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TRANSPORT

A COMPUTER PROGRAM FOR DESIGNING

CHARGED PARTICLE BEAM TRANSPORT SYSTEMS

K.L. Brown, D.C. Carey, Ch. Iselin and F. Rothacker

G E N E V A

1980

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*⁾ This report supersedes the earlier version CERN 73-16.

ABSTRACT

TRANSPORT is a computer program for first-order and second-order matrix multiplication, intended for the design of static-magnetic beam-transport systems. It has been in existence in various versions since 1963. The first part of the report is a user's manual, and supersedes the earlier report CERN 73-16. The second part is a reproduction of the Fermilab document "TRANSPORT appendix", by the same authors, which describes the theory of charged-particle beam optics and the applications of transformation matrices for numerical computation of beam trajectories and properties, as applied in the program.

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INTRODUCTION

TRANSPORT is a first- and second-order matrix multiplication computer program intended for the design of static-magnetic beam transport systems. It has been in existence in various evolutionary versions since 1963. The present version, described in this manual, includes both first- and second-order fitting capabilities.

Many people from various laboratories around the world have contributed either directly or indirectly to the development of TRANSPORT. The first-order matrix methods were developed by the AGS machine theorists¹⁾ followed by a paper by Penner²⁾. The extension of the first-order matrix methods to include second and higher orders was conceived and developed by Brown, Belbeoch and Bounin³⁾ in Orsay, France, in 1958-59. The original first-order TRANSPORT computer program was written in BALGOL by C.H. Moore at SLAC in collaboration with H.S. Butler and S.K. Howry in 1963. The second-order portion of the program was developed and debugged by Howry and Brown⁴⁾, also in BALGOL. The resulting BALGOL version was translated into FORTRAN by S. Kowalski at MIT and later debugged and improved by Kear, Howry and Brown at SLAC⁵⁾. In 1971-72, D. Carey at FNAL completely rewrote the program and developed an efficient second-order fitting routine using the coupling coefficient (partial derivatives) of multipole components to the optics as derived by Brown⁶⁾. This version was implemented at SLAC by F. Rothacker in the early spring of 1972 and subsequently carried to CERN in April, 1972, by K.L. Brown. C. Iselin of CERN made further contributions to the program structure and improved the convergence capabilities of the first-order fitting routines.

A standard version of the resulting program has now been adopted at SLAC, FNAL, and CERN. This manual describes the use of this standard version and is not necessarily applicable to other versions of TRANSPORT. Copies of this manual may be obtained from

- 1) Scientific Information Service, CERN, 1211 Geneva 23, Switzerland (Ref. CERN 80-04).
- 2) The Reports Office, Stanford Linear Accelerator Center, P.O. Box 4349, Stanford, CA 94305, USA (Ref. SLAC-91, without the Appendix).
- 3) The Reports Office, Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, IL 60510, USA (Ref. NAL-91, "TRANSPORT Appendix" available under separate cover).

The program may be obtained from:

1) IBM Version:

Frank Rothacker
TRANSPORT Program Librarian
Mail Bin 88
Stanford Linear Accelerator Center
P.O. Box 4349
Stanford, CA 94305, USA

2) IBM, CDC or PDP10 Versions:

David C. Carey
Fermi National Accelerator Laboratory
P.O. Box 500
Batavia, IL 60510, USA

3) IBM or CDC Versions:

Program Library
Division DD
CERN
CH 1211 Genève 23
Switzerland

The present authors assume responsibility for the contents of this manual, but in no way imply that they are solely responsible for the entire evolution of the program.

In order to make this report available without delay, the Appendix has been reproduced directly as published by FNAL.

MATHEMATICAL FORMULATION OF "TRANSPORT" *)

General conventions

A beam line is comprised of a set of magnetic elements placed sequentially at intervals along an assumed reference trajectory. The reference trajectory is here taken to be a path of a charged particle passing through idealized magnets (no fringing fields) and having the central design momentum of the beam line.

In TRANSPORT, a beam line is described as a sequence of elements. Such elements may consist not only of magnets and the intervals between them, but also of specifications of the input beam, calculations to be done, or special configurations of the magnets. A certain relation, described below, of the magnets and their fields to the assumed reference trajectory is considered normal. Alternative configurations can be described by means of elements provided for such purposes.

The two coordinates transverse to the initial reference trajectory are labelled as horizontal and vertical. A bending magnet will normally bend in the horizontal plane. To allow for other possibilities a coordinate rotation element is provided. Because of such other possibilities, when describing bending magnets we shall often speak of the bend and non-bend planes. The transverse coordinates will also often be labelled x and y , while the longitudinal coordinate will be labelled z .

All magnets are normally considered "aligned" on the central trajectory. A particle following the central trajectory through a magnet experiences a uniform field which begins and ends abruptly at the entrance and exit faces of the idealized magnet. Therefore, through a bending magnet the reference trajectory is the arc of a circle, while through all other magnetic elements it is a straight line. To accommodate a more gradual variation of field at the ends of a bending magnet a fringing field element is provided. In order to represent an orientation with respect to the reference trajectory other than normal of a magnet or section of a beam line, a misalignment element also exists.

The magnetic field of any magnet, except a solenoid, is assumed to have midplane symmetry. This means that the scalar potential expanded

*) For a more complete description of the mathematical basis of TRANSPORT, refer to SLAC-75⁴⁾, and to other references listed at the end of this manual.

in transverse coordinates about the reference trajectory is taken to be an odd function of the vertical coordinate. If a coordinate rotation is included, then the potential is odd in the coordinate to which the vertical has been rotated. For a bending magnet this will always be in the non-bend plane.

The program TRANSPORT will step through the beam line, element by element, calculating the properties of the beam or other quantities, described below, where requested. Therefore one of the first elements is a specification of the phase space region occupied by the beam entering the system. Magnets and intervening spaces and other elements then follow in the sequence in which they occur in the beam line. Specifications of calculations to be done or of configurations other than normal are placed in the same sequence, at the point where their effect is to be made.

The transfer matrix R

The following of a charged particle through a system of magnetic lenses may be reduced to a process of matrix multiplication. At any specified position in the system an arbitrary charged particle is represented by a vector (single column matrix) X, whose components are the positions, angles, and momentum of the particle with respect to the reference trajectory, i.e.

$$X = \begin{bmatrix} x \\ \theta \\ y \\ \phi \\ \ell \\ \delta \end{bmatrix}$$

Definitions:

- x = the horizontal displacement of the arbitrary ray with respect to the assumed central trajectory.
- θ = the angle this ray makes in the horizontal plane with respect to the assumed central trajectory.
- y = the vertical displacement of the ray with respect to the assumed central trajectory.
- ϕ = the vertical angle of the ray with respect to the assumed central trajectory.

ℓ = the path length difference between the arbitrary ray and the central trajectory.

δ = $\Delta p/p$ is the fractional momentum deviation of the ray from the assumed central trajectory.

This vector, for a given particle, will henceforth be referred to as a ray. The magnetic lens is represented to first order by a square matrix R , which describes the action of the magnet on the particle coordinates. Thus the passage of a charged particle through the system may be represented by the equation

$$X(1) = RX(0) , \quad (1)$$

where $X(0)$ is the initial coordinate vector and $X(1)$ is the final coordinate vector of the particle under consideration. The same transformation matrix R is used for all such particles traversing a given magnet [one particle differing from another only by its initial coordinate vector $X(0)$].

The traversing of several magnets and interspersing drift spaces is described by the same basic equation, but with R now being replaced by the product matrix $R(t) = R(n) \dots R(3)R(2)R(1)$ of the individual matrices of the system elements. This cumulative transfer matrix is automatically calculated by the program and is called TRANSFORM 1. It may be printed where desired, as described in later sections.

This formalism may be extended to second order by the addition of another term⁴⁾. The components of the final coordinate vector, in terms of the original, are now given as

$$X_i(1) = \sum_j R_{ij} X_j(0) + \sum_{jk} T_{ijk} X_j(0) X_k(0) ,$$

where T is the second-order transfer matrix. It too is accumulated by the program as one traverses a series of elements. At each point the series is again truncated to second order. Normally the program will calculate only the first-order terms and their effect. If it is desired to include second-order effects in a beam line, an element is provided which specifies that a second-order calculation is to be done. For more information on the T matrix, see the references at the end of the manual.

The following of a charged particle via TRANSPORT through a system of magnets is thus analogous to tracing rays through a system of optical lenses. The difference is that TRANSPORT is a matrix calculation which truncates the problem to either first- or second-order in a Taylor's expansion about a central trajectory. For studying beam optics to greater precision than a second-order TRANSPORT calculation permits, ray-tracing programs which directly integrate the basic differential equation of motion are recommended⁷⁾.

The beam matrix σ

In accelerator and beam transport systems, the behaviour of an individual particle is often of less concern than is the behaviour of a bundle of particles (the beam), of which an individual particle is a member. An extension of the matrix algebra of Eq. (1) provides a convenient means for defining and manipulating this beam. TRANSPORT assumes that the beam may be correctly represented in phase space by an ellipsoid in the six-dimensional coordinate system described above. Particles in a beam are assumed to occupy the volume enclosed by the ellipsoid, each point representing a possible ray. The sum total of all phase points, the phase space volume, is commonly referred to as the "phase space" occupied by the beam.

The validity and interpretation of this phase ellipse formalism must be ascertained for each system being designed. However, in general, for charged particle beams in, or emanating from accelerators, the first-order phase ellipse formalism of TRANSPORT is a reasonable representation of physical reality. For other applications, such as charged particle spectrometers, caution is in order in its use and interpretation.

The equation of an n-dimensional ellipsoid may be written in matrix form as follows:

$$X(0)^T \sigma(0)^{-1} X(0) = 1, \quad (2)$$

where $X(0)^T$ is the transpose of the coordinate vector $X(0)$, and $\sigma(0)$ is a real, positive definite, symmetric matrix.

The volume of the n-dimensional ellipsoid defined by sigma is $[\pi^{n/2} / \Gamma(n/2 + 1)] (\det \sigma)^{1/2}$. The area of the projection in one plane is

$A = \pi(\det \sigma_1)^{\frac{1}{2}}$, where σ_1 is the submatrix corresponding to the given plane. This is the "phase space" occupied by the beam.

As a particle passes through a system of magnets, it undergoes the matrix transformation of Eq. (1). Combining this transformation with the equation of the initial ellipsoid, and using the identity $RR^{-1} = I$ (the unity matrix), it follows that

$$X(0)^T (R^T R^{T^{-1}}) \sigma(0)^{-1} (R^{-1} R) X(0) = 1$$

from which we derive

$$[RX(0)]^T [R\sigma(0)R^T]^{-1} [RX(0)] = 1 . \quad (3)$$

The equation of the new ellipsoid after the transformation becomes

$$X(1)^T \sigma(1)^{-1} X(1) = 1 , \quad (4)$$

where

$$\sigma(1) = R\sigma(0)R^T . \quad (5)$$

It can readily be shown that the square roots of the diagonal terms of the sigma matrix are a measure of the "beam size" in each coordinate. The off-diagonal terms determine the orientation of the ellipsoid in n-dimensional space (for TRANSPORT $n = 6$)^{*}). Thus, we may specify the beam at any point in the system via Eq. (5), given the initial "phase space" represented by the matrix elements of $\sigma(0)$.

The initial beam is specified by the user as one of the first elements of the beam line. Normally it is taken to be an upright ellipse centred on the reference trajectory; that is, there are no correlations between coordinates. Both correlations and centroid displacements may be introduced via additional elements.

^{*}) See the Appendix of this report, or the Appendix of Ref. 5, for a derivation of these statements.

The phase ellipse may be printed wherever desired. For an interpretation of the parameters printed see the section under type code 1.0.

When a second-order calculation is specified the second-order matrix elements are included in the beam matrix. For details on how this is done see the Appendix to this manual.

Fitting

Several types of physical elements have been incorporated in the program to facilitate the design of very general beam transport systems. Included are an arbitrary drift distance, bending magnets, quadrupoles, sextupoles, solenoids, and an accelerator section (to first-order only). Provision is made in the program to vary some of the physical parameters of the elements comprising the system and to impose various constraints on the beam design. In a first-order run one may fit either the TRANSFORM (R) matrix representing the transformation of an arbitrary ray through the system and/or the phase ellipse (sigma) matrix representing a bundle of rays by the system as transformed. In a second-order run one may fit either the second-order TRANSFORM (T) matrix or minimize the net contribution of second-order terms to the beam (sigma) matrix.

The program will normally make a run through the beam line using values for the physical parameters as specified by the user and printing the results. If constraints and parameters to be varied are indicated, it will attempt to fit. To do this it will make an additional series of runs through the beam line. Each time it will calculate corrections to be made from the previous step to the varied parameters to try to satisfy the indicated constraints. When the constraints are satisfied (or the fitting procedure has failed) the program will make a final run through the beam line again printing the results. In this final run the values of the physical parameters used are those which are the result of the fitting procedure.

Thus, in principle, the program is capable of searching for and finding the first- or second-order solution to any physically realizable problem. In practice, life is not quite so simple. The user will find that an adequate knowledge of geometric magnetic optics principles is a

necessary prerequisite to the successful use of TRANSPORT. He (or she) should possess a thorough understanding of the first-order matrix algebra of beam transport optics and of the physical interpretation of the various matrix elements.

In other words, the program is superb at doing the numerical calculations for the problem but not the physics. The user must provide a reasonable physical input if he (or she) expects complete satisfaction from the program. For this reason a list of pertinent reprints and references are included at the end of this manual. They should provide assistance to the inexperienced as well as the experienced user.

INPUT FORMAT FOR TRANSPORT

By the TRANSPORT input DATA SET is meant the totality of data read by the program in a single job. A DATA SET may consist of one or more problems placed sequentially. A problem specifies a calculation or set of calculations to be done on a given beam line.

A problem, in turn, may consist of one or more problem steps. The data in the first step of a problem specify the beam line and the calculations to be made. The data in succeeding steps of the same problem specify only changes to the data given in the first step.

A common example of a problem with several steps is sequential fitting. In the first step one may specify that certain parameters are to be varied to satisfy certain constraints. Once the desired fit has been achieved the program will then proceed to the next step. The data in this step now need specify only which new parameters to vary, or old ones no longer to vary, or which constraints to add or delete. The values of the varied parameters that are passed from one step to the next one are those that are the result of the fitting procedure.

A problem step contains three kinds of DATA cards: the TITLE card, the INDICATOR card, and the ELEMENT cards.

The TITLE card contains a string of characters and blanks enclosed by single quotes. Whatever is between the quotes will be used as a heading in the output of a TRANSPORT run.

The second card of the input is the INDICATOR card. If the data which follow describe a new problem, a zero (0) is punched in any column on the card. If the data which follow describe changes to be made in the previous problem step, a one (1) or two (2) is punched in any column on the card. For further explanation read the Indicator Card section of this manual.

The remaining cards in the deck for a given problem step contain the DATA describing the beam line and the calculations to be done. The DATA consist of a sequence of elements whose order is the same as encountered as one proceeds down the beam line. Each element specifies a magnet or

portion thereof or other piece of equipment, a drift space, the initial beam phase space, a calculation to be done, or a print instruction. Calculation specifications, such as misalignments and constraints, are placed in sequence with the other beam line elements where their effect is to take place. The input format of the cards is "free-field", which is described below. The data for a given problem step are terminated by the word SENTINEL, which need not be punched on a separate card.

Each element, in turn, is given by a sequence of items (mostly numbers), separated by spaces and terminated by a semicolon. The items, in order, are a type code number, a vary field, the physical parameters, and an optional label.

The type code number identifies the element, indicating what sort of entity (such as a magnet, drift space, constraint, etc.) is represented. It is an integer (number) followed by a decimal point. The interpretation of the physical parameters which follow is therefore dependent on the type code number. The type code numbers and their meanings are summarized in Table 1. If the type code number is negative, the element will be ignored in the given problem step. However, storage for that element will be allocated by the program, so that the element may be introduced in a later step of the same problem. Storage space for any element in any problem step must be allocated in the first step of the problem.

The vary field indicates which physical parameters of the element are to be adjusted if there is to be any fitting. It is punched immediately (no intervening blanks) to the right of the decimal point of the type code number. See the section under type code 10.0 for an explanation of the use of vary codes.

The physical parameters are the quantities which describe the physical element represented. Such parameters may be lengths, magnetic fields, apertures, rotation angles, beam dimensions, or other quantities, depending on the type code number. The meanings for the physical parameters for each type code are described thoroughly in the section for that type code. A summary, indicating the order in which the physical parameters should be punched, is given in Table 1. For any element the first physical parameter is the second entry in Table 1 or the second parameter

in the section describing a given element. In some cases the parameters of an element do not really refer to physical quantities, but will nevertheless be referred to as such in this manual.

The label, if present, contains one to four characters and is enclosed by single quotes, slashes or equal signs. During the calculation the elements will be printed in sequence and the label for a given element will be printed with that element. Labels are useful in problems with many elements and/or when sequential fitting is used. They must be used to identify any element to be changed in succeeding steps of a given problem.

Provision has been made in the program to allow the user to introduce comments before any type code entry in the data deck. This is accomplished by enclosing the comments made on each card within single parentheses.

Each element must be terminated by a semicolon (;). Optionally a semicolon may be replaced by an asterisk (*) or a dollar sign (\$). Spaces before and after the semicolon are allowed but not required. If the program encounters a semicolon, dollar sign or asterisk before the expected number of parameters has been read in and if the indicator card was a zero (0), the remaining parameters are set to zero. If the indicator card was a one (1) or two (2), then the numbers indicated on the card are substituted for the numbers remaining from the previous solution; all other numbers are unchanged.

The "free-field" input format of the data cards makes it considerably easier to prepare input than the standard fixed-field formats of FORTRAN. Numbers may be punched anywhere on the card and must simply be in the proper order. They must be separated by one or more blanks. Several elements may be included on the same card and a single element may continue from one card to the next. A single number must be all on one card, it may not continue from one card to the next. The program storage is limited to a total of 2000 locations (including type codes and those parameters not punched but implied equal to zero) and 500 elements.

A decimal number (e.g. 2.47) may be represented in any of the following ways:

2.47
.00247+3
.0247E+02
247E-2
247000-5

The sample problem below contains two problem steps, each beginning with title and indicator cards and terminating with a SENTINEL. The first step causes TRANSPORT to do a first-order calculation with fitting. The second initiates a second-order calculation with the data that is a result of that fitting. Corresponding elements between the two steps are identified by having the same label.

The type ten element which specifies the fitting condition is labelled FIT1. It is active for the first-order calculation, but is turned off for the second-order calculation. The vary codes for elements DR1 are set to zero for the second-order problem. The second-order element, SEC1, is ineffective during the fitting, but causes the program to compute the second-order matrices in the second calculation.

An Example of a TRANSPORT Input Deck

'FORTRAN H CHECK ON BETA FIT'

Title card

0

Indicator card

1 .5 1 .5 1 .5 1 1 ;

-17 'SEC1' ; 13 3 ;

3.3 2.745 'DR1' ;

2 0 ; 4 9.879 10 .5 ; 2 0 ;

3.3 2.745 'DR1' ;

13 4 ;

10 -1 2 0 .0001 'FIT1' ;

Elements

*First problem
step*

SENTINEL

'SECOND ORDER'

Title card

1

Indicator card

17 'SEC1' ;

3 'DR1' ;

-10 'FIT1' ;

*Elements to be
changed*

*Second problem
step*

SENTINEL

SENTINEL *Second sentinel signifies end of run.*

As many problems and problem steps as one wishes may be stacked in one job.

Note that in previous versions of TRANSPORT a decimal point was required with every numerical entry except the indicator card (which must not have a decimal point in any version of TRANSPORT).

The use of labels

The use of labels is available for identification of individual elements. When inserted for the user's convenience, the association of a label with a given element is optional. If the parameters of an element are to be changed between steps of a given problem, a label is required. The label identifies the element in the earlier step to which the changes specified in the later step are to apply.

The label may be placed anywhere among the parameters of a given element. It should be enclosed in quotes, slashes, or equal signs. Blanks within a label are ignored. The maximum length of a label is four non-blank characters.

As an example, the following all denote the same drift space:

```
'DRF'      3.      1.5      ;  
3/DRF/15-1*  
3.      .15E1      = D R F =      $
```

On a 15.0 type code element the label may not be the third item. This is to avoid ambiguities with the unit name. Thus the following are not equivalent:

```
15.  1.  'FT'  'CM'  ;  
15.  1.  'CM'  'FT'  ;
```

This entry is used as the units symbol.
↓
This entry is used as a label.
↓

If the parameters describing an element are to differ in succeeding steps of a given problem the element must be included in both steps, having the same label each time. All elements which appear in a problem must be included in the first step (indicator card 0) of that problem. Only those to be changed in later steps need to be labelled. In later steps (indicator card 1) of a problem only those elements to be changed are specified. The elements to be changed are identified by their labels.

If the type code number of an element is negative in a given step of a problem, that element will be ignored when the calculation is performed. However, storage space in the computer will be allocated for the element for possible activation in later steps of the problem. In the later step, only those parameters to be changed need to be specified.

The storage space allocated for the parameters of a given element is determined only by the type code. The sole exceptions are the continuation codes for type codes 1.0 and 14.0.

For example, if a fitting constraint is to be ignored in the first step of a problem, but activated in a later step, it should be indicated in both steps. In the first step such an element might appear as

```
-10.   'FIT'   ;
```

In the later step one would then insert

```
10.   1.   2.   0.0   .001   'FIT'   ;
```

causing a waist constraint to be imposed on the beam. Alternatively one can specify the physical parameters in the first step and then, in the later step, merely indicate that the element is now to be activated. The above procedure is therefore equivalent to placing the element:

```
-10.   1.   2.   0.0   .001   'FIT'   ;
```

in the first problem step, and the element

```
10.   'FIT'   ;
```

in the later step.

Vary codes may also be inserted or removed in passing from one problem step to the next. For instance, one might wish to vary the field of a quadrupole in one step of a problem and then use the fitted value as data in the following step. The first step might then contain the element:

```
5.01   5.0   10.0   5.0   'QUAD'   ;
```

and the following step would contain the element

```
5.   'QUAD'   ;
```

Since, in the second step, the first item on the card contains no vary code the vary code is deleted. All other parameters, not being re-specified, are left unchanged.

Several elements may have the same label. If, as in the above example, one wished to vary the field of several quadrupoles in one step, then pass the final values to the next step, one could give all such

elements the same label. There might be four quadrupoles, all labelled 'QUAD', being varied simultaneously. If the data for the next step contain the single element

```
5.   'QUAD'   ;
```

the vary code on all elements labelled 'QUAD' will be deleted.

The physical parameters of an element may be changed between steps of a problem. In the first step a bending magnet may be given a length of 5 metres.

```
4.   5.0   10.0   0.0   'BEND'   ;
```

In a succeeding step, its length could be increased to 10 metres by inserting the element

```
4.   10.0   'BEND'   ;
```

All parameters, up to and including the one to be changed, must be specified. The remaining, if omitted, will be left unchanged from the previous step.

Table 1: Summary of TRANSPORT type codes

PHYSICAL ELEMENT	TYPE CODE	2nd ENTRY	3rd ENTRY	4th ENTRY	5th ENTRY	6th ENTRY	7th ENTRY	8th ENTRY	9th ENTRY
BEAM	1.vvvvv0	x(cm)	0(mr)	y(cm)	φ(mr)	t(cm)	δ(percent)	ρ _s	0
r.m.s. ADDITION TO BEAM ENVELOPE	1.vvvvv00	Δx(cm)	Δ0(mr)	Δy(cm)	Δφ(mr)	Δt(cm)	Δδ(percent)	Δρ (GeV/c)	
POLE FACE ROTATION	2.v	ANGLE OF ROTATION (degrees)							
DRIFT	3.v	LENGTH (metres)							
BENDING MAGNET	4.vvv	LENGTH (metres)	FIELD (kG)	FIELD GRADIENT (n-value)					
QUADRUPOLE	5.vv0	LENGTH (metres)	FIELD (kG)	HALF-APERTURE (cm)					
TRANSFORM 1 UPDATE	6.0	0.0	1.0						
TRANSFORM 2 UPDATE	6.0	0.0	2.0						
BEAM CENTROID SHIFT	7.0	SHIFT (x)(cm)	SHIFT (0) (mr)	SHIFT (y)(cm)	SHIFT (φ)(mr)	SHIFT (t)(cm)	SHIFT (δ)percent)		
ALIGNMENT TOLERANCE	8.vvvvv0	DISPLACEMENT (x)(cm)	ROTATION (θ)(mr)	DISPLACEMENT (y)(cm)	ROTATION (φ)(mr)	DISPLACEMENT (z)(cm)	ROTATION (α)(mr)	CODE NUMBER	
REPEAT CONTROL	9.0	NUMBER OF REPEATS							
FITTING CONSTRAINTS	10.0	±I	J	DESIRED VALUE OF (I,J) MATRIX ELEMENTS	ACCURACY OF FIT				

Note: +I is used for fitting a beam (σ) matrix element. -I is used for fitting an R1 matrix element.
 - (I + 20) is used for fitting an R2 matrix element.

ACCELERATOR	11.0	LENGTH (metres)	E (energy gain) (GeV)	φ (phase lag) (degrees)	(WAVELENGTH) (cm)				
BEAM (Rotated Ellipse)	12.0	THE FIFTEEN CORRELATIONS AMONG THE SIX ELEMENTS (This entry must be preceded by a type code 1.0 entry.)							
INPUT/OUTPUT OPTIONS	13.0	CONTROL CODE NUMBER							
ARBITRARY R MATRIX	14.vvvvv0	R(J,1)	R(J,2)	R(J,3)	R(J,4)	R(J,5)	R(J,6)	J	
UNITS CONTROL (Transport Dimensions)	15.0	CODE	UNIT SYMBOL	SCALE FACTOR (if required)					
QUADRATIC TERM OF BENDING FIELD	16.0v	1.0	$c(1) = B(\frac{1}{\rho_0})^2$	ρ_0 in units of transverse length (cm)					
MASS OF PARTICLES IN BEAM	16.0	3.0	M/m (dimensionless)	m = mass of electron					
HALF-APERTURE OF BENDING MAGNET IN x-PLANE	16.0	4.0	w/2 (cm)						
HALF-APERTURE OF BENDING MAGNET IN y-PLANE (gap)	16.0	5.0	g/2 (cm)						
LENGTH OF SYSTEM	16.0	6.0	L (metres)						
FRINGE FIELD CORRECTION COEFFICIENT	16.0	7.0	K ₁ (dimensionless)						
FRINGE FIELD CORRECTION COEFFICIENT	16.0	8.0	K ₂ (dimensionless)						
CURVATURE OF ENTRANCE FACE OF BENDING MAGNET	16.0v	12.0	(1/R ₁) (1/metres)						
CURVATURE OF EXIT FACE OF BENDING MAGNET	16.0v	13.0	(1/R ₂) (1/metres)						
FOCAL PLANE ROTATION	16.0	15.0	Angle of focal plane rotation (degrees). See type code 16.0 for details.						
INITIAL BEAM LINE x-COORDINATE	16.0	16.0	x ₀						
INITIAL BEAM LINE y-COORDINATE	16.0	17.0	y ₀						
INITIAL BEAM LINE z-COORDINATE	16.0	18.0	z ₀						
INITIAL BEAM LINE HORIZONTAL ANGLE	16.0	19.0	θ ₀						
INITIAL BEAM LINE VERTICAL ANGLE	16.0	20.0	φ ₀						
SECOND-ORDER CALCULATIONS	17.0								
SEXTUPOLE	18.0v	LENGTH (metres)	FIELD (kG)	HALF-APERTURE (cm)					
SOLENOID	19.vv	LENGTH (metres)	FIELD (kG)						
BEAM ROTATION	20.v	ANGLE OF ROTATION (degrees)							
STRAY FIELD	21.0	See later section of report.							

Note: The v's following the type codes indicate the parameters which may be varied. See section under type code 10.0 for a detailed explanation of Vary Codes. The units are standard TRANSPORT units (as shown) unless changed via type code 15.0 entries.

OUTPUT FORMAT

General appearance

Here we give a brief description of the general appearance of the output and its meaning. The user may refer to the sample output shown on pages 22 through 27. It is the printed output resulting from the sample data shown in the section on input format. In a simple example it is not possible to show each of the different type codes. Several of the type codes produce output which is not characteristic of all other type codes. We therefore refer the user to the sections on the various type codes for an explanation of any features peculiar to a given type code.

The output for each step of a given problem is printed separately. The printing for one step is completed before that for the next step is begun. Therefore we will describe the output for a single problem step. The output shown below is from a problem with two steps.

Initial listing

For each problem step, the program begins by printing out the user's input records.

Listing during the calculation

The program now begins the calculation. If there is no fitting, one listing of the beam line will be made. If there is fitting there will normally be two listings. The first will represent the beam line before any fitting has occurred. The second will be based on the new values of the physical parameters which were altered by the fitting process. If sequential fitting is employed and an indicator card of two (2) is used the first run will be omitted. The user should read the section describing the indicator card for further explanation.

In any listing the elements are printed in order with their labels and physical parameters. Elements with negative type code numbers are ignored. Each listed element is preceded by the name of that type of element, enclosed in asterisks. All physical elements are listed in this way. Some of the other elements are not explicitly listed but produce their effect in either the calculated quantities or the listing of the beam line. For descriptions of individual cases, the reader should consult the sections on the type codes.

Calculated quantities appear in the listing as requested in the input data. Important cases will be described in greater detail below. The physical parameters for each element are printed with the appropriate units. For some elements a calculated quantity, not in the input data, will appear, enclosed in parenthesis. Such quantities are explained in the sections under the individual type codes.

Calculated quantities

The important cases of calculated quantities which appear in the output are the transfer matrices, the beam matrix, the layout coordinates, and the results of the fitting procedure. The transfer and beam matrices and layout coordinates appear as requested in the listing of the beam line. The results of the fitting procedure appear between the two listings. All these quantities are explained in greater detail below.

The transfer and beam matrices appear only where requested. A request for printing of layout coordinates should be made at the beginning of the beam line. The coordinates will then be printed after each physical element. In all cases the quantities printed are the values at the

interface between two elements. They are evaluated at a point after the element listed above them and before the element listed below. For further explanation of calculated quantities the user should read the section on the mathematical formulation of TRANSPORT, the appendix to the manual, and the section on the appropriate type code. For the transfer matrix the appropriate type code is thirteen; for the beam matrix it is one, and for the coordinate layout it is again thirteen.

Quantities relevant to the fitting appear between the two listings of the beam line. At each iteration of the fitting procedure a line is printed containing the value of the relaxation factor used, the value of chi-squared before the iteration was made, and the corrections made to each of the varied parameters. Once the fitting is complete the final chi-squared and the covariance matrix are printed. For further details the user should read the section on type code 10.0, and the section on fitting in the appendix.

#FORTRAN H CHECK ON BETA FIT #

0									
1.000000	#SEC1# †	.50000	1.00000	.50000	1.00000	1.00000	1.00000	1.00000	1.00000
-17.		3.00000							
13.	#DR1 #	2.74500							
3.3		0.00000							
2.0		9.87900							
4.000		0.00000	10.00000	.50000					
2.0	#DR1 #	0.00000							
3.3		2.74500							
13.		4.00000							
10.0	#FIT1#	-1.00000	2.00000	0.00000	.00010				

SENTINEL

FORTRAN H CHECK ON BETA FIT

```

*BEAM*          1.          1.00000 GEV
0.000 M

*DRIFT*        3.  #DR1 "    2.74500 M
VARY CODE = 3
2.745 M

*ROTAT*        2.          0.00000 DEG
2.745 M

*BEND*         4.          9.87900 M      10.00000 KG      .50000
12.624 M      ( 3.336 M

*ROTAT*        2.          0.00000 DEG
12.624 M

*DRIFT*        3.  #DR1 "    2.74500 M
VARY CODE = 3
15.369 M

*TRANSFORM I*
-1.00383  -.00418  0.00000  0.00000  0.00000  0.00000  13.36811
-1.83605  -1.00383  0.00000  0.00000  0.00000  0.00000  12.24881
0.00000  0.00000  -1.00383  -.00418  0.00000  0.00000  0.00000
0.00000  0.00000  -1.83605  -1.00383  0.00000  0.00000  0.00000
-1.22488  -1.33681  0.00000  0.00000  1.00000  -11.58647
0.00000  0.00000  0.00000  0.00000  0.00000  1.00000
*FIT*         10.0 #FIT1*  -1.    2.      0.00000 / .00010 ( -.00418 )

*LENGTH*      15.36900 M

*CORRECTIONS*
.10000E+01 ( .17464E+04)  -.0208
.10000E+01 ( .63283E-02)  -.0000

*COVARIANCE (FIT .84364E-13 )
.000

```

0.000	.500 CM	0.000	0.000	0.000	0.000	0.000
0.000	1.000 MR	0.000	0.000	0.000	0.000	0.000
0.000	.500 CM	0.000	0.000	0.000	0.000	0.000
0.000	1.000 MR	0.000	0.000	0.000	0.000	0.000
0.000	.500 CM	0.000	0.000	0.000	0.000	0.000
0.000	1.000 PC	0.000	0.000	0.000	0.000	0.000
0.000	.570 CM	.481	0.000	0.000	0.000	0.000
0.000	1.000 MR	0.000	0.000	0.000	.481	0.000
0.000	.570 CM	0.000	0.000	0.000	0.000	0.000
0.000	1.000 MR	0.000	0.000	0.000	0.000	0.000
0.000	.500 CM	0.000	0.000	0.000	0.000	0.000
0.000	1.000 PC	0.000	0.000	0.000	0.000	0.000
0.000	.570 CM	.481	0.000	0.000	0.000	0.000
0.000	1.000 MR	0.000	0.000	0.000	0.000	0.000
0.000	.570 CM	0.000	0.000	0.000	.481	0.000
0.000	1.000 MR	0.000	0.000	0.000	0.000	0.000
0.000	.500 CM	0.000	0.000	0.000	0.000	0.000
0.000	1.000 PC	0.000	0.000	0.000	0.000	0.000
.50000	(3.336 M					
0.000	10.013 CM	.993	0.000	0.000	0.000	0.000
0.000	12.324 MR	0.000	0.000	0.000	0.000	0.000
0.000	.369 CM	0.000	0.000	0.000	0.000	0.000
0.000	1.360 MR	-.992	0.000	0.000	0.000	0.000
0.000	11.690 CM	.999	.994	0.000	0.000	-.991
0.000	1.000 PC	.993	0.000	0.000	0.000	0.000
0.000	10.013 CM	0.000	0.000	0.000	0.000	0.000
0.000	12.324 MR	0.000	0.000	0.000	0.000	0.000
0.000	.369 CM	0.000	0.000	0.000	0.000	0.000
0.000	1.360 MR	-.992	0.000	0.000	0.000	0.000
0.000	11.690 CM	.999	.994	0.000	0.000	-.991
0.000	1.000 PC	.993	0.000	0.000	0.000	0.000
0.000	13.378 CM	.996	0.000	0.000	0.000	0.000
0.000	12.324 MR	0.000	0.000	0.000	0.000	0.000
0.000	.502 CM	0.000	0.000	0.000	.681	0.000
0.000	1.360 MR	-.988	-.972	0.000	0.000	0.000
0.000	11.690 CM	.999	.994	0.000	0.000	-.991
0.000	1.000 PC	.999	0.000	0.000	0.000	0.000

#SECOND ORDER#

1	#SEC1#	‡
17.	#DRI#	‡
3.0	#FIT1#	‡
-10.		

#SECOND ORDER#

1	#SEC1#	‡						
* 17.			.50000	1.00000	.50000	1.00000	1.00000	1.00000‡
13.	#DRI#	‡	3.00000‡					
* 3.0			2.72415‡					
2.0	#FIT1#	‡	0.00000‡					
4.0000			9.87900	10.00000	.50000‡			
2.0	#DRI#	‡	0.00000‡					
* 3.0			2.72415‡					
13.	#FIT1#	‡	4.00000‡	2.00000	0.00000		.00010‡	
* -10.			-1.00000					

SENTINEL


```

*TRANSFORM I *
-1.00000 -1.00000 0.00000 0.00000 0.00000 13.34256
-1.83605 -1.00000 0.00000 0.00000 0.00000 12.24881
0.00000 0.00000 -1.00000 0.00000 0.00000 0.00000
0.00000 0.00000 -1.83605 -1.00000 0.00000 0.00000
-1.22488 -1.33426 0.00000 0.00000 1.00000 -11.58647
0.00000 0.00000 0.00000 0.00000 0.00000 1.00000

*2ND ORDER TRANSFORM*
1 11 1.124E-03
1 12 1.225E-03
1 13 0.
1 14 0.
1 15 0.
1 16 2.065E-02
2 11 6.119E-07
2 12 1.333E-06
2 13 0.
2 14 0.
2 15 0.
2 16 3.097E-02
3 11 0.
3 12 0.
3 13 -7.500E-04
3 14 -2.042E-03
3 15 0.
3 16 0.
4 11 0.
4 12 0.
4 13 5.503E-03
4 14 -2.999E-03
4 15 0.
4 16 0.

1 22 6.673E-04
1 23 0.
1 24 0.
1 25 0.
1 26 3.066E-02
2 22 6.126E-04
2 23 0.
2 24 0.
2 25 0.
2 26 3.564E-02
3 22 0.
3 23 -2.042E-03
3 24 -2.224E-03
3 25 0.
3 26 0.
4 22 0.
4 23 5.994E-03
4 24 -2.042E-03
4 25 0.
4 26 0.

1 33 -4.871E-03
1 34 -2.042E-03
1 35 0.
1 36 0.
2 33 -5.504E-03
2 34 -2.999E-03
2 35 0.
2 36 0.
3 33 0.
3 34 0.
3 35 0.
3 36 -1.064E-02
4 33 0.
4 34 0.
4 35 0.
4 36 5.756E-03

1 44 -1.112E-03
1 45 0.
1 46 0.
2 44 -1.021E-03
2 45 0.
2 46 0.
3 44 0.
3 45 0.
3 46 -3.414E-03
4 44 0.
4 45 0.
4 46 4.367E-03

1 55 0.
1 56 0.
2 55 0.
2 56 0.
3 55 0.
3 56 0.
4 55 0.
4 56 0.

1 66 7.908E-02
2 66 3.834E-02
3 66 0.
4 66 0.

*LENGTH*
15.32729 M

```

TITLE CARD

The title card is the first card in every problem step of a TRANSPORT data set. The title card is always required and must be followed by a 0, a 1 or a 2 card (see next section) to indicate whether the data to follow is new (0 card) or a continuation of a previous data set (a 1 card or a 2 card).

