - (2n-1)} dz .$$

PROPOSITION 4.1.   (i)   *There are constants C_1 and C_2 such that for $1 < p < \infty$ one has*

(4.20) $$C_1 \|F\|_{p, w} \le \|g(F)\|_{p, w} \le C_2 \|F\|_{p, w}$$

for all radial functions F in $L^p(C^n)$,

   (ii)   *If $p > 2$ and $k > n$, then for all radial F in $L^p(C^n)$ one also has*

(4.21) $$\|g_k^*(F)\|_{p, w} \le C \|F\|_{p, w} .$$

PROOF.   We first claim that (i) implies (ii).   To see this let h be a non-negative function. Then for $k > n$ it is clear that

(4.22) $$\int_{C^n} (g_k^*(F, z))^2 h(z) dz \le C \int_{C^n} (g(F, z))^2 (Mh)(z) dz ,$$

where Mh is the Hardy-Littlewood maximal function of h. For $p > 2$ let $q = p/2$ and define

$$h_1(z) = h(z)|z|^{(2n-1)(1 - 2/p)} .$$

Then, it follows that

(4.23) $$\int_{C^n} (g_k^*(F, z))^2 |z|^{(2n-1)(1 - 2/p)} h(z) dz \le C \int_{C^n} (g(F, z))^2 (Mh_1)(z) dz .$$

Now we write

(4.24) $$\int_{C^n} (g(F, z))^2 (Mh_1)(z) dz = \int_{C^n} (g(F, z))^2 |z|^{(2n-1)(1 - 2/p)} |z|^{-(2n-1)/q'} (Mh_1)(z) dz ,$$

where q' is the conjugate index of q. By applying Hölder's inequality we obtain

(4.25) $$\int_{C^n} (g(F, z))^2 (Mh_1)(z) dz \le \left(\int_{C^n} (g(F, z))^p |z|^{(n-1/2)p - (2n-1)} dz \right)^{2/p}$$
$$\times \left(\int_{C^n} |z|^{-(2n-1)} (Mh_1)(z))^{q'} dz \right)^{1/q'} .$$

Since $q' > 1$, $-2n < -2n+1 < 2n(q'-1)$ so that the function $|z|^{-(2n-1)}$ is in

Muckenhoupt's $A_{q'}$ class and consequently

$$(4.26) \quad \left(\int_{C^n} |z|^{-(2n-1)} (Mh_1(z))^{q'} dz \right)^{1/q'} \le C \left(\int_{C^n} (h_1(z))^{q'} |z|^{-(2n-1)} dz \right)^{1/q'} \le C \|h\|_{q'} .$$

On the other hand, for F radial the right hand side inequality of (4.20) gives

$$\int_{C^n} |g(F,z)|^p |z|^{(n-1/2)p-(2n-1)} dz \le C \int_{C^n} |F(z)|^p |z|^{(n-1/2)p-(2n-1)} dz .$$

Hence in view of this (4.26) and (4.25) we get

$$\int_{C^n} (g_k^*(F,z))^2 |z|^{(2n-1)(1-2/p)} h(z) dz \le C \|F\|_{p,w}^2 \|h\|_{q'} .$$

Now taking supremum over all h with $\|h\|_{q'} = 1$ we obtain

$$\|g_k^*(F)\|_{p,w} \le C \|F\|_{p,w} , \qquad F \text{ radial} .$$

Therefore, it remains to prove Part (i).

   In order to establish Part (i) we first observe that $g(F)$ is radial whenever F is. In fact, an easy calculation shows that

$$(4.27) \qquad\qquad (F \times \varphi_k)(z) = C_n (h, \varphi_k^{n-1}) \varphi_k^{n-1} \left(\frac{r}{\sqrt{2}} \right) r^{-n+1/2}$$

where F and h are related by

$$(4.28) \qquad\qquad h(r) = F(\sqrt{2} \, r) r^{n-1/2} .$$

Therefore,

$$(4.29) \qquad\qquad T^t F(z) = C_n \tilde{T}^t h \left(\frac{r}{\sqrt{2}} \right) r^{-n+1/2}$$

where $\tilde{T}^t$ is the semigroup defined by

$$(4.30) \qquad\qquad \tilde{T}^t f(r) = \sum_{k=0}^{\infty} e^{-(2k+n)t} (f, \varphi_k^{n-1}) \varphi_k^{n-1}(r) .$$

This gives the relation

$$(4.31) \qquad\qquad g(F,z) = C_n \tilde{g} \left(h, \frac{r}{\sqrt{2}} \right) r^{-n+1/2} ,$$

where $\tilde{g}$ is defined for the semigroup $\tilde{T}^t$. From the above relation it follows that

$$\int_{C^n} (g(F,z))^p |z|^{(n-1/2)p-(2n-1)} dz = C_n \int_0^\infty (\tilde{g}(h,r))^p dr ,$$

and also

$$\int_{C^n} |F(z)|^p |z|^{(n-1/2)p-(2n-1)}dz = C_n \int_0^\infty |h(r)|^p dr .$$

Thus, Part (i) of Proposition 4.1 is equivalent to the following.