The title must be enclosed within either quotation marks ('), slashes (/), or equal signs (=) on a single card. The string may begin and end in any column (free field format), for example

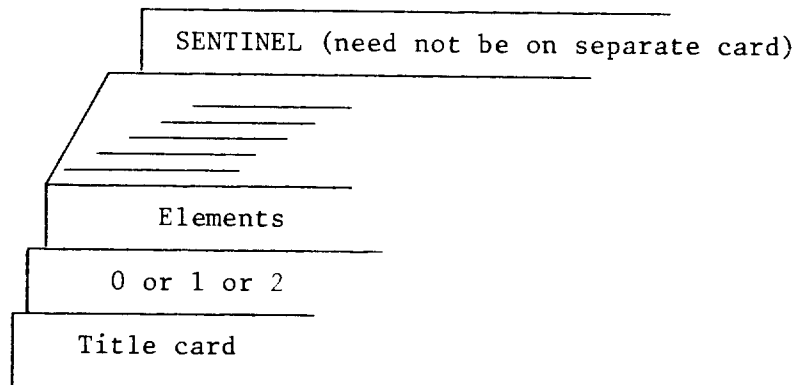
'SLAC 20 GEV/C SPECTROMETER'

or

/SLAC 20 GEV SPECTROMETER/

Note that whichever character is used to enclose the title must not be used again within the title itself.

Example of a DATA SET for a single problem step



INDICATOR CARD

The second card of the input for each step of a problem is the indicator card. If the data which follow describe a new problem, a zero (0) is punched in any column on the card. If the data which follow describe changes to be made in the previous step of a given problem, a one (1) or two (2) is punched in any column on the card.

If a given problem step involves fitting, the program will normally list the beam line twice, printing each time the sequence of elements along with transfer or beam matrices where specified. The first listing uses the parameters of each element before any fitting has taken place. The second shows the results of the fitting. If a problem involving fitting has several steps, the second run of a given step often differs little from the first run of the following step.

If the second or subsequent step of a problem involves fitting and one wishes to print both runs through the beam line, a one (1) is punched on the indicator card. If the first listing is to be suppressed a two (2) is punched. If no fitting is involved, the program will ignore the two and will do one single run through the system.

If the initial listing is to be deleted, 10 is added to the indicator to give 10, 11, or 12. In order to be consistent with earlier versions of TRANSPORT, an indicator of minus one (-1) is interpreted as a two (2), but nine (9) is not interpreted as twelve (12).

The sample problem input shown on page 14 causes TRANSPORT to do a first-order calculation with fitting (0 indicator card) and then to do a second-order calculation (1 indicator card) with the data that is the result of the fitting.

COMMENT CARDS

Comment cards may be introduced anywhere in the deck where an element would be allowed by enclosing the comments made on each card within single parentheses. No parentheses are allowed within the parentheses of any comment card. The comments are not stored, but appear only in the initial listing of the given problem step.

Example of the use of comment cards in a data set

```
'Title Card'  
0  
(THIS IS A TEST PROBLEM TO ILLUSTRATE THE)  
(USE OF COMMENT CARDS)  
elements  
(COMMENTS MAY ALSO BE MADE BETWEEN)  
(TYPE CODE ENTRIES)  
elements  
SENTINEL
```


LISTING OF AVAILABLE TRANSPORT TYPE CODE ENTRIES

INPUT BEAM: Type code 1.0

The phase space and the average momentum of the input beam for a TRANSPORT calculation are specified by this element. The input is given in terms of the semi-axes of a six-dimensional erect^{*)} beam ellipsoid representing the phase space variables x , θ , y , ϕ , ℓ , and δ . Each of these six parameters is entered as a positive quantity, but should be thought of as $\pm x$, $\pm\theta$, etc; hence, the total beam width is $2x$, the total horizontal beam divergence is 2θ and so forth.

Usually the BEAM card is the third card in the deck. If other than standard TRANSPORT units are to be used, the units specification cards (type code 15.0) should precede the BEAM card. Standard TRANSPORT units for x , θ , y , ϕ , ℓ , and δ are cm, mr, cm, mr, cm and percent. The standard unit for the momentum $p(0)$ is GeV/c. Also if a beam line coordinate layout is desired, the card specifying that a layout is to be made (a 13.0 12.0 element), and any initial coordinates (see type code 16.0) all precede the BEAM card.

There are eight entries (all positive) to be made on the BEAM card.

- 1 - The type code 1.0 (specifies a BEAM entry follows).
- 2 - One-half the horizontal beam extent (x) (cm in standard units).
- 3 - One-half the horizontal beam divergence (θ) (mr).
- 4 - One-half the vertical beam extent (y) (cm).
- 5 - One-half the vertical beam divergence (ϕ) (mr).
- 6 - One-half the longitudinal beam extent (ℓ) (cm).
- 7 - One-half the momentum spread (δ) (in units of percent $\Delta p/p$).
- 8 - The momentum of the central trajectory [$p(0)$] (GeV/c).

All eight entries must be made even if they are zero (0). As for all other type codes, the last entry must be followed by a semicolon, dollar sign, or asterisk. Thus a typical BEAM entry might be

1. 0.5 2. 1.3 2.5 0. 1.5 10. ' ' ;

Label if
↓ desired.

*) For a rotated (non-erect) phase ellipsoid input, see type code 12.0.

meaning, $x = \pm 0.5$ cm, $\theta = \pm 2.0$ mr, $y = \pm 1.3$ cm, $\phi = \pm 2.5$ mr, $l = \pm 0.0$ cm, $\delta = \pm 1.5$ percent $\Delta p/p$, and the central momentum $p(0) = 10.0$ GeV/c.

The units of the tabulated matrix elements in either the first-order R or sigma matrix or second order T matrix of a TRANSPORT print-out will correspond to the units chosen for the BEAM card. For the above example, the $R(12) = (x/\theta)$ matrix element will have the dimensions of cm/mr; and the $T(236) = (\theta/y\delta)$ matrix element will have the dimensions mr/(cm * percent) and so forth.

The longitudinal extent l is useful for pulsed beams. It indicates the spread in length of particles in a pulse. It does not interact with any other component and may be set to zero if the pulse length is not important.

The phase ellipse (sigma matrix) beam parameters may be printed as output after every physical element if activated by a (13. 3. ;) element. Alternatively, individual printouts may be activated by a (13. 1. ;) element. The projection of the semi-axes of the ellipsoid upon each of its six coordinates axes is printed in a vertical array, and the correlations among these components indicating the phase ellipse orientations are printed in a triangular array (see the following pages).

The phase ellipse beam matrix

The beam matrix carried in the computer has the following construction:

	x	θ	y	φ	ℓ	δ
x	σ(11)					
θ	σ(21)	σ(22)				
y	σ(31)	σ(32)	σ(33)			
φ	σ(41)	σ(42)	σ(43)	σ(44)		
ℓ	σ(51)	σ(52)	σ(53)	σ(54)	σ(55)	
δ	σ(61)	σ(62)	σ(63)	σ(64)	σ(65)	σ(66)

The matrix is symmetric so that only a triangle of elements is needed.

In the printed output this matrix has a somewhat different format for ease of interpretation:

			x	θ	y	φ	ℓ
x	$\sqrt{\sigma(11)}$	CM					
θ	$\sqrt{\sigma(22)}$	MR	r(21)				
y	$\sqrt{\sigma(33)}$	CM	r(31)	r(32)			
φ	$\sqrt{\sigma(44)}$	MR	r(41)	r(42)	r(43)		
ℓ	$\sqrt{\sigma(55)}$	CM	r(51)	r(52)	r(53)	r(54)	
δ	$\sqrt{\sigma(66)}$	PC	r(61)	r(62)	r(63)	r(64)	r(65)

where:
$$r(ij) = \frac{\sigma(ij)}{[\sigma(ii)\sigma(jj)]^{\frac{1}{2}}}$$

As a result of the fact that the σ matrix is positive definite, the r(ij) satisfy the relation

$$|r(ij)| \leq 1 .$$

The full significance of the $\sigma(ij)$ and the $r(ij)$ are discussed in detail in the Appendix ("Description of Beam Matrix"). The units are always printed with the matrix.

In brief, the meaning of the $\sqrt{\sigma(ii)}$ is as follows:

$\sqrt{\sigma(11)} = x_{\max}$ = the maximum (half)-width of the beam envelope in the x(bend)-plane at the point of the print-out.

$\sqrt{\sigma(22)} = \theta_{\max}$ = the maximum (half)-angular divergence of the beam envelope in the x(bend) plane.

$\sqrt{\sigma(33)} = y_{\max}$ = the maximum (half)-height of the beam envelope.

$\sqrt{\sigma(44)} = \phi_{\max}$ = the maximum (half)-angular divergence of the beam envelope in the y(non-bend)-plane.

$\sqrt{\sigma(55)} = \ell_{\max}$ = one-half the longitudinal extent of the bunch of particles.

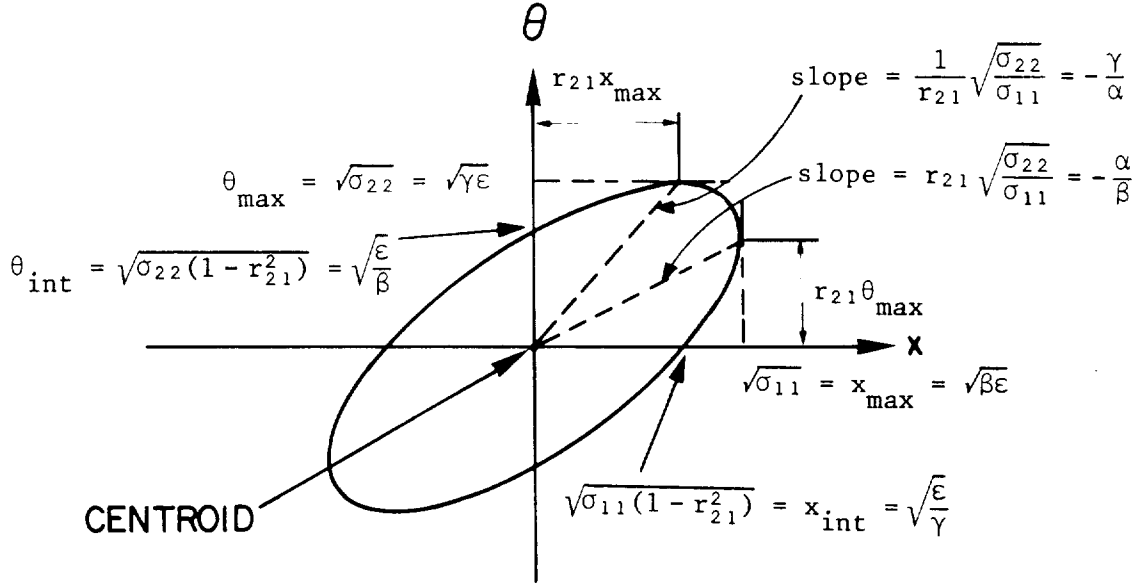
$\sqrt{\sigma(66)} = \delta$ = the half-width ($\frac{1}{2} \Delta p/p$) of the momentum interval being transmitted by the system.

The units appearing next to the $\sqrt{\sigma(ii)}$ in the TRANSPORT print-out are the units chosen for coordinates x, θ , y, ϕ , ℓ and $\delta = \Delta p/p$, respectively.

To the immediate left of the listing of the beam envelope size in a TRANSPORT print-out, there appears a column of numbers whose values will normally be zero. These numbers are the coordinates of the centroid of the beam phase ellipse (with respect to the initially assumed central trajectory of the system). They may become non-zero under one of three circumstances:

- 1) when the misalignment (type code 8.0) is used,
- 2) when a beam centroid shift (type code 7.0) is used, or
- 3) when a second-order calculation (type code 17.0) is used.

To aid in the interpretation of the phase ellipse parameters listed above, an example of an (x, θ) plane ellipse is illustrated below. For further details the reader should refer to the Appendix of this report.



A TWO-DIMENSIONAL BEAM PHASE ELLIPSE

The area of the ellipse is given by:

$$A = \pi(\det \sigma)^{\frac{1}{2}} = \pi x_{\max} \theta_{\text{int}} = \pi x_{\text{int}} \theta_{\max}$$

The equation of the ellipse is:

$$\gamma x^2 + 2\alpha x\theta + \beta\theta^2 = \epsilon$$

where

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \epsilon \begin{bmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{bmatrix}$$

and

$$\beta\gamma - \alpha^2 = 1, \quad r_{21} = r_{12} = \frac{-\alpha}{\sqrt{1+\alpha^2}} = \frac{-\alpha}{\sqrt{\beta\gamma}}$$

r.m.s. addition to the BEAM

To allow for physical phenomena such as multiple scattering, provision has been made in the program to permit an r.m.s. addition to the beam envelope. There are nine entries to be included:

- 1 - Type code 1.0 (specifying a BEAM entry follows).
- 2 - The r.m.s. addition to the horizontal beam extent (Δx) (cm).
- 3 - The r.m.s. addition to the horizontal beam divergence ($\Delta\theta$) (mr).
- 4 - The r.m.s. addition to the vertical beam extent (Δy) (cm).
- 5 - The r.m.s. addition to the vertical beam divergence ($\Delta\phi$) (mr).
- 6 - The r.m.s. longitudinal beam extent (Δz) (cm).
- 7 - The r.m.s. momentum spread ($\Delta\delta$) (in percent $\Delta p/p$).
- 8 - The momentum change in the central trajectory [$\Delta p(0)$] in (GeV/c).
- 9 - The code digit 0. indicating an r.m.s. addition to the BEAM is being made.

The units for the r.m.s. addition are the same as those selected for a regular BEAM type code 1.0 entry. Thus a typical r.m.s. addition to the BEAM would appear as follows:

1. .1 .2 .15 .3 0. .13 -0.1 0. ;

where the last entry (0.) preceding the semicolon signifies an r.m.s. addition to the BEAM is being made and the next to the last entry indicates a central momentum change of -0.1 GeV/c.

FRINGING FIELDS and POLE-FACE ROTATIONS for bending magnets:

Type code 2.0

To provide for fringing fields and/or pole-face rotations on bending magnets, the type code 2.0 element is used.

There are two parameters:

- 1 - Type code 2.0.
- 2 - Angle of pole-face rotation (degrees).

The type code 2.0 element must either immediately precede a bending magnet (type code 4.0) element (in which case it indicates an entrance fringing field and pole-face rotation) or immediately follow a type code 4.0 element (exit fringing field and pole-face rotation) with no other data entries between^{*)}. A positive sign of the angle on either entrance or exit pole-faces corresponds to a non-bend plane focusing action and bend plane defocusing action.

For example, a symmetrically oriented rectangular bending magnet whose total bend is 10 degrees would be represented by the three entries
2. 5. ; 4. --- ; 2. 5. ;

The angle of rotation may be varied. For example, the element
2.1 5. ; would allow the angle to vary from an initial guess of 5 degrees to a final value which would, say, satisfy a vertical focus constraint imposed upon the system. See the type code 10.0 section for a complete discussion of vary codes.

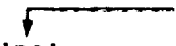
Even if the pole-face rotation angle is zero, 2. 0. ; entries must be included in the data set before and after a type code 4.0 entry if fringing-field effects are to be calculated.

A single type code 2.0 entry that follows one bending magnet and precedes another will be associated with the latter.

*) It is extremely important that no data entries be made between a type code 2.0 and a type code 4.0 entry. If this occurs, it may result in an incorrect matrix multiplication in the program and hence an incorrect physical answer. If this rule is violated, an error message will be printed.

Should it be desired to misalign such a magnet, an update element must be inserted immediately before the first type 2.0 code entry and the convention appropriate to misalignment of a set of elements applied, since, indeed, three separate transformations are involved. See section under type code 8.0 for a discussion of misalignment calculations and the section under type code 6.0 for a discussion of updates.

The type code signifying a rotated pole-face is 2.0. The input format is:

2. β . 'RO' ;  Label (if desired)

The units for β are degrees.

Pole-face rotation matrix

The first-order R matrix for a pole-face rotation used in a TRANSPORT calculation is as follows:

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{\tan \beta}{\rho_0} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\tan(\beta-\psi)}{\rho_0} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Definitions: β = angle of rotation of pole face (see figure on following page for sign convention of β)

ρ_0 = bending radius of central trajectory

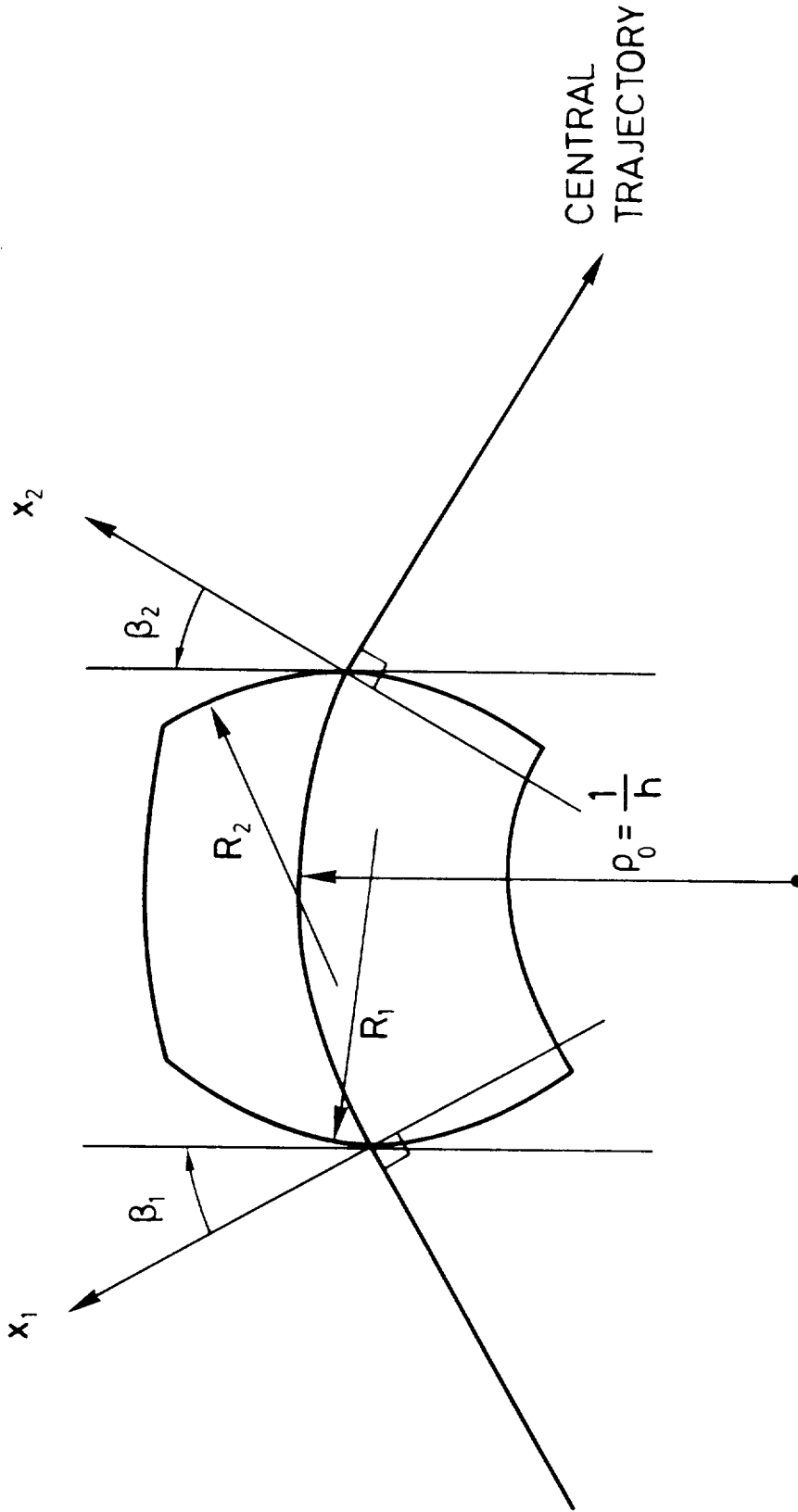
g = total gap of magnet

ψ = correction term resulting from spatial extent of fringing fields^{**}).

where
$$\psi = K_1 \left(\frac{g}{\rho_0} \right) \left(\frac{1 + \sin^2 \beta}{\cos \beta} \right) \left[1 - K_1 K_2 \left(\frac{g}{\rho_0} \right) \tan \beta \right]^*$$

^{*}) See type code 16.0 for input formats for g , K_1 , and K_2 TRANSPORT entries.

^{**}) See SLAC-75⁴) (page 74) for a discussion of ψ .



FIELD BOUNDARIES FOR BENDING MAGNETS

The TRANSPORT sign conventions for x , β , R and h are all positive as shown in the figure. The positive y direction is out of the paper. Positive β 's imply transverse focusing. Positive R 's (convex curvatures) represent negative sextupole components of strength $S = (-h/2R) \sec^3 \beta$. (See SLAC-75, page 71.)

DRIFT: Type code 3.0

A drift space is a field-free region through which the beam passes. There are two parameters:

- 1 - Type code 3.0 (specifying a drift length).
- 2 - (Effective) drift length (metres). The length of a drift space may be varied in either first- or second-order fitting.

Typical input format for a DRIFT:

3. 6. 'D1' ;

Label (if desired) (not to exceed 4 spaces between quotes).

DRIFT space matrix

The first-order R matrix for a drift space is as follows:

$$\begin{pmatrix} 1 & L & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where

L = the length of the drift space.

The dimensions of L. are those chosen for longitudinal length via a

15. 8. ' ' ; type code entry (if used) preceding the BEAM (type code 1.0) card. If no 15. 8. entry is made, the units of L. will automatically be in metres (standard TRANSPORT units).

units symbol

scale factor (if needed)

WEDGE BENDING MAGNET: Type code 4.0

A wedge bending magnet implies that the central trajectory of the beam enters and exits perpendicularly to the pole-face boundaries (to include fringing-field effects and non-perpendicular entrance or exit boundaries -- see type codes 2.0 and 16.0).

There are four first-order parameters to be specified for the wedge magnet via type code 4.0:

- 1 - Type code 4.0 (specifying a wedge bending magnet).
- 2 - The (effective) length L of the central trajectory in metres.
- 3 - The central field strength B(0) in kG,

$$B(0) = 33.356 (p/\rho_0),$$
 where p is the momentum in GeV/c and ρ_0 is the bending radius of the central trajectory in metres.
- 4 - The field gradient (n-value, dimensionless); where n is defined by the equation

$$B_y(x,0,t) = B_y(0,0,t) (1 - nhx + \dots) ,$$

where

$$h = 1/\rho_0. \text{ See SLAC-75 (page 31) }^4).$$

The quantities L, B(0), and n may be varied for first-order fitting (see type code 10.0 for a discussion of vary codes).

The bend angle in degrees and the bend radius in metres are printed in the output.

A typical first-order TRANSPORT input for a wedge magnet is

```

4.  L.  B.  n.  ' ' ;

```

Label (not to exceed
4 spaces)

If fringing field effects are to be included, a type code 2.0 entry must immediately precede and follow the pertinent type code 4.0 entry (even if there are no pole-face rotations). Thus a typical TRANSPORT input for a bending magnet including fringing fields might be:

```

2.  0.  ' ' ;
4.  L.  B(0).  n.  ' ' ;
2.  0.  ' ' ;

```

Labels (not to exceed
4 spaces) if
desired

For non-zero pole-face rotations a typical data input might be

```
2. 10. ; 4. L. B(0). n. ; 2. 20. ;
```

Note that the use of labels is optional and that all data entries may be made on one line if desired.

The sign conventions for bending magnet entries are illustrated in the following figure. For TRANSPORT a positive bend is to the right looking in the direction of particle travel. To represent a bend in another sense, the coordinate rotation matrix (type code 20.0) should be used as follows:

A bend up is represented by rotating the (x, y) coordinates by -90.0 degrees about the (z) beam axis as follows:

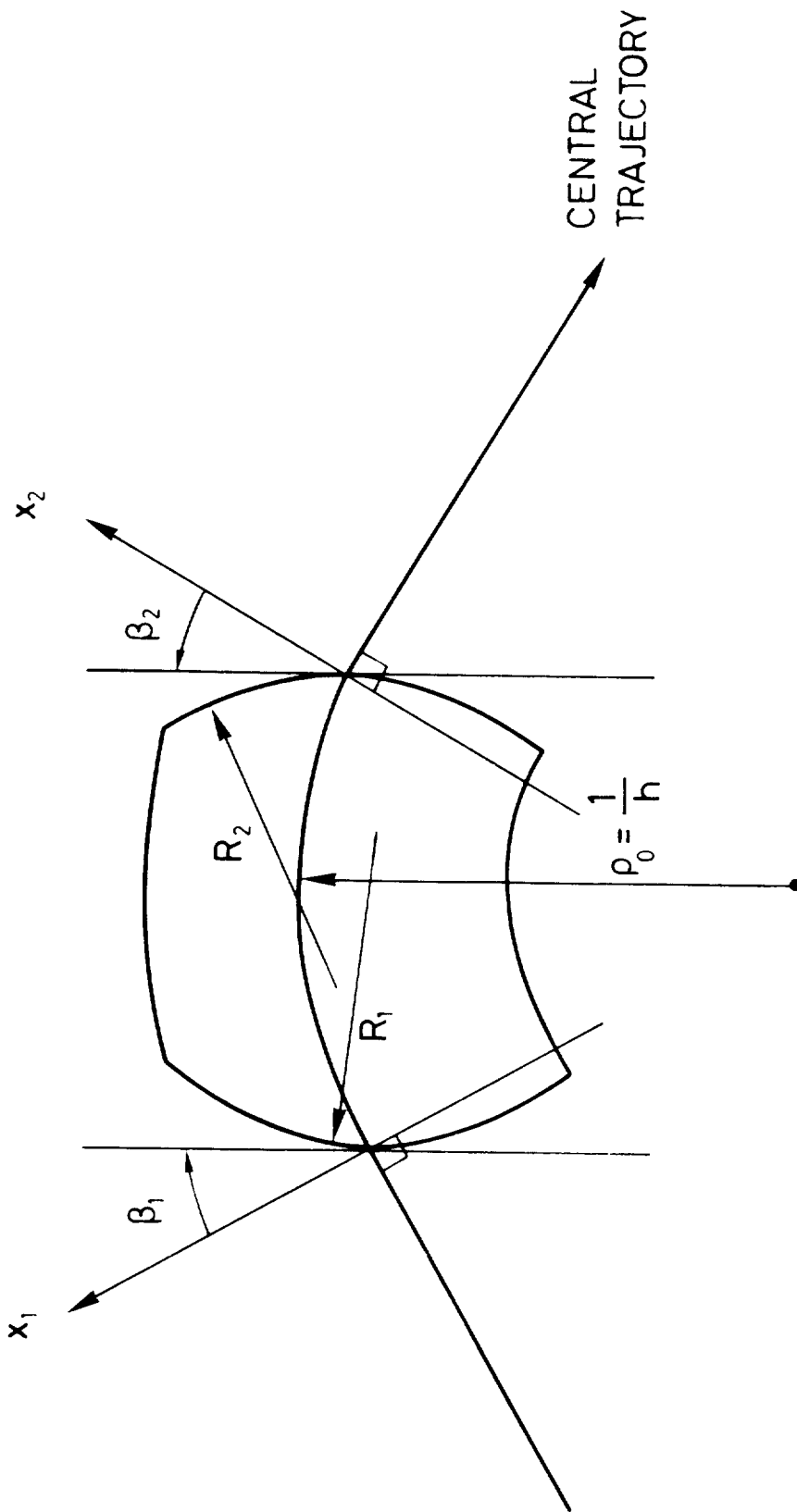
```
                Labels (not to exceed 4 spaces)
                |
                v
20.  -90.  ' ' ;
2.   β(1).  ' ' ;
4.   L.   B.   n.   ' ' ;
2.   β(2).  ' ' ;
20.  +90.  ' ' ; (returns coordinates to their initial
                  orientation)
```

A bend down is accomplished via:

```
20.  +90.  ' ' ;
2.
4.
2.
20.  -90.  ' ' ;
```

A bend to the left (looking in the direction of beam travel) is accomplished by rotating the x, y coordinates by 180 degrees, e.g.

```
20.  180.  ' ' ;
2.
4.
2.
20.  -180  ' ' ;
```



FIELD BOUNDARIES FOR BENDING MAGNETS

The TRANSPORT sign conventions for x , β , R and h are all positive as shown in the figure. The positive y direction is out of the paper. Positive β 's imply transverse focusing. Positive R 's (convex curvatures) represent negative sextupole components of strength $S = (-h/2R) \sec^3 \beta$. (See SLAC-75, page 71.)

First-order wedge bending magnet matrix

$\cos k_x L$	$\frac{1}{k_x} \sin k_x L$	0	0	0	$\frac{h}{k_x^2} (1 - \cos k_x L)$
$-k_x \sin k_x L$	$\cos k_x L$	0	0	0	$\frac{h}{k_x} \sin k_x L$
0	0	$\cos k_y L$	$\frac{1}{k_y} \sin k_y L$	0	0
0	0	$-k_y \sin k_y L$	$\cos k_y L$	0	0
$-\frac{h}{k_x} \sin k_x L$	$-\frac{h}{k_x^2} (1 - \cos k_x L)$	0	0	0	$-\frac{h^2}{k_x^3} (k_x L - \sin k_x L)$
0	0	0	0	0	1

Definitions: $h = L/\rho_0$, $k_x^2 = (1 - n)h^2$, $k_y^2 = nh^2$

$\alpha = hL$ = the angle of bend

L = path length of the central trajectory.

The field expansion for the midplane of a bending magnet is taken from Eq. (18) page 31 of SLAC-75, thereby defining the dimensionless quantities n and β as follows:

$$B_y(x,0,t) = B_y(0,0,t) [1 - nhx + \beta h^2 x^2 + \gamma h^3 x^3 + \dots] .$$

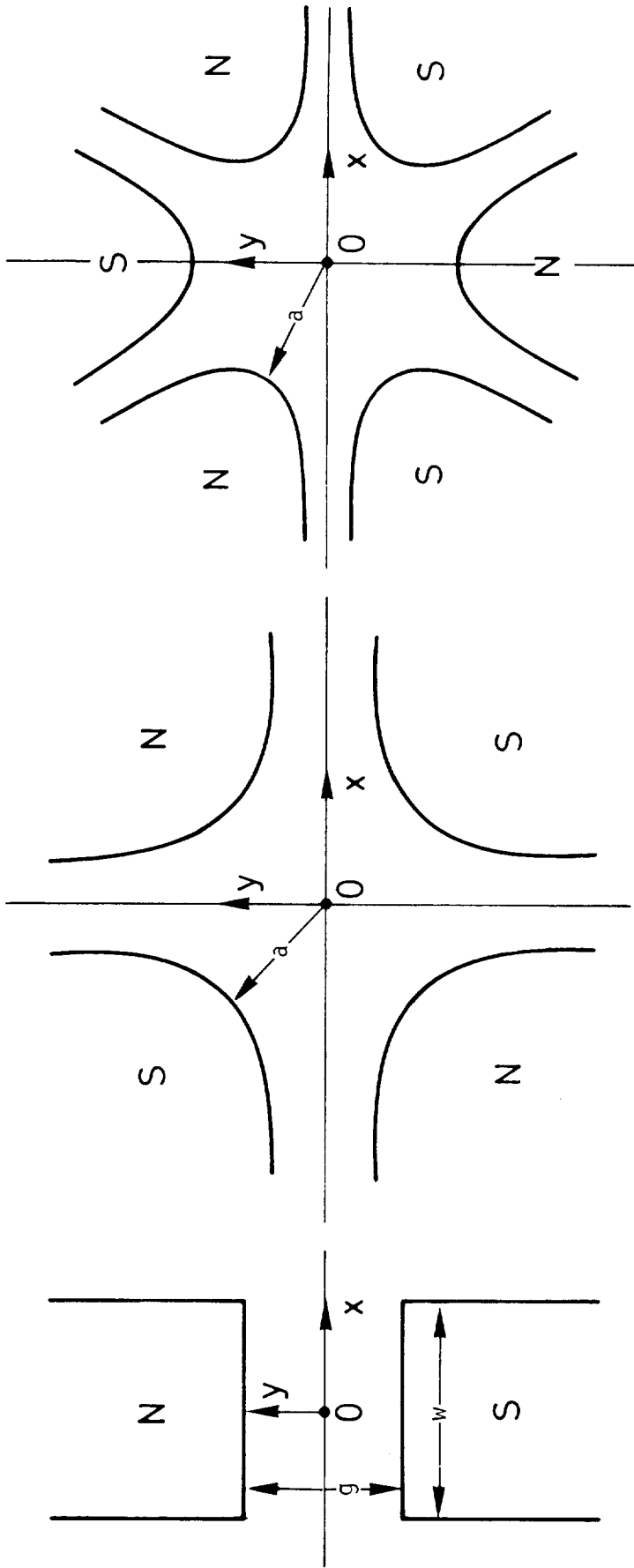
The type code signifying a BEND is 4.0. The input format for a TRANSPORT calculation is:

4. L. B. n. ' ' ;

└── Label (not to exceed 4 spaces)
↓

If n is not included in the data entry, the program assumes it to be zero. A β entry for a second-order calculation is made via the 16.0 1.0 element. (Do not confuse this β with a pole-face rotation.)

The standard units for L and B are metres and kG. If desired, these units may be changed by 15.0 8.0 and 15.0 9.0 type code entries preceding the BEAM Card.



DIPOLE

QUADRUPOLE

SEXTUPOLE

ILLUSTRATION OF THE MAGNETIC MIDPLANE (x AXIS) FOR DIPOLE, QUADRUPOLE AND SEXTUPOLE ELEMENTS. THE MAGNET POLARITIES INDICATE MULTIPOLE ELEMENTS THAT ARE POSITIVE WITH RESPECT TO EACH OTHER.

QUADRUPOLE: Type code 5.0

A quadrupole provides focusing in one transverse plane and defocusing in the other.

There are four parameters to be specified for a TRANSPORT calculation:

- 1 - Type code 5.0 (specifying a quadrupole).
- 2 - (Effective) magnet length L (in metres).
- 3 - Field at pole tip B (in kG). A positive field implies horizontal focusing; a negative field, vertical focusing.
- 4 - Half-aperture a (in cm). Radius of the circle tangent to the pole tips.

The length and field of a quadrupole may be varied in first-order fitting. The aperture may not be.

The strength of the quadrupole is computed from its field, aperture and length. The horizontal focal length is printed in parentheses as output. A positive focal length indicates horizontal focusing and a negative focal length indicates horizontal defocusing. The quantity actually printed is the reciprocal of the (θ/x) transfer matrix element $(1/R_{21})$ for the quadrupoles. Thus two identical quadrupoles of opposite polarity will have different horizontal focal lengths due to the difference between the sine and the hyperbolic sine.

The type code for a QUAD is 5.0. The input format for a typical data set is:

5. L. B. a. ' ' ;

Label (if desired) not to exceed
4 spaces between quotes

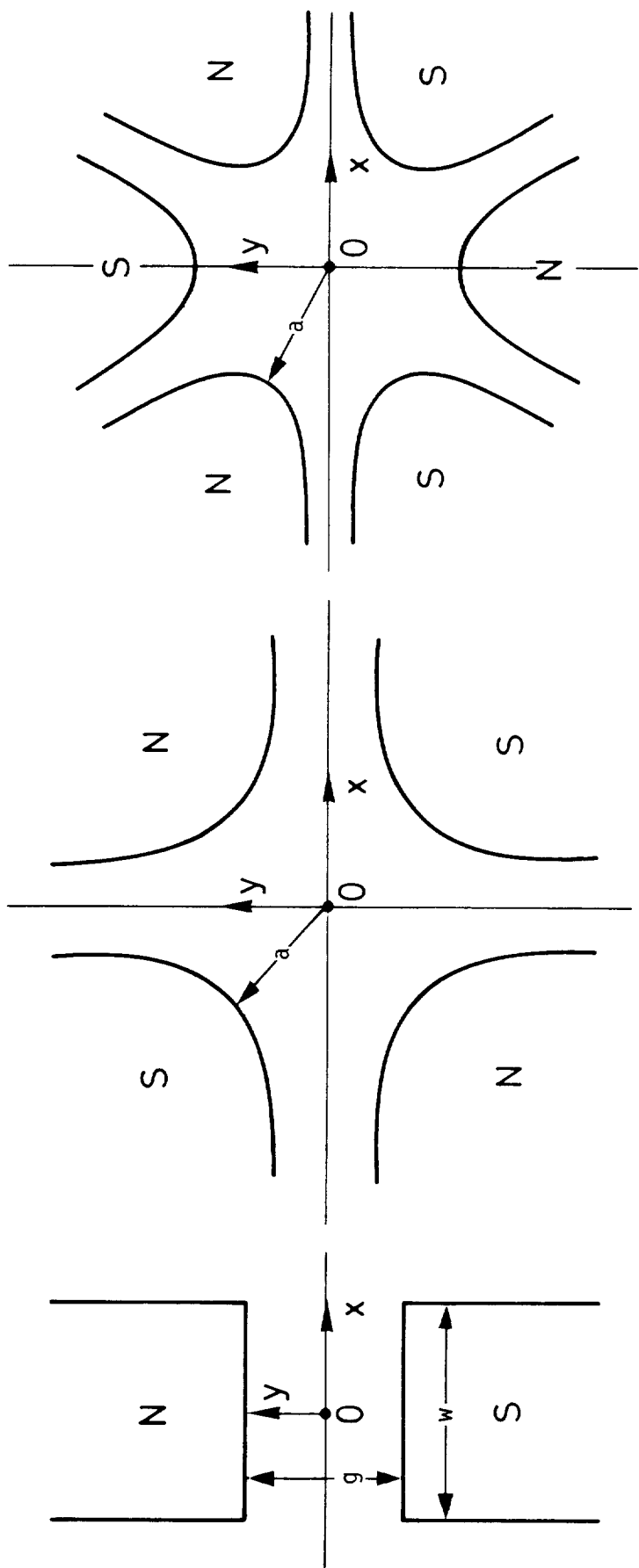
The standard TRANSPORT units for L, B, and a are metres, kG, and cm, respectively. If other units are desired they must be chosen via the appropriate 15.0 type code entries preceding the BEAM (type code 1.0) card.

First-order quadrupole matrix

$\cos k_q L$	$\frac{1}{k_q} \sin k_q L$	0	0	0	0
$-k_q \sin k_q L$	$\cos k_q L$	0	0	0	0
0	0	$\cosh k_q L$	$\frac{1}{k_q} \sinh k_q L$	0	0
0	0	$k_q \sinh k_q L$	$\cosh k_q L$	0	0
0	0	0	0	1	0
0	0	0	0	0	1

These elements are for a quadrupole which focuses in the horizontal (x) plane (B positive). A vertically (y-plane) focusing quadrupole (B negative) has the first two diagonal submatrices interchanged.