PROPOSITION 4.2. *There are constants C_1 and C_2 such that*

$$C_1\|h\|_p \le \|\tilde{g}(h)\|_p \le C_2\|h\|_p$$

for h in $L^p(\mathbf{R}_+)$, $1 < p < \infty$.

PROOF. The L^2-estimate

(4.32) $$\|\tilde{g}(h)\|_2 = 2^{-1}\|h\|_2$$

follows directly from the orthonormality of the functions φ_k^{n-1} and the Plancherel theorem. We will show that

(4.33) $$\|\tilde{g}(h)\|_p \le C\|h\|_p \quad \text{for} \quad 1 < p < \infty .$$

Then from this and (4.32) the reverse inequality follows by standard arguments (see Stein [10]). The proof of (4.33) is similar to the proof of the L^p-boundedness of the g function associated to the Hermite series. So we merely sketch the proof referring to [14] for details.

We consider $\tilde{g}$ as a singular integral operator whose kernel $K_t(r, s)$ takes values in the Hilbert space $L^2(\mathbf{R}_+, t\,dt)$. The kernel is explicitly given by

(4.34) $$K_t(r, s) = -\sum_{k=0}^\infty (2k+n)e^{-(2k+n)t}\varphi_k^{n-1}(r)\varphi_k^{n-1}(s) .$$

For this kernel it is not difficult to establish the following estimates:

(4.35) $$\left(\int_0^\infty |K_t(r, s)|^2 t\,dt\right)^{1/2} \le C|r-s|^{-1}$$

(4.36) $$\left(\int_0^\infty |\partial_r K_t(r, s)|^2 t\,dt\right)^{1/2} \le C|r-s|^{-2} .$$

These estimates show that K_t is a Calderón-Zygmund kernel taking values in $L^2(\mathbf{R}_+, t\,dt)$. Hence by appealing to the theory of vector valued singular integrals we obtain (4.33). This completes the proof of Proposition 4.2 modulo the estimates (4.35) and (4.36).

The kernel $K_t(r, s)$ can be explicitly calculated. In fact, we have the generating function relation

(4.37) $\displaystyle\sum_{k=0}^{\infty} \varphi_k^{n-1}(r)\varphi_k^{n-1}(s)z^{2k} = 2(rs)^{1/2}z^{-n+1}(1-z^2)^{-1}$

$$\times \exp\left(-\frac{(1+z^2)(r^2+s^2)}{2(1-z^2)}\right)e^{-i(n-1)\pi/2}J_{n-1}\left(\frac{2irsz}{1-z^2}\right),$$

where J_k is the Bessel function of order k. By taking $z = e^{-2t}$ in (4.37) and differentiating with respect to t we obtain an expression for the kernel $K_t(r, s)$. If we set $I_k(z) = e^{-ik\pi/2}J_k(iz)$ then the following are well known (see [8]).

(4.38) $2I_k'(z) = I_{k-1}(z) + I_{k+1}(z)$, $k \geq 1$,

(4.38)' $I_0'(z) = I_1(z)$,

(4.39) $|I_k(z)| \leq Cz^{-1/2}e^z$.

Using these relations and estimates it is not difficult to establish the following:

(4.40) $|K_t(r, s)| \leq Ct^{-3/2}e^{-a|r-s|^2/t}$,

(4.41) $|\partial_r K_t(r, s)| \leq Ct^{-2}e^{-a|r-s|^2/t}$,

where a is a positive constant.

Such estimates have been obtained for the Hermite case in [19]. The proof is not difficult. If we use (4.38) and (4.39) it is easy to see that the same arguments go through in our present case also. The estimates (4.35) and (4.36) are then immediate from (4.40) and (4.41).

This completes the proof of the multiplier theorem.

5. Proof of Proposition 3.2. The two ingredients used by Kanjin [5] in establishing the inequality

(5.1) $\|T_{\alpha,\varphi}^{\alpha+k+i\theta}f\|_p \leq M(\theta)\|f\|_p$

are the following.