Definitions: L = the effective length of the quadrupole
a = the radius of the aperture
 B_0 = the field at radius a
 $k_q^2 = (B_0/a)(1/B\rho_0)$, where $(B\rho_0)$ = the magnetic rigidity (momentum) of the central trajectory.



SEXTUPOLE

QUADRUPOLE

DIPOLE

ILLUSTRATION OF THE MAGNETIC MIDPLANE (x AXIS) FOR DIPOLE, QUADRUPOLE AND SEXTUPOLE ELEMENTS. THE MAGNET POLARITIES INDICATE MULTIPOLE ELEMENTS THAT ARE POSITIVE WITH RESPECT TO EACH OTHER.

TRANSFORM 1 update^{*}): Type code 6. 0. 1.

To re-initialize the matrix TRANSFORM 1 (the product of the R matrices, R1) use type code 6.0. A (6. 0. 1. ;) card effects an update of the R1 matrix and initiates the accumulation of a new product matrix at the point of the update. This facility is often useful for misaligning a set of magnets or fitting only a portion of a system.

The matrix R1 is updated by no other element. It is not used in the calculation of the beam matrix. The beam matrix is calculated from the auxiliary transfer matrix R2 described on the next page.

A TRANSFORM 1 matrix will be printed at any position in the data set where a (13. 4. ;) entry is inserted.

See the following section for the introduction of an auxiliary transformation matrix R2 (TRANSFORM 2) to avoid the need for TRANSFORM 1 updates.

The (6. 0. 1. ;) card also causes an update of the R2 matrix.

*) By "updating" we mean initiating a new starting point for the accumulation (multiplication) of the R matrix. At the point of update the previous accumulation is discontinued. When the next element possessing a transfer matrix is encountered, the accumulated transfer matrix R1 is set equal to the individual transfer matrix R for that element. Accumulation is resumed thereafter.

AUXILIARY TRANSFORMATION MATRIX (R2): Type code 6. 0. 2.

The R1 matrix represents the accumulated transfer matrix from either the beginning of the beam line or the last explicit R1 update (6. 0. 1. ;). However several elements in TRANSPORT which affect the beam matrix cannot be represented in any transfer matrix. To avoid update complications with R1 an auxiliary transfer matrix R2 exists. The beam matrix is then calculated from the R2 matrix and the beam matrix at the last R2 update.

Both the R1 and R2 matrices are normally available for printing. However there is no redundancy in computer use, since, internally to the program only R2 is calculated at each element. The matrix R1 is calculated from R2 only as needed.

The R2 matrix is updated explicitly via a (6. 0. 2. ;) entry. It may be printed by a (13. 24. ;) entry. Constraints on R2 are imposed similarly to those on R1. For details see the section describing type code 10.0.

The complete list of elements which update TRANSFORM 2 is:

- 1) a beam type code 1.0 entry
- 2) the (6. 0. 1. ;) entry
- 3) the (6. 0. 2. ;) entry
- 4) a centroid shift type code 7.0 entry
- 5) a misalignment type code 8.0 entry
- 6) a stray field type code 21.0 entry.

Please note that automatic updates of TRANSFORM 2 occur when an align element (type code 8.) is inserted specifying the misalignment of all subsequent bending magnets. These TRANSFORM 2 updates take place immediately before and immediately after any bending magnet which has either the entrance or exit fringe fields specified via a type code 2 entry.

SHIFT IN THE BEAM CENTROID: Type code 7.0

Sometimes it is convenient to redefine the beam centroid^{**}) such that it does not coincide with the TRANSPORT reference trajectory. Provision has been made for this possibility via type code 7.0. Seven parameters are required:

- 1 - Type code 7.0.
- (2 to 7) - the coordinates x , θ , y , ϕ , l , and δ defining the shift in the location of the beam centroid with respect to its previous position. The units for x , θ , y , ϕ , l , δ are the same as those chosen for the BEAM (type code 1.0 entry), normally cm, mr, cm, mr, cm, and percent.

Any or all of the six beam centroid shift parameters may be varied in first-order fitting. The centroid position may then be constrained at any later point in the beam line by this procedure.

The transformation matrix R2 is updated by this element.

In order for this code to function properly, the initial BEAM entry (type code 1.0) must have a non-zero phase space volume, for example a

1. 0 0 0 0 0 0 p(0). ;

BEAM entry is not permissible when calculating a shift in the beam centroid; whereas a

1. 1. 1. 1. 1. 1. 1. p(0). ;

entry (non-zero phase volume) is acceptable.

^{**}) By "beam centroid" we mean the centre of the beam ellipsoid.

MAGNET ALIGNMENT TOLERANCES: Type code 8.0

The first-order effects of the misalignment of a magnet or group of magnets are a shift in the centroid of the beam and a change in the beam focusing characteristics. Two varieties of misalignment are commonly encountered: 1) the magnet is displaced and/or rotated by a known amount; or 2) the actual position of the magnet is uncertain within a given tolerance. TRANSPORT has the capability of simulating the misalignment of either single magnets or entire sections of a beam line. Any combination of the above alternatives may be simulated through the use of the "align" element. The results may be displayed in either the printed output of the beam (sigma) matrix or tabulated in a special misalignment table (described below).

There are eight parameters to be specified:

- 1 - Type code 8.0 (specifying a misalignment).
- 2 - The magnet displacement in the horizontal direction (cm).
- 3 - A rotation about the horizontal axis (mr).
- 4 - A displacement in the vertical direction (cm).
- 5 - A rotation about the vertical axis (mr).
- 6 - A displacement in the beam direction (cm).
- 7 - A rotation about the beam direction (mr).
- 8 - A three-digit code number (defined below) specifying the type of misalignment.

The three displacements and three rotations comprise the six degrees of freedom of a rigid body and are used as the six misalignment coordinates. The coordinate system employed is that to which the beam is referred at the point it enters the magnet. For example, a rotation of a bending magnet about the beam direction (parameter 7 above) is referred to the direction of the beam where it enters the magnet. The units employed are the standard TRANSPORT units shown above, unless redefined by type code 15. entries. If the units are changed, the units of the misalignment displacements are those determined by the 15. 1. type code entry; the units for the misalignment rotations are those determined by the 15. 2. type code entry.

The misalignment of any physical element or section of a beam line may be simulated. Misaligned sections of a beam line may be nested. A

beam line rotation (type code 20.) may be included in a misaligned section. Thus, for example, one can simulate the misalignment of magnets that bend vertically. The arbitrary matrix (type code 14.) may not be included in a misaligned section. A misalignment must never be included in a second-order run (type code 17.).

A misalignment element may indicate that a single magnet or section of the beam line is to be misaligned, or it may indicate that all subsequent magnets of a given type (quadrupoles and/or bending magnets) are to be misaligned. The type of misalignment is specified in the three-digit code number, and the location of the type code 8. align element depends on the type of misalignment.

If a misalignment pertains to a single magnet or a single section of the beam line, then the misalignment element (type code 8.) must directly follow that magnet or section of the beam line. If a misalignment element indicates that all subsequent magnets of a given type are to be misaligned, it must precede the first of such magnets. Further description of the available types of misalignment is given in the table below.

The results of the misalignment may be displayed in either the beam (sigma) matrix or in a misalignment table. If the results are displayed in the beam (sigma) matrix, then that matrix is altered by the effects of the misalignment. The effects of additional misalignments cause further alterations, so that at any point along the beam line, the beam (sigma) matrix will contain the combined effects of all previous misalignments.

The misalignment table can be used to show independently the effects on the beam matrix of a misalignment in each degree of freedom of each misaligned magnet. Each new misalignment to be entered in the table creates a new set of six duplicates of the beam matrix. Printed for each duplicate beam matrix are the centroid displacement and the beam half width in each of the six beam coordinates. Each of the six matrices shows the combined result of the undisturbed beam matrix and the effect of the misalignment in a single coordinate of a single magnet or section of the beam line. In a single TRANSPORT run the results of misaligning up to ten magnets or sections of the beam line may be included in the misalignment table. Further requests for entry in the misalignment table will be ignored. Examples of such a table and the input which generated it are shown below.

When the user specifies that the actual position of the magnet(s) is uncertain within a given tolerance, the printout will show a change in the beam (sigma) matrix resulting from the effects of the misalignment(s). Thus, if one wishes to determine the uncertainty in the beam centroid resulting from uncertainties in the positioning of the magnets, the initial beam dimensions should be set to zero, i.e. the beam card entry at the beginning of the system should appear as follows:

1. 0. 0. 0. 0. 0. 0. p(0).

If it is desired to know the effect of an uncertainty in position on the beam focusing characteristics, then a non-zero initial phase space must be specified. The printout will then show the envelope of all possible rays, including both the original beam and the effects of the misalignment.

If the misalignment is a known amount, it may affect the beam centroid as well as the beam dimensions. Therefore one should place on the BEAM card the actual dimensions of the beam entering the system. For a known misalignment, the program requires that the initial beam specified by type code 1 must be given a non-zero phase volume, to insure a correct printout.

An align element pertaining to a single magnet or section of the beam line updates the BEAM (sigma) matrix and the R2 matrix, but not the R1 matrix. A misalignment element which indicates misalignment of all subsequent magnets of a given type will update the BEAM (sigma) matrix and the R2 matrix before each bending magnet with fringe fields and after each misaligned magnet of any type.

The tolerances may be varied. Thus, type-vary code 8.111111 permits any of the six parameters (2 through 7 above) to be adjusted to satisfy whatever BEAM constraints may follow. For fitting, a misalignment must pertain to a single magnet or single section of the beam line, and the results must be displayed in the beam (sigma) matrix. (See the section under type code 10. for a discussion of the use of vary codes.)

The meaning of the options for each digit of the three-digit code number is given in the following table.

A. The units position specifies the magnet(s) or section of the beam line to be misaligned.

CODE NUMBER	INTERPRETATION
XX0.	The <u>single</u> magnet (type code element) immediately preceding the align card it to be misaligned. A bending magnet with fringe fields should be misaligned using one of the options described below.
XX1.	The last R1 matrix update (the start of the beam line or a 6. 0. 1. ; type code entry) marks the beginning of the section to be misaligned. The misalignment element itself marks the end. The section is treated as a unit and misaligned as a whole. The misalignments of quadrupole triplets and other combinations involving more than two quadrupoles may be studied using this code digit.
XX2.	<p>The last R2 matrix update (see type code 6. for a list of elements which update R2) marks the beginning of the misaligned section. The misalignment element marks the end. This option makes use of the fact that R2 matrix updates do not affect the R1 matrix.</p> <p>A bending magnet with fringing fields or pole face rotations (type code 2.) should be misaligned using this option. See examples 1 and 2 below for an illustration of this.</p> <p>An array of quadrupoles provides another example of the use of this option. By successive application of align elements, the elements of a quadrupole triplet could be misaligned relative to each other, and then the triplet as a whole could be misaligned. See example 3 below for an illustration of this.</p>
XX3.	All subsequent bending magnets and quadrupoles are independently misaligned by the amount specified. This option is useful in conjunction with the tabular display of the misalignment results (see below). A bending magnet, with fringing fields included, is treated as a single unit and misaligned accordingly.

- XX4. All subsequent bending magnets, including fringing fields, are independently misaligned by the amount specified. See XX3 above for further comments.
- XX5. All subsequent quadrupoles are independently misaligned by the amount specified. See example 4 below for an illustration of this. See XX3 above for further comments.
- B. The tens position defines the mode of display of the results of the misalignment.
- XOX. The beam matrix contains the results of the misalignment. The beam matrix is printed wherever a 13. 1. ; card is encountered. The beam matrix will then contain contributions from all previous misalignments.
- XIX. A table is used to store the results of misalignments. The effect of up to ten independently misaligned magnets may be shown in the table in a single run. The table is printed via a 13. 8. ; card, and may be compared with the undisturbed beam matrix (printed by a 13. 1. ; card) at any point. An example of such a table is shown below.

C. The hundreds position distinguishes between an uncertainty in position (OXX.) or a known displacement (1XX.).

Any combination of digits may be used to define the exact circumstances intended. Thus, code 111. indicates the deliberate displacement of a set of magnets (referred to the point where the beam enters the set), with the results to be stored in a table.

Example No. 1: A bending magnet with a known misalignment

A bending magnet (including fringe fields) misaligned by a known amount might be represented as follows:

```
3. L(1). ;  
6. 0. 2. ;  
2. 0. ; 4. 1. B. n. ; 2. 0. ;  
8. 0. 0. 0. 0. 0. 2. 102. ;  
3. L(2). ;
```

This represents a known rotation of the bending magnet about the incoming beam direction (z axis) by 2.0 mr. The result of this misalignment will be a definite shift in the beam centroid, and a mixing of the horizontal and vertical coordinates. The use of the 6. 0. 2. ; transform 2 update and the misalignment code number XX2 is necessary because the magnetic array (bending magnet + fringing fields) consists of three type code elements instead of one.

Example No. 2: A bending magnet with an uncertain position

A bending magnet having an uncertainty of 2 mr in its angular positioning about the incoming beam (z axis) would be represented as follows:

```
3. L(1). ;
6. 0. 2. ;
2. 0. ; 4. L. B. n. ; 2. 0. ;
8. 0. 0. 0. 0. 0. 2.0 002. ;
3. L(2). ;
```

To observe the uncertainty in the location of the outgoing beam centroid, the input BEAM card should have zero phase space dimensions as follows:

```
1. 0. 0. 0. 0. 0. 0. p(0). ;
```

If the beam dimensions on the input BEAM card are non-zero, the resultant beam (sigma) matrix will show the envelope of possible rays, including both the input beam and the effect of the misalignment.

Example No. 3: A misaligned quadrupole triplet

One typical use of both the R1 and R2 matrices is to permit the misalignment of a triplet. For example, an uncertainty in the positions within the following triplet

```
5. 1.  -8. 10.  ;
5. 2.   7. 10.  ;
5. 1.  -8. 10.  ;
```

may be induced by appropriate 8. elements as noted:

```
6. 0.   1.  ;
5. 1.  -8. 10.  ;
6. 0.   2.  ;
5. 2.   7. 10.  ;
5. 1.  -8. 10.  ;
8. --- --- --- --- --- --- 000.  ;
8. --- --- --- --- --- --- 002.  ;
8. --- --- --- --- --- --- 001.  ;
```

The first 8. card in the list refers to the misalignment of the third magnet only. The second 8. card refers to the misalignment of the second and third magnets as a single unit via the R2 matrix update (the 6. 0. 2. ; entry). The last 8. card refers to the misalignment of the whole triplet as a single unit via the R1 matrix update (the 6. 0. 1. ; entry).

The comments about the BEAM card (type code 1. entry) in example 2 above are applicable here also.

Example No. 4: Misaligned quadrupoles in a triplet

Individual uncertainties in the positions of the quadrupoles in the triplet in example no. 3 above may be induced by a single misalignment as follows:

8. --- --- --- --- --- 015. ;
5. 1. -8. 10. ;
5. 2. 7. 10. ;
5. 1. -8. 10. ;

The effect of each misalignment coordinate on each quadrupole will be stored separately in a table. This table is printed wherever a 13. 8. ; type code is inserted.

REPETITION: Type code 9.0

Many systems include a set of elements that are repeated several times. To minimize the chore of input preparation, a 'repeat' facility is provided.

There are two parameters:

- 1 - Type code 9.0
- 2 - Code digit. If non-zero, it states the number of repetitions desired from the point it appears. If zero it marks the end of a repeating unit.

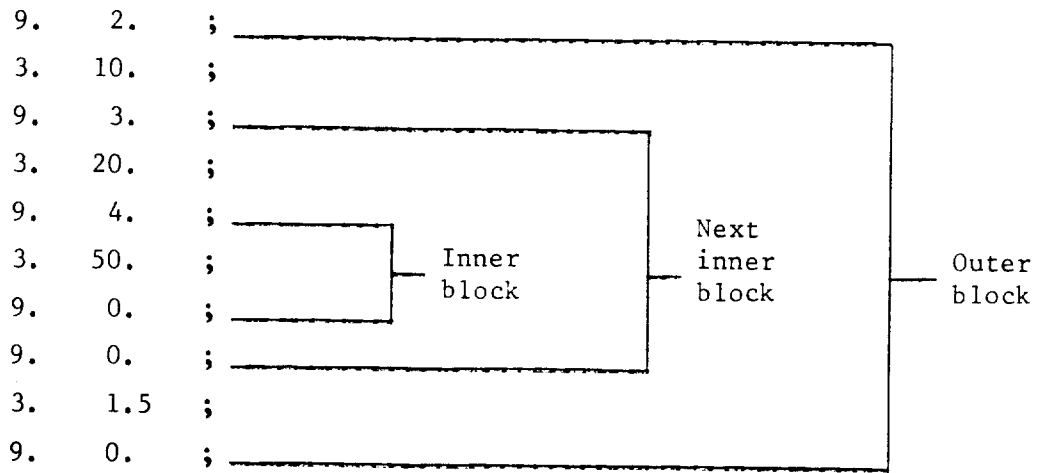
For example, a total bend of 12 degrees composed of four 3-degree bending magnets each separated by 0.5 metres could be represented by 9. 4. ; 4. --- ; 3. .5 ; 9. 0. ; Those elements (in this case a bend and drift) between the 9. 4. ; and 9. 0. ; would be employed four times.

There is no indication of the 9.0 cards in the printed TRANSPORT output when calculating except for the repeated listing of the elements they control.

Vary codes may be used within a repeating unit in the usual fashion. However all repetitions of a given varied element will be coupled.

Repeat cards may be nested four deep. By "nesting" we mean a repeat within a repeat. An example is given below.

Example of Nesting



The total length of this sequence is:

$$2*(10. + 3*(20 + 4* 50) + 1.5) = 1343.$$

VARY CODES and FITTING CONSTRAINTS: Type code 10.0

Some (not all) of the physical parameters of the elements comprising a beam line may be varied in order to fit selected matrix elements. In a first-order calculation one might fit elements of the R1 or R2 transformation matrices or the beam (sigma) matrix. In second order one might constrain an element of the second-order matrix T1 or minimize the net contribution of aberrations to a given beam coordinate. Special constraints are also available.

One may not mix orders in fitting. First order vary codes and constraints must be inserted only in a first-order calculation, and similarly for second order.

The physical parameters to be varied are selected via 'Vary Codes' attached to the type codes of the elements comprising the system. The fitting constraints on matrix elements are selected via type code 10.0 entries placed in the system where the constraint is to be imposed.

Vary codes

Associated with each physical element in a system is a vary code which specifies which physical parameters of the element may be varied. This code occupies the fraction portion of the type code specifying the element. It has one digit for each parameter, the digits having the same order in the code as the physical parameters have on the card. A '0' indicates the parameter may not be varied; a '1' that it may be. For instance, 3.0 is the combined type (3) and vary code (0) for a drift length which is to remain fixed; 3.1 indicates a drift length that may be varied (by the virtue of the .1). The type code 4.010 indicates a bending magnet with a variable magnetic field. In punching the code 3.0, the zero need not be punched. In punching the 4.010 code, the first zero must be punched but the second zero need not be.

First-order vary codes

In a first-order run the following parameters marked v may be varied, those marked 0 may not be varied.

BEAM.....	1.vvvvvv0 - All components of the input beam may be varied, except the momentum.
R.M.S. ADDITION..	1.vvvvvv00 - All components of an r.m.s. addition may be varied except the momentum change Δp .
ROTAT.....	2.v - The pole face angle of a bending magnet may be varied.
DRIFT.....	3.v - The drift length may be varied.
BEND.....	4.vvv - The length, the field, and/or the n-value may be varied.
QUAD.....	5.vv0 - The length may be varied; the field may be, the aperture may <u>not</u> be.
AXIS SHIFT..	7.vvvvvv - Any of the axis shift parameters may be varied.
ALIGN.....	8.vvvvvv0 - Any of the alignment parameters may be varied.
INITIAL COORDINATES..	16.0v - Any of the three initial position floor coordinates or two angle coordinates may be varied.
MATRIX.....	14.vvvvvv0 - Any of the first order matrix elements may be varied.
SOLENOID...	19.vv - The length and/or field may be varied.
BEAM ROTATION....	20.v - The angle of rotation may be varied.

The use of the permissive 'may' rather than the imperative 'will' in discussing variables is meaningful. The program will choose the parameters it will vary from among those that it may vary. In general it chooses to vary those parameters that have the greatest influence upon the conditions to be fit.

Second-order vary codes

In a second-order run the following parameters may be varied:

- DRIFT..... 3.v - The drift length may be varied. Variation of a drift length should be done with caution as it may affect the first-order properties of the beam line. But inversely coupled drift spaces straddling a sextupole will, for example, show only second-order effects.
- $\epsilon(1)$ 16.0v 1. - The normalized quadratic term (sextupole component) in the midplane expansion for the field of a bending magnet may be varied.
- 1/R1..... 16.0v 12. - The pole face curvature of a bending magnet entrance may be varied.
- 1/R2..... 16.0v 13. - The pole face curvature of a bending magnet exit may be varied.
- SEXTUPOLE 18.0v - The field strength may be varied.

The special parameter cards (type code 16.0) once introduced apply to all subsequent magnets in a beam line until another type code 16.0 specifying the same parameter is encountered. Thus if such a parameter is varied, the variation will apply simultaneously to all subsequent magnets to which it pertains. The variation will persist until the parameter or vary code attached to the parameter is changed by the introduction of another type code 16.0 card specifying the same parameter.

Coupled vary codes

It is possible to apply the same correction to each of several variables. This may be done by replacing the digit 1 in the vary code with one of the digits 2 through 9, or a letter A through Z. All such variables whose vary digits are the same, regardless of position will receive the same correction. For example, the three type-vary codes (5.0A, 5.01, 5.0A) might represent a symmetric triplet. The same correction will be made to the first and third quadrupoles, guaranteeing that the triplet will remain symmetric.

If a vary digit is immediately preceded by a minus sign, the computed correction will be subtracted from, rather than added to, this variable. Thus parameters with the same vary digit, one of them being preceded by a minus sign, will be inversely coupled. For example the type-vary code sequence (3.B, 5.01, 3.-B) will allow the quadrupole to move without altering the total system length.

Vary digits may also be immediately preceded by a plus sign without changing their meaning. Thus 5.0A is the same as 5.0+A. For historical reasons, the vary digits (9 and 4), (8 and 3), and (7 and 2) are also inversely coupled. Inverse coupling may not be used with type codes 1.0 or 8.0.

The total number of independent variables in a first-order run is limited to 20 by reasons of the mathematical method of fitting and to 10 for a second-order run. So far as this limit is concerned, variables that are tied together count as one. Variables within repeat elements (type code 9.0) also count only one.

Possible fitting constraints

A variety of possible constraints is available. Fitting may be done in either first- or second-order, but not in both simultaneously. The order of the constraint must be appropriate to the order of the run. A list of constraints available is given below. They are explained more fully on later pages.

First-order constraints

- 1) An element of the first-order transfer matrix R1.
- 2) An element of the auxiliary first-order matrix R2.
- 3) A σ (BEAM) matrix element.
- 4) The correlations r in the beam coordinates.
- 5) The first moments of the beam.
- 6) The total system length.
- 7) An AGS machine constraint.
- 8) The reference trajectory floor coordinates.

Second-order constraints

- 1) An element of the second-order transfer matrix T1.
- 2) An element of the second-order auxiliary transfer matrix T2.
- 3) The net contributions of aberrations to a given coordinate of the beam matrix σ .
- 4) The strength of sextupoles used in the system.

The second-order matrices are actually computed using the auxiliary matrix T2. Therefore, when activating second-order fitting, one must not include any element which causes an update of the R2 matrix. For a complete list of such elements see type code 6.0.

The present value of the constrained quantity, as well as the desired value, is printed in the output. In the case of transfer matrix elements this value may be checked by printing the transfer matrix itself. Certain other constrained quantities may be checked similarly. Exceptions are noted in the explanations following.

R1 matrix fitting constraints

There are five parameters to be specified when imposing a constraint upon the (i, j) element of an R1 matrix.

- 1 - Type code 10.n(specifying that a fitting constraint follows).
- 2 - Code digit (-i).
- 3 - Code digit (j).
- 4 - Desired value of the (i, j) matrix element.
- 5 - Desired accuracy of fit (standard deviation).

Note that any fitting constraint on an R1 matrix element is from the preceding update of the R1 matrix. An R1 matrix is updated only by a (6. 0. 1. ;) entry.

The symbol (n) is normally zero or blank. If n = 1, then entry 4 is taken to be a lower limit on the matrix element. If n = 2, entry 4 is taken to be an upper limit.

Some typical R1 matrix constraints are as follows:

Desired optical condition	Typical fitting constraint
<u>Point to point imaging:</u>	
Horizontal plane $R(12) = 0$	10. -1. 2. 0. .0001 'F1';
Vertical plane $R(34) = 0$	10. -3. 4. 0. .0001 'F2';
<u>Parallel to point focus:</u>	
Horizontal plane $R(11) = 0$	10. -1. 1. 0. .0001 'F3';
Vertical plane $R(33) = 0$	10. -3. 3. 0. .0001 'F4';
<u>Point to parallel transformation:</u>	
Horizontal plane $R(22) = 0$	10. -2. 2. 0. .0001 'F5';
Vertical plane $R(44) = 0$	10. -4. 4. 0. .0001 'F6';
<u>Achromatic beam:</u>	
Horizontal plane	10. -1. 6. 0. .0001 'F7';
$R(16) = R(26) = 0$	10. -2. 6. 0. .0001 'F8';
<u>Zero dispersion beam:</u>	
Horizontal plane $R(16) = 0$	10. -1. 6. 0. .0001 'F9';
<u>Simultaneous point to point and waist to waist imaging:</u>	
Horizontal plane	10. -1. 2. 0. .0001 'F10' ;
$R(12) = R(21) = 0$	10. -2. 1. 0. .0001 'F11';
Vertical plane	10. -3. 4. 0. .0001 'F12';
$R(34) = R(43) = 0$	10. -4. 3. 0. .0001 'F13';
<u>Simultaneous parallel to point and waist to waist transformation:</u>	
Horizontal plane	10. -1. 1. 0. .0001 'F14';
$R(11) = R(22) = 0$	10. -2. 2. 0. .0001 'F15';
Vertical plane	10. -3. 3. 0. .0001 'F16';
$R(33) = R(44) = 0$	10. -4. 4. 0. .0001 'F17';

R2 matrix fitting constraints

There are five parameters to be specified when imposing a constraint upon the (i, j) element of an R2 matrix.

- 1 - Type code 10.n
- 2 - Code digit -(20 + i).
- 3 - Code digit (j).
- 4 - Desired value of the (i, j) matrix element.
- 5 - Desired accuracy of fit (standard deviation).

Some typical R2 matrix constraints are as follows:

The symbol (n) is normally zero or blank. If n = 1, then entry 4 is taken to be a lower limit on the matrix element. If n = 2, entry 4 is taken to be an upper limit.

Desired optical condition	Typical fitting constraint
<u>Point to point imaging:</u>	
Horizontal plane R(12) = 0	10. -21. 2. 0. .001 'F1' ;
Vertical plane R(34) = 0	10. -23. 4. 0. .001 'F2' ;
<u>Parallel to point focus:</u>	
Horizontal plane R(11) = 0	10. -21. 1. 0. .001 'F1' ;
Vertical plane R(33) = 0	10. -23. 3. 0. .001 'F2' ;
<u>Achromatic beam:</u>	
Horizontal plane	10. -21. 6. 0. .001 'F3' ;
R(16) = R(26) = 0	10. -22. 6. 0. .001 'F4' ;

See type code 6.0 for a complete list of elements which update the R2 matrix.

σ (BEAM) matrix fitting constraints

There are five parameters to be specified when imposing a constraint upon the (i, j) element of a σ (BEAM) matrix.

- 1 - Type code 10.n
- 2 - Code digit (i). ($i \geq j$)
- 3 - Code digit (j).
- 4 - Desired value of the (i, j) matrix element.
- 5 - Desired accuracy of fit (standard deviation).

The symbol (n) is normally zero or blank. If $n = 1$, then entry 4 is taken to be a lower limit on the matrix element. If $n = 2$, entry 4 is taken to be an upper limit. If $i = j$, then the value inserted in entry 4 is the desired beam size $(\sigma(ii))^{1/2}$ e.g. $x(\max) = (\sigma(11))^{1/2}$ etc.

Some typical σ matrix constraints are as follows:

Desired optical condition	Typical fitting constraint
Horizontal waist $\sigma(21) = 0$	10. 2. 1. 0. .001 'F1' ;
Vertical waist $\sigma(43) = 0$	10. 4. 3. 0. .001 'F2' ;
Fit beam size to $x(\max) = 1$ cm	10. 1. 1. 1. .001 'F3' ;
Fit beam size to $y(\max) = 2$ cm	10. 3. 3. 2. .001 'F4' ;
Limit max beam size to $x = 2$ cm	10.2 1. 1. 2. .01 'F5' ;
Limit min beam size to $y = 1$ cm	10.1 3. 3. 1. .01 'F6' ;

In general, it will be found that achieving a satisfactory 'beam' fit with TRANSPORT is more difficult than achieving an R matrix fit. When difficulties are encountered, it is suggested that the user 'help' the program by employing sequential (step by step) fitting procedures when setting up the data for his problem. More often than not a "failure to fit" is caused by the user requesting the program to find a physically unrealizable solution. An often encountered example is a violation of Liouville's theorem.

Beam correlation matrix (r) fitting constraints

Five parameters are needed for a constraint on the (i, j) element of the beam correlation matrix.

- 1 - Type code 10.n
- 2 - Code digit (10 + i).
- 3 - Code digit (j).
- 4 - Desired value of the (i, j) matrix element.
- 5 - Desired accuracy of fit (standard deviation).

TRANSPORT does not print the beam (σ) matrix directly. Instead it prints the beam half widths and represents the off-diagonal elements by the correlation matrix. If one wishes to fit an element of this matrix to a non-zero value it is convenient to be able to constrain the matrix element directly.

Some typical r matrix constraints are as follows:

Desired optical condition	Typical fitting constraint
Horizontal waist $r(21) = 0$	10. 12. 1. 0. .001 'F1' ;
yy' correlation = $r(34) = 0.2$	10. 13. 4. 0.2 .001 'F2' ;

First moment constraint

In first order, known misalignments and centroid shifts cause the centre (centroid) of the phase ellipsoid to be shifted from the reference trajectory, i.e., they cause the beam to have a non-zero first moment. The first moments appear in a vertical array to the left of the vertical array giving the $\sqrt{\sigma(ii)}$. The units of the corresponding quantities are the same.

It is perhaps helpful to emphasize that the origin always lies on the reference trajectory. First moments refer to this origin. However, the ellipsoid is defined with respect to its centre, so the covariance matrix, as printed, defines the second moment about the mean.

First moments may be fitted. The code digits are $i = 0$ and j , where j is the index of the quantity being fit. Thus 10. 0. 1. .1 .01; constrains the horizontal (1.) displacement of the ellipsoid to be 0.1 ± 0.01 cm.

This constraint is useful in deriving the alignment tolerances of a system or in warning the system designer to offset the element in order to accommodate a centroid shift.

System length constraint

A running total of the lengths of the various elements encountered is kept by the program and may be fit. The code digits are $i = 0.$, $j = 0.$

Thus the element (10. 0. 0. 150. 5. ;) would make the length of the system prior to this element equal to 150 ± 5 metres. Presumably there would be a variable drift length somewhere in the system. By redefining the cumulative length via the (16. 6. L. ;) element, partial system lengths may be accumulated and fit.

AGS machine constraint *)

Provision has been made in the program for fitting the betatron phase shift angle μ , associated with the usual AGS treatment of magnet systems.

In the horizontal plane: use code digits $i = -11.$, $j = 2.$, and specify:

$$\Delta = \frac{1}{2\pi} \cos^{-1} \left[0.5 (R_{11} + R_{22}) \right] = \frac{\mu}{2\pi} \text{ (horiz)}$$

= freq./ (No. of periods).

In the vertical plane: $i = -13.$, $j = 4.$, and

$$\Delta = \frac{1}{2\pi} \cos^{-1} \left[0.5 (R_{33} + R_{44}) \right] = \frac{\mu}{2\pi} \text{ (vert) .}$$

For example, if there are 16 identical sectors to a proposed AGS machine and the betatron frequencies per revolution are to be 3.04 and 2.14 for the horizontal and vertical planes respectively, then the last element of the sector should be followed by the constraints:

10. -11. 2. .190 .001 ;
10. -13. 4. .134 .001 ;

i.e. $\frac{3.04}{16} = 0.190$ and $\frac{2.14}{16} = 0.134$.

For example: A typical data listing might be:

5.01 --- ;
3. --- ;
5.01 --- ;
3. --- ;

10. -11. 2. 0.190 .001 ;
10. -13. 4. 0.134 .001 ;

*) See Courant and Snyder¹⁾. Also note that this constraint is valid only when the unit cell structure and the corresponding beta functions are both periodic.

Floor coordinate fitting constraint

Five parameters are needed to specify a floor coordinate constraint:

- 1 - Type code 10.
- 2 - Code digit 8.
- 3 - Code digit (j).
- 4 - Desired value of floor coordinate.
- 5 - Desired accuracy of fit (standard deviation).

The code digit (j) indicates the floor coordinate to be constrained. Its possible values are 1 to 6 indicating the floor x, y, z, theta, phi, and psi, respectively. Theta is the angle which the floor projection of the reference trajectory makes with the floor z axis. Phi is the vertical pitch. Psi is a rotation about the reference trajectory. This is also the order in which coordinates are printed in the floor layout activated by the 13. 12. ; element. Initial coordinates are given on type codes 16. 16. ; through 16. 20. ; and type code 20.

The floor coordinates are actually zero-th rather than first order properties of a beam line. However, in TRANSPORT, they may be constrained in a first-order fitting run, and therefore are included here.

T1 matrix fitting constraints

Five parameters are needed for a constraint on the (i, j, k) element of the second-order transfer matrix T1.

- 1 - Type code 10.0
- 2 - Code digit (-i).
- 3 - Code digit (10j + k).
- 4 - Desired value of the (i, j, k) matrix element.
- 5 - Desired accuracy of the fit (standard deviation).

Note that upper and lower limit constraints are not available for second order fitting.

Some typical T1 matrix constraints are as follows:

Desired optical condition	Typical fitting constraint
Geometric aberration $T_{122} = 0$	10. -1. 22. .0 .001 'F1' ;
Chromatic aberration $T_{346} = .5$	10. -3. 46. .5 .001 'F2' ;

There must be no updates of the R2 matrix when constraining an element of the T1 matrix. There is no limit on the number of constraints which may be imposed.

If no drift lengths are varied the problem will be linear and the absolute size of the tolerances will be unimportant. Only their relative magnitude will be significant. Sometimes only a subset of the elements of the matrix T_{ijk} which give significant contributions to beam dimensions need be eliminated. In such cases one may wish to minimize the effect of this subset, by weighing each matrix element according to its importance. One does this by including a constraint for each such matrix element, and setting its tolerance equal to the inverse of the phase space factor which the matrix element multiplies. For a matrix element T_{ijk} acting on an uncorrelated initial phase space, the tolerance factor would be $1/(x_{0j}x_{0k})$, where x_{0j} and x_{0k} are the initial beam half widths specified by the type code 1.0 card.

T2 matrix fitting constraints

Five parameters are needed for a constraint on the (i, j, k) element of the second order auxiliary transfer matrix T2.

- 1 - Type code 10.0
- 2 - Code digit - (20 + i).
- 3 - Code digit (10j + k).
- 4 - Desired value of the (i, j, k) matrix element.
- 5 - Desired accuracy of the fit (standard deviation)

Note that upper and lower limit constraints are not available for second-order fitting.

Some typical T2 matrix constraints are as follows:

Desired optical condition	Typical fitting constraint
Geometric aberration $T_{122} = 0$	10. -21. 22. .0 .001 'F1' ;
Chromatic aberration $T_{346} = .5$	10. -23. 46. .5 .001 'F2' ;

By using a T2 constraint the user may fit an element of the second-order transfer matrix which pertains to any section of the beam. One causes an R2 update at the beginning of the section with a 6. 0. 2. ; element. One then places the T2 constraint at the end of the section. Any number of such constraints may be imposed. This is the only second-order constraint that may be used in conjunction with an R2 update.

If a printing of the T1 matrix is requested via a 13. 4. ; element it will be the second-order transfer matrix from the last R1 update. The comments about phase space weighting, made in connection with the T1 constraint, are equally valid for the T2 constraint, provided the phase space factors are obtained from the beam matrix at the position of the R2 update.

Second-order σ (BEAM) matrix fitting constraint

Five parameters must be specified for a constraint on the second-order contributions to a beam matrix diagonal element σ_{ii} .

- 1 - Type code 10.0
- 2 - Code digit (i).
- 3 - Code digit (i).
- 4 - The number 0.
- 5 - Desired accuracy of the fit (standard deviation).

If, for example, one wished to minimize the net contributions of second-order aberrations to the horizontal divergence, one would insert the following card:

10. 2. 2. .0 .01 ;

The quantity that is minimized is the net increase due to second-order terms in the second moment of the beam about the origin. This quantity is treated as the chi-squared of the problem, so the only meaningful desired value for the fit is zero. The square root of this quantity is printed in the output. It is computed using the R_2 matrix. Therefore, once again, one must not include any element which updates the R_2 matrix. Centroid shifts must not be inserted when doing second-order fitting, even immediately following the beam card.

The second-order image of the initial beam centroid at some later point in the beam is not necessarily the beam centroid at the later point. The parameters printed by TRANSPORT are the new centroid position and the beam matrix about the new centroid. One must therefore look at both of these to observe the effects of the fitting procedure. It may even happen that an improvement in one parameter will be accompanied by a slight deterioration in the other.

The beam profile at any point is a function of the initial beam parameters. One may therefore impose weights on the effect of the various aberrations by the choice of parameters on the BEAM card. One might, for example, adjust the strength of the correction of the chromatic aberrations by the choice of the $\Delta p/p$ parameter. In particular, when using a BEAM constraint, one should not attempt to minimize or eliminate chromatic aberrations if $\Delta p/p$ is set equal to zero on the beam card (type code 1.0).

Correlations (the 12.0 card) may also be included in the initial beam specification.

Sextupole strength constraints

Five parameters must be specified for a constraint on sextupole strength.

- 1 - Type code 10.0
- 2 - Code digit 18.
- 3 - Code digit 0.
- 4 - The number 0.
- 5 - Desired maximum sextupole field strength.

A single sextupole constraint card applies to all sextupoles which follow. The maximum field strength is treated as a standard deviation and may be exceeded on an optimal fit.

One can employ this constraint to find the optimal locations for sextupoles. By placing inversely coupled drift lengths before and after the sextupole its longitudinal position may be varied. By constraining the field strength the sextupole can be slid to a position where the coupling coefficients to the aberrations will be largest. One will need to experiment with adjusting the maximum field strength to achieve the best configuration.

Internal constraints

A set of upper and lower bounds on the value of each type of parameter is in the memory of the program. If a correction is computed for a parameter which would take its value outside this range, it is reset to the limit of the range. The current limits are:

Type code	Limits
1.0	0 < input beam
2.0	-60 < pole-face rotation < 60 (deg)
3.0	0 < drift
4.0	0 < magnet length
5.0	0 < quad length
20.0	-360 < beam rotation < 360 (deg).

These limits apply only when a parameter is being varied. Fixed values that exceed this range may be used as desired.

These constraints were included to avoid physically meaningless solutions.

Corrections and covariance matrix

When the program is fitting, it makes a series of runs through the beam line. From each run it calculates the chi-squared and the corrections to be made to the varied parameters. For each iteration a single line is printed containing these quantities.

The program calculates the corrections to be made using a matrix inversion procedure. However, because some problems are difficult, it proceeds with caution. The corrections actually made are sometimes a fixed fraction of those calculated. This fraction, used as a scaling factor, is the first item appearing on the line of printed output. The second factor is the chi-squared before the calculated corrections are made. Following are the corrections to be made to the varied parameters. They are in the order in which they appear in the beam line. If several parameters are coupled, they are considered as one and their position is determined by the first to appear.

When convergence has occurred, the final value of the chi-squared and the covariance matrix are printed. The covariance matrix is symmetric, so only a triangular matrix is shown. The diagonal elements give the change in each varied parameter needed to produce a unit increase in the chi-squared. The off-diagonal elements give the correlations between the varied parameters.

The appearance of the chi-squared and covariance matrix is:

*COVARIANCE (FIT χ^2)

$$\begin{array}{ccccccc} \sqrt{C_{11}} & & & & & & \\ r_{12} & \sqrt{C_{22}} & & & & & \\ \cdot & & \cdot & & & & \\ \cdot & & & \cdot & & & \\ \cdot & & & & \cdot & & \\ r_{1n} & \cdot & \cdot & \cdot & r_{n,n-1} & \sqrt{C_{nn}} & \end{array}$$

For more details on the mathematics of the fitting, the user should consult the Appendix. For an example of the output of the program he (or she) should refer to the section on output format.

ACCELERATION: Type code 11.0

An energy gain is reflected in both the divergence and the width of the beam. This element provides a simulation of a travelling wave linear accelerator energy gain over a field free drift length (i.e. no externally applied magnetic field).

There are five parameters:

- 1 - Type code 11.0
- 2 - Accelerator length (metres).
- 3 - Energy gain (GeV).
- 4 - ϕ (phase lag in degrees).
- 5 - λ (wavelength in cm).

The new beam energy is printed as output.

The energy of the reference trajectory is assumed to increase linearly over the entire accelerator length. If this is not the case, an appropriate model may be constructed by combining separate 11.0 elements. An 11.0 element with a zero energy gain is identical to a drift length.

None of the parameters may be varied.

Second-order matrix elements have not been incorporated in the program for the accelerator section.

The units of parameters 2, 3, and 5 are changed by 15. 8., 15. 11., and 15. 5. type code entries respectively.

Accelerator section matrix

$$\begin{array}{ccccccc}
 1 & \left[L \frac{E_0}{\Delta E \cos \phi} \ln \left(1 + \frac{\Delta E \cos \phi}{E_0} \right) \right] & 0 & 0 & 0 & 0 & 0 \\
 0 & \frac{E_0}{E_0 + \Delta E \cos \phi} & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & \left[L \frac{E_0}{\Delta E \cos \phi} \ln \left(1 + \frac{\Delta E \cos \phi}{E_0} \right) \right] & 0 & 0 & 0 \\
 0 & 0 & 0 & \frac{E_0}{E_0 + \Delta E \cos \phi} & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & \left(\frac{\Delta E \sin \phi}{E_0 + \Delta E \cos \phi} \right) \left(\frac{2\pi}{\lambda} \right) & \frac{E_0}{E_0 + \Delta E \cos \phi}
 \end{array}$$

Definitions: L = effective length of accelerator sector

E_0 = particle energy at start of sector

ΔE = energy gain over sector length

ϕ = phase lag of the reference particle behind the crest of the accelerating wave, i.e. if ϕ is positive then for some $\ell > 0$ the particles having this value are riding the crest of the wave; the units of ϕ are degrees

λ = wavelength of accelerating wave; the units of λ are those of ℓ (normally cm).

This matrix element assumes that $E_0 \gg m_0 c^2$ (fully relativistic).

BEAM (rotated ellipse): Type code 12.0

To allow the output beam from some point in a system to become the input beam of some succeeding system, provision has been made for re-entering the correlation matrix which appears as a triangular matrix in the beam output. (See section under type code 1.0 and/or the Appendix for definitions.)

There are 16 parameters:

- 1 - Type code 12.0
- 2 to 16 - The 15 correlations ($r(ij)$) among the 6 beam components - in the order printed (by rows).

Several cards may be used to insert the 15 correlations, if necessary.

Since this element is solely an extension of the beam input, a 12.0 element must immediately be preceded by a 1.0 (BEAM) element entry.

The effect of this element in the printed output is shown only in the beam matrix. If the beam matrix is printed automatically, it is not printed directly after the BEAM element but only after the correlation matrix has been inserted.

Output PRINT CONTROL instructions: Type code 13.0

A number of control codes which transmit output print instructions to the program have been consolidated into a single type code:

There are two parameters:

1 - Type code 13.0

2 - Code number.

The effects of the various code numbers will be described below (not in numerical order).

Several codes are available to control various aspects of the printed output. Most type codes produce a line of output that advertises their existence. Those that do not, usually have an obvious effect upon the remainder of the output and thus make their presence clear.

Beam matrix print controls 1., 2., 3.

(13. 1. ;): The current beam (σ) matrix is printed by this code.

(13. 3. ;): The beam (σ) matrix will be printed after every physical element which follows this code.

(13. 2. ;): The effect of a previous (13. 3. ;) code is cancelled and the beam (σ) matrix is printed only when a (13. 1. ;) code is encountered or when another (13. 3. ;) code is inserted. The suppression of the beam matrix is the normal default.

Transformation matrix print controls 4., 5., 6., 24.

(13. 4. ;): The current transformation matrix R1 (TRANSFORM 1) is printed by this code. If the program is computing a second-order matrix, this second-order transformation matrix will be included in the print-out. This matrix is cumulative from the last R1 (TRANSFORM 1) update. The units of the elements of the printed matrix are consistent with the input units associated with the type code 1.0 (BEAM) entry.

(13. 6. ;): The transformation matrix R1 will be printed after every physical element which follows this code. The second-order matrix will be printed automatically only if the one-line form (code 13. 19. ;) of the transformation is selected. The second-order matrix will, however, be printed at each location of a (13. 4. ;) element. The first-order matrix will not be repeated.

(13. 5. ;) : The automatic printing of R1 will be suppressed and R1 will be printed only when subsequently requested.

(13. 24. ;) : The TRANSFORM 2 matrix, R2, will be printed by this code. The format and units of R2 are identical with those of R1, which is printed by the (13. 4. ;) code. For a list of elements which update the R2 matrix, see type code 6.

The units of the tabulated matrix elements in either the first-order R or sigma matrix or second-order T matrix of a TRANSPORT print-out will correspond to the units chosen for the BEAM card. For example, the $R(12) = (x/\theta)$ matrix element will normally have the dimensions of cm/mr; and the $T(236) = (\theta/y\delta)$ matrix element will have the dimensions mr/(cm-percent $\Delta p/p$) and so forth.

Misalignment table print control 8.

The misalignment summary table is printed wherever a (13. 8. ;) element is inserted. Its contents are the effects of all previously specified misalignments whose results were to be stored in a table. A full description of the table and its contents is to be found in the section on the align element (type code 8.).

Coordinate layout control 12.

One can produce a layout of a beam line in any Cartesian coordinate system one chooses. The coordinates printed represent the x, y, and z position, and the angles theta, phi, and psi, respectively, of the reference trajectory at the interface between two elements. Theta is the angle which the floor projection of the reference trajectory makes with the floor z axis. Phi is the vertical pitch. Psi is a rotation about the reference trajectory. In the printed output the values given are those at the exit of the element listed above and at the entrance of the element listed immediately below.

A request for a layout is specified by placing a (13. 12. ;) card before the beam card. If no additional cards are inserted the reference trajectory of the beam line will be assumed to start at the origin and proceed along the positive z-axis. The y-axis will point up and the x-axis to the left. One can also specify other starting coordinates and orientations by placing certain other cards before the beam card. For a description of such cards see type code 16.0 (special parameters).

The calculation of the coordinates is done from the parameters of the physical elements as given in the data. Therefore, if effective lengths are given for magnetic elements, the coordinates printed will be those at the effective field boundary. The effects of fringing fields in bending magnets are not taken into account.

General output format controls 17., 18., 19.

(13. 17. ;): The subsequent printing of the physical parameters of all physical elements will be suppressed. Only the type code and the label will remain. This element is useful in conjunction with the (13. 19. ;) element which restricts the beam (σ) matrix and the transformation (R) matrix each to a single row. The elements of these matrices then appear in uninterrupted columns in the output, similar to the TRAMP computer code used at the Rutherford Lab, CERN, and elsewhere.

(13. 18. ;): Only varied elements and constraints will be printed. This element, in conjunction with the various options on the indicator card, can produce a very abbreviated output. The entire output of a multistep problem can now easily be printed on a teletype or other terminal.

(13. 19. ;): The beam (σ) and transformation (R1 or R2) matrices, when printed, will occupy a single line. Only those elements are printed which will be non-zero if horizontal midplane symmetry is maintained. The second-order transformation matrix will obviously occupy several lines. This element, in conjunction with the 13. 17. ; element and either the 13. 3. ; element or the 13. 6. ; element, will produce output in which the printed matrix elements will occupy single uninterrupted columns. For visual appearances it is recommended that, if both beam (σ) and transformation matrices are desired, they be printed in separate steps of a given problem.

Punched output controls 29., 30., 31., 32., 33., 34., 35., 36.

If the control is equal to 29, all of the terms in the first-order matrix and the x and y terms of the second-order matrix are punched.

If the control is equal to 30, all of the terms of the first-order matrix and all second-order matrix elements are punched out.

If the control, n , is greater than 30, all of the first-order terms are punched and the second-order matrix elements which correspond to $(n-30.)$, i.e. if $n = 32$, the second-order theta matrix elements are punched out. If $n = 31$, the second-order x matrix elements are punched, and so forth.

ARBITRARY TRANSFORMATION input: Type code 14.0

To allow for the use of empirically determined fringing fields and other specific (perhaps non-phase-space-conserving) transformations, provision has been made for reading in an arbitrary transformation matrix. The first-order 6×6 matrix is read in row by row.

There are eight parameters for each row of a first-order matrix entry:

- 1 - Type code 14.0
- 2 to 7 - The six numbers comprising the row. The units must be those used to print the transfer matrix; in other words, consistent with the BEAM input/output.
- 8 - Row number (1. to 6.)

A complete matrix must be read and applied one row at a time. Rows that do not differ from the unit transformation need not be read.

For example, (14. -.1 .9 0. 0. 0. 0. 2. ;) introduces a transformation matrix whose second row is given but which is otherwise a unit matrix. Note that this transformation does not conserve phase space because $R(22) = 0.9$, i.e. the determinant of $R \neq 1$.

Any of the components of a row may be varied; however, there are several restrictions.

Type code 14.0 elements that immediately follow one another will all be used to form a single transformation matrix. If distinct matrices are desired, another element must be inserted to separate the type code 14.0 cards. Several do-nothing elements are available; for example, a zero length drift (3. 0. ;) is a convenient one.

When the last of a sequence of type code 14.0 cards is read, the assembled transformation matrix will be printed in the output.

Note that

$$\begin{pmatrix} 1 & 0 \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Hence, a matrix formed by successive 14. (3. 0. ;), 14. - elements is not always equal to the one formed by leaving out the (3. 0. ;) element.

If components of a 14.0 card are to be varied it must be the last 14.0 card in its matrix. This will force a matrix to be split into factors if more than one row has variable components.

If it is desired to read in the second-order matrix coefficients for the i^{th} row, then the following 22 additional numbers may be read in ^{**}).

9 - continuation code 0.

10 to 30 - the 21 coefficients:

T(i11) T(i12) T(i13) T(i14) T(i15) T(i16)
T(i22) T(i23) T(i24) T(i25) T(i26) T(i33)
T(i34) T(i35) T(i36) T(i44) T(i45) T(i46)
T(i55) T(i56) T(i66)

in that order, where i is the row number. It is necessary to read in the first-order matrix row which corresponds to the second-order matrix row being read in.

As in the first-order case, full rows not different from the identity matrix [i.e., $R(ii) = 1$, all other $R(ij) = 0$, and all $T(ijk) = 0$] need not be read in.

^{**}) This feature frees the user from making repetitive, expensive, second-order runs through a fixed portion of his system while experimenting with other magnets. This is done by reading the full matrix of this portion (obtained from a previous run) back into the machine as a single "arbitrary matrix."

Input-output UNITS: Type code 15.0

TRANSPORT is designed with a standard set of units that have been used throughout this manual. However, to accommodate other units conveniently, provision has been made for redefining the units to be employed. This is accomplished by insertion of one or more of the following elements.

There are four parameters to be specified:

- 1 - Type code 15.0
- 2 - Code digit.
- 3 - The abbreviation of the unit (see examples below).

This will be printed on the output listing. It must be enclosed in single quotes and is a maximum of three characters long (four for energy). The format for insertion is the same as for labels.

- 4 - The scale factor (if needed).

The scale factor is the size of the new unit relative to the standard TRANSPORT unit. For example, if the new unit is inches and the standard TRANSPORT unit cm, the scale factor is (2.54).

The various units that may be changed are:

Code Digit	Quantity	Standard TRANSPORT Unit	Symbols used in SLAC-75
1.0	horizontal and vertical transverse dimensions, magnet apertures and misalignment displacements.	cm	x,y
2.0	horizontal and vertical angles and misalignment rotation angles	mr	θ, ϕ
3.0	vertical beam extent (only) ^{*)} and bending magnet gap height	cm	y
4.0	vertical beam divergence ^{*)} (only)	mr	ϕ
5.0	pulsed beam length and wave length in accelerator	cm	λ
6.0	momentum spread	percent (PC)	δ
7.0	bend, pole face rotation, and coordinate layout angles	degrees (DEG)	
8.0	length (longitudinal) of elements, layout coordinates and bending magnet pole face curvatures	metres (M)	t
9.0	magnetic fields	kG	B
10.0	mass	electron mass	m
11.0	momentum and energy gain in accelerator section	GeV/c GeV	p(0) ΔE

*) These codes should not be used if the coordinate rotation (20.0) type code is used anywhere in the system.

Units are not normally restored at the end of a problem step. Once changed, they remain the same for all succeeding problem steps in an input deck until a 0 indicator card is encountered, at which time they are reset to standard TRANSPORT units. The units may be reset to standard units by inserting a (15. ;) type code entry.

The 15.0 elements are the first cards in a deck (immediately following the title card and the 0 or 1 indicator card) and should not be inserted in any other location. They produce no printed output during the calculation, their effect being visible only in the output from other elements.

Example: To change length to feet, width to inches, and momentum to MeV/c, add to the front of the deck the elements

```
15.  8.  ' FT'  0.3048;  
15.  1.  ' IN'  2.54;  
15. 11.  'MEV'  0.001;
```

The scale factor, 0.3048, multiplies a length expressed in the new unit, feet, to convert it to the reference unit, metres, etc.

For the conventional units listed below, it is sufficient to stop with the unit name (the conversion factor is automatically inserted by the program). If units other than those listed below are desired, then the unit name and the appropriate conversion factor must be included. If the automatic feature is used with older versions of the program, there must be no blank spaces between the quotes and the unit name.

Input-output units: Type code 15.0
 (Conversion factors for dimension changes versus code digit and label)

LABEL	CODE DIGIT										
	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.
'CM'	1.	---	1.	---	1.	---	---	.01	---	---	---
'M'	100.	---	100.	---	100.	---	---	1.	---	---	---
'IN'	2.54	---	2.54	---	2.54	---	---	.0254	---	---	---
'FT'	30.48	---	30.48	---	30.48	---	---	.3048	---	---	---
'MM'	.1	---	.1	---	.1	---	---	.001	---	---	---
'R'	---	1000.	---	1000.	---	---	---	---	---	---	---
'MR'	---	1.	---	1.	---	---	---	---	---	---	---
'PC'	---	---	---	---	---	1.	---	---	---	---	---
'P/10'	---	---	---	---	---	.1	---	---	---	---	---
'N'	---	---	---	---	---	100.	---	---	---	---	---
'MEV'	---	---	---	---	---	---	---	---	---	---	.001
'GEV'	---	---	---	---	---	---	---	---	---	---	1.
'KG'	---	---	---	---	---	---	---	---	1.	---	---
'G'	---	---	---	---	---	---	---	---	.001	---	---

PC is an abbreviation for percent
 P/10 means one-tenth of a percent
 N means 100 percent

SPECIAL INPUT PARAMETERS: Type code 16.0

A number of constants are used by the program which do not appear as parameters in elements of any other type code. A special element has been provided to allow the designer to set their values. These special parameter entries must always precede the physical element(s) to which they apply. Once introduced, they apply to all succeeding elements in the beam line unless reset to zero or to new values.

There are three parameters:

- 1 - Type code 16.0
- 2 - Code digit.
- 3 - Value of the constant.

A number of such constants have been defined in this manner. All have a normal value that is initialized at the beginning of each run.

Code digits for special parameters

- 1. $\epsilon(1)$ - a second-order measure of magnetic field inhomogeneity in bending magnets. If

$$B(x) = B(0) \left[1 - n \left(\frac{x}{\rho_0} \right) + \beta \left(\frac{x}{\rho_0} \right)^2 - \dots \right]$$

is the field expansion in the median ($y = 0$) plane, then

$\epsilon(1)$ is defined as

$$\epsilon(1) = \beta \left(\frac{1}{\rho_0} \right)^2$$

(where ρ_0 is measured in units of horizontal beam width - normally cm). This parameter affects second-order calculations only. Normally the value is 0. It may be varied in second-order fitting.

- 3. (M/m) - Mass of the particles comprising the beam, in units of the electron mass; normally 0. A non-zero mass introduces the dependence of pulse length on velocity, an important effect in low-energy pulsed beams.
- 4. W/2 - Horizontal half-aperture of bending magnet, in the same units as horizontal beam width, normally 0 (i.e. effect of horizontal half aperture is ignored).

5. $g/2$ - Vertical half-aperture of bending magnet, in the same units as vertical beam height; this parameter must be inserted if the effect of the spatial extent of the fringing fields upon transverse focusing is to be taken into account. (See type codes 2.0 and 4.0 as a cross reference) normally 0.
6. L - Cumulative length of system, in the same units as system length. It is set to zero initially, then increased by the length of each element, and finally printed at the end of the system. This element allows the cumulative length to be reset as desired.
7. K_1 - An integral related to the extent of the fringing field of a bending magnet. See section under type code 2.0 and SLAC-75 page 74 for further explanation.
If the (16. 5. $g/2$. ;) element has been inserted, the program inserts a default value of $K_1 = \frac{1}{2}$ unless a (16. 7. K_1 . ;) element is introduced, in which case the program uses the K_1 value selected by the user. The table below shows typical values for various types of magnet designs.
8. K_2 - A second integral related to the extent of the fringing field. Default value of $K_2 = 0$ unless specified by a (16. 8. K_2 . ;) entry.

Typical values of K_1 and K_2 are given below for four types of fringing field boundaries:

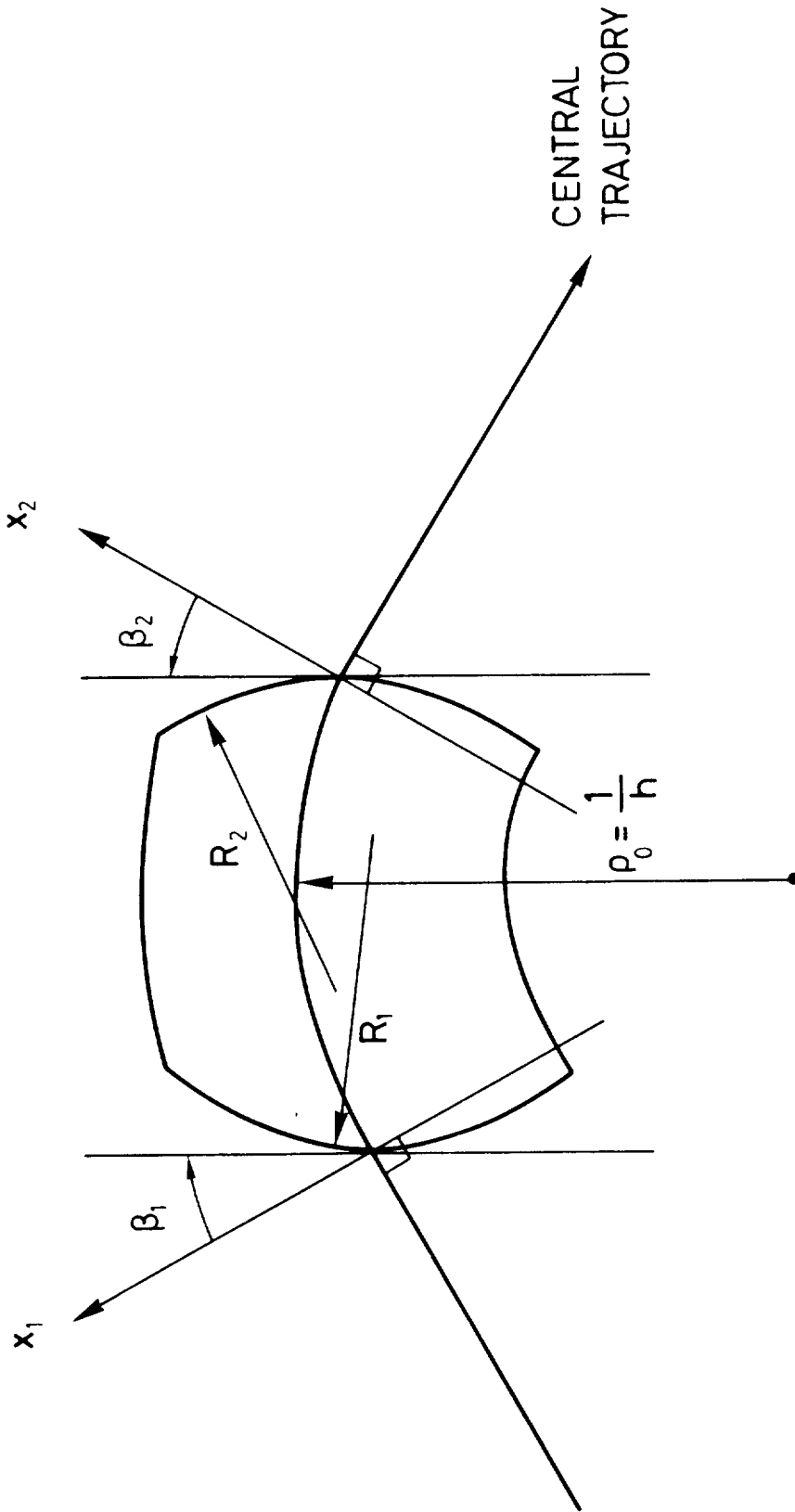
- a) a linear drop-off of the field,
- b) a clamped "Rogowski" fringing field,
- c) an unclamped "Rogowski" fringing field,
- d) a "square-edged" non-saturating magnet

Model	K_1	K_2^*
Linear	1/6	3.8
Clamped Rogowski	0.4	4.4
Unclamped Rogowski	0.7	4.4
Square-edged magnet	0.45	2.8

12. $1/R_1$ - Where R_1 is the radius of curvature (in units of longitudinal length, normally metres) of the entrance face of bending magnets. (See figure on p.
13. $1/R_2$ - Where R_2 is the radius of curvature (in units of longitudinal length, normally metres) of the exit face of bending magnets. (See figure on p.

The pole face curvatures ($1/R_1$) and ($1/R_2$) affect the system only in second-order, creating an effective sextupole component in the neighbourhood of the magnet. If the parameters are not specified, they are assumed to be zero, i.e. no curvature and hence no sextupole component. Either parameter (or both) may be varied in second-order fitting.

*) For most applications K_2 is unimportant. If you find it is important to your result you should probably be making a more accurate calculation with a ray-tracing program (see References at the end of the manual.)



FIELD BOUNDARIES FOR BENDING MAGNETS

The TRANSPORT sign conventions for x , β , R and h are all positive as shown in the figure. The positive y direction is out of the paper. Positive β 's imply transverse focusing. Positive R 's (convex curvatures) represent negative sextupole components of strength $S = (-h/2R) \sec^3 \beta$. (See SLAC-75, page 71.)

Tilt-to-focal plane (16. 15. α . ;) element

Very often it is desired to have a listing of the second-order aberrations along the focal plane of a system rather than perpendicular to the optic axis, i.e. along the x coordinate. If the focal plane makes an angle α with respect to the x axis (measured clockwise) then provision has been made to rotate to this focal plane and print out the second-order aberrations. This is achieved by the following procedures:

Alpha is the focal-plane tilt angle (in degrees) measured from the perpendicular to the optic axis (α is normally zero).

The programming procedure for a tilt in the x(bend)-plane (rotation about y axis) is:

```
16. 15.  $\alpha$ . ;  
   3.  0. ; (a necessary do-nothing element)  
13.  4. ;  
16. 15.  $-\alpha$ . ; (rotate back to zero)  
   3.  0. ; (a necessary do-nothing element)  
16. 15. 0. ; (to turn off rotation element)
```

The programming procedure for a tilt in the y-plane (rotation about x-axis) is:

```
16. 15.  $\alpha$ . ;  
20. 90. ;  
3. 0. ;  
20. -90. ;  
13. 4. ;  
16. 15.  $-\alpha$ . ; (rotate back to zero)  
3. 0. ;  
16. 15. 0. ; (to turn off rotation element)
```

Initial beam line coordinates and direction

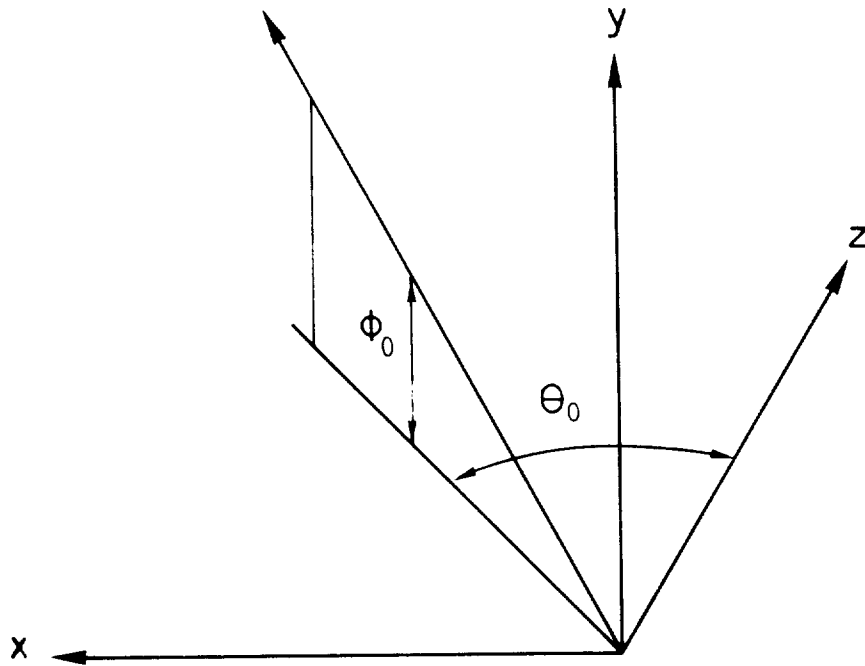
When requesting a beam line coordinate layout via a (13. 12. ;) element one can employ any coordinate system one desires. The position and direction of the beginning of the reference trajectory in this coordinate system are given on elements 16. 16. through 16. 20. Such cards should be placed before the beam card, but after any units changes. Their meanings are as follows:

16.	16.	}	$x_0, y_0,$ and $z_0,$ respectively, the coordinates of
16.	17.		the initial point of the reference trajectory
16.	18.		in the units chosen for longitudinal length.
16.	19.	}	θ_0 and ϕ_0 the initial horizontal and vertical angles
16.	20.		of the reference trajectory in degrees.

When specifying the initial orientation of the reference trajectory via the two angles, one must give the horizontal angle first. The meaning of the two angles is given in the following figure. Any of the above five parameters not explicitly specified will be taken to equal zero.

The initial coordinates may be varied in first-order fitting. Their values will affect only the beam line floor coordinates and not any beam or transfer matrix element.

REFERENCE
TRAJECTORY



SPECIFICATION OF INITIAL ANGLES θ_0 AND ϕ_0 FOR BEAM LINE LAYOUT.

SECOND-ORDER CALCULATION: Type code 17.0

A second-order calculation may be obtained provided no alignments are employed. A special element instructs the program to calculate the second-order matrix elements. It must be inserted immediately following the beam (1. element).

Only one parameter should be specified:

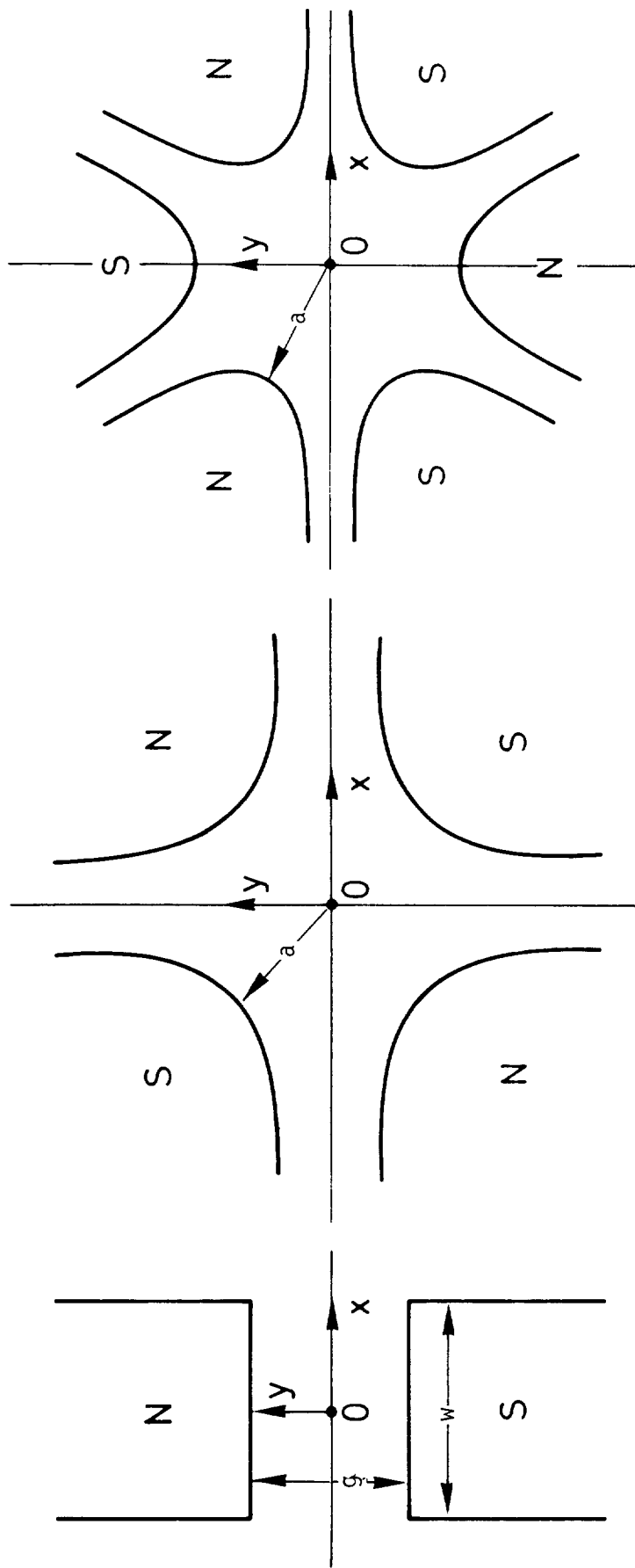
1 - Type code 17.0 (signifying a second-order calculation is to be made).

To print out the second order T1 matrix terms at a given location in the system, the (13. 4. ;) print control card is used. For T2, the (13. 24. ;) print control card is used. The update rules are the same as those for the corresponding first-order R matrix. See SLAC-75 for definitions of subscripts in the second order T(ijk) matrix elements.

The values of the BEAM (sigma) matrix components may be perturbed from their first-order value by the second-order aberrations. In a second-order TRANSPORT calculation, the initial beam is assumed to have a Gaussian distribution. For exact details the reader should consult the Appendix. For the beam matrix to be calculated correctly, there should be no elements which update the R2 matrix. If a centroid shift is present, it must immediately follow the beam (type code 1.0) or beam rotated ellipse (type code 12.0) card.

Only second-order fitting may be done in a second-order run. See the section on type code 10.0 for a list of quantities that may be constrained in a second-order run. If a beam constraint is to be imposed in second-order, there must be no centroid shifts present anywhere.

Second-order matrices are included in the program for quadrupoles, bending magnets (including fringing fields), the arbitrary matrix, sextupoles, and solenoids. They have not been calculated for the acceleration (type code 11.0) element.



SEXTUPOLE

QUADRUPOLE

DIPOLE

ILLUSTRATION OF THE MAGNETIC MIDPLANE (x AXIS) FOR DIPOLE, QUADRUPOLE AND SEXTUPOLE ELEMENTS. THE MAGNET POLARITIES INDICATE MULTIPOLE ELEMENTS THAT ARE POSITIVE WITH RESPECT TO EACH OTHER.

SOLENOID: Type code 19.0

The solenoid is most often used as a focusing element in systems passing low-energy particles. Particles in a solenoidal field travel along helical trajectories. The solenoid fringing field effects necessary to produce the focusing are included.

There are three parameters:

- 1 - Type code 19.0
- 2 - Effective length of the solenoid (metres).
- 3 - The field (kG). A positive field, by convention, points in the direction of positive z for positively charged particles.

The length and the field may be varied in first-order fitting. Both first- and second-order matrix calculations are available for the solenoid.

A typical input format is:

19. L. B. ' ' ;

└── Label if desired (not to exceed
4 spaces)

First-order solenoid matrix

Solenoid R matrix

Definitions: L = effective length of solenoid

$K = B(0)/(2B\rho_0)$, where B(0) is the field inside the solenoid and $(B\rho_0)$ is the (momentum) of the central trajectory.

$$C = \cos KL$$

$$S = \sin KL$$

For a derivation of this tranformation see report SLAC-4 by R. Helm.

Alternate forms of matrix representation of the solenoid:

$$R(\text{Solenoid}) = \begin{pmatrix} C^2 & \frac{1}{K}SC & SC & \frac{1}{K}S^2 & 0 & 0 \\ -KSC & C^2 & -KS^2 & SC & 0 & 0 \\ -SC & -\frac{1}{K}S^2 & C^2 & \frac{1}{K}SC & 0 & 0 \\ KS^2 & -SC & -KSC & C^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Rotating the transverse coordinates about the z axis by an angle
= -KL decouples the x and y first-order terms, i.e.

$$R(-KL) \cdot R(\text{Solenoid}) = \begin{pmatrix} C & \frac{1}{K}S & 0 & 0 & 0 & 0 \\ -KS & C & 0 & 0 & 0 & 0 \\ 0 & 0 & C & \frac{1}{K}S & 0 & 0 \\ 0 & 0 & -KS & C & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

COORDINATE ROTATION: Type code 20.0

The transverse coordinates x and y may be rotated through an angle α about the z axis (the axis tangent to the central trajectory at the point in question)^{**}). Thus a rotated bending magnet, quadrupole, or sextupole may be inserted into a beam transport system by preceding and following the element with the appropriate coordinate rotation. (See examples below.) The positive sense of rotation is clockwise about the positive z axis.

There are two parameters to be specified for a coordinate rotation:

- 1 - Type code 20.0 (signifying a beam coordinate rotation).
- 2 - The angle of rotation α (degrees).

The angle of rotation may be varied in a first-order fitting (see type code 10.0).

Note

This transformation assumes that the units of (x and y) and (θ and ϕ) are the same. This is always true unless a 15.0 3.0 or a 15.0 4.0 type code has been used.

⁴⁾ **) See SLAC-75, page 45 and Fig. 4, page 12 for definitions of x , y , and z coordinates.

Examples:

For a bending magnet, the beam rotation matrix may be used to specify a rotated magnet.

Example No. 1

A bend up is represented by rotating the x, y coordinates by -90.0 degrees as follows:

Labels (not to exceed 4 spaces) if desired

```
20. -90. ' ' ;
2.  β(1). ' ' ;
4.  L. B. n. ' ' ;
2.  β(2). ' ' ;
20. +90. ' ' ; (returns coordinates to their initial
                orientation)
```

A bend down is accomplished via a +90 degree rotation.

```
20. +90. ' ' ;
2.
4.
2.
20. -90. ' ' ;
```

A bend to the left (looking in the direction of beam travel) is accomplished by rotating the x, y coordinates by 180 degrees, e.g.

```
20. 180. ' ' ;
2.
4.
2.
20. -180. ' ' ;
```

Example No. 2

A quadrupole rotated clockwise by 60 degrees about the positive z axis would be specified as follows:

20. 60. ' ' ;
 5. L. B. a. ' ' ;
 20. -60. ' ' ;

Beam rotation matrix

$$R = \begin{pmatrix} C & & S & & & & \\ & C & & S & & & \\ -S & & C & & & & \\ & -S & & C & & & \\ & & & & 1 & & \\ & & & & & 1 & \end{pmatrix}$$

where $C = \cos \alpha$,
 $S = \sin \alpha$,
 $\alpha =$ angle of coordinate rotation about the beam axis,
 blank spaces are zeros.

e.g. for $\alpha = +90$ degrees, this matrix interchanges rows 1 and 2 with 3 and 4 of the accumulated R matrix as follows:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} R(11) & R(12) & R(13) & R(14) \\ R(21) & R(22) & R(23) & R(24) \\ R(31) & R(32) & R(33) & R(34) \\ R(41) & R(42) & R(43) & R(44) \end{bmatrix} \begin{bmatrix} R(31) & R(32) & R(33) & R(34) \\ R(41) & R(42) & R(43) & R(44) \\ -R(11) & -R(12) & -R(13) & -R(14) \\ -R(21) & -R(22) & -R(23) & -R(24) \end{bmatrix}$$

(The rest of the matrix is unchanged.)