    (i) The L^p-boundedness of Calderón–Zygmund singular integral operators,

    (ii) Hardy's inequality

(5.2) $\displaystyle\int_0^\infty \left|\int_x^\infty f(t)t^{-1}dt\right|^p dx \leq C\int_0^\infty |f(t)|^p dt$.

In order to establish a weighted version of (5.1) we need to use the following.

    (i) Weighted norm inequalities for the singular integral operators. When $w(x)$ is a weight function in Muckenhoupt's A_p class, we have

(5.3) $\displaystyle\int_{-\infty}^\infty |Tf(x)|^p w(x)dx \leq C\int_{-\infty}^\infty |f(x)|^p w(x)dx$,

whenever T is a Calderón-Zygmund singular integral operator. For $1 < p < \infty$, the

function $w(x) = |x|^{p/4 - 1/2}$ is in A_p and hence (5.3) is valid for this particular weight.

(ii)   The weighted version of Hardy's inequality we need is the following. If $\alpha + 1 > 0$ then

$$(5.4) \qquad \int_0^\infty \left| \int_x^\infty f(t) t^{-1} dt \right|^p x^\alpha dx \leq C \int_0^\infty |f(x)|^p x^\alpha dx .$$

This can be proved from (5.2).

The proof of the inequality

$$(5.5) \qquad \| T_{\alpha, \varphi}^{\alpha + k + i\theta} f \|_{p, w} \leq M(\theta) \| f \|_{p, w}$$

is very similar to that (5.1). We only need to use (5.3) and (5.4) wherever the boundedness of singular integrals and (5.2) are used. Therefore, we give only a sketch of the proof. Another remark we want to make is the following. Throughout this section $C(\theta)$ stands for an admissible function which may change from inequality to inequality. By keeping track of them it is not difficult to check that they are admissible. As usual C will stand for a generic constant which may change with each occurrence. We now turn to the proof of Proposition 3.2.

We first consider the estimation of $T_{\alpha, \varphi}^{\alpha + k + i\theta} f$ where $\alpha \geq 0$ and $k = 0, 2$. For $\varepsilon > 0$ we define $G_\varepsilon^\theta f$ as in [5] by

$$(5.6) \qquad G_\varepsilon^\theta f(x) = \sum_{k=0}^\infty \varphi(k) \omega_k^\alpha (f, \mathscr{L}_k^{\alpha + \varepsilon + i\theta}) \mathscr{L}_k^\alpha .$$

For this we establish the following.

PROPOSITION 5.1.   For $\alpha \geq 0$, $0 < \varepsilon < 1$ and $1 < p < \infty$ one has the inequality

$$(5.7) \qquad \| G_\varepsilon^\theta f \|_{p, w} \leq C(\theta)(\| f(x) x^{\varepsilon/2} \|_{p, w} + \| f(x) x^{-\varepsilon/2} \|_{p, w}) .$$

Before going to the proof of this proposition we observe that $G_\varepsilon^\theta f$ converges to $T_{\alpha, \varphi}^{\alpha + i\theta}$ as $\varepsilon \to 0$. Therefore, in view of Fatou's lemma and the dominated convergence Theorem (5.7) implies (5.5) with $k = 0$.

Coming to the proof of Proposition 5.1, proceeding as in [5] we first obtain the following integral representation for $G_\varepsilon^\theta f$. For $\alpha > -1$, $\varepsilon > 0$ and $-\infty < \theta < \infty$

$$(5.8) \qquad G_\varepsilon^\theta f(x) = \frac{e^{x/2}}{\Gamma(\varepsilon + i\theta)} \int_0^1 v^{\alpha/2 - 1} (1 - v)^{\varepsilon - 1 + i\theta} f\left(\frac{x}{v}\right) e^{-x/2v} \left(\frac{x}{v}\right)^{(\varepsilon + i\theta)/2} dv .$$

From this one obtains by making a change of variables

$$(5.9) \qquad G_\varepsilon^\theta f(x) = I_\varepsilon^\theta f(x) + J_\varepsilon^\theta f(x) ,$$

where

$$(5.10) \qquad I_\varepsilon^\theta f(x) = \frac{1}{\Gamma(\varepsilon + i\theta)} \int_x^\infty \frac{f(t)}{t} e^{-(t - x)/2} t^{(\varepsilon + i\theta)/2} \left(1 - \frac{x}{t}\right)^{\varepsilon - 1 + i\theta} dt ,$$