STRAY MAGNETIC FIELD: Type code 21.0

1 - Element No. 21.0

2 - Code No. n. n = 4: horizontal deflection

n = 2: vertical deflection.

3 - $\langle \overline{BL} \rangle$ mean value of $\int Bdz$.

4 - $\pm \langle \sigma BL \rangle$ +: Gaussian random number generator;
affects beam first moment.

-: uncertainty in $\int Bdz$ - affects beam
second moment.

Uses the misalignment element (8.) to calculate an angular deflection

equal to $\int \frac{Bdz}{(B\rho)}$

This type code is not functioning in the present version of the
program.

SENTINEL

Each step of every problem in a TRANSPORT data set must be terminated with the word SENTINEL. The word SENTINEL need not be on a separate card. For a description of the form of a TRANSPORT data set see the section on input format.

An entire run, consisting of one or several problems, is indicated by an additional card containing the word SENTINEL. Thus, at the end of the entire data set the word SENTINEL will appear twice.

Acknowledgements

R. Helm's suggestions and criticisms at SLAC have been invaluable throughout the development of the program and the underlying theory. R. Pordes has ably assisted D. Carey at FNAL during the more recent developments of the program.

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7. Suggested Ray-Tracing Programs to supplement TRANSPORT:

David C. Carey, "TURTLE (Trace Unlimited Rays Through Lumped Elements)", Fermilab Report No. NAL-64 (1971). This is a computer program using TRANSPORT notation and designed to be run using the same data cards as for a previous TRANSPORT run.

K.L. Brown and Ch. Iselin, "DECAY TURTLE (Trace Unlimited Rays Through Lumped Elements)", CERN Report 74-2 (1974). This is an extension of TURTLE to include particle decay calculations.

H. Enge and S. Kowalski have developed a Ray-Tracing program using essentially the same terminology as TRANSPORT. Any experienced user of TRANSPORT should find it easy to adapt to the M.I.T. program.
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TRANSPORT APPENDIX

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December 1977

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APPENDIX

Introduction

This appendix has been included as an addition to the manual in an attempt to better acquaint the user with 'what TRANSPORT does', and with the notation and mathematical formalism used in a TRANSPORT calculation.

The first section (Beam Transport Optics - Part I and Part II) is a rewrite of two lectures given to members of the SLAC technical staff on the elementary matrix algebra of optics. We include them here for the benefit of the new user who may need a brief refresher course on charged particle optics and/or has a need to become familiar with TRANSPORT notation. The new user should also acquaint himself with the contents of the books and other publications listed under 'references' at the end of the manual. References 1 and 2 are essential if the user is to obtain the maximum value from TRANSPORT.

The second section of this appendix was written to introduce the mathematical formalism of the first-order R matrix and Sigma matrix (phase ellipsoid) beam optics used in a TRANSPORT calculation and to correlate this with the printed output.

Section three discusses second-order calculations and, in particular, a procedure for calculating the "Sextupole" strengths required to minimize and/or eliminate second-order aberrations in a beam transport system.

Section four is a brief derivation of the mathematical formalism used by TRANSPORT for calculating magnet alignment tolerances.

Section five deals with the first-order parameter optimization code of TRANSPORT and includes a brief explanation of the covariance matrix that is printed after each first-order fit routine.

BEAM TRANSPORT OPTICS

Section I

Beam Transport Optics - Part I

(K. L. Brown)

1. Introduction

A convenient starting point for this lecture is the equation relating the magnetic rigidity of a particle ($B\rho$) to the particle momentum P .

$$B\rho = \frac{10^2}{2.99793} P \quad \text{or} \quad B\rho = 33.356 P$$

where

B is in kilogauss

ρ is the bending radius in meters

P is the particle's momentum in BeV/c.

A note of caution: When using this equation for a TRANSPORT calculation, it is necessary to use at least 5 significant figures for the constant to avoid round-off errors in the readout.

2. Geometric Light Optics vs. Magnetic Optics

To relate geometrical light optics to charged particle optics, we begin with the thin lens. Figure 1 shows a thin lens with a ray leaving a focal point A at an angle θ_0 , impinging on the lens at x_0 . As the ray leaves the lens, it is at x_1 and going toward a focal point B at an angle of θ_1 .

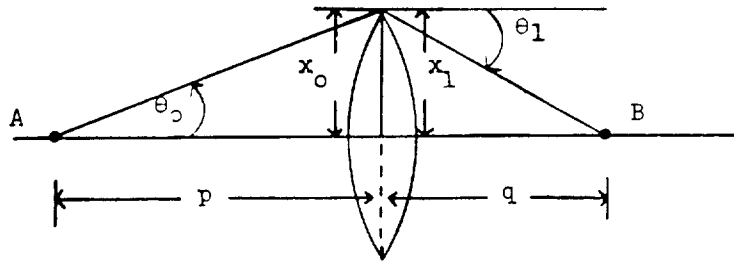


Figure 1

Thin lens optics says that $1/p + 1/q = 1/f$. Using this equation it is readily verified that the matrix transformation for the lens action between principal planes is

$$\begin{bmatrix} x_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ \theta_0 \end{bmatrix}$$

The transformation for a drift distance L is

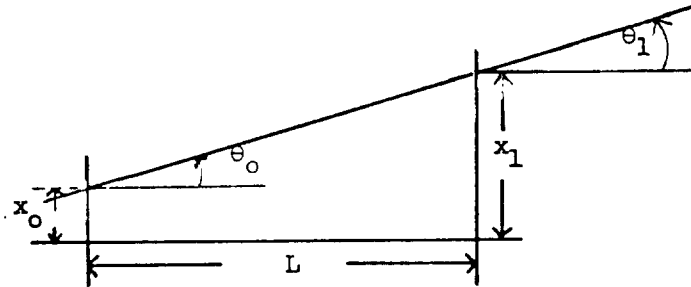


Figure 2

$$\begin{bmatrix} x_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ \theta_0 \end{bmatrix}$$

Note that the determinant of the matrix in both examples is equal to unity. This is always the case as will be proved formally later. That this is so is a manifestation of Liouville's theorem of conservation of phase space area.

Consider now a thick lens, as illustrated in Figure 3.

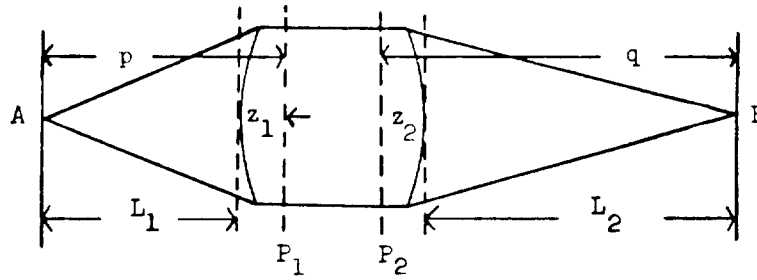


Figure 3

If L_1 is the object distance to the face of the lens and L_2 is the corresponding image distance, then, in general,

$$1/L_1 + 1/L_2 \neq 1/f .$$

If, however, we introduce two planes P_1 and P_2 located at a distance z_1 and z_2 from the entrance and exit faces of the lens, it is always possible to find a z_1 and a z_2 such that the equation

$$1/p + 1/q = 1/f \text{ is valid.}$$

where

$$p = L_1 + z_1$$

$$q = L_2 + z_2 .$$

When this is so, P_1 and P_2 are called the principal planes of the lens.

Now, relating the above statement to matrix formalism, the matrix transformation for a thick lens between the input and output faces of the lens has the general form:

$$\begin{bmatrix} x_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} x_0 \\ \theta_0 \end{bmatrix} \quad (1)$$

where as before the $\det R = 1$. For a general transformation, R_{12} is not necessarily equal to 0 and R_{11} and R_{22} are not necessarily equal to 1.

The principal planes may be located by the transformation

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \begin{bmatrix} 1 & z_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & z_1 \\ 0 & 1 \end{bmatrix} \quad (2)$$

Using the relation

$$\begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -z \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \text{ (the unit matrix)}$$

the previous equation may be manipulated into the form

$$\left[\begin{array}{c|c} R_{12} - z_1 R_{11} & \\ \hline R_{11} - z_2 R_{21} & -z_2 (R_{22} - z_1 R_{21}) \\ \hline R_{21} & R_{22} - z_1 R_{21} \end{array} \right] = \begin{bmatrix} 1 & -z_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} 1 & -z_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1/f \end{bmatrix} \quad (3)$$

Solving for z_1 and z_2 , we find

$$z_1 = \frac{R_{22} - 1}{R_{21}}$$

$$z_2 = \frac{R_{11} - 1}{R_{21}}$$

where z_1 and z_2 are the location of the principal planes as shown in Fig. 3. The principal planes of any system may be determined by this method.

Note that $R_{21} = -1/f$ is not affected by the transformation and that the upper right hand matrix element is zero if $\det R = 1$. The principal planes may coincide, may be close together, be far apart; or in many systems, may be located external to all of the elements comprising the system. An example of the latter case is a quadrupole pair.

Some examples of principal plane locations for simple systems follow:

A quadrupole singlet:

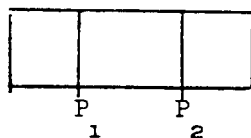


Figure 4

The principal planes in a single quadrupole are located very close to each other and very near the center of the lens. As such, a quadrupole singlet may be considered as a thin lens if the object and image distances are measured to the center of the lens.

A simple uniform-field wedge magnet:

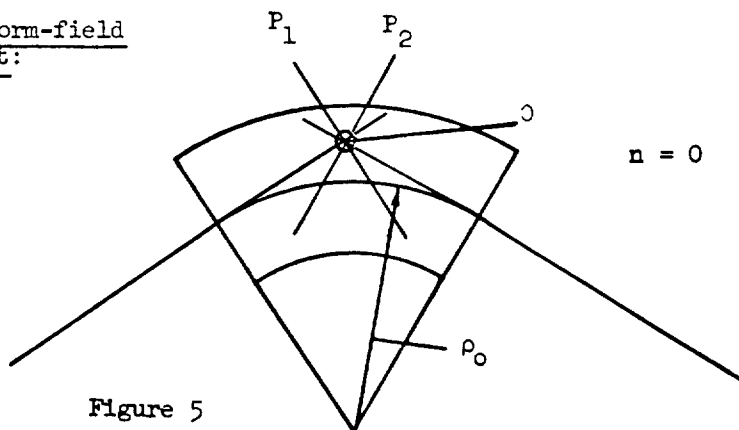
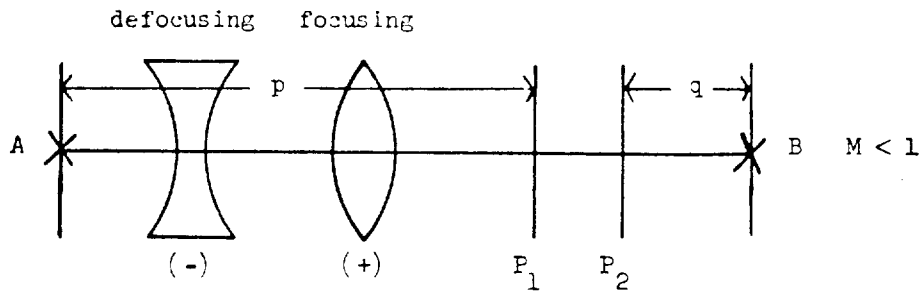


Figure 5

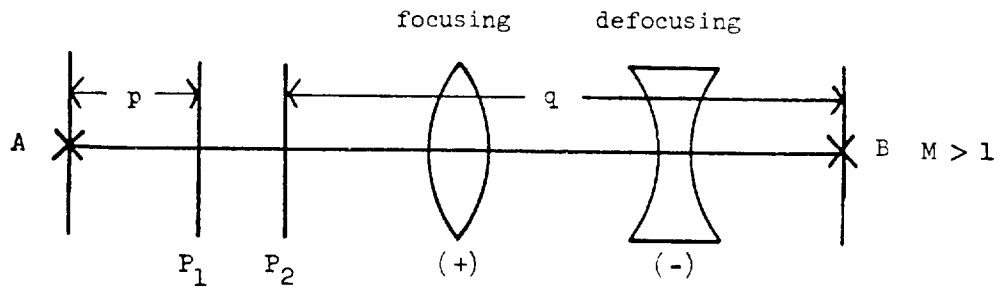
If the optic axis enters and exits perpendicularly to the pole boundary, the principal planes are at the "center" of the magnet, as shown in Figure 5. From this, we conclude that a simple wedge bending magnet may be considered as a "thin" lens if the object and image distances are measured to the lens center O.

A quadrupole pair:



x plane

Figure 6



y plane

Figure 7

For a quadrupole pair, the principal planes are displaced toward and, usually, beyond the focusing element of the pair, as shown in Figures 6 and 7.

For any lens system, no matter how many elements are involved

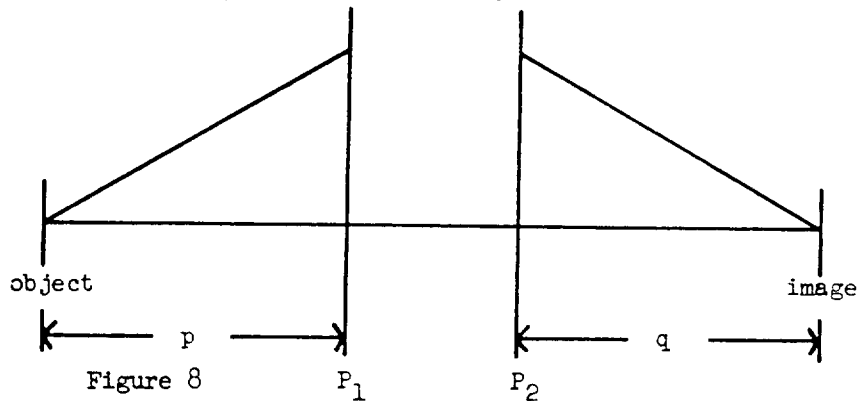


Figure 8

$1/p + 1/q = 1/f$, if p and q are distances measured to the principal planes. Then the magnification between object and image planes is $M = q/p$.

Since the quadrupole pair is different in the two planes, (x) and (y), both situations must be examined. The interesting result turns out to be that in the x plane, the principal planes are to the right (Figure 6) and in the y plane, they are to the left (Figure 7). Therefore, in the y plane

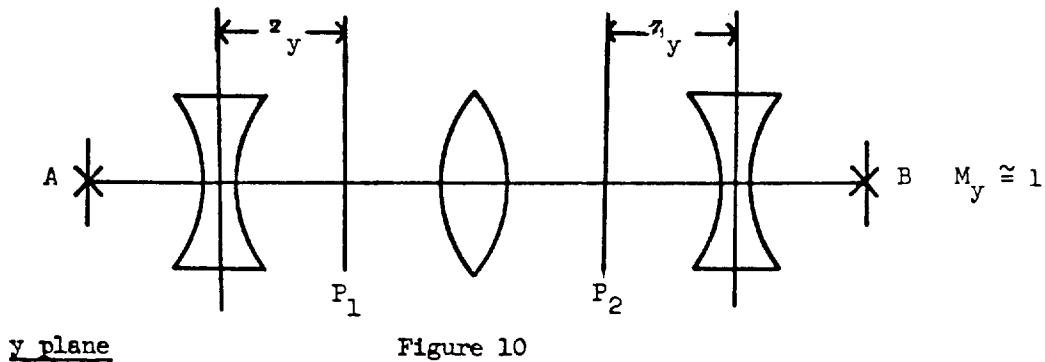
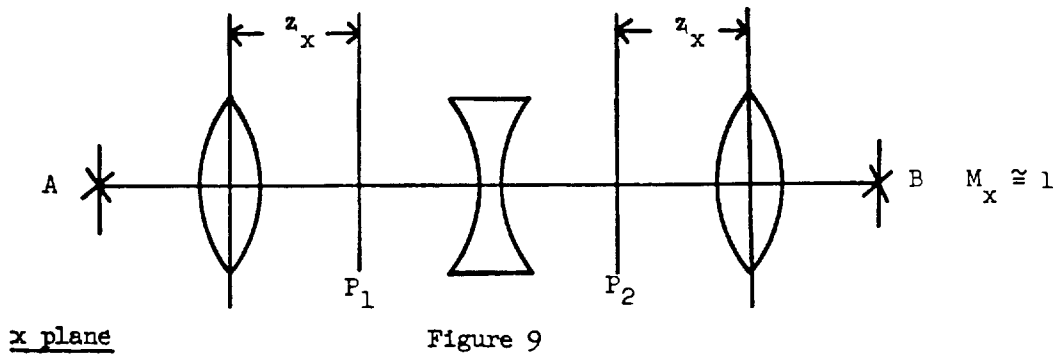
the magnification is greater than 1, and in the x plane the magnification is less than 1. Typically, for a quadrupole pair the ratio of

$$M_y/M_x$$

may be as high as 20:1 and such cases can be disastrous if not recognized beforehand. This is a first-order image distortion. For example, if the source is a circular spot at A, the image at B will appear as a long thin line.

The situation is different for the

Quadrupole triplet:



In the symmetric triplet, as shown in Figures 9 and 10, the principal planes are located symmetrically about the center of the system, although $z_x > z_y$. This is, perhaps, the dominant reason why quadrupole triplets are used. The magnification is approximately equal in both planes; consequently, a circular spot can be imaged through the system with much less first-order image distortion than is the case for the doublet.

3. Introduction of Momentum Dispersion into the Matrix Formalism

The foregoing discussion and examples dealt only with monoenergetic first-order effects. First-order dispersion may be taken into account by introducing a 3×3 matrix as follows:

Consider two particles of momentum p_0 and $p_0 + \Delta p$ passing through the midplane of a static magnetic field, as illustrated in Figure 11.

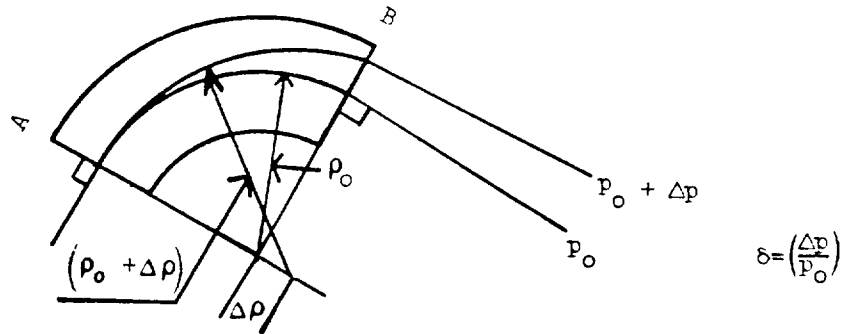


Figure 11

Since the scalar momentum of a particle is constant in a static magnetic field, the transport equation from A to B may be expressed as:

$$\begin{bmatrix} x_1 \\ \theta_1 \\ \delta \end{bmatrix}_B = \begin{bmatrix} R_{11} & R_{12} & d \\ R_{21} & R_{22} & d' \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ \theta_0 \\ \delta \end{bmatrix}_A$$

where

$$\delta = \Delta p / p_0$$

d = the spatial momentum dispersion

d' = the derivative of the dispersion (the angular momentum dispersion)

and

1 = a carrying term to generate a square matrix and denote a constant momentum.

The determinant of the matrix $|R|$ is equal to 1 as for the 2×2 matrix. However, because of the zeros in the bottom row, the fact that $|R| = (R_{11} R_{22} - R_{12} R_{21}) = 1$ only checks the 2×2 matrix and not the terms containing d and d' .

Consider now a general system from an object point A to an image point B.

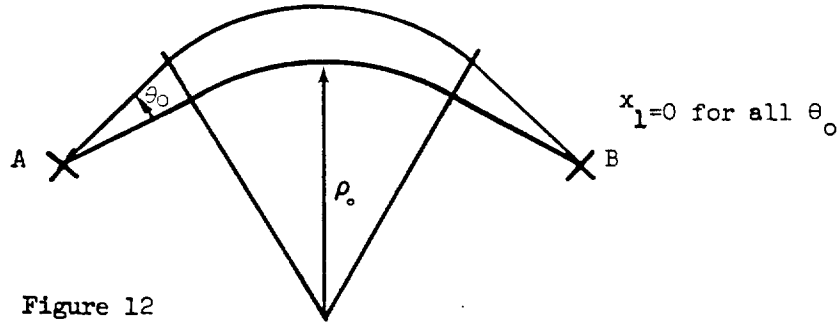


Figure 12

The above matrix equation is still valid for midplane trajectories. If A is a source point and if $R_{12} = 0$ (i.e., x is independent of θ_0), then B is an image point for monoenergetic particles. Under these circumstances:

$$R_{11} = M_x \text{ is the } x \text{ plane magnification}$$

$$R_{22} = + 1/M_x \text{ because } \det R = 1$$

and
$$R_{21} = - 1/f_x$$

In fact, $R_{21} = - 1/f_x$ for the system between A and B, even if A and B are not foci.

It is now convenient to develop a more general definition of the matrix elements R_{ij} and, at the same time, introduce the first-order matrix transformation for the y (non-bend) plane. Consider, again, a general system where the projection of the central trajectory is allowed to bend in the x plane but is a straight line in the y plane. The x plane and y plane matrix transformations may be written as follows:

For the x plane $x_1 = R_x x_0$

$$\text{or } \begin{bmatrix} x_1 \\ \theta_1 \\ \delta \end{bmatrix} = \begin{bmatrix} c_x(t) & s_x(t) & d_x(t) \\ c'_x(t) & s'_x(t) & d'_x(t) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ \theta_0 \\ \delta \end{bmatrix}$$

Similarly, for the y plane, $y_1 = R_y y_0$

$$\text{or } \begin{bmatrix} y_1 \\ \phi_1 \\ \delta \end{bmatrix} = \begin{bmatrix} c_y(t) & s_y(t) \\ c'_y(t) & s'_y(t) \end{bmatrix} \begin{bmatrix} y_0 \\ \phi_0 \end{bmatrix}$$

The c and s functions may be defined in terms of their initial conditions.

Let τ be the distance measured along the

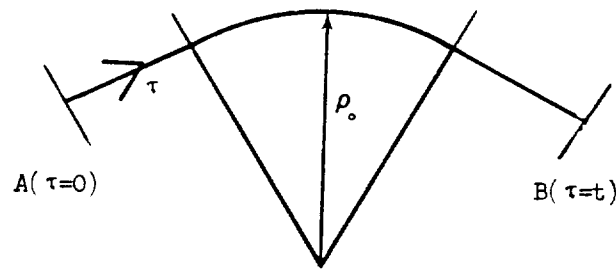


Figure 13

central trajectory. Then:

$$\begin{aligned} s(0) = 0 & \quad s'(0) = 1 & \quad \text{where} & \quad s'(\tau) = \frac{ds(\tau)}{d\tau} \\ c(0) = 1 & \quad c'(0) = 0 & \quad c'(\tau) = \frac{dc(\tau)}{d\tau} = -\frac{1}{f} \end{aligned}$$

Within an "ideal" magnet, where the bending radius ρ_0 is constant, s and c are sine and cosine or else sinh and cosh functions. Because of

this, the terminology s = a sine-like function and c = a cosine-like function has been adopted for describing the general case where $\rho_0 = \rho_0(\tau)$ is a function of τ .

By analogy with previous discussions, we observe that whenever $s(\tau) = 0$, we are at an image of point A. Also, $c(\tau)$ at the position where $s(\tau) = 0$ is the magnification of point A at that image.

$c'(t) = -1/f$ where f is the focal length of the system between A and B. The dispersion d_x may be derived from the general differential equations of motion of a charged particle in a static magnetic field.⁽¹⁾ The results may always be expressed as a function of s_x and c_x as follows:

$$d_x(t) = s_x(t) \int_0^t c_x(\tau) d\alpha - c_x(t) \int_0^t s_x(\tau) d\alpha$$

and

$$d'_x(t) = s'_x(t) \int_0^t c_x(\tau) d\alpha - c'_x(t) \int_0^t s_x(\tau) d\alpha$$

where

$$d\alpha = \frac{d\tau}{\rho_0(\tau)}$$

is the differential angle of bend of the central trajectory. At an image point $[s(t) = 0]$ note that

$$d_x(t) = -c(t) \int_0^t s(\tau) d\alpha$$

This approach to the problem may be generalized to include all of the second-order aberrations of a system. When this is done, it is always

(1) See SLAC-75 for a derivation of these equations.

possible to express these aberrations as functions of the first-order matrix elements c_x, s_x, d_x, c_y and s_y .

Having developed the above physical concepts and mathematical tools, we are now in a position to study more complicated systems. As an example, we consider the general system shown in Figure 14.

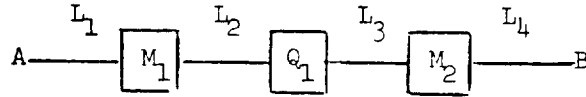


Figure 14

L = drift elements

= magnetic elements

The matrix formalism states that in the x plane, the transformation from A to B is given by the following matrix equation.

$$\begin{bmatrix} x_1 \\ \theta_1 \\ \delta \end{bmatrix} = R_{L_4} R_{M_2} R_{L_3} R_{Q_1} R_{L_2} R_{M_1} R_{L_1} \begin{bmatrix} x_0 \\ \theta_0 \\ \delta \end{bmatrix}$$

As in all matrix calculations, the order of writing down the elements comprising the system is from right to left. The individual matrix elements must be derived from the solution of the equation of motion within each element. If this has been done, then the calculation for the total system is carried out in the fashion shown by the above equation.

4. Second-Order Matrix Formalism (1)

It is possible to extend the 3×3 matrix formalism to solve simultaneous sets of power series by generating a second-order matrix equation

as follows:

$$\begin{bmatrix} x_1 \\ \theta_1 \\ \delta \\ x_1^2 \\ x_1 \theta_1 \\ x_1 \delta \\ \vdots \end{bmatrix} = \begin{bmatrix} \begin{matrix} 3 \times 3 \\ \text{first-order terms} \\ "R_1" \end{matrix} & \begin{matrix} \text{second-order} \\ \text{terms} \end{matrix} \\ \hline \begin{matrix} \text{all zero} \end{matrix} & \begin{matrix} "R_1^2" \end{matrix} \end{bmatrix} \begin{bmatrix} x_0 \\ \theta_0 \\ \delta \\ x_0^2 \\ x_0 \theta_0 \\ x_0 \delta \\ \vdots \end{bmatrix} \left. \vphantom{\begin{bmatrix} x_0 \\ \theta_0 \\ \delta \\ x_0^2 \\ x_0 \theta_0 \\ x_0 \delta \\ \vdots \end{bmatrix}} \right\} \begin{matrix} \text{all other} \\ \text{order} \\ \text{terms} \end{matrix}$$

The " R_1^2 " term is obtained by squaring the upper left corner (3×3) matrix so as to obtain second-order equations for x_1^2 , $x_1 \theta_1$, $x_1 \delta$, etc., as functions of products of the initial first-order variables x_0 , θ_0 , and δ .

This is, then, a convenient mathematical formalism for keeping all the terms desired and dropping those undesired. In the above example, all first- and second-order terms are retained and all higher-order terms are automatically dropped by the matrix multiplication.

5. Transformations Involving Many Trajectories

All of the discussion to this point relates to the transformation of a single trajectory (in addition to the central trajectory) through a static magnetic system. We wish now to extend the discussion to include "bundles" of rays. To accomplish this, we take advantage of Liouville's Theorem, which states that the "phase space" is conserved through the system. While the Theorem is strictly true to all orders, a convenient mathematical transformation

has only been developed to first-order. A manifestation of Liouville's Theorem is the fact that $|R| = 1$.

Now, so long as there is no coupling mechanism between the x plane and y plane of a magnetic-optical system (which is the case if the midplane symmetry prevails throughout the system) then, the phase space area in a given plane is also conserved. Consider a bundle of rays represented by the parallelogram, shown in Figure 15(a), representing the phase space distribution of the rays at some initial position. If we now look at the phase space distribution of the same bundle after it has drifted down stream, we observe the the θ_{max} boundary and the x intercept x_1 remain unchanged. In other words, the area of the parallelogram is the same or "phase space area has been conserved."

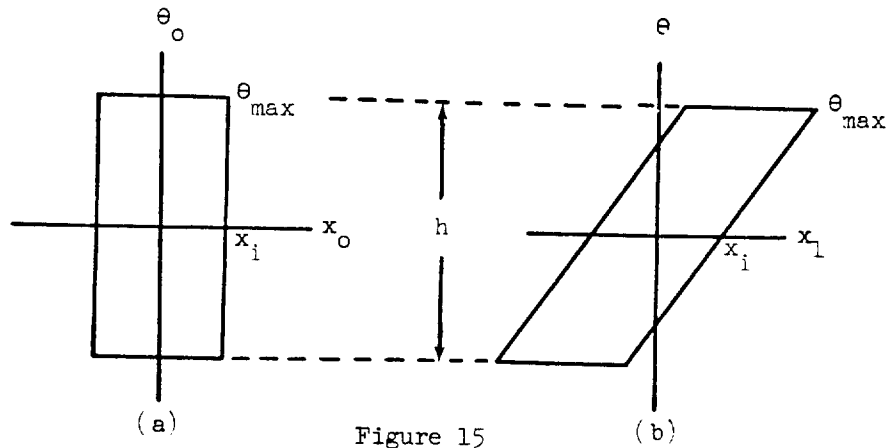


Figure 15

For mathematical convenience, the parallelogram is rather difficult to work with and, hence, a phase ellipse is usually used.

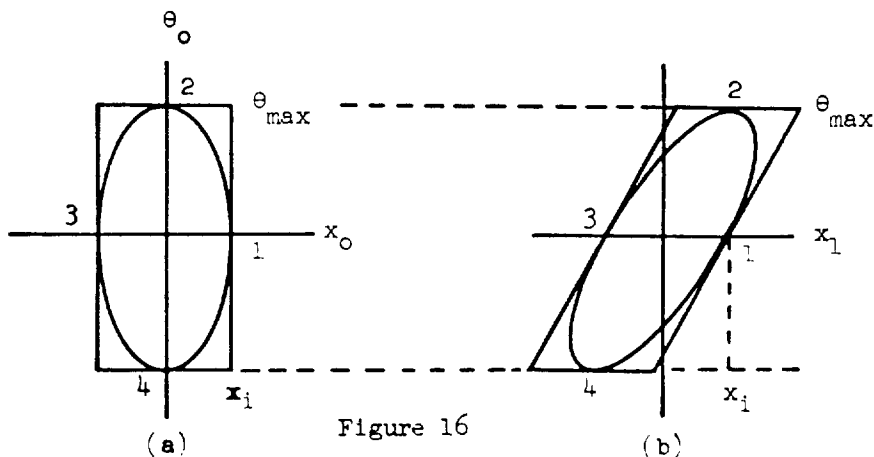


Figure 16

The phase ellipse transformation for a drift distance is illustrated in Figure 16. Figure 16(a) corresponds to a beam which is at its minimum width (a "waist") and Figure 16(b) shows the same beam after it has drifted downstream from the waist position. The physical meaning of this is that particles entering at $\theta = 0$ are parallel to the optic axis and, therefore, cannot change their relative positions with respect to the optic axis; that is, all particles on the x_0 axis act in this manner. Those that enter at a given angle continue at the same angle.

The phase ellipse transformation for a thin lens is illustrated in Figure 17. In passing through a thin lens, θ changes and the x dimension remains constant for a given trajectory.

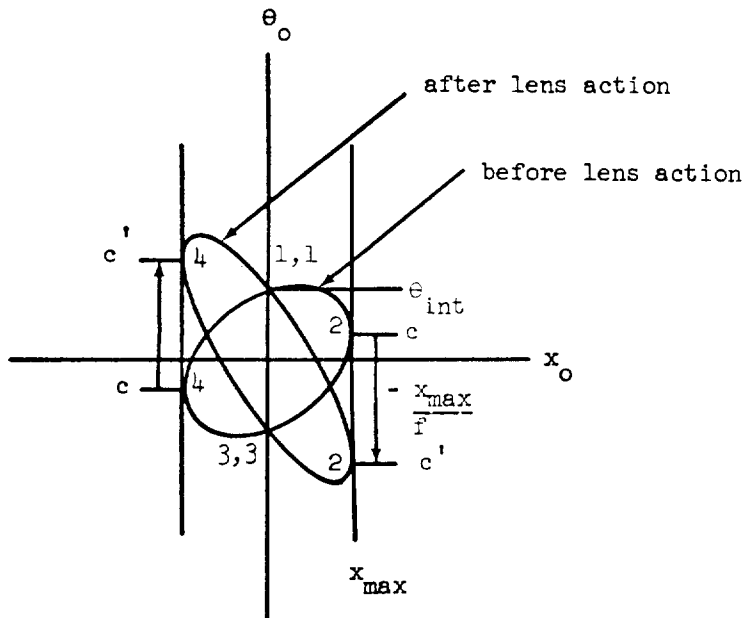


Figure 17

Stated in other terms, x_{\max} remains constant, as does θ_i (the θ intercept). It is apparent, in the given example, that the spot size now becomes, or can become, smaller at the new image because θ_{\max} is larger. This can be related to the physics of the system by saying that the x magnification is less than unity. This fact is observed directly by comparison of the x intercept of the ellipse before and after the lens action. It is interesting to observe that a particle initially at c is transformed to c' and that particles entering at $x = 0$ do not change their direction (θ_i is constant). If the particles are now allowed to drift, the ellipse rotates clockwise; when the ellipse is vertical, the spot size is at a minimum, namely, $x_{\max} = x_i$, as was illustrated in Figure 16.

Beam Transport Optics - Part II
(K. L. Brown)

1. Introduction

In Part I, the basic concepts of beam transport optics were established. Starting from the essentials of geometric optics, the methods of matrix algebra were introduced with the example of calculating the principal planes of a thick lens. The 3×3 matrix for the first-order beam transport calculations were introduced to take into account the particle momenta.

2. First Order Transformation Matrix

Figure 1 shows a general region containing a magnetic field.

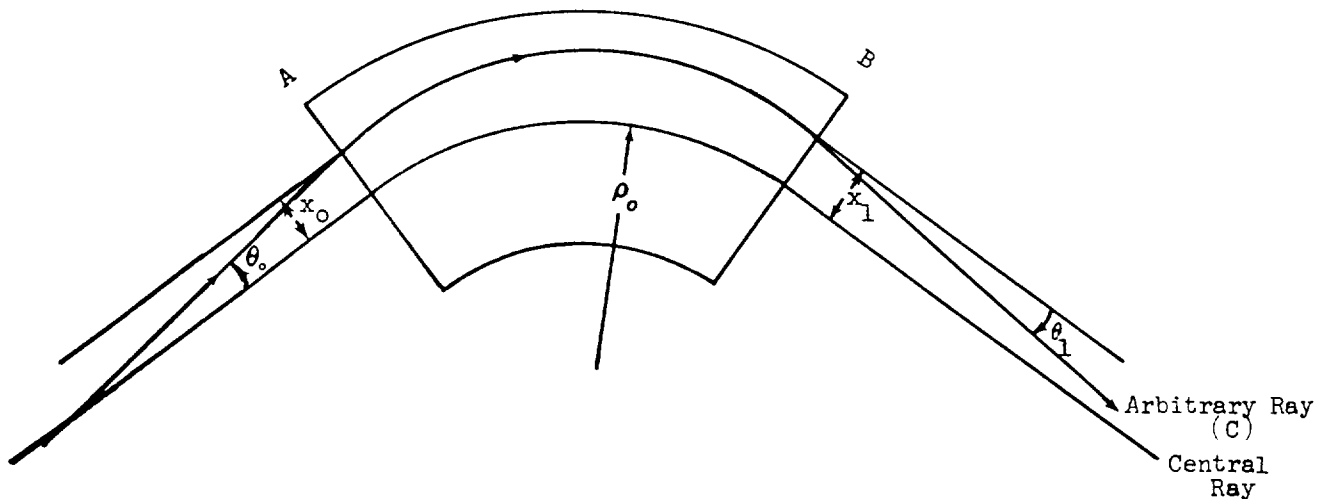


Figure 1: General Magnetic Field Configuration

The matrix presents a convenient way of writing the family of equations which describe the transformation from surface A to surface B. If x_0 , θ_0 and δ represent the conditions of a ray entering the system at A, then the conditions of the ray at B are x_1 , θ_1 and δ . Here x_0 is the distance from the central ray to the ray C, θ_0 is the angle between C and the parallel to the central ray and δ is the ratio $\Delta p/p$ where Δp is the difference between the momentum of C and the momentum of the central ray.

The linear transformation equations are:

$$\begin{aligned} x_1 &= c_x x_0 + s_x \theta_0 + d_x \delta \\ \theta_1 &= c'_x x_0 + s'_x \theta_0 + d'_x \delta \\ \delta &= 0 + 0 + \delta \end{aligned} \tag{1}$$

Expressed as a matrix, Eq. (1) are:

$$\begin{bmatrix} x_1 \\ \theta_1 \\ \delta \end{bmatrix} = \begin{bmatrix} c_x & s_x & d_x \\ c'_x & s'_x & d'_x \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ \theta_0 \\ \delta \end{bmatrix} \tag{2}$$

The equation $\delta = \delta$ expresses the fact that the magnetic field cannot change the scalar momentum of the particle. The δ terms in the x and θ equations express the momentum dispersion of the system.

If it happens that A is an object point and B is an image point of the system, then x_1 is independent of θ_0 , thus $s_x = 0$. In this case, c_x is given by $c_x = x_1/x_0 = M_x$ = the magnification in x plane, (for $\delta = 0$). if $\theta_0 = \delta = 0$, then $\theta_1 = c'_x x_0 = -x_0/f$ or $c'_x = -1/f$. It must always be true that the determinant of the matrix $|R|$, is unity. Thus for this special case of $s_x = 0$, it follows that $s'_x = 1/M_x$.

3. Beam Switchyard

As an example of a system which can be calculated with the matrix method, we next consider the beam switchyard of the two-mile accelerator. Figure 2 shows the three essential elements, two bending magnets and a quadrupole lens. In common with many beam transport systems, this one is designed to be achromatic. Mathematically, this means that the matrix elements, d_x and d'_x should be zero, so that there is no x or θ dependence on the momentum of the particles.

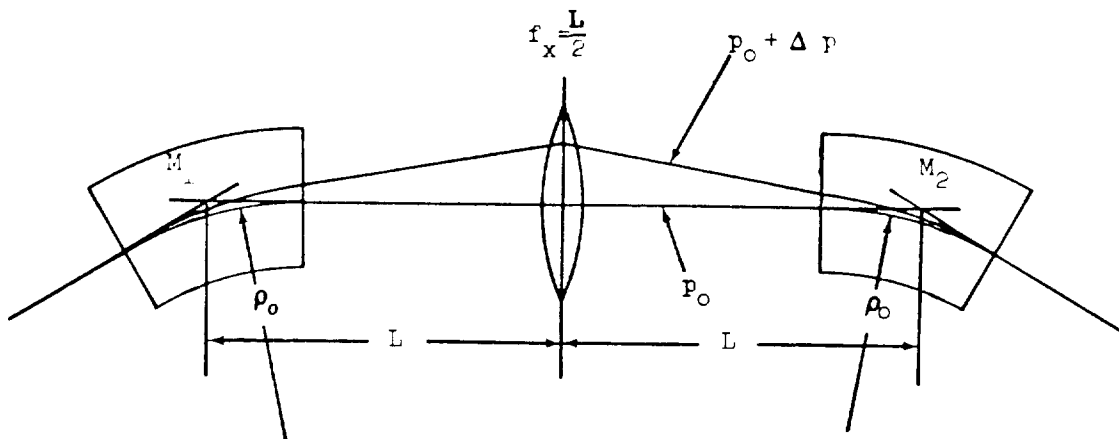


Figure 2: The Essentials of the Beam Switchyard

As a preliminary step, we will find the matrix expression for a bending magnet when measured from the principal planes. The matrix for a bending magnet, when measured from the ends of the poles, is given by:

$$R_{\text{bend}} = \begin{bmatrix} c & s & 1-c \\ -s & c & s \\ 0 & 0 & 1 \end{bmatrix} \quad (3)$$

where $c = \cos \alpha$ and $s = \sin \alpha$ and α is the deflection angle of the central ray. This expression has been normalized by setting the bending radius equal to unity. To restore ordinary units it is only necessary to insert the bending radius wherever a length is needed dimensionally. In this case, the matrix then becomes

$$R_{\text{bend}} = \begin{bmatrix} c & \rho s & \rho(1-c) \\ -s/\rho & c & s \\ 0 & 0 & 1 \end{bmatrix} \quad (4)$$

If the distance from the entrance plane to the first principal plane is z_1 and the distance from the second principal plane to the exit plane is z_2 , we can find the values z_1 and z_2 by solving the following matrix

equation:

$$\begin{bmatrix} 1 & -z_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c & s & 1-c \\ -s & c & s \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -z_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & d_x \\ -1/f & 1 & d'_x \\ 0 & 0 & 1 \end{bmatrix} \quad (5)$$

The matrix multiplication need only be done for the 2×2 matrices as outlined. To illustrate matrix multiplication the indicated operations will be given below in natural stages as follows:

$$\begin{bmatrix} 1 & -z_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c & -cz_1 + s \\ -s & +sz_1 + c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$$

$$\begin{bmatrix} c + z_2 s & -cz_1 + s \\ -s & +sz_1 + c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \quad (6)$$

Note that these transformations do not change the focal length expression, $-1/f = -s$. In order for two matrices to be equal, each individual element must be equal to its counterpart in the other matrix. Thus we have

$$\begin{aligned} c + z_2 s &= 1 \\ c + z_1 s &= 1 \end{aligned} \quad (7)$$

which when solved for z_1 and z_2 yield

$$z_2 = (1-c)/s \quad \text{and} \quad z_1 = (1-c)/s .$$

If we substitute the trigonometric equivalents, and apply standard identities, we have $z_2 = z_1 = \tan(\alpha/2)$ which can be seen from Figure 3 to indicate that the two principal planes are coincident with the symmetry plane in the middle of the magnet.

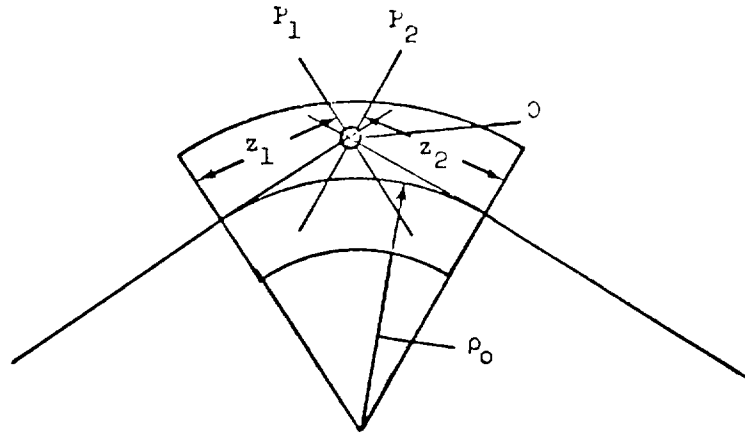


Figure 3: The Principal Planes of a Simple Bending Magnet are Coincident with the Center Plane

The simplified matrix for a bending magnet measured to the principal planes is then:

$$R_{\text{bend}} = \begin{bmatrix} 1 & 0 & 0 \\ -s & 1 & s \\ 0 & 0 & 1 \end{bmatrix} \quad (8)$$

To calculate the transformation matrix for the entire Beam Switchyard system as shown in Fig. 2, we write the matrices in opposite order from that in which the beam passes through the elements. That this must be true can be seen from the way in which one element alone is calculated by

$$\begin{bmatrix} x_1 \\ \theta_1 \\ \delta \end{bmatrix} = R_1 \begin{bmatrix} x_0 \\ \theta_0 \\ \delta \end{bmatrix} \quad (9)$$

Then for a second element we have

$$\begin{bmatrix} x_2 \\ \theta_2 \\ \delta \end{bmatrix} = R_2 \begin{bmatrix} x_1 \\ \theta_1 \\ \delta \end{bmatrix} = R_2 R_1 \begin{bmatrix} x_0 \\ \theta_0 \\ \delta \end{bmatrix} \quad (10)$$

and so forth.

If we allow the system to be symmetrical, i.e. $s_1 = s_2$ and $L_1 = L_2$; the complete series of matrices for Fig. 2 are

$$R_{BSY} = \begin{bmatrix} 1 & 0 & 0 \\ -s & 1 & s \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1/f & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -s & 1 & s \\ 0 & 0 & 1 \end{bmatrix} \quad (11)$$

We will show the step-by-step multiplication of the matrices to get the d_x and d'_x terms.

$$R_{BSY} = \begin{bmatrix} 1 & L & 0 \\ -s & 1-sL & s \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1/f & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1-sL & L & Ls \\ -s & 1 & s \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{BSY} = \begin{bmatrix} (1-\frac{L}{f}) & & \\ -Ls(2-\frac{L}{f}) & L(2-\frac{L}{f}) & Ls(2-\frac{L}{f}) \\ (Ls-1) \left[2s + \frac{1}{f}(1-Ls) \right] & (1-\frac{L}{f}) & s(1-Ls) (2-\frac{L}{f}) \\ 0 & 0 & 1 \end{bmatrix} \quad (12)$$

To obtain the required condition that $d_x = d'_x = 0$, we set

$$\left(2 - \frac{L}{f} \right) = 0 \quad \text{or} \quad f = \frac{L}{2}$$

then

$$R_{BSY} = \begin{bmatrix} -1 & 0 & 0 \\ -\frac{2}{L}(1-Ls) & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (13)$$

Thus the quadrupole acts as a lens to refocus rays from the center of the first bending magnet to the center of the last one. For the serious student, it is a worthwhile exercise to do the BSY problem without the simplification which resulted from introducing the principal planes.

4. TRANSPORT

As an aid to solving beam transport problems, a computer program TRANSPORT has been developed at SLAC which takes the greatest amount of labor out of this work. The program operates in about the way as the BSY example above was calculated, but with some important exceptions. Most importantly:

1. TRANSPORT has the ability to find the best first-order solution given a certain set of constraints;
2. TRANSPORT also calculates the transformation of a whole family of rays as found in a beam by means of the concept of "phase space" which was introduced in Part I;
3. TRANSPORT can, as an option, calculate the second order effects on the beam. By second order is meant, for example, terms which depend not linearly on the displacement x_0 , but on x_0^2 or $x_0 \theta_0$, etc.

To aid in the discussion of TRANSPORT and of the second order terms we now introduce an abbreviated notation. By writing out the complete equations for x and y , to second order, we will adequately have displayed the new notation.

$$\begin{aligned}
 x_1 = & (x|x_0)x_0 + (x|\theta_0)\theta_0 + (x|\delta)\delta \\
 & + (x|x_0^2)x_0^2 + (x|x_0\theta_0)x_0\theta_0 + (x|x_0\delta)x_0\delta \\
 & + (x|\theta_0^2)\theta_0^2 + (x|\theta_0\delta)\theta_0\delta + (x|\delta^2)\delta^2 \\
 & + (x|y_0^2)y_0^2 + (x|y_0\phi_0)y_0\phi_0 + (x|\phi_0^2)\phi_0^2
 \end{aligned} \tag{14}$$

$$\begin{aligned}
 y_1 = & (y|y_0) + (y|\phi_0)\phi_0 \\
 & + (y|x_0y_0)x_0y_0 + (y|x_0\phi_0)x_0\phi_0 + (y|\theta_0y_0)\theta_0y_0 \\
 & + (y|\theta_0\phi_0)\theta_0\phi_0 + (y|y_0\delta)y_0\delta + (y|\delta\phi_0)\delta\phi_0
 \end{aligned} \tag{15}$$

The absence of certain terms which might otherwise be expected in Eqs. (14) and (15) is due to the fact that horizontal mid-plane symmetry has been assumed in the derivation. That is, the field on the horizontal mid-plane is normal to the plane. Thus there can be no $(y|x)$ or $(y|\theta)$ term. Similarly, there can only be even powers of y and ϕ , such as $(x|y_0^2)$ and $(x|y_0\phi_0)$, in the x equation. Also, note that there is no $(y|\delta)$ or $(y|\delta^2)$ term if there is mid-plane symmetry.

TRANSPORT uses a numerical notation to signify the six basic coordinates:

$$\begin{array}{cccccc} x & \theta & y & \phi & \ell & \delta \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array} \quad (16)$$

The ℓ term has not been introduced here before. Its significance is the preservation of the bunch length of a beam such as the SIAC electron beam.

The first order output from TRANSPORT is a 6×6 matrix printout of the R matrix where the labels are implied by row and column position of the elements. For example the element appearing at the intersection of row 3 and column 4 is the coefficient $(y|\phi_0)$ etc.

$$\begin{array}{cccccc} & x_0 & \theta_0 & y_0 & \phi_0 & \ell_0 & \delta_0 \\ x & - & - & - & - & - & - \\ \theta & - & - & - & - & - & - \\ y & - & - & - & - & - & - \\ \phi & - & - & - & - & - & - \\ \ell & - & - & - & - & - & - \\ \delta & - & - & - & - & - & - \end{array}$$

The second order terms are labelled by the convention indicated in Eqs. (14), (15) and (16). For example, $(x|x_0^2)$ becomes 1 11 and $(x|\theta_0\delta)$ becomes 1 26.

5. Second Order Matrix

Normally the matrix method is expected only to apply to the solution

of linear, i.e. first order, equations. However, the method has been extended to include second order terms as discussed in Part I.

For a more extensive discussion of the second-order matrix formalism, the reader is referred to SLAC report number 75 by K.L. Brown.

FIRST-ORDER R MATRIX FORMALISM FOR TRANSPORT

Section II

Beam transport optics may be reduced to a process of matrix multiplication(1,2). To first-order, this is represented by the matrix equation (using the notation of SLAC-75).

$$x_i(t) = \sum_{j=1}^6 R_{ij} x_j(0) \quad (1)$$

where

$$x_1=x \quad x_2=\theta \quad x_3=y \quad x_4=\varphi \quad x_5=l \quad \text{and} \quad x_6=\delta$$

The determinant $|R| = 1$. This is a direct consequence of the basic equation of motion for a charged particle in a static magnetic field and is a manifestation of Liouville's theorem of conservation of phase space volume. (See SLAC-75, page 41 for a proof that $|R| = 1$.)

For static magnetic systems possessing midplane symmetry, the six simultaneous linear equations represented by Eq. (1) may be expanded in matrix form as follows:

$$\begin{bmatrix} x(t) \\ \theta(t) \\ y(t) \\ \varphi(t) \\ l(t) \\ \delta(t) \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & 0 & 0 & 0 & R_{16} \\ R_{21} & R_{22} & 0 & 0 & 0 & R_{26} \\ 0 & 0 & R_{33} & R_{34} & 0 & 0 \\ 0 & 0 & R_{43} & R_{44} & 0 & 0 \\ R_{51} & R_{52} & 0 & 0 & 1 & R_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ \theta_0 \\ y_0 \\ \varphi_0 \\ l_0 \\ \delta_0 \end{bmatrix} \quad (2)$$

where the transformation is from an initial position $\tau = 0$ to a final position $\tau = t$ measured along the assumed central reference trajectory.

Thus at any specified position in a system, an arbitrary charged particle is represented by a vector (single column matrix), X , whose components are the positions, angles, and momentum of the particle with respect to a specified reference trajectory.

$$\text{i.e. } X = \begin{bmatrix} x \\ \theta \\ y \\ \varphi \\ l \\ \delta \end{bmatrix}$$

where:

x = the radial displacement of the arbitrary ray with respect to the assumed central trajectory.

θ = the angle this ray makes in the radial plane with respect to the assumed central trajectory.

y = the transverse displacement of the ray with respect to the assumed central trajectory.

φ = the transverse angle of the ray with respect to the assumed central trajectory.

l = the path length difference between the arbitrary ray and the central trajectory.

$\delta = \Delta P/P$ is the fractional momentum deviation of the ray from the assumed central trajectory.

The magnetic lens is represented by the square matrix, R , which describes the action of the magnet on the particle coordinates. Thus the passage of a charged particle through the system may be represented by the matrix equation:

$$X(1) = R X(0) \quad (3)$$

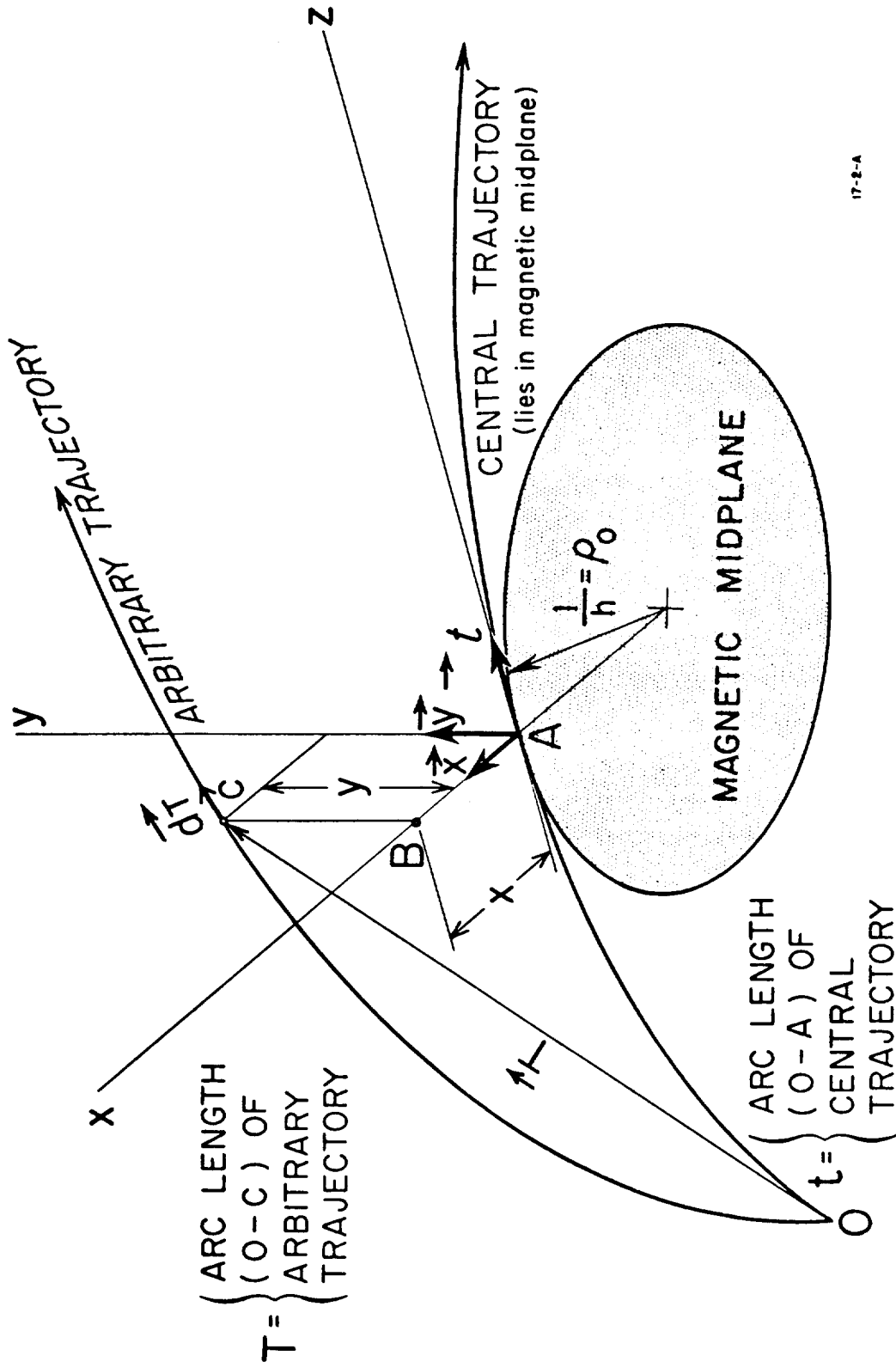
where $X(0)$ is the initial coordinate vector and $X(1)$ is the final coordinate vector of the particle under consideration; R is the transformation matrix for all such particles traversing the system (one particle differing from another only by its initial coordinate vector $X(0)$).

The traversing of several magnets and interspersing drift spaces is described by the same basic equation but with R now being the product matrix $R = R(n) \dots R(3)R(2)R(1)$ of the individual matrices of the system elements. TRANSPORT calculates and tabulates the product matrix R representing the system.

The zero elements $R_{13} = R_{14} = R_{23} = R_{24} = R_{31} = R_{32} = R_{41} = R_{42} = R_{36} = R_{46} = 0$ in the R matrix are a direct consequence of midplane symmetry. If midplane symmetry is destroyed, these elements will in general become non-zero. The zero elements in column five occur because the variables x , θ , y , φ , and δ are independent of the path length difference l . The zero's in row six result from the fact that we have restricted the problem to static magnetic fields, i.e., the scalar momentum is a constant of the motion.

In SLAC report 75 (Ref. 1), a physical significance has been attached to the non-zero matrix elements in the first four rows in terms of their identification with characteristic first-order trajectories. We include figures showing these characteristic functions as a convenient reference.

We now wish to relate the elements appearing in column six and those in row five in terms of simple integrals of the characteristic first-order matrix elements $c_x(t) = R_{11}$ and $s_x(t) = R_{12}$. In order to do this, we make use of the Green's function integral, Eq. (43), Section II of SLAC-75, and of the expression for the differential path length in curvilinear coordinates



17-2-A

FIG. 1--CURVILINEAR COORDINATE SYSTEM USED IN DERIVATION OF EQUATIONS OF MOTION.

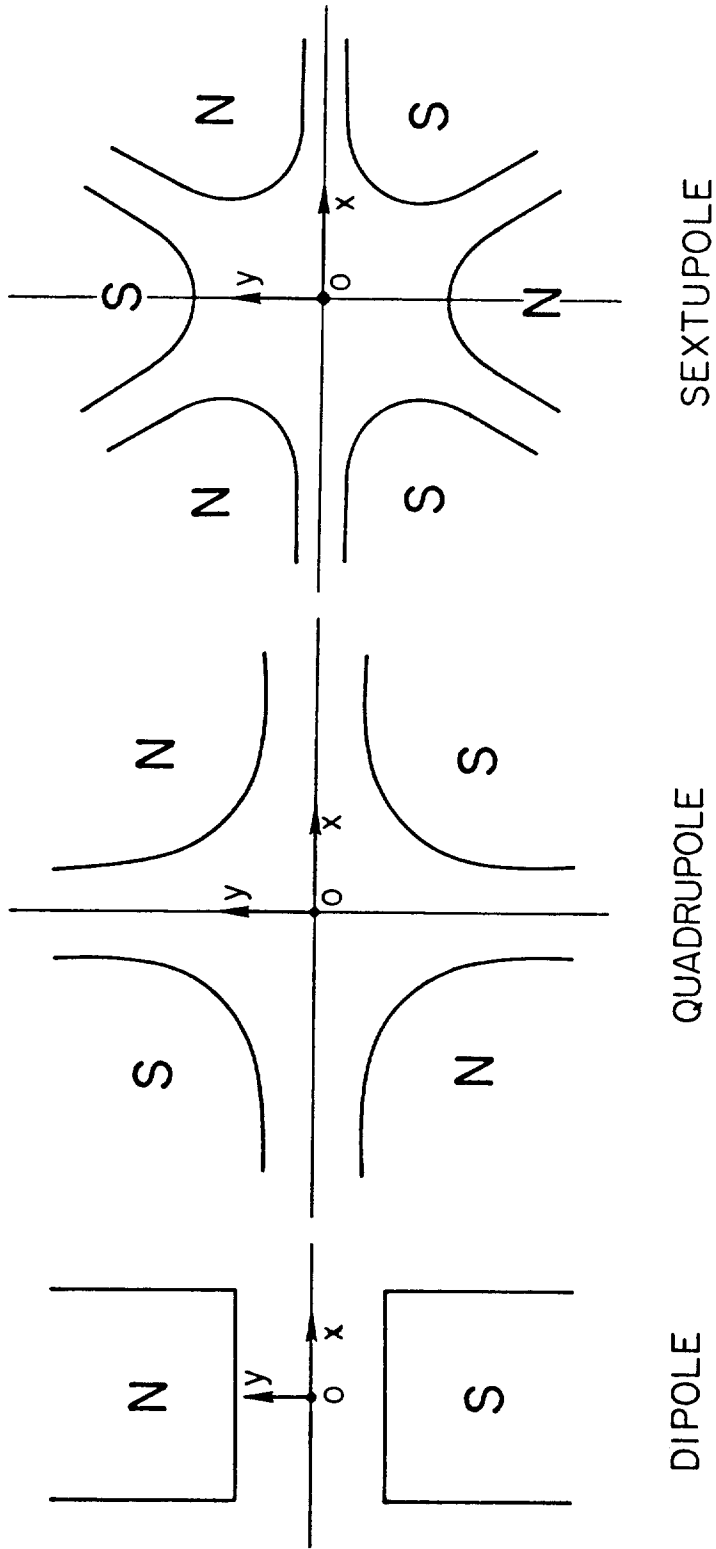


FIG. 2--ILLUSTRATION OF THE MAGNETIC MIDPLANE (X AXIS) FOR DIPOLE, QUADRUPOLE AND SEXTUPOLE ELEMENTS. THE MAGNETIC POLARITIES INDICATE MULTIPOLE ELEMENTS THAT ARE POSITIVE IN RESPECT TO EACH OTHER.

748A10

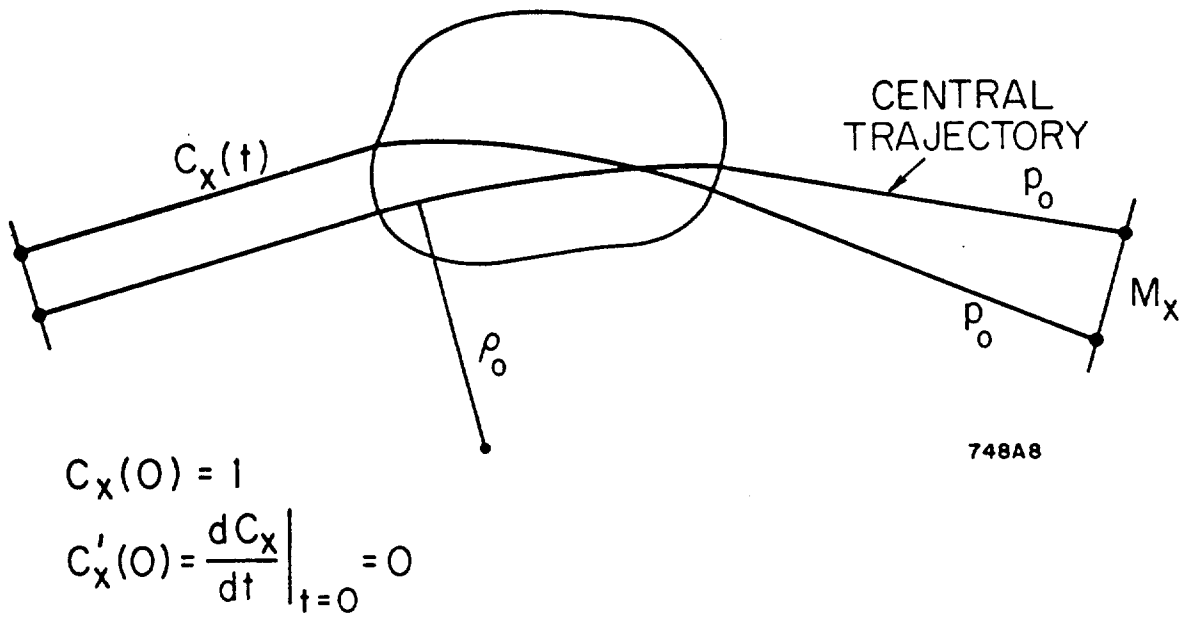


FIG. 3--COSINE-LIKE FUNCTION $c_x(t) = R_{11}$ IN MAGNETIC MIDPLANE. $c'_x(t) = R_{21}$.

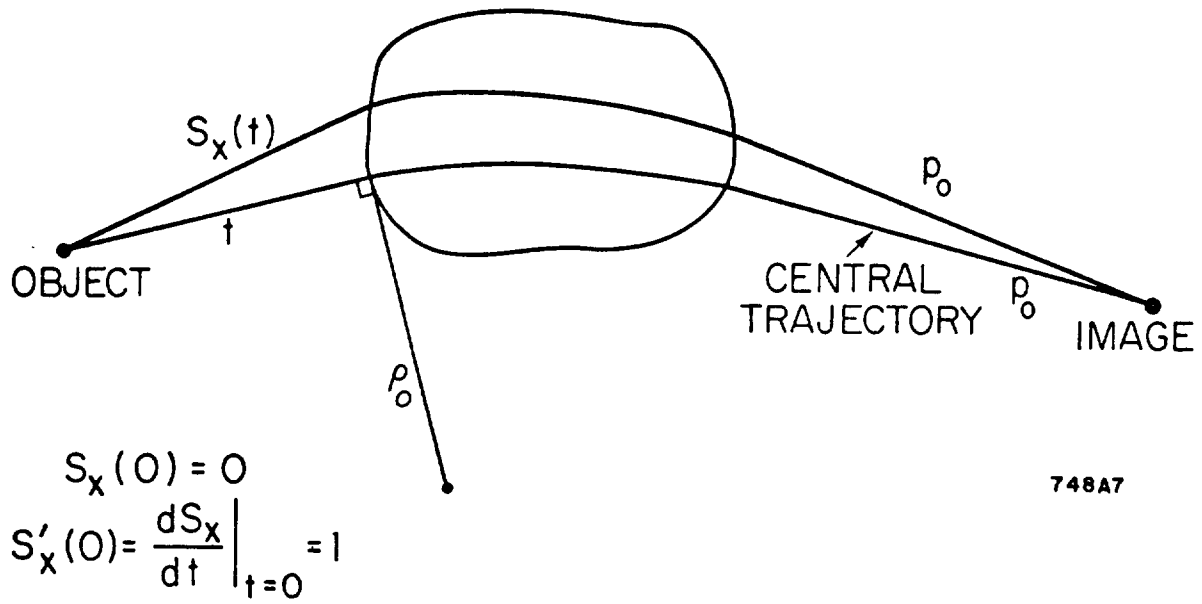
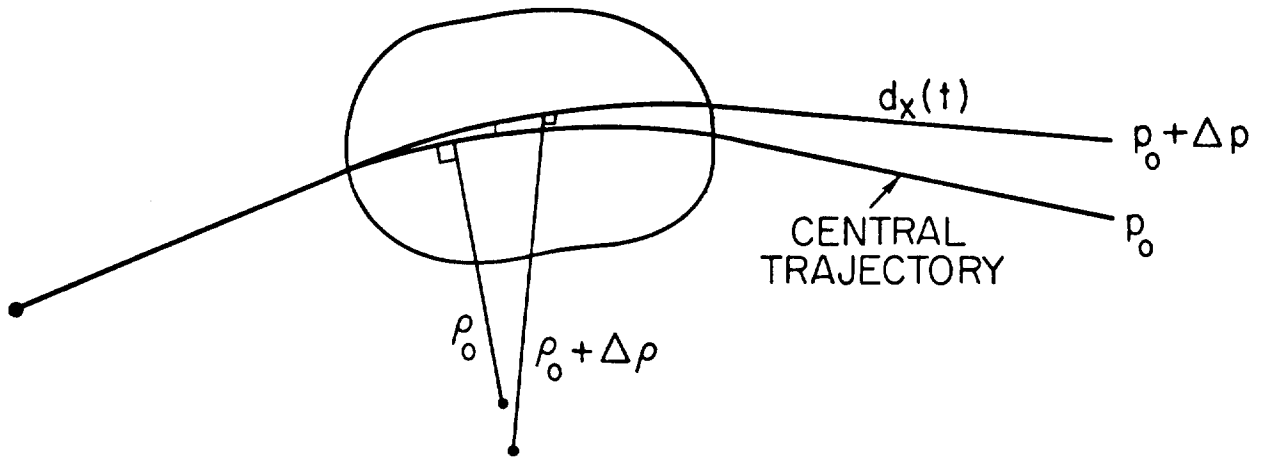


FIG. 4--SINE-LIKE FUNCTION $s_x(t) = R_{12}$ IN MAGNETIC MIDPLANE. $s'_x(t) = R_{22}$.

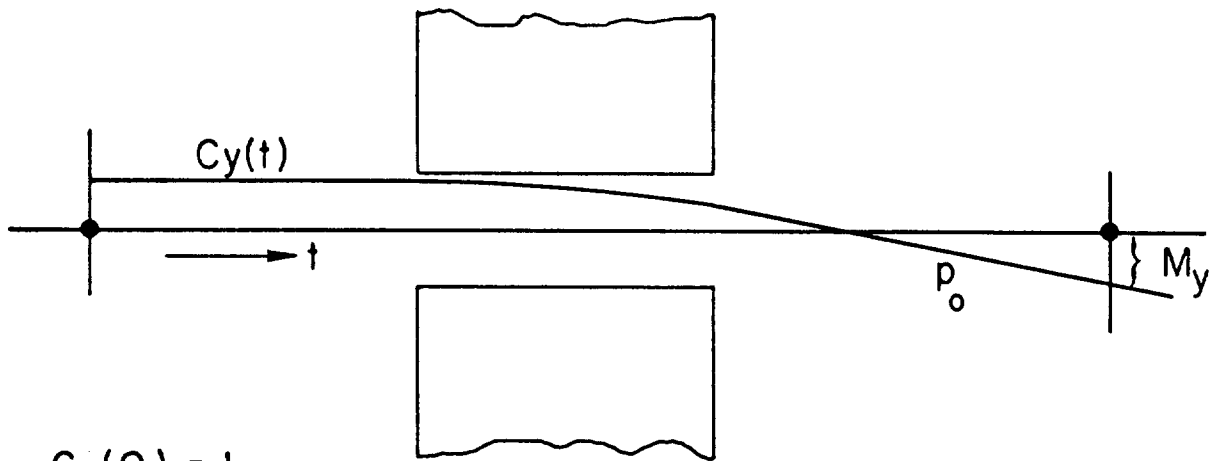


$$d_x(0) = 0$$

$$d'_x(0) = \left. \frac{d(d_x)}{dt} \right|_{t=0} = 0$$

748A9

FIG. 5--DISPERSION FUNCTION $d_x(t) = R_{16}$ IN MAGNETIC MIDPLANE. $d'_x(t) = R_{26}$.



$$C_y(0) = 1$$

$$C'_y(0) = \left. \frac{dC_y}{dt} \right|_{t=0} = 0$$

748A6

FIG. 6--COSINE-LIKE FUNCTION $c_y(t) = R_{33}$ IN THE NON-BEND (y) PLANE. $c'_y(t) = R_{43}$.

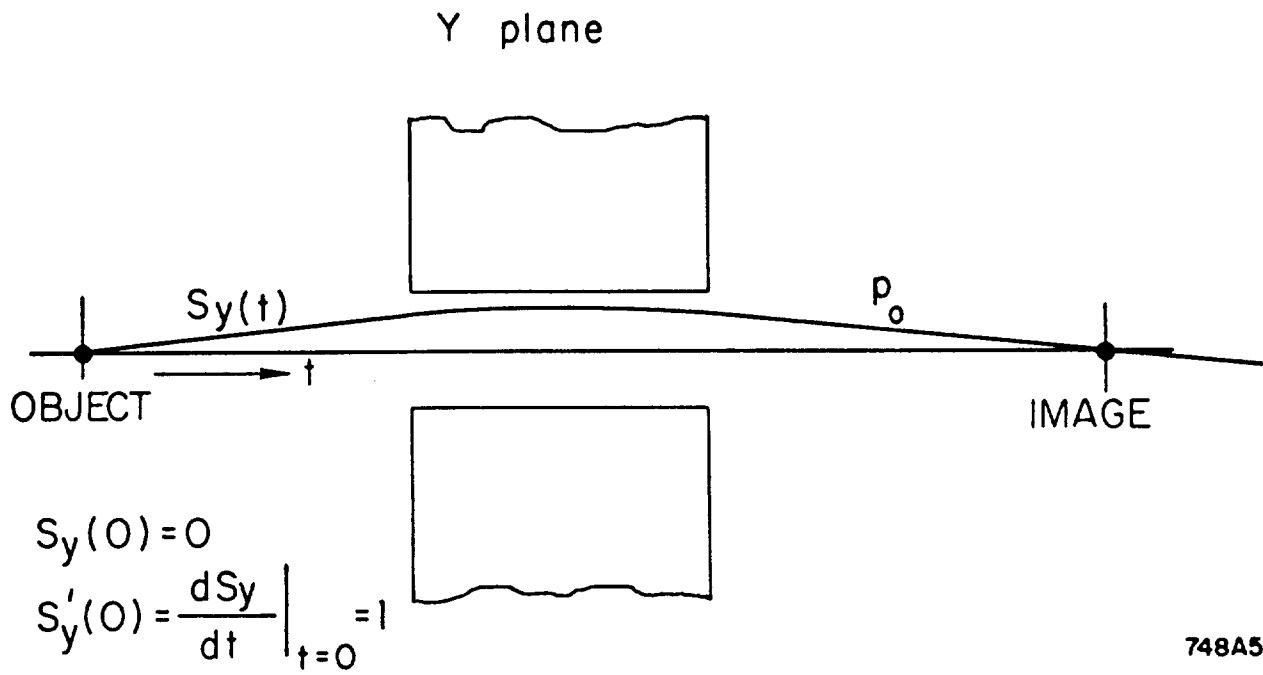


FIG. 7--SINE-LIKE FUNCTION $s_y(t) = R_{34}$ IN NON-BEND (y) PLANE. $s_y'(t) = R_{44}$.

$$dT = \left[(dx)^2 + (dy)^2 + (1+hx)^2 (dt)^2 \right]^{1/2} \quad (4)$$

used in the derivation of the equation of motion.

First-Order Dispersion

The spatial dispersion $d_x(t) = R_{16}$ of a system at position t is derived using the Green's function integral, and the driving term $f(\tau) = h(\tau) = \frac{1}{\rho_o(\tau)}$ for the dispersion (see Table I of SLAC-75). The result is

$$d_x(t) = R_{16} = s_x(t) \int_0^t c_x(\tau) h(\tau) d\tau - c_x(t) \int_0^t s_x(\tau) h(\tau) d\tau \quad (5)$$

where τ is the variable of integration. Note that $h(\tau)d\tau = d\alpha$ is the differential angle of bend of the central trajectory at any point in the system. Thus first-order dispersion is generated only in regions where the central trajectory is deflected (i.e., in dipole elements.) The angular dispersion is obtained by direct differentiation of $d_x(t)$ with respect to t , the curvilinear distance along the central trajectory.

$$d'_x(t) = R_{26} = s'_x(t) \int_0^t c_x(\tau) h(\tau) d\tau - c'_x(t) \int_0^t s_x(\tau) h(\tau) d\tau \quad (6)$$

where

$$c'_x(t) = R_{21} \text{ and } s'_x(t) = R_{22}$$

First-Order Path Length

The first-order path length difference is obtained by expanding and integrating Eq. (4) and retaining only the first-order term, i.e.,

$$l - l_o = (T - t) = \int_0^t x(\tau) h(\tau) d\tau + \text{higher order terms}$$

from which

$$\begin{aligned}
 l &= x_0 \int_0^t c_x(\tau) h(\tau) d\tau + \theta_0 \int_0^t s_x(\tau) h(\tau) d\tau + l_0 + \delta \int_0^t d_x(\tau) h(\tau) d\tau \\
 &= R_{51} x_0 + R_{52} \theta_0 + l_0 + R_{56} \delta
 \end{aligned} \tag{7}$$

Inspection of Eqs. (5), (6), and (7) yields the following useful theorems:

Achromaticity: A system is defined as being achromatic if $R_{16} = R_{26} = 0$, i.e., if $d_x(t) = d'_x(t) = 0$. Therefore it follows from Eq's. (5) and (6) that the necessary and sufficient conditions for achromaticity are that

$$\int_0^t s_x(\tau) h(\tau) d\tau = \int_0^t c_x(\tau) h(\tau) d\tau = 0 \tag{8}$$

By comparing Eq. (7) with Eq. (8), we note that if a system is achromatic, all particles of the same momentum will have equal (first-order) path lengths through the system.

Isochronicity: It is somewhat unfortunate that this word has been used in the literature to mean equal path lengths since equal path lengths only imply equal transit times for highly relativistic particles. Nevertheless, from Eq. 7, the necessary and sufficient conditions that the first-order path length of all particles (independent of their initial momenta) will be the same through a system are that $R_{51} = R_{52} = R_{56} = 0$, i.e., if

$$\int_0^t c_x(\tau) h(\tau) d\tau = \int_0^t s_x(\tau) h(\tau) d\tau = \int_0^t d_x(\tau) h(\tau) d\tau = 0 \tag{9}$$

First-Order Imaging

First-order point-to-point imaging in the x plane occurs when $x(t)$ is independent of the initial angle θ_0 . This can only be so when

$$s_x(t) = R_{12} = 0. \tag{10}$$

Similarly first-order point-to-point imaging occurs in the y plane when

$$s_y(t) = R_{34} = 0. \quad (11)$$

First-order parallel-to-point imaging occurs in the x plane when $x(t)$ is independent of the initial particle position x_0 . This will occur only if

$$c_x(t) = R_{11} = 0. \quad (12)$$

and correspondingly in the y plane, parallel-to-point imaging occurs when

$$c_y(t) = R_{33} = 0. \quad (13)$$

A parallel ray entering a system exits parallel to the central trajectory if

$$c'_x(t) = R_{21} = 0. \quad (14)$$

in the x plane; and if

$$c'_y(t) = R_{43} = 0. \quad (15)$$

in the y plane.