(5.11) $J_\varepsilon^\theta f(x) = \frac{1}{\Gamma(\varepsilon + i\theta)} \int_x^\infty \frac{f(t)}{t} e^{-(t-x)/2} t^{(\varepsilon + i\theta)/2} \left(\left(\frac{x}{t}\right)^{\alpha/2} - 1 \right) \left(1 - \frac{x}{t}\right)^{\varepsilon - 1 + i\theta} dt .$

Since $\alpha \geq 0$, for $J_\varepsilon^\theta f(x)$ we obtain

$$| J_\varepsilon^\theta f(x) | \leq \frac{C}{\Gamma(\varepsilon + i\theta)} \int_x^\infty | f(t) | t^{\varepsilon/2 - 1} dt .$$

Therefore, by weighted Hardy's inequality (5.4) we get

(5.12) $\int_0^\infty | J_\varepsilon^\theta f(x) |^p x^{p/4 - 1/2} dx \leq C(\theta) \int_0^\infty | f(x) x^{\varepsilon/2} |^p x^{p/4 - 1/2} dx ,$

where $C(\theta)$ is independent of ε.

To obtain a similar inequality for $I_\varepsilon^\theta f(x)$ we proceed as follows. Consider the operator

$$\tilde{I}_\varepsilon^\theta \tilde{f}(x) = \int_{-\infty}^\infty \tilde{f}(t) Q(x - t) dt ,$$

where Q is the kernel

$$Q(u) = \frac{1}{\Gamma(\varepsilon + i\theta)} e^{-|u|/2} | u |^{\varepsilon - 1 + i\theta} \chi_{(-\infty, 0)}(u) .$$

Then one verifies that $\tilde{I}_\varepsilon^\theta$ is a Calderón-Zygmund singular integral operator. Hence by the weighted inequality (5.3) we have

$$\int_{-\infty}^\infty | \tilde{I}_\varepsilon^\theta \tilde{f}(x) |^p | x |^{p/4 - 1/2} dx \leq C(\theta) \int_{-\infty}^\infty | \tilde{f}(x) |^p | x |^{p/4 - 1/2} dx .$$

Since $I_\varepsilon^\theta f(x) = \tilde{I}_\varepsilon^\theta \tilde{f}(x)$ for $x > 0$ where $\tilde{f}(x) = f(x) x^{-(\varepsilon + i\theta)/2} \chi_{(0, \infty)}(x)$ we obtain

(5.13) $\int_0^\infty | I_\varepsilon^\theta f(x) |^p x^{p/4 - 1/2} dx \leq C(\theta) \int_0^\infty | f(x) x^{-\varepsilon/2} |^p x^{p/4 - 1/2} dx .$

Combining (5.12) and (5.13) we obtain (5.7).

In order to estimate $T_{\alpha, \varphi}^{\alpha + 2 + i\theta} f$ for $\alpha \geq -1/2$ we define ρ_k and σ_k as in [5] and set

(5.14) $U^\theta f(x) = \sum_{k=0}^\infty \frac{\rho_k}{\sigma_k} (f, \mathscr{L}_k^{\alpha + 2 + i\theta}) \mathscr{L}_k^\alpha ,$

(5.15) $V^\theta f(x) = \sum_{k=0}^\infty \frac{k}{\sigma_k} (f, \mathscr{L}_k^{\alpha + 2 + i\theta}) \mathscr{L}_k^\alpha .$

Then it is easy to see that

(5.16) $T_{\alpha, \varphi}^{\alpha + 2 + i\theta} f(x) = U^\theta f(x) + V^\theta f(x) .$

We first deal with $U^\theta f(x)$. From the definition of $G_\varepsilon^\theta f(x)$ and the multiplier m_λ^α (see (3.8)) it follows that

$$(5.17) \qquad\qquad m_\lambda^\alpha G_2^\theta f(x) = U^\theta f(x)$$

with $\lambda(2k+1) = \rho_k$. Therefore, in order to prove

$$(5.18) \qquad\qquad \|U^\theta f(x)\|_{p,w} \le C(\theta)\|f\|_{p,w},$$

it is enough to prove the following:

$$(5.19) \qquad\qquad \|m_\lambda^\alpha f\|_{p,w} \le C(\theta)\|f\|_{p,w},$$

$$(5.20) \qquad\qquad \|G_2^\theta f\|_{p,w} \le C(\theta)\|f\|_{p,w}.$$

To establish (5.19) we require the following result of Butzer, Nessel and Trebels [3].