Point-to-parallel imaging occurs in the x plane if

$$s'_x(t) = R_{22} = 0. \quad (16)$$

and in the y plane if

$$s'_y(t) = R_{44} = 0. \quad (17)$$

Focal Lengths

A simple ray diagram of a "thick" lens demonstrates that R_{21} and R_{43} have the following physical interpretations

$$c'_x(t) = R_{21} = -\frac{1}{f_x} \quad \text{and} \quad c'_y(t) = R_{43} = -\frac{1}{f_y} \quad (18)$$

where f_x and f_y are the system focal lengths in the x and y planes respectively.

Zero Dispersion

For point-to-point imaging, using Eq's. (5) and (10), the necessary and sufficient condition for zero dispersion at an image is

$$d_x(t) = R_{16} = \int_0^t s_x(\tau) h(\tau) d\tau = 0 \quad (19)$$

For parallel-to-point imaging, (i.e., $c_x(t) = 0$), the condition for zero dispersion at the image is

$$d_x(t) = R_{16} = \int_0^t c_x(\tau) h(\tau) d\tau = 0. \quad (20)$$

Magnification

For monoenergetic point-to-point imaging in the x-plane, the magnification is given by

$$M_x = \frac{x(t)}{x_0} = R_{11} = c_x(t)$$

and in the y plane by

$$M_y = R_{33} = c_y(t) \quad (21)$$

where a negative number means an inverted image.

First-Order Momentum Resolution

For point-to-point imaging the first-order momentum resolving power R_1 (not to be confused with the matrix R) is the ratio of the momentum dispersion to the total image size. Thus if $2x_0$ is the total source size then

$$R_1 = \frac{P}{\Delta P} = \left| \frac{R_{16}}{2x_0 R_{11}} \right| = \left| \frac{d_x(t)}{2x_0 c_x(t)} \right|$$

For point-to-point imaging $s_x(t) = 0$. Using Eq. (5), the dispersion at an image is

$$d_x(t) = -c_x(t) \int_0^t s_x(\tau) h(\tau) d\tau \quad (22)$$

from which the first-order momentum resolving power R_1 becomes

$$2x_o R_1 = \left| \frac{d_x(t)}{c_x(t)} \right| = \left| \int_0^t s_x(\tau) h(\tau) d\tau \right| = |R_{52}| \quad (23)$$

Equation (23) for the first-order resolving power of a system may be expressed in a number of useful forms. If we consider a ray (particle) originating at the source with $x_o = 0$ and $\delta = \frac{\Delta P}{P} = 0$ and lying in the midplane. (i.e., a mono-energetic point source), the first-order equation representing the midplane displacement, x , of this trajectory is

$$x(t) = s_x(t) \theta_o \quad (24)$$

We may then rewrite Equation (23) as follows:

$$2x_o R_1 = \int_0^t s_x(\tau) h(\tau) d\tau = \frac{1}{\theta_o} \int_0^t x(\tau) h(\tau) d\tau = \frac{(\ell - \ell_o)}{\theta_o} = |R_{52}| \quad (25)$$

or we may also write it in the form

$$R_1 = \frac{1}{2x_o \theta_o} \int_0^t B \frac{x(t) d\tau}{B \rho_o} = \left(\frac{1}{2x_o \theta_o} \right) \left(\frac{1}{B \rho_o} \right) \int_0^t B dA \quad (26)$$

where $\int B dA$ is the magnetic flux inclosed between the central trajectory and the ray represented by Eq. (24), and $B\rho$ is the magnetic rigidity (momentum) of the central trajectory. Please note, however, that if the ray crosses the central trajectory or the sign of B changes, this changes the sign of the integration.

Some important observations may be made from Eq's. (25) and (26).

1) Resolving particles of different momentum requires that a path length difference must exist between the central trajectory and the trajectory defined by Eq. (22). The greater the path length difference, the greater the resolving power.

2) From Eq. (24), we may define resolving power as the magnetic flux inclosed per unit phase space area per unit momentum (BP) of the central ray.

First-Order σ Matrix (Phase Ellipse) Formalism for TRANSPORT

In accelerators and beam transport systems, the behavior of an individual particle is often of less concern than is the behavior of a bundle of particles (the BEAM) of which an individual particle is a member. An extension of the matrix algebra of Eq. (3) provides a convenient means for defining and manipulating this BEAM. TRANSPORT assumes that the bundle of rays constituting a BEAM may correctly be represented in coordinate phase-space by an ellipsoid whose coordinates are the position, angle, and momentum coordinates of the arbitrary rays in the beam about an assumed central trajectory. TRANSPORT is a matrix calculation that truncates the problem to either first- or second-order in a Taylor's expansion about the central trajectory. Particles in a BEAM are assumed to lie within the boundaries of the ellipsoid with each point within the ellipsoid representing a possible ray. The sum total of all phase points, the phase space volume, is commonly referred to as the "phase space" occupied by the BEAM. The validity and interpretation of this phase ellipse formalism must be ascertained for each system being designed. However, in general, for charged particle beams in, or emanating, from accelerators, the first-order phase ellipse formalism of TRANSPORT is a reasonable representation of physical reality; but for other applications, such as charged particle spectrometers, caution is in order in its use and interpretation.

The equation of an n-dimensional ellipsoid may be written in matrix form as follows:

$$X(0)^T \sigma(0)^{-1} X(0) = 1 \quad (27)$$

where $X(0)^T$ is the transpose of the coordinate vector $X(0)$, and $\sigma(0)$ is a real, positive definite, symmetric matrix.

The volume of the n-dimensional ellipsoid defined by sigma is

$$\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)} (\det \sigma)^{1/2}, \text{ the area of the projection in one plane is } A = \pi(\det \sigma)^{1/2}.$$

This is the "phase space" occupied by the beam.

As a particle passes through a system of magnets, it undergoes the matrix transformation of Eq. (3). Combining this transformation with the equation of the initial ellipsoid, and using the identity $RR^{-1} = I$ (the unity matrix), it follows that:

$$X(0)^T (R^T R^{-1}) \sigma(0)^{-1} (R^{-1} R) X(0) = 1$$

from which:

$$(RX(0))^T (R\sigma(0) R^T)^{-1} (RX(0)) = 1 \quad (28)$$

The equation of the ellipsoid representing the "BEAM" at the end of the system is thus:

$$X(1)^T \sigma(1)^{-1} X(1) = 1 \quad (29)$$

where the equation for the sigma matrix at the end may be related to that at the beginning by:

$$\sigma(1) = R \sigma(0) R^T \quad (30)$$

In addition to calculating the product matrix R, TRANSPORT also computes the sigma "BEAM" matrix at the end of each physical element via Eq. (30).

All of the important physical parameters of the BEAM ellipsoid may be expressed as functions of the matrix elements of the sigma matrix at the location

in question. In particular the square roots of the diagonal elements ($\sqrt{\sigma_{ii}}$'s) are the projection of the ellipse upon the coordinate axes and thus represent the maximum extent of the BEAM in the various coordinate directions. The correlation between components (the orientation of the ellipse) is determined by the off-diagonal terms (the σ_{ij} 's). An illustration of this is given below for a 2-dimensional ellipse.

Description of the Sigma BEAM Matrix

Consider a 2-dimensional (x, θ) plane projection of the general 6-dimensional ellipsoid. Let

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

be a real, positive definite, symmetric matrix; the inverse of which is

$$\sigma^{-1} = \frac{1}{\epsilon^2} \begin{bmatrix} \sigma_{22} & -\sigma_{21} \\ -\sigma_{21} & \sigma_{11} \end{bmatrix}$$

where ϵ^2 is the determinant of σ .

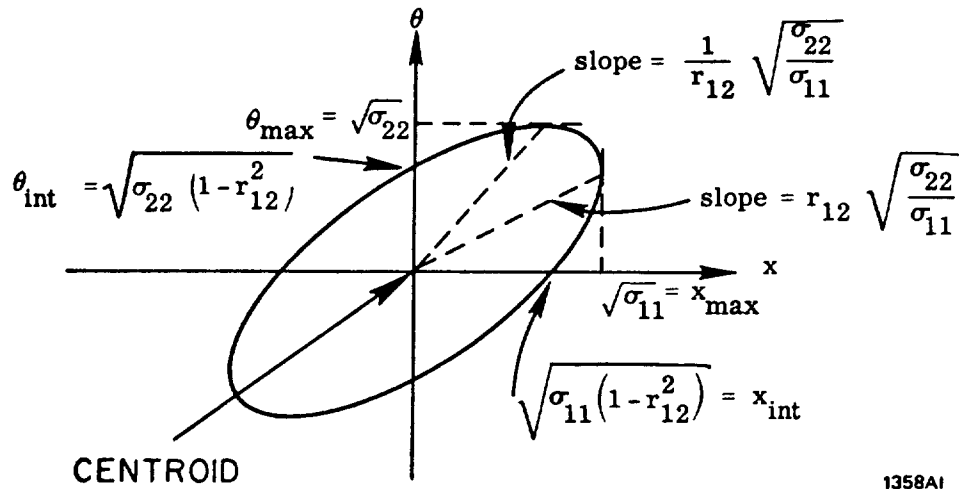
The 2-dimensional coordinate vector (column matrix) and its transpose are:

$$X = \begin{pmatrix} x \\ \theta \end{pmatrix} \text{ and } X^T = (x \ \theta)$$

The expansion of the matrix equation $X^T \sigma^{-1} X = 1$ is the equation of the ellipse

$$\sigma_{22} x^2 - 2\sigma_{21} x\theta + \sigma_{11} \theta^2 = \epsilon^2 = \det \sigma \tag{31}$$

The (x, θ) plane BEAM ellipse represented by Eq. (31) is shown in the following figure along with the physical meaning of the sigma matrix elements.



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The area of the ellipse is given by:

$$A = \pi(\det \sigma)^{1/2} = \pi x_{\max} \theta_{\text{int}} = \pi x_{\text{int}} \theta_{\max} \quad (32)$$

A Two Dimensional BEAM Phase Ellipse

The correlation between x and θ (the orientation of the ellipse) depends upon the off-diagonal term σ_{21} . This correlation is defined as

$$r_{21} = r_{12} = \frac{\sigma_{21}}{\sqrt{\sigma_{11} \sigma_{22}}}$$

So defined r always falls in the range

$$-1 \leq r \leq 1$$

The correlation, r , measures the tilt of the ellipse and the intersection of the ellipse with the coordinate axes.

Since the $\det R = 1$ for all static magnetic beam transport elements, it follows that the determinant of $\sigma(1)$ and $\sigma(0)$ are identical under the transformation of Eq. (30). Hence the "phase space" area is an invariant under the transformation of Eq. (30). This is a statement of Liouville's Theorem for the magnetostatic fields employed and results from the fact that the $\det R = 1$.

It is perhaps worthwhile noting that this 2-dimensional representation of the BEAM matrix has a one to one correspondence with the Courant-Snyder treatment of the theory of the Alternating Gradient Synchrotron* as follows:

$$\begin{bmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \epsilon \begin{bmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{bmatrix} \quad (33)$$

The Phase Ellipse Beam Matrix used by TRANSPORT

For static magnetic systems possessing midplane symmetry, the (x, θ) plane and (y, φ) plane trajectories are decoupled in first-order, i.e. there is no mixing of phase space between the two planes. However for mathematical simplicity and to allow for the possibility of more general systems, the sigma BEAM matrix used in a TRANSPORT calculation has the following general 6-dimensional construction.

	x	θ	y	φ	l	δ
x	$\sigma(11)$					
θ	$\sigma(21)$	$\sigma(22)$				
y	$\sigma(31)$	$\sigma(32)$	$\sigma(33)$			
φ	$\sigma(41)$	$\sigma(42)$	$\sigma(43)$	$\sigma(44)$		
l	$\sigma(51)$	$\sigma(52)$	$\sigma(53)$	$\sigma(54)$	$\sigma(55)$	
δ	$\sigma(61)$	$\sigma(62)$	$\sigma(63)$	$\sigma(64)$	$\sigma(65)$	$\sigma(66)$

* E. D. Courant and H. S. Snyder, "Theory of the Alternating Gradient Synchrotron", Annals of Physics 3, pp 1-48 (1958).

The matrix is symmetric so that only a triangle of elements is needed.

In the printed output this matrix has a somewhat different format for ease of interpretation:

			x	θ	y	φ	l
x	$\sqrt{\sigma(11)}$	CM					
θ	$\sqrt{\sigma(22)}$	MR	r(21)				
y	$\sqrt{\sigma(33)}$	CM	r(31)	r(32)			
φ	$\sqrt{\sigma(44)}$	MR	r(41)	r(42)	r(43)		
l	$\sqrt{\sigma(55)}$	CM	r(51)	r(52)	r(53)	r(54)	
δ	$\sqrt{\sigma(66)}$	PC	r(61)	r(62)	r(63)	r(64)	r(65)

where:
$$r(ij) = \frac{\sigma(ij)}{[\sigma(ii) \sigma(jj)]^{1/2}} \quad (34)$$

As a result of the fact that the matrix is positive definite, the $r(ij)$'s satisfy the relation

$$|r(ij)| \leq 1 \quad (35)$$

The physical meaning of the $\sqrt{\sigma(ii)}$'s is as follows:

$\sqrt{\sigma(11)}$ = x_{\max} = The maximum (half)-width of the beam envelope in the x plane at the point of the printout.

$\sqrt{\sigma(22)}$ = θ_{\max} = The maximum (half)-angular divergence of the beam envelope in the x plane.

$\sqrt{\sigma(33)}$ = y_{\max} = The maximum (half)-height of the beam envelope.

$\sqrt{\sigma(44)}$ = φ_{\max} = The maximum (half)-angular divergence of the beam envelope in the y plane.

$\sqrt{\sigma(55)}$ = l_{\max} = 1/2 the longitudinal extent of the bunch of particles.

$\sqrt{\sigma(66)}$ = δ = The half-width 1/2 ($\Delta P/P$) of the momentum interval being transmitted by the system.

The units appearing next to the $\sqrt{\sigma(ii)}$'s in the TRANSPORT printout sheet are the units chosen for coordinates x, θ, y, φ, l and $\delta = \Delta P/P$ respectively.

To the immediate left of the listing of the beam envelope size in a TRANSPORT printout, there appears a column of numbers whose values will normally be zero. These numbers are the coordinates of the centroid of the beam phase ellipse (with respect to the initially assumed central trajectory of the system). They may become non-zero under one of three circumstances:

- 1) When the misalignment (Type Code 8.) is used.
 - 2) When a Beam Centroid shift (Type Code 7.) is used.
- or
- 3) When a 2nd-order calculation (Type Code 17.) is used.

Physical Interpretation of Various Projections of the 2-dimensional BEAM Ellipse

Consider again Eq. (30) $\sigma(1) = R \sigma(0) R^T$ and expand it in it's most general form for the 2-dimensional (x, θ) plane case.

$$\sigma(1) = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} \sigma_{11}(0) & \sigma_{21}(0) \\ \sigma_{21}(0) & \sigma_{22}(0) \end{pmatrix} \begin{pmatrix} R_{11} & R_{21} \\ R_{12} & R_{22} \end{pmatrix} = \begin{pmatrix} \sigma_{11}(1) & \sigma_{21}(1) \\ \sigma_{21}(1) & \sigma_{22}(1) \end{pmatrix}$$

the result is:

$$\sigma(1) = \left[\begin{array}{c|c} R_{11}^2 \sigma_{11}(0) + 2R_{11}R_{12} \sigma_{21}(0) + R_{12}^2 \sigma_{22}(0) & R_{11}R_{21} \sigma_{11}(0) + (R_{11}R_{22} + R_{12}R_{21}) \sigma_{21}(0) + R_{12}R_{22} \sigma_{22}(0) \\ \hline R_{21}^2 \sigma_{11}(0) + 2R_{21}R_{22} \sigma_{21}(0) + R_{22}^2 \sigma_{22}(0) & \end{array} \right] \quad (36)$$

In the special case when the initial ellipse is erect i.e., $\sigma_{21}(0)=0$, $\sigma(1)$ reduces to:

$$\sigma(1) = \left[\begin{array}{c|c} R_{11}^2 \sigma_{11}(0) + R_{12}^2 \sigma_{22}(0) & R_{11}R_{21} \sigma_{11}(0) + R_{12}R_{22} \sigma_{22}(0) \\ \hline R_{21}^2 \sigma_{11}(0) + R_{22}^2 \sigma_{22}(0) & \end{array} \right] \quad (37)$$

Similar results are, of course, obtained for the (y, φ) plane.

If an arbitrary beam transport system is reduced to the most elementary first-order form of representing it as an initial drift distance, followed by a lens action between two principal planes, and a final drift distance; then we observe that for the 2-dimensional representation there are only two basic phase ellipse transformations of interest.

- (1) An arbitrary DRIFT distance and
- (2) A LENS action

Each of these elementary cases are illustrated on Fig.8 for both a parallelogram as well as ellipse phase space transformations. Note that a DRIFT followed by a LENS action is not necessarily equal to a LENS action followed by a DRIFT; i.e., the matrices do not necessarily commute.

The phase ellipse transformations for a DRIFT and for a LENS action (between principal planes) as shown in Fig. (8) may be readily calculated using the results of Eq. (37).

The 2-dimensional R matrix representing a drift of distance L is:

$$R(\text{Drift}) = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \quad (38)$$

Substituting into Eq. (37) we find

$$\alpha(1) = \left[\begin{array}{c|c} \sigma_{11}(0) + L^2\sigma_{22}(0) & L\sigma_{22}(0) \\ \hline L\sigma_{22}(0) & \sigma_{22}(0) \end{array} \right] = \left[\begin{array}{c|c} \sigma_{11}(1) & \sigma_{21}(1) \\ \hline \sigma_{21}(1) & \sigma_{22}(1) \end{array} \right] \quad (39)$$

Attaching the physical meaning to the matrix elements yields the following interpretations:

$$\alpha_{11}(1) = \alpha_{11}(0) + L^2\sigma_{22}(0)$$

or

$$(\mathbf{x}_1^2)_{\max} = (\mathbf{x}_0^2)_{\max} + L^2 (\theta_0^2)_{\max} \quad (40)$$

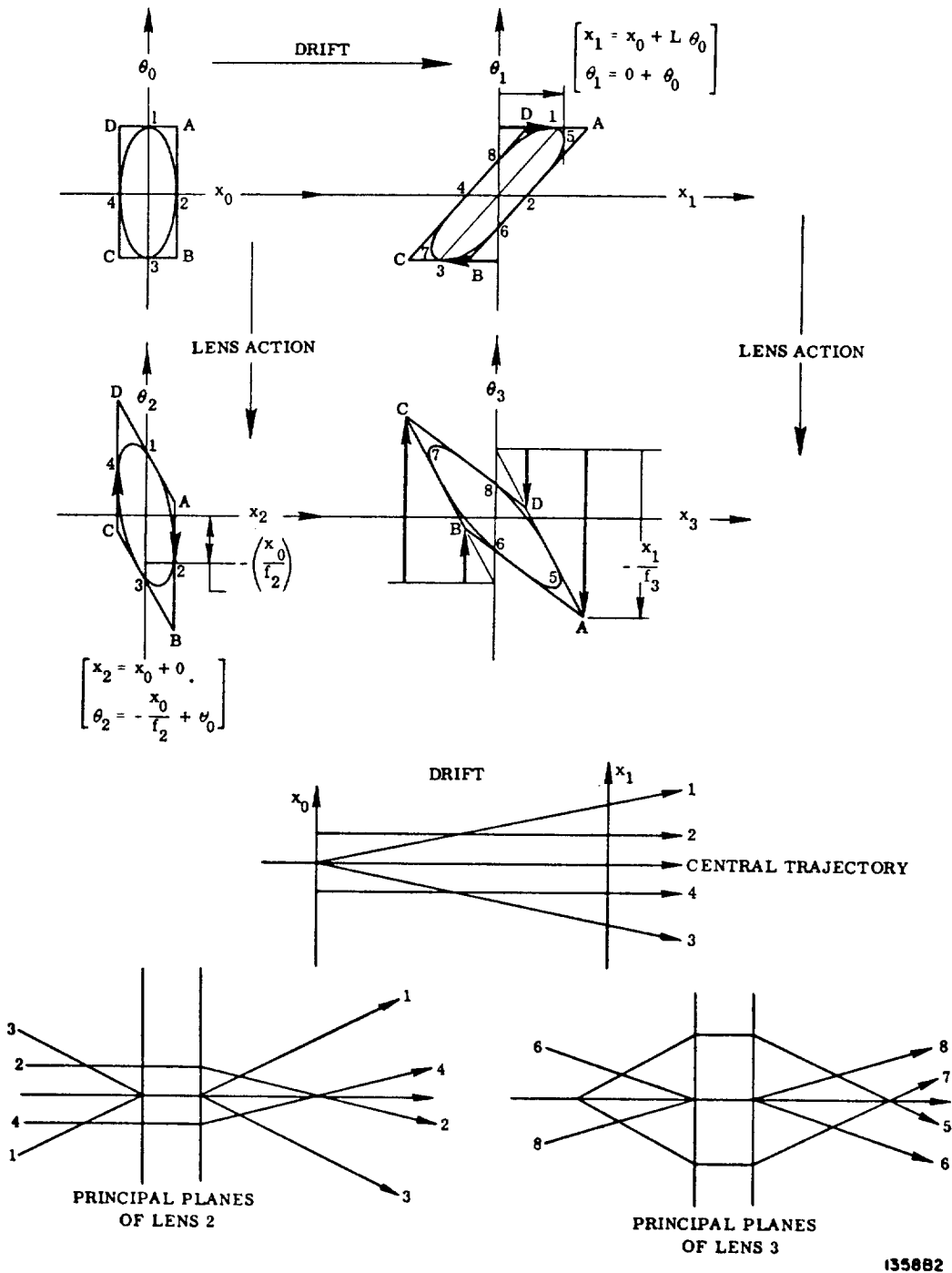


Fig. 8

similarly

$$\sigma_{22}(1) = \sigma_{22}(0)$$

or

$$(\theta_1^2)_{\max} = (\theta_0^2)_{\max} \quad (41)$$

Note that this transformation assumes that the initial phase ellipse is erect, i.e., $\sigma_{21}(0) = 0$.

The 2-dimensional R matrix for a lens actions (between principal planes)

is

$$R(\text{Lens}) = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \quad (42)$$

Substitution into Eq. (37) yields

$$\sigma(2) = \left[\begin{array}{c|c} \sigma_{11}(0) & -\frac{\sigma_{11}(0)}{f} \\ \hline -\frac{\sigma_{11}(0)}{f} & \frac{\sigma_{11}(0)}{f^2} + \sigma_{22}(0) \end{array} \right] = \left[\begin{array}{c|c} \sigma_{11}(2) & \sigma_{21}(2) \\ \hline \sigma_{21}(2) & \sigma_{22}(2) \end{array} \right] \quad (43)$$

Again attaching physical meaning to the matrix elements we have:

$$\sigma_{11}(2) = \sigma_{11}(0)$$

or

$$(x_2^2)_{\max} = (x_0^2)_{\max} \quad (44)$$

and

$$\sigma_{22}(2) = \frac{\sigma_{11}(0)}{f^2} + \sigma_{22}(0)$$

or

$$(\theta_2^2)_{\max} = \frac{1}{f^2} (x_0^2)_{\max} + (\theta_0^2)_{\max} \quad (45)$$

Note the change in sign of the σ_{21} elements for the Drift and the Lens actions indicating the different sense of orientation of the resulting ellipses as illustrated in Fig. 8.

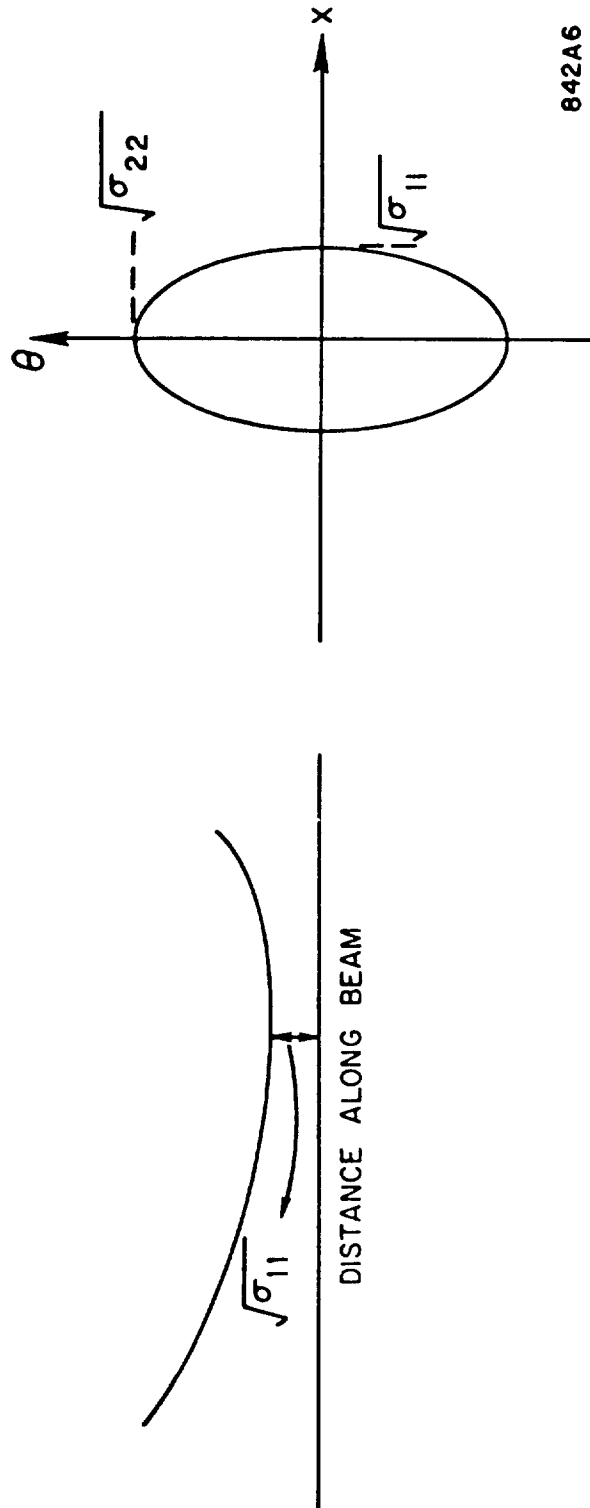
The Upright Ellipse:

A case of particular interest in any 2-dimensional phase ellipse projection (e.g., the (x,θ) plane or the (y,φ) plane) is when the off-diagonal correlation matrix elements are equal to zero; i.e., an erect ellipse. In a field-free region this corresponds to a so-called "waist" in the BEAM as illustrated in Fig. 9.

It is important to understand correctly the meaning of a waist: For an existing beam, it is the location of the minimum beam size in a given region of the system. Although the waist is the minimum beam size in any given beam, the minimum beam size achievable at a fixed target position by varying the focal strength of the preceding lens system is not the same as the above defined waist. See Fig. 10. In a field-free region, the minimum beam spot size achievable at a fixed target position will occur when the preceding lens system is adjusted such that a waist precedes the target position. Only in the limit of zero phase space area do these quantities occur at the same location. A useful criterion that determines the physical proximity of these quantities is the following: If the system has been adjusted for the smallest spot size at a fixed position and if the size of the beam at the principal planes of the optical system is large compared to its size at the waist, or at the minimum spot size, then the location of these quantities, the waist and the minimum, will closely coincide; if, on the other hand, the size of the beam does not change substantially throughout the system, then the locations of a waist and the minimum beam size may (and usually will) differ substantially. The numerical proximity of these two quantities will be discussed in greater detail later in the report.

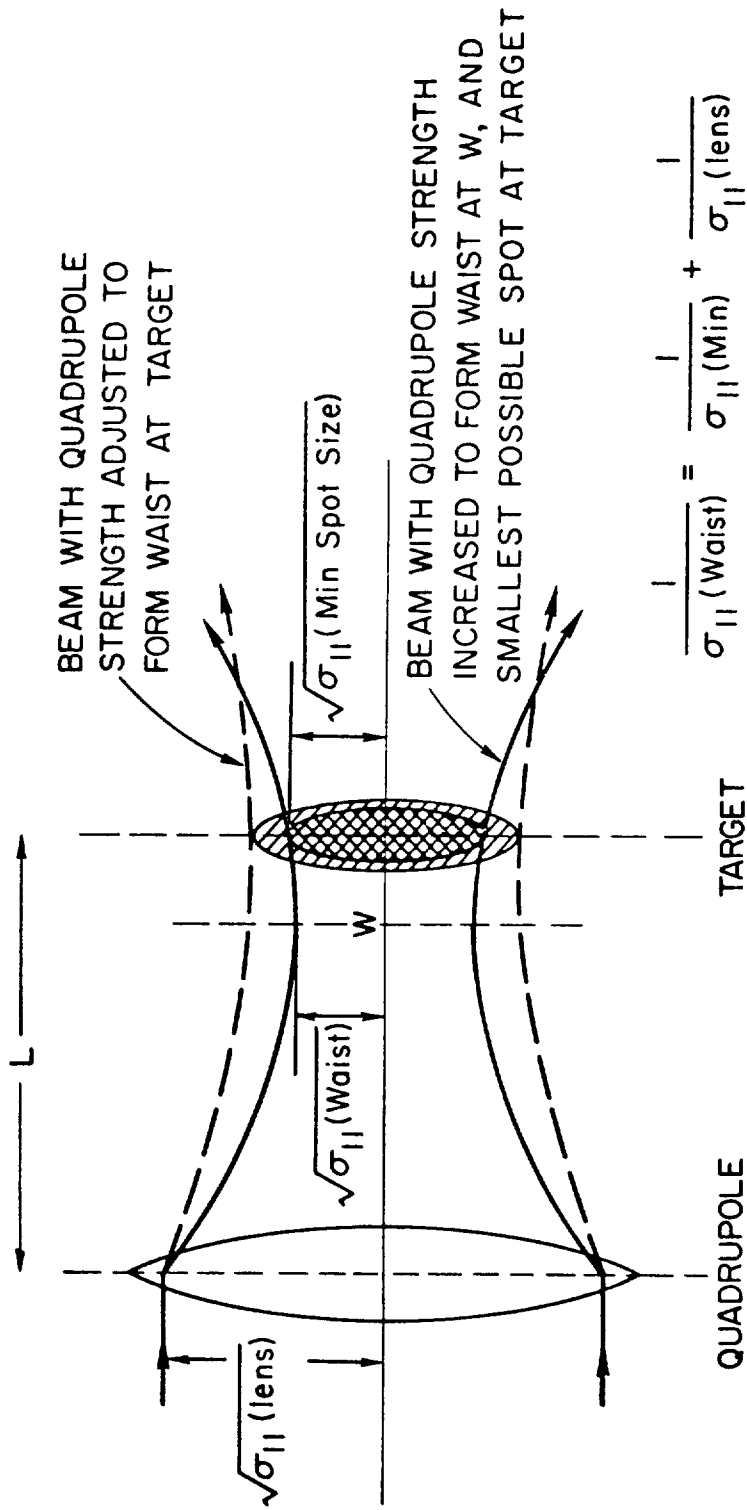
In a field free region (i.e., a Drift), the distance to a waist from any location may be readily calculated if the σ matrix at the location is known. Using Eq. (36) and the R matrix for a Drift (Eq. 38) we have for the (x,θ) plane:

$$\sigma_{21}(1) = \sigma_{21}(0) + L \sigma_{22}(0) = 0 \text{ (specifying that } \sigma(1) \text{ shall be at a waist)}$$



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FIG. 9--BEAM WAIST.



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Fig. 10

or the distance to the waist is:

$$L = -\frac{\sigma_{21}(0)}{\sigma_{22}(0)} = -r_{21} \sqrt{\frac{\sigma_{11}}{\sigma_{22}}} \quad (46)$$

Similarly for the (y, ϕ) plane the distance to a waist is:

$$L = -\frac{\sigma_{43}(0)}{\sigma_{44}(0)} = -r_{43} \sqrt{\frac{\sigma_{33}}{\sigma_{44}}} \quad (47)$$

Relationship between a Waist and a Parallel-to-Point Image

A parallel-to-point image in the (x, θ) plane occurs when $R_{11} = 0$. The R matrix corresponding to this is

$$R = \left(\begin{array}{c|c} 0 & R_{12} \\ \hline R_{21} & R_{22} \end{array} \right) = \left(\begin{array}{c|c} 0 & s_x \\ \hline c_x & s_x \end{array} \right) = \left(\begin{array}{c|c} 0 & f_x \\ \hline -\frac{1}{f_x} & s_x \end{array} \right) \quad (48)$$

Since $|R| = 1$, $R_{12}R_{21} = -1$ for this situation.

If we assume an erect ellipse $\sigma(0)$ as the beginning of the system, the final beam matrix $\sigma(1)$ is given by substitution of Eq. (48) into Eq. (37) as follows:

$$\sigma(1) = \left[\begin{array}{c|c} R_{12}^2 \sigma_{22}(0) & R_{12} R_{22} \sigma_{22}(0) \\ \hline R_{12} R_{22} \sigma_{22}(0) & R_{21}^2 \sigma_{11}(0) + R_{22}^2 \sigma_{22}(0) \end{array} \right] \quad (49)$$

for parallel-to-point imaging.

Several conclusions may be extracted from this result: The first observation is that a waist and a parallel-to-point image will coincide if $R_{11} = R_{22} = 0$. This is equivalent to requiring that the object and image distances (measured to the principal planes) are both equal to the focal length f of the system.

The distance to a waist in this example is:

$$L = -\frac{\sigma_{21}(1)}{\sigma_{22}(1)} = -\frac{R_{22}R_{12}\sigma_{22}(0)}{R_{21}^2\sigma_{11}(0)+R_{22}^2\sigma_{22}(0)} = -\frac{s_x s_x' \sigma_{22}(0)}{(c_x')^2 \sigma_{11}(0) + (s_x')^2 \sigma_{22}(0)} \quad (50)$$

If $s_x s_x' = R_{12}R_{22} = 0$, a waist and a parallel-to-point image coincide. If $s_x s_x' = R_{12}R_{22} > 0$, the waist precedes the image; and if $s_x s_x' = R_{12}R_{22} < 0$, the waist follows the image; unless $\sigma_{22}(0) = 0$ (zero phase space area) in which case a waist and an image always coincide.

The size of the beam at the image is:

$$(x_1^2)_{\max} = \sigma_{11}(1) = R_{12}^2 \sigma_{22}(0) = f^2 (\theta_0^2)_{\max} \quad (51)$$

independent of the source size x_0 and of the object distance.

The size of the beam at the waist is:

$$(\text{size at waist})^2 = \frac{|\sigma(0)|}{\sigma_{22}(1)} = \frac{\sigma_{11}(0)\sigma_{22}(0)}{R_{21}^2\sigma_{11}(0) + R_{22}^2\sigma_{22}(0)} \quad (52)$$

If $R_{22} = s_x' = 0$, the two sizes are equal as expected, otherwise the size at the waist is always smaller.

Relationship between a Waist and a Point-to-Point Image

A point-to-point first-order image in the (x, θ) plane occurs when

$R_{12} = s_x = 0$. The R matrix representing this case is:

$$R = \left(\begin{array}{c|c} R_{11} & 0 \\ \hline R_{21} & R_{22} \end{array} \right) = \left(\begin{array}{c|c} c_x & 0 \\ \hline c_x & s_x \end{array} \right) = \left(\begin{array}{c|c} M & 0 \\ \hline -\frac{1}{f_x} & \frac{1}{M} \end{array} \right) \quad (53)$$

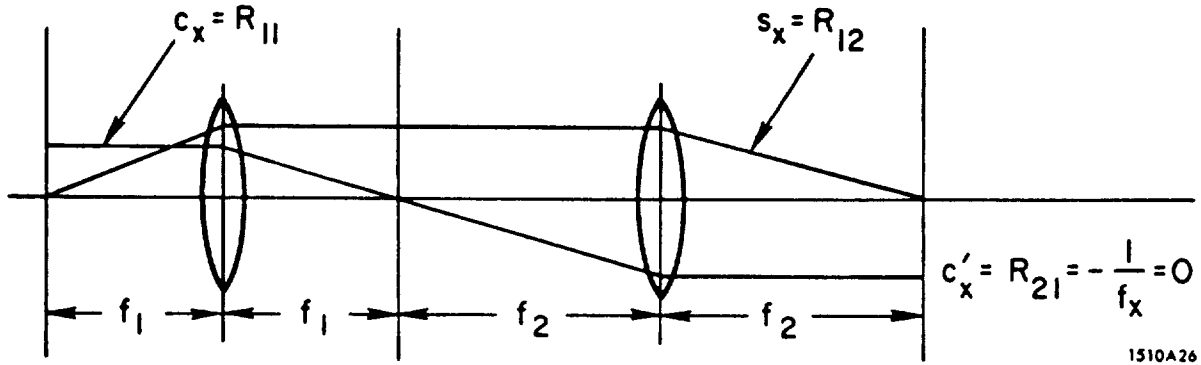
where $|R| = 1 = R_{11}R_{22}$, and M is the magnification.

If we again assume an erect ellipse $\sigma(0)$ as the beginning, the final beam matrix $\sigma(1)$ is given by Eq. (37) as:

$$\sigma(1) = \left[\begin{array}{c|c} R_{11}^2 \sigma_{11}(0) & R_{11} R_{21} \sigma_{11}(0) \\ \hline R_{11} R_{21} \sigma_{11}(0) & R_{21}^2 \sigma_{11}(0) + R_{22}^2 \sigma_{22}(0) \end{array} \right] \quad (54)$$

for point-to-point imaging.

Our first observation is that except for a zero source size, an image and a waist will coincide only if $R_{12} = R_{21} = 0$. Clearly this is not possible with a single lens; at least two lenses are needed. Such an optical situation is as follows:



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The distance to a waist is

$$L = - \frac{\sigma_{21}(1)}{\sigma_{22}(1)} = - \frac{R_{11} R_{21} \sigma_{11}(0)}{R_{21}^2 \sigma_{11}(0) + R_{22}^2 \sigma_{22}(0)} \quad (55)$$

So if $R_{11} R_{21} = c_x c'_x = 0$, a waist and a point-to-point image coincide. If $c_x c'_x > 0$, the waist precedes the image and if $c_x c'_x < 0$, the waist follows the image.