Consider the series $f = \sum_{k=0}^\infty (f, \varphi_k^\alpha)\varphi_k^\alpha$ and let the multiplier $\tilde{m}_\lambda^\alpha f$ and Cesàro means $\sigma_N f$ be defined by

$$(5.21) \qquad\qquad \tilde{m}_\lambda^\alpha f = \sum_{k=0}^\infty \lambda(2k+1)(f, \varphi_k^\alpha)\varphi_k^\alpha,$$

$$(5.22) \qquad\qquad \sigma_N f = \frac{1}{N}\sum_{k=0}^N \left(1 - \frac{k}{N}\right)(f, \varphi_k^\alpha)\varphi_k^\alpha.$$

Then one has:

THEOREM (Butzer-Nessel-Trebels). *Assume that σ_N are uniformly bounded on $L^p(\mathbf{R}_+)$ and $(\lambda(k))$ be a bounded quasi-convex sequence in the sense that*

$$(5.23) \qquad \|\lambda\|_{bqc} = \sum_{k=0}^\infty (k+1)(\lambda(k+2) - 2\lambda(k+1) + \lambda(k)) + \lim_{k\to\infty} \lambda(k)$$

is finite. Then $\tilde{m}_\lambda^\alpha f$ is bounded on $L^p(\mathbf{R}_+)$:

$$(5.24) \qquad\qquad \|\tilde{m}_\lambda^\alpha f\|_p \le C\|\lambda\|_{bqc}\|f\|_p.$$

For the sequence $\lambda(2k+1) = \rho_k$ it is easy to verify that $\|\lambda\|_{bqc} \le C(1 + |\theta|)$ as in [5]. For the Cesàro means σ_N it follows from the work of Markett [7] that they are uniformly bounded on $L^p(\mathbf{R}_+)$, $1 \le p \le \infty$ provided $\alpha \ge -1/2$. Therefore, the above theorem of Butzer-Nessel-Trebels applies to $\tilde{m}_\lambda^\alpha f$ and we get for $1 < p < \infty$, $\alpha \ge -1/2$

$$(5.25) \qquad\qquad \|\tilde{m}_\lambda^\alpha f\|_p \le C\|f\|_p.$$

It is easy to see that (5.25) is equivalent to the estimate (5.19). Thus it remains to prove (5.20).

From the expression (5.8) with $\varepsilon = 2$ it follows that

$$(5.26) \qquad |G_2^\theta f(x)| \le C(\theta) \int_0^1 v^{\alpha/2 - 2}(1-v)\left|f\left(\frac{x}{v}\right)\right| e^{x(1 - 1/v)/2}x\,dv.$$

By applying Minkowski's integral inequality to (5.26) we get

$$\|G_2^\theta f(x)\|_{p,w} \le C(\theta) \int_0^1 v^{\alpha/2-2}(1-v)dv \left\{ \int_0^\infty \left| f\left(\frac{x}{v}\right) e^{x(1-1/v)/2} x \right|^p x^{p/4-1/2} dx \right\}^{1/p}$$

$$\le C(\theta) \int_0^1 v^{\alpha/2+1/2p+1/4-1}(1-v)dv \left\{ \int_0^\infty |f(t)e^{t(v-1)/2}t|^p t^{p/4-1/2} dt \right\}^{1/p}.$$

Since $e^{t(v-1)/2}t \le 2e^{-1}(1-v)^{-1}$ for $t>0$ and $0<v<1$ it follows that

$$\|G_2^\theta f\|_{p,w} \le C(\theta)\|f\|_{p,w} \int_0^1 v^{\alpha/2+1/2p+1/4-1}dv .$$

Since the last integral is finite for $\alpha \ge -1/2$ we obtain (5.20). This takes care of the estimation of $U^\theta f$.

Next we turn our attention towards the estimation of $V^\theta f$. We define for $\varepsilon > 0$

$$V_\varepsilon^\theta f(x) = \sum_{k=0}^\infty \frac{k}{\sigma_k} \omega_k^{\alpha+2}(f, \mathscr{L}_k^{\alpha+2+i\theta}) \mathscr{L}_k^\alpha ,$$

and as before it is enough to prove

(5.27) $$\|V_\varepsilon^\theta f\|_{p,w} \le C(\theta)\{\|f(x)x^{\varepsilon/2}\|_{p,w} + \|f(x)x^{-\varepsilon/2}\|_{p,w}\} .$$

We observe that $V_\varepsilon^\theta f$ is $G_\varepsilon^\theta f$ with $(k/\sigma_k)\omega_k^{\alpha+2}$ and $\varepsilon+2$ in place of $\varphi_k\omega_k^\alpha$ and ε, respectively. Therefore, there is an integral representation for V_ε^θ also. Proceeding as in [5] we establish the following:

(5.28) $$V_\varepsilon^\theta f(x) = D_\varepsilon^\theta f(x) - E_\varepsilon^\theta f(x) - F_\varepsilon^\theta f(x)$$

with D_ε^θ, E_ε^θ and F_ε^θ being defiend as in [5].

To estimate $E_\varepsilon^\theta f$ we first observe that it is defined just like $G_\varepsilon^\theta f$ with $\alpha+2$ in place of α. Since $\alpha+2 \ge 0$ we can apply (5.7) to conclude that

(5.29) $$\|E_\varepsilon^\theta f\|_{p,w} \le C(\theta)\|f\|_{p,w} .$$

For the term $D_\varepsilon^\theta f$ we obtain the representation

$$D_\varepsilon^\theta f(x) = \frac{\alpha+1}{\Gamma(\varepsilon+1+i\theta)} \int_x^\infty f(y)e^{-(y-x)/2}y^{(\varepsilon+i\theta)/2}\left(\frac{x}{y}\right)^{\alpha/2}\left(1-\frac{x}{y}\right)^{\varepsilon+i\theta}y^{-1}dy ,$$

which gives

$$|D_\varepsilon^\theta f(x)| \le C(\theta) \int_x^\infty |f(y)|y^{\varepsilon/2}\left(\frac{x}{y}\right)^{\alpha/2}\frac{dy}{y} .$$

The proof of this is given in [5]. Therefore, by Hardy's inequality

$$(5.30) \quad \int_0^\infty |D_\varepsilon^\theta f(x)|^p x^{p/4 - 1/2} dx \le C(\theta) \int_0^\infty \left| \int_x^\infty |f(y)| y^{(\varepsilon - \alpha)/2} y^{-1} dy \right|^p x^{p/4 + \alpha p/2 - 1/2} dx$$

$$\le C(\theta) \int_0^\infty |f(x) x^{\varepsilon/2}|^p x^{p/4 - 1/2} dx \,,$$

since $p/4 + \alpha p/2 + 1/2 \ge 0$ for $\alpha \ge -1/2$. Hence we have taken care of D_ε^θ also.

Finally for the term $F_\varepsilon^\theta f$ we can prove that

$$(5.31) \quad |F_\varepsilon^\theta f(x)| \le C(\theta) \int_x^\infty |f(y) y^{\varepsilon/2}| \left(\frac{x}{y} \right)^{\alpha/2} e^{-(y-x)/2} dy \,.$$

When $\alpha \ge 0$ it is clear that

$$(5.32) \quad |F_\varepsilon^\theta f(x)| \le C(\theta) \int_x^\infty |f(y) y^{\varepsilon/2}| e^{-(y-x)/2} dy \,,$$

which is simply the convolution $(K * \tilde{f})(x)$ for $x > 0$ where

$$K(u) = e^{-|u|/2} \chi_{(-\infty, 0]}(u)$$

and $\tilde{f}(x) = f(x) x^{\varepsilon/2} \chi_{(0, \infty]}(x)$. Since $|x|^{p/4 - 1/2}$ is in A_p and K is a regular kernel, it follows that

$$\|K * \tilde{f}\|_{p, w} \le C(\theta) \|f\|_{p, w} \,.$$

This proves that for $\alpha \ge 0$

$$(5.33) \quad \|F_\varepsilon^\theta f\|_{p, w} \le C(\theta) \|f\|_{p, w} \,.$$

When $\alpha < 0$ we can write

$$|F_\varepsilon^\theta f(x)| \le C(\theta)(K * f^*)(x)| x|^{\alpha/2} \,,$$

where $f^*(x) = f(x) x^{(\varepsilon - \alpha)/2} \chi_{(0, \infty]}(x)$. For $-1/2 \le \alpha \le 0$, $|x|^{\alpha p/2 + p/4 - 1/2}$ is in A_p. Since K is a regular kernel it follows that

$$\|F_\varepsilon^\theta f\|_{p, w} \le C(\theta) \|f\|_{p, w}$$

for $-1/2 \le \alpha \le 0$ also. Hence we have established that (5.27) is true.

This completes the proof of Proposition 3.2.

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