The size of the beam at the image is

$$(\mathbf{x}_1^2)_{\max} = \sigma_{11}(1) = R_{11}^2 \sigma_{11}(0) = (Mx_0)^2 \quad (56)$$

and the size of the beam at the waist is:

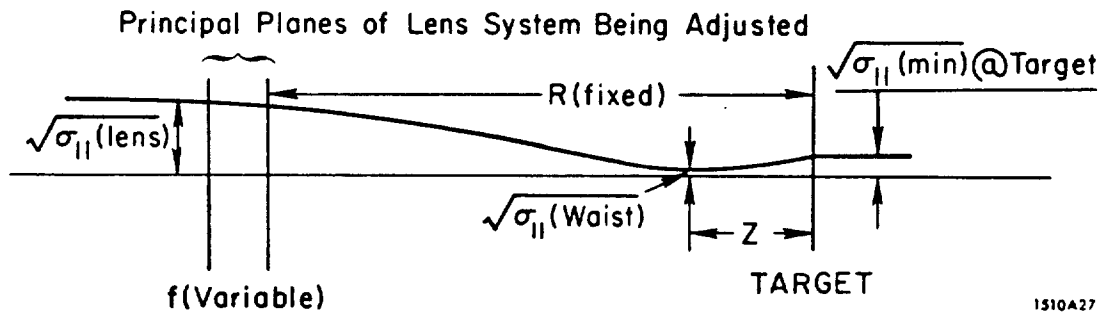
$$(\text{size at waist})^2 = \frac{|\sigma(0)|}{\sigma_{22}(1)} = \frac{\sigma_{11}(0)\sigma_{22}(0)}{R_{21}^2 \sigma_{11}(0) + R_{22}^2 \sigma_{22}(0)} \quad (57)$$

Thus if $R_{21} = 0$, the two sizes are equal since $|R| = R_{11}R_{22} = 1$.

Otherwise the size at the waist is smaller than the image size.

Relationship between a Waist and the Smallest Spot Size Achievable at a Fixed Target Position

Consider the following general situation:



Assume that the size of the beam $\sqrt{\sigma_{11}(\text{lens})}$ at the principal planes of the lens system being adjusted is held constant (i.e., that no other preceding optical elements of the system are being varied); and that the remainder of the system may be represented by a general matrix R which is also held constant. The focal length f is then varied until a minimum spot size $\sqrt{\sigma_{11}(\text{min})}$ is achieved at the target location. The sigma beam matrix at the target position then has the

following unique form independent of the orientation of the initial beam ellipse at the lens.

$$\sigma(\text{at target}) \text{ for a minimum spot size at target} = \left[\begin{array}{c|c} \frac{R_{12}^2 |\sigma|}{\sigma_{11}(\text{lens})} & \frac{R_{12} R_{22} |\sigma|}{\sigma_{11}(\text{lens})} \\ \hline \frac{\sigma_{11}(\text{lens})}{R_{12}^2} + \frac{R_{22}^2 |\sigma|}{\sigma_{11}(\text{lens})} & \end{array} \right] = \left[\begin{array}{c|c} \sigma_{11}(\text{min}) & \sigma_{21}(\text{min}) \\ \hline & \sigma_{22}(\text{min}) \end{array} \right] \quad (58)$$

thus

$$\sigma_{11}(\text{min}) = \frac{R_{12}^2 |\sigma|}{\sigma_{11}(\text{lens})}$$

or

$$x_{\text{min}} = \frac{R_{12} |\sigma|^{1/2}}{x(\text{lens})} \quad (59)$$

If the position of the waist and the minimum beam size both fall within the same field-free region, then the distance to the waist from the target is:

$$z = - \frac{\sigma_{21}(\text{min})}{\sigma_{22}(\text{min})} = - \frac{R_{12} R_{22}}{R_{22}^2 + \frac{\sigma_{11}(\text{lens})}{\sigma_{11}(\text{min})}} = - R_{12} R_{22} \frac{\sigma_{11}(\text{waist})}{\sigma_{11}(\text{lens})} \quad (60)$$

So if $s_x s_x' = R_{12} R_{22} = 0$, the waist and the minimum spot size coincide. If $R_{12} R_{22} > 0$, the waist precedes the target; and if $R_{12} R_{22} < 0$, the waist occurs after the target position.

If the waist and the target positions fall within the same field-free region, the following simple relationship exists between the beam size at the lens $\sqrt{\sigma_{11}(\text{lens})}$, at the waist $\sqrt{\sigma_{11}(\text{waist})}$, and at the target $\sqrt{\sigma_{11}(\text{min})}$.

$$\frac{1}{\sigma_{11}(\text{waist})} = \frac{1}{\sigma_{11}(\text{min})} + \frac{R_{22}^2}{\sigma_{11}(\text{lens})} \quad (61)$$

If now the lens system is readjusted to form a waist at the target position as shown by the dotted lines in Fig. 10, the relative size of this waist and the minimum spot size achieved by the previous lens setting is:

$$\frac{\sigma_{11}(\text{min at target})}{\sigma_{11}(\text{waist at target})} = 1 - \frac{R_{22}^2 \sigma_{11}(\text{waist at target})}{\sigma_{11}(\text{lens})} \quad (62)$$

Again we observe that the two quantities approach each other if the size of the beam at the lens is large compared to the beam size at the target.

There are several cases of special interest that may be derived from the above equations:

1) If $R_{22} = 0$ at the target position, then a minimum spot size at the target is also a waist. This corresponds to point-to-parallel imaging from the principal planes of the variable lens system to the target position. Beyond the last lens in the field-free region preceding the target, $R_{12} = \text{a constant}$ if $R_{22} = 0$; thus we conclude from Eq. (59) that in this field-free region, the minimum spot size achievable at a target is a waist and is independent of the target position. Such a system is a "Zoom" lens.

2) If there are no lenses beyond the variable lens system, i.e., R is an entirely field-free region (a drift), then R is of the form:

$$R = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}$$

In this situation $R_{12}R_{22} = L$ is a positive quantity, consequently the waist always precedes a minimum spot size at a target. A case of particular interest is when the minimum spot size achievable is equal to the initial beam size at the lens. It then follows from Eq. (60) that $Z = -L/2$, i.e., a waist occurs midway between the

lens and the target. From Eq. (61), the ratio of the size of the beam at the lens and at the waist is:

$$\frac{x(\text{lens})}{x(\text{waist})} = \sqrt{\frac{\sigma_{11}(\text{lens})}{\sigma_{11}(\text{waist})}} = \sqrt{2} \quad (63)$$

Combining this result with Eq. (59),

$$L = R_{12} = \frac{x(\text{lens})x(\text{min})}{|\sigma|^{1/2}} = \frac{x^2(\text{lens})}{x(\text{waist})\theta(\text{waist})}$$

or

$$L = \frac{2 x(\text{waist})}{\theta(\text{waist})} \quad (64)$$

where L is the longest distance a beam can drift without exceeding its initial size at the lens.

Imaging from an Erect Ellipse to an Erect Ellipse

The general sigma matrix for imaging from an erect ellipse to an erect ellipse may be derived by inspection from Eq. (36) by setting $\sigma_{21}(1) = \sigma_{21}(0) = 0$.

The result is:

$$\sigma(1) = \left[\begin{array}{c|c} R_{11}^2 \sigma_{11}(0) + R_{12}^2 \sigma_{22}(0) & 0 \\ \hline 0 & R_{21}^2 \sigma_{11}(0) + R_{22}^2 \sigma_{22}(0) \end{array} \right] = \left[\begin{array}{c|c} \sigma_{11}(1) & 0 \\ \hline 0 & \sigma_{22}(1) \end{array} \right] \quad (65)$$

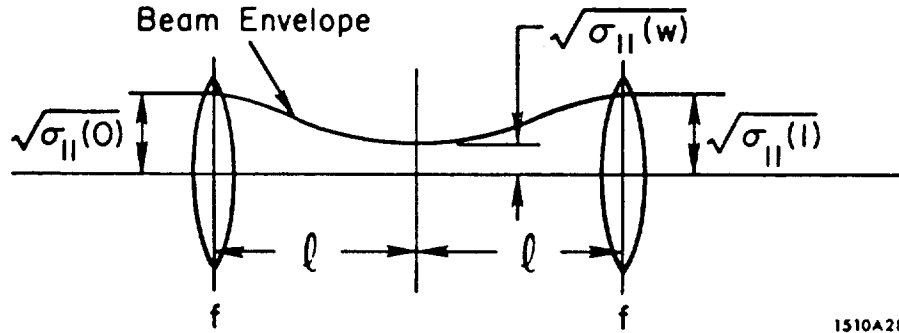
For symmetric magnetic systems $R_{11} = R_{22}$. Using this property and the fact that $|R| = 1$, it follows that $R_{12}R_{21} = (R_{11}^2 - 1)$. So for symmetric magnetic systems Eq. (65) reduces to:

$$\sigma(1) = \left[\begin{array}{c|c} \left[\begin{array}{c} R_{12}|\sigma| \\ -R_{21} \end{array} \right]^{1/2} & 0 \\ \hline 0 & \left[\begin{array}{c} R_{21}|\sigma| \\ -R_{12} \end{array} \right]^{1/2} \end{array} \right] = \left[\begin{array}{c|c} \sigma_{11}(1) & 0 \\ \hline 0 & \sigma_{22}(1) \end{array} \right] \quad (66)$$

The above equations may be used to calculate the optimum design parameters for periodic beam transport systems.

Example No. 1:

Consider a unit-cell of a periodic focusing array consisting of focusing elements only as indicated below.



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The R matrix for the unit-cell, i.e., from the principal planes of the first lens to the principal planes of the second lens, is

$$R = \begin{bmatrix} R_{11} & 2l \\ -\frac{1}{f} + \frac{l}{2f^2} & R_{11} \end{bmatrix} \quad (67)$$

If now we require that the beam envelope possess symmetry coincident with the lens symmetry i.e., that erect ellipses occur at the principal planes of each lens and a waist midway in between, and furthermore that the beam size at the second lens be kept to a minimum and equal to the beam size at the first lens: then substituting Eq.(67) into Eq. (66) and setting $\sigma_{11}(1)$ to be a minimum yields:

$$\begin{aligned} \sqrt{\sigma_{11}(1)} &= \sqrt{\sigma_{11}(0)} \\ f = l &= \frac{1}{2} \sqrt{\frac{\sigma_{11}(0)}{\sigma_{22}(0)}} = \sqrt{\frac{\sigma_{11}(w)}{\sigma_{22}(w)}} \end{aligned} \quad (68)$$

where $\sigma_{11}(0)$ and $\sigma_{22}(0)$ are measured at the principal planes of the first lens.

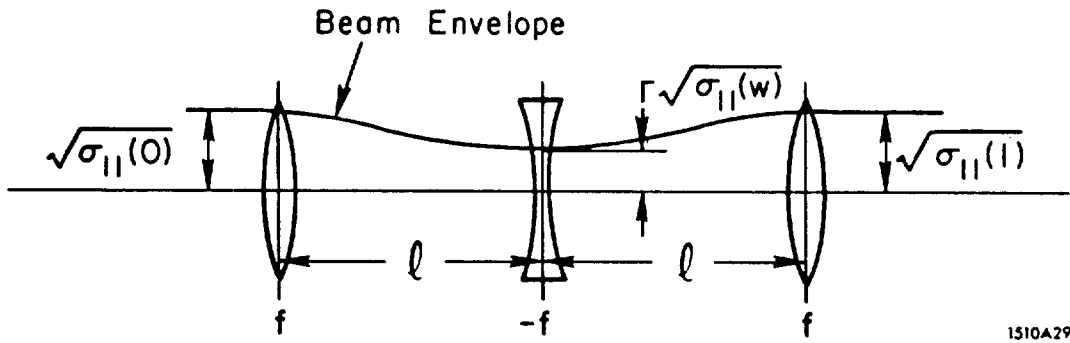
and finally

$$\frac{x(\max)}{x(\min)} = \sqrt{\frac{\sigma_{11}(0)}{\sigma_{11}(w)}} = \sqrt{2} \quad (69)$$

Note that the ratio of the maximum to the minimum beam size (Eq. 69) is independent of the phase space area and of the lens spacing.

Example No. 2:

If the unit cell is a FODO array as follows:



The R matrix for the unit cell (from the principal planes of the first lens to the principal planes of the third lens) is:

$$R = \begin{bmatrix} R_{11} & l(2 + \frac{l}{f}) \\ -\frac{l}{4f^2}(2 - \frac{l}{f}) & R_{11} \end{bmatrix} \quad (70)$$

If we now impose the symmetry requirements that erect ellipses occur at the principal planes of each lens; and that the beam size $\sqrt{\sigma_{11}(1)}$ at lens 3 be a minimum and equal to the beam size $\sqrt{\sigma_{11}(0)}$ at lens 1, then it follows that:

$$\begin{aligned} \sqrt{\sigma_{11}(1)} &= \sqrt{\sigma_{11}(0)} \\ \frac{l}{f} &= \frac{4}{1+\sqrt{5}} \approx 1.236 \end{aligned} \quad (71)$$

$$\frac{\sigma_{11}(0)}{\sigma_{11}(w)} = -\frac{3 + \sqrt{5}}{1 - \sqrt{5}} \approx 4.23$$

or

$$\frac{x(\max)}{x(\min)} = \sqrt{\frac{\sigma_{11}(0)}{\sigma_{11}(w)}} \approx 2.055 \quad (72)$$

and finally

$$l \approx 0.3003 \sqrt{\frac{\sigma_{11}(0)}{\sigma_{22}(0)}} \quad (73)$$

where $\sigma_{11}(0)$ and $\sigma_{22}(0)$ are measured at the principal planes of the first lens in the FODO array.

For a FODO quadrupole array where the field strength is held constant for all elements rather than the focal lengths, the results are somewhat different than those above. This case may be readily calculated via TRANSPORT using the above results as initial guesses in the calculation.

Relationship between a First-Order Point-to-Point Image and the Minimum Spot Size Achievable at a Fixed Target Position

This problem is not as easy to explore as were the preceding ones because the question arises "the first-order image of what?" If, however, we restrict the discussion to a thick or thin lens system that does not have intermediate images between the source and the image under consideration, then the following comments are applicable.

The ratio of the minimum beam size to the size of a first-order image at a fixed target position may be calculated using Eq.'s (56) and (59). From Eq. (59) we have:

$$\sigma_{11}(\min) = \frac{L^2 |\sigma|}{\sigma_{11}(\text{lens})}$$

and from Eq. (56) the size of a first-order image at the target position is:

$$\sqrt{\sigma_{11}(\text{image})} = |M| \sqrt{\sigma_{11}(\text{object})} = \left(\frac{L}{p}\right) \sqrt{\sigma_{11}(\text{object})}$$

where M is the magnification of the first-order image, p is the object distance measured to the principal planes, and L is the distance to the target measured from the principal planes.

The ratio of sizes is

$$\frac{\sigma_{11}(\text{min})}{\sigma_{11}(\text{1st order image})} = \frac{p^2 |\sigma|}{\sigma_{11}(\text{object}) \sigma_{11}(\text{lens})} \quad (74)$$

Using Eq. (36), we may write

$$\sigma_{11}(\text{lens}) = \sigma_{11}(\text{object}) + 2p \sigma_{21}(\text{object}) + p^2 \sigma_{22}(\text{object}) \quad (75)$$

and since

$$|\sigma| = \sigma_{11}(\text{object}) \sigma_{22}(\text{object}) - \sigma_{21}^2(\text{object})$$

it follows that the first-order image will coincide with the smallest spot size only if the orientation of the initial beam ellipse at the object is such that

$$p \sigma_{21}(\text{object}) = - \sigma_{11}(\text{object}) \quad (76)$$

or if $\sigma_{11}(\text{object}) = 0$ i.e., for a point source.

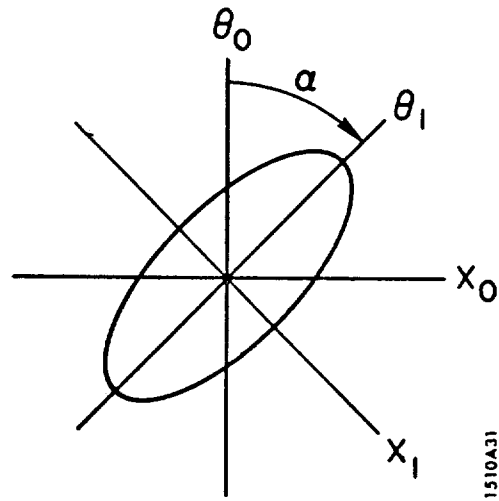
For an erect ellipse at the source and the lens adjusted to provide a minimum spot size at the target, it can be shown that the first-order image will always follow the target position (the minimum spot size) by a distance

$$Z = L|M| \frac{\sigma_{11}(\text{object})}{\sigma_{11}(\text{lens})} = L|M| \frac{x_o^2}{x^2(\text{lens})} \quad (77)$$

where L is the distance to the target position from the principal planes of the lens system, x_o is the source size, and M is the magnification of the first-order image. Again we observe that the ratio of the beam size at the source and the beam size at the "lens" is the criterion determining the proximity of these two quantities.

Orientation of the Major Axes of a Phase Space Ellipse

The matrix equation for a coordinate rotation as shown in Fig. 11 is



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Fig. 11

$$\begin{pmatrix} x_1 \\ \theta_1 \end{pmatrix} = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} x_0 \\ \theta_0 \end{pmatrix} \quad (78)$$

or

$$X_1 = M X_0$$

The equation of an ellipse in either set of coordinates is

$$X^T \sigma^{-1} X = 1 \quad \text{where} \quad X = \begin{pmatrix} x \\ \theta \end{pmatrix}; \quad X^T = (x \ \theta)$$

and the transformation from $\sigma(0)$ to $\sigma(1)$ is

$$\sigma(1) = M \sigma(0) M^T \quad (79)$$

provided $|M| = 1$, which it does.

If we assume a general ellipse for $\sigma(0)$ and an "erect" ellipse for $\sigma(1)$,

i.e.,

$$\sigma(0) = \begin{bmatrix} \sigma_{11}(0) & \sigma_{21}(0) \\ \sigma_{21}(0) & \sigma_{22}(0) \end{bmatrix} \quad \text{and} \quad \sigma(1) = \begin{bmatrix} \sigma_{11}(1) & 0 \\ 0 & \sigma_{22}(1) \end{bmatrix}$$

It follows from Eq. (72) that:

$$\sigma_{21}(1) = 0 = M_{11} M_{21} \sigma_{11}(0) + (M_{11} M_{22} + M_{21} M_{12}) \sigma_{12}(0) + M_{12} M_{22} \sigma_{22}(0)$$

from which

$$\tan 2\alpha = \frac{2\sigma_{21}(0)}{\sigma_{22}(0) - \sigma_{11}(0)} \quad (80)$$

or using the definition

$$r_{21} = \frac{\sigma_{21}}{\sqrt{\sigma_{11} \sigma_{22}}}$$

an alternate form of expressing the ellipse orientation is

$$\tan 2\alpha = \frac{2r_{21} \sqrt{\sigma_{11} \sigma_{22}}}{\sigma_{22} - \sigma_{11}} = \frac{2r_{21}}{\sqrt{\frac{\sigma_{22}}{\sigma_{11}}} - \sqrt{\frac{\sigma_{11}}{\sigma_{22}}}} \quad (81)$$

Clearly α is dependent upon the units chosen for σ_{11} and σ_{22} except in the obvious case of $\alpha = 0$; i.e., an erect ellipse.

SECOND-ORDER ABERRATIONS

Section III

TRANSPORT has the capability of calculating the second-order matrix elements (aberration coefficients) of any static-magnetic beam transport system composed of combinations of bending magnets, quadrupoles, solenoids, sextupoles and interspersed drift spaces. It is assumed that mid-plane symmetry prevails for any given magnetic element in a system (except for solenoids) but not necessarily for the system as a whole. The notation used in a TRANSPORT printout is described in reference 1 (SLAC-75) beginning on page 46. The subscript notation is the same as that used for first-order where the subscript 1 means x, 2 means θ , 3 means y, 4 means φ , 5 means l , and 6 means δ .

The symbol R_{ij} has been used to signify a first-order matrix element and the symbol T_{ijk} will be used to signify a second-order matrix element. Thus we may write the second-order Taylor expansion representing the deviation of an arbitrary trajectory from the central trajectory as:

$$x_i(t) = \sum_{j=1}^6 R_{ij} x_j(0) + \sum_{j=1}^6 \sum_{k=1}^6 T_{ijk} x_j(0) x_k(0)$$

where $x_1=x$ $x_2=\theta$ $x_3=y$ $x_4=\varphi$ $x_5=l$ and $x_6=\delta$ denotes the subscript notation. In an actual computer printout, the T_{ijk} 's are abbreviated as (i jk); for example $T_{126} = (x|\theta_0 \delta)$ would appear in a printout as (1 26) followed by the computed value of the aberration coefficient for the system being designed.

In order to modify the magnitude of any given aberration coefficient, it is necessary to introduce multipole component(s) of the magnetic field of order equal to or less than the order of the aberration. Thus sextupole, quadrupole and dipole components of the field may all be used to modify any given second-order aberration. But, in practice, the second-order aberrations are usually minimized by only introducing sextupole components so as not to disturb the first-order optics of the system. It should always be kept in mind, however, that it may be

beneficial to go back and change the first-order solution (optical mode) so as to provide a more favorable situation for correcting aberrations; a wise selection of the first-order optical mode may in many instances be the deciding factor between the success or failure of a design.

For a fixed location of a sextupole component, the partial derivative of any second-order aberration coefficient T_{ijk} with respect to the strength S_2 of a given sextupole component is a constant. i.e.,

$$\frac{\partial T_{ijk}}{\partial S_2} = \text{a constant} = \text{the coupling coefficient of } S_2 \text{ to } T_{ijk}$$

Thus minimizing a selected group of aberrations is a straight forward problem of solving a set of simultaneous linear equations once the coupling coefficients are known.

The strengths of the sextupole components may be determined directly by TRANSPORT. The user may either constrain certain second order matrix elements to certain values, or may minimize the net second-order contributions to a given component of the beam ellipsoid

Second-Order Phase Ellipsoid Formalism

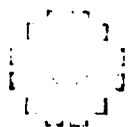
It will be noted by the user, that a second-order TRANSPORT calculation modifies the phase-ellipsoid printout. In a second-order run, TRANSPORT calculates and prints out the second-moments of the phase space distribution function in the $\sqrt{\sigma_{ii}}$ columns. In addition, it also calculates and prints out the new coordinates of the centroid (first-moment) of the phase space distribution function and tabulates this result to the left of the $\sqrt{\sigma_{ii}}$ columns in the same manner as it does for a magnet misalignment run.

Caution should be used in the use and interpretation of the second-order phase ellipsoid results especially if it is known or suspected that the phase space

distribution resulting from a second-order run is not symmetrical about the beam centroid. To be certain of the situation in any given design, it would be wise to calculate the actual distribution function by using the Monte-Carlo computer program TURTLE¹.

The actual method used in TRANSPORT by which the second-order terms are included in the beam ellipse is described in the following report. The reader should bear in mind that the derivation is based on a gaussian initial beam distribution. For any other initial distribution the second order effects on the beam ellipsoid should be regarded only as an approximation.

¹ D. C. Carey, "TURTLE, A Computer Program for Simulating Charged Particle Beam Transport Systems", N.A.L. Report No. 64, Fermi National Accelerator Laboratory, Batavia, Illinois (1971).



Second Order Contributions to Beam Dimensions

David C. Carey

May, 1972

I. Introduction

The phase space region occupied by an aggregate of charged particles in a beam line is often represented by a higher dimensional ellipsoid. Given no further information, one might interpret such an ellipsoid as an envelope inside of which particles are distributed uniformly, or as giving the scale dimensions of a gaussian distribution. The latter case has the advantage that it is easily adapted to include higher order effects of the beam line. In either case the parameters of the ellipsoid are simply related to the first and second moments and therefore the width of the distribution in any coordinate. In first order an ellipsoid at any point in a beam line is transformed into another ellipsoid at any other location in a beam line. In second and higher orders a transformation from one location in a beam line to another will cause the ellipsoid to become distorted. One can still, however, calculate the first and second moments of the distribution, and thereby obtain a measure of its dimensions in any coordinate.

Below we elaborate on the methods for calculating the ellipsoid parameters at any point in the beam line. Much of the first order theory can be found in the work of Brown and Howry.¹ It is included here for completeness.



II. The Ellipsoid Formalism

The position and motion of a particle in a beam line may be represented via a six-dimensional vector.

$$\mathbf{x} = \begin{pmatrix} x \\ \theta \\ y \\ \phi \\ \ell \\ \delta \end{pmatrix} \quad (1)$$

The coordinates x and y represent respectively the horizontal and vertical displacements at the position of the particle, θ and ϕ , the angles with the axis of the beam line in the same planes. The quantity ℓ represents the longitudinal position of the particle relative to a particle traveling on the magnetic axis of the system with the central momentum designed for the system. The remaining quantity $\delta = \frac{\Delta p}{p}$ gives the fractional deviation of the momentum of the particle from the central design momentum of the system.

An ellipsoidal hypersurface in this six-dimensional space may be represented by the equation:

$$\mathbf{x}^T \sigma^{-1} \mathbf{x} = 1 \quad (2)$$

where σ^{-1} is a symmetric positive definite matrix. We represent this matrix as an inverse for reasons which will become apparent later. At this stage the center of the ellipsoid is assumed to lie at the origin of the coordinate system. The ellipsoid may be taken to be the envelope of a uniform distribution, or the scale in a gaussian distribution, giving a particle density:

$$\rho = C \exp(-\frac{1}{2} \mathbf{x}^T \sigma^{-1} \mathbf{x}) \quad (3)$$

For any real symmetric matrix there exists a coordinate system in which that matrix is diagonal and an orthogonal transformation to that coordinate system.² Let us represent the orthogonal transformation by the matrix O , so that:

$$x_i = \sum_j O_{ij} \tilde{x}_j \quad (4)$$

where \tilde{x}_j are the coordinates in the frame where the transform of σ^{-1} and therefore that of σ are diagonal. Calling the matrix σ transformed to the new frame $\tilde{\sigma}$ we now have:

$$\sigma_{ij} = \sum_{kl} O_{ik} \tilde{\sigma}_{kl} O_{jl} \quad (5)$$

and equation (1) becomes

$$\tilde{x}^T \tilde{\sigma}^{-1} \tilde{x} = 1 \quad (1a)$$

Specializing to the gaussian distribution, it is now an easy matter to calculate the second moments in the new frame since the coordinates are decoupled. We arrive at:

$$\overline{\tilde{x}_i \tilde{x}_j} = \tilde{\sigma}_{ij} = \delta_{ij} \tilde{\sigma}_{jj} \quad (6)$$

The second moments in the old frame are now:

$$\begin{aligned} \overline{x_i x_j} &= \overline{\sum_{kl} O_{ik} O_{jl} \tilde{x}_k \tilde{x}_l} = \sum_{kl} O_{ik} O_{jl} \overline{\tilde{x}_k \tilde{x}_l} \\ &= \sum_{kl} O_{ik} O_{jl} \tilde{\sigma}_{kl} = \sigma_{ij}. \end{aligned} \quad (7)$$

Therefore in this case the elements of the matrix σ give the second moments of the distribution in the original coordinate system. The density function, properly normalized, now becomes:

$$\rho = \frac{N_0}{\sqrt{\det(\sigma)} (2\pi)^3} \exp(-\frac{1}{2} x^T \sigma^{-1} x) \quad (8)$$

where N_0 is the total number of particles. Since the matrix O is orthogonal the determinants of σ and $\tilde{\sigma}$ are equal.

The elements of the matrix σ may be put in more convenient form for interpretation. The square roots of the diagonal elements may be taken as giving the half widths x_{oi} of the distribution in a given coordinate while the off-diagonal elements may be related to the correlations r_{ij} , so

$$\begin{aligned}
 x_{oi} &= \sqrt{\sigma_{ii}} \\
 r_{ij} &= \sigma_{ij} / \sqrt{\sigma_{ii}\sigma_{jj}}
 \end{aligned}
 \tag{9}$$

Since, for any positive definite symmetric matrix σ , we have:²

$$\sigma_{ii} \sigma_{jj} - \sigma_{ij}^2 > 0
 \tag{10}$$

the correlations must all obey the inequality

$$|r_{ij}| < 1
 \tag{11}$$

If the ellipsoid is interpreted as describing the envelope of a uniform distribution, then the x_{oi} represent the maximum extents of the beam in the given coordinates.

III. The Effect of a Beam Line

A. First Order

If we now let $x_j^{(1)}$ be the coordinates of a ray at the initial point in a beam line, and $x_i^{(2)}$ the coordinates at some later point, the two are related by the equation:

$$x_i^{(2)} = \sum_j R_{ij} x_j^{(1)}
 \tag{12}$$

If we continue to assume a distribution centered at the origin the first moments at both initial and final point will be zero. The second moments will now be given by:

$$\begin{aligned} \sigma_{ij}^{(2)} &= \overline{x_i^{(2)} x_j^{(2)}} = \sum_{kl} R_{ik} R_{jl} \overline{x_k^{(1)} x_l^{(1)}} \\ &= \sum_{kl} R_{ik} R_{jl} \sigma_{kl}^{(1)} \end{aligned} \quad (13)$$

or more concisely

$$\sigma^{(2)} = R \sigma^{(1)} R^T \quad (14)$$

To first order an ellipsoid at the initial point will transform into an ellipsoid at the final point, so that the equation:

$$x^{(2)T} \sigma^{(2)-1} x^{(2)} = 1 \quad (15)$$

will give the envelope of the particle distribution at the later point.

B. Second Order

In second order the transformation on the coordinates effected by the beam line is given by:

$$x_i^{(2)} = \sum_j R_{ij} x_j^{(1)} + \sum_{jk} T_{ijk} x_j^{(1)} x_k^{(1)}. \quad (16)$$

We employ here a symmetric T matrix whose off-diagonal elements are half those of the T matrix used by Brown. The first and second moments of the distribution at the final point are now given by:

$$\begin{aligned} \overline{x_i^{(2)}} &= \sum_j R_{ij} \overline{x_j^{(1)}} + \sum_{jk} T_{ijk} \overline{x_j^{(1)} x_k^{(1)}} \\ \overline{x_i^{(2)} x_j^{(2)}} &= \sum_{kl} R_{ik} R_{jl} \overline{x_k^{(1)} x_l^{(1)}} \\ &+ \sum_{klm} \left[R_{ik} T_{jlm} + T_{ikl} R_{jm} \right] \overline{x_k^{(1)} x_l^{(1)} x_m^{(1)}} \\ &+ \sum_{klmn} T_{ikl} T_{jmn} \overline{x_k^{(1)} x_l^{(1)} x_m^{(1)} x_n^{(1)}} \end{aligned} \quad (17)$$

For a symmetric, on-axis initial distribution, the first and third moments vanish. The problem now reduces to determining the fourth moments of the initial distribution.

As an extension of previous notation we now denote the fourth moments of the distribution about the initial point by $\sigma_{ijkl}^{(1)}$. We consider the coordinate system in which the matrix of second moments σ_{ij} is diagonalized, denoting the moments in this frame by $\tilde{\sigma}$. Then from equation (7) we have:

$$\begin{aligned} \sigma_{ij} &= \sum_{kl} O_{ik} O_{jl} \tilde{\sigma}_{kl} \\ &= \sum_k O_{ik} O_{jk} \tilde{\sigma}_{kk} \end{aligned} \tag{18}$$

We continue to specialize to a gaussian distribution so that the fourth moments will be directly derivable from the second moments. In the diagonal frame the coordinates separate, and the fourth moments are easily calculated. The only ones which are non-zero are $\tilde{\sigma}_{iijj}$, $\tilde{\sigma}_{ijij}$, or $\tilde{\sigma}_{ijji}$ for $i \neq j$, and $\tilde{\sigma}_{iiii}$ with:

$$\begin{aligned} \tilde{\sigma}_{iijj} &= \tilde{\sigma}_{ii} \tilde{\sigma}_{jj} \\ \tilde{\sigma}_{ijij} &= \tilde{\sigma}_{ii} \tilde{\sigma}_{jj} \\ \tilde{\sigma}_{ijji} &= \tilde{\sigma}_{ii} \tilde{\sigma}_{jj} \\ \tilde{\sigma}_{iiii} &= 3 \tilde{\sigma}_{ii} \tilde{\sigma}_{ii} \end{aligned} \tag{19}$$

so that in general:

$$\begin{aligned} \tilde{\sigma}_{ijkl} &= \delta_{ij} \delta_{kl} \tilde{\sigma}_{ii} \tilde{\sigma}_{kk} + \delta_{ik} \delta_{jl} \tilde{\sigma}_{ii} \tilde{\sigma}_{jj} \\ &+ \delta_{il} \delta_{jk} \tilde{\sigma}_{ii} \tilde{\sigma}_{jj}. \end{aligned} \tag{20}$$

Now if under the transformation O, the fourth moments transform as:

$$\sigma_{ijkl} = \sum_{mnop} O_{im} O_{jn} O_{ko} O_{lp} \tilde{\sigma}_{mnop} \quad (21)$$

then from equation (1a) we finally arrive at:

$$\sigma_{ijkl} = \sigma_{ij} \sigma_{kl} + \sigma_{ik} \sigma_{jl} + \sigma_{il} \sigma_{jk} \quad (22)$$

Substituting into equation (17) we determine that:

$$\begin{aligned} \overline{x_i^{(2)}} &= \sum_{jk} T_{ijk} \sigma_{jk}^{(1)} \quad (23) \\ \overline{x_i^{(2)} x_j^{(2)}} &= \sum_{kl} R_{ik} R_{jl} \sigma_{kl}^{(1)} + \left(\sum_{kl} T_{ikl} \sigma_{kl}^{(1)} \right) \left(\sum_{mn} T_{jmn} \sigma_{mn}^{(1)} \right) \\ &\quad + 2 \sum_{lm} \sum_k \left(\sum T_{ikl} \sigma_{km}^{(1)} \right) \left(\sum T_{jmn} \sigma_{ln}^{(1)} \right) \end{aligned}$$

Note that, because of the symmetry properties of both T and σ that the two expressions in parentheses in the last term of the second equation represent the same array. From a practical standpoint this means that it needs to be calculated only once.

We see from equation (23) that the centroid of the distribution at the final point no longer coincides with the beam axis. Letting $\sigma^{(2)}$ represent the matrix of second moments about the new centroid we now have:

$$\begin{aligned} \sigma_{ij}^{(2)} &= \overline{x_i^{(2)} x_j^{(2)}} - \overline{x_i^{(2)}} \overline{x_j^{(2)}} \quad (24) \\ &= \sum_{kl} R_{ik} R_{jl} \sigma_{kl}^{(1)} \\ &\quad + 2 \sum_{lm} \sum_k \left(\sum T_{ikl} \sigma_{km}^{(1)} \right) \left(\sum T_{jmn} \sigma_{ln}^{(1)} \right) \end{aligned}$$

IV. Off-Axis Initial Distribution

Now consider a gaussian distribution whose center does not

coincide with the beam axis. Letting the coordinates of the centroid by $\overline{x_i^{(1)}}$, we have for the coordinates of a ray:

$$x_i^{(1)} = \overline{x_i^{(1)}} + \xi_i^{(1)} \quad (25)$$

We let the matrix σ represent the moments of the distribution about its centroid so that:

$$\overline{\xi_i^{(1)} \xi_j^{(1)}} = \sigma_{ij}^{(1)} \quad (26)$$

$$\overline{\xi_i^{(1)} \xi_j^{(1)} \xi_k^{(1)} \xi_l^{(1)}} = \sigma_{ijkl}^{(1)}$$

Equation (17) continues to hold for the moments of the distribution about the beam axis, while equation (22) holds for the moments about the centroid. We must therefore express one set of moments in terms of the other.

Using equations (22), (25), and (26) and applying the first part of equation (24) to the initial distribution, the initial third and fourth moments are given in terms of the initial first and second moments as follows:

$$\begin{aligned} \overline{x_i^{(1)} x_j^{(1)} x_k^{(1)}} &= \overline{x_i^{(1)} x_j^{(1)} x_k^{(1)}} + \overline{x_j^{(1)} x_i^{(1)} x_k^{(1)}} \\ &+ \overline{x_k^{(1)} x_i^{(1)} x_j^{(1)}} - 2 \overline{x_i^{(1)} x_j^{(1)} x_k^{(1)}} \\ \overline{x_i^{(1)} x_j^{(1)} x_k^{(1)} x_l^{(1)}} &= \overline{x_i^{(1)} x_j^{(1)} x_k^{(1)} x_l^{(1)}} + \overline{x_i^{(1)} x_k^{(1)} x_j^{(1)} x_l^{(1)}} \\ &+ \overline{x_i^{(1)} x_l^{(1)} x_j^{(1)} x_k^{(1)}} - 2 \overline{x_i^{(1)} x_j^{(1)} x_k^{(1)} x_l^{(1)}} \end{aligned} \quad (27)$$

Substituting into equation (17) and rearranging terms we arrive at the following expressions for the first and second moments of the distribution at the final point.

$$\begin{aligned} \overline{x_i^{(2)}} &= \sum_j R_{ij} \overline{x_j^{(1)}} + \sum_{jk} T_{ijk} \overline{x_j^{(1)}} \overline{x_k^{(1)}} \\ \overline{x_i^{(2)} x_j^{(2)}} &= \sum_{kl} R_{ik} R_{jl} \overline{x_k^{(1)} x_l^{(1)}} + \overline{x_i^{(2)} x_j^{(2)}} - 2 \overline{x_i^{(2)} x_j^{(2)}} \\ &+ 2 \sum_{lm} (R_{il} \overline{x_m^{(1)}} + \sum_k T_{ikl} \overline{x_k^{(1)} x_m^{(1)}}) (R_{jm} \overline{x_l^{(1)}} + \\ &\sum_n T_{jmn} \overline{x_l^{(1)} x_n^{(1)}}) - (\sum_k R_{ik} \overline{x_k^{(1)}}) (\sum_m R_{jm} \overline{x_m^{(1)}}) \end{aligned} \tag{28}$$

where

$$\overline{x_i^{(2)}} = \sum_k R_{ik} \overline{x_k^{(1)}} + \sum_{kl} T_{ikl} \overline{x_k^{(1)} x_l^{(1)}}$$

is the image of the original centroid.

We may now again use equations (9) and (24) to relate this matrix of second moments to the final beam half widths and correlations.

References

1. Karl L. Brown, Sam K. Howry, SLAC Report No. 91 (1970).
2. F. R. Gantmacher, The Theory of Matrices, Chelsea Publishing Co., New York (1959).

A Systematic Procedure for Designing High Resolving Power Beam Transport Systems or Charged Particle Spectrometers

The following is a report submitted to the Third International Magnet Symposium held in Hamburg, Germany - May 1970. It is a general description of a suggested procedure for designing systems to any order and includes the derivation of the coupling coefficient of an n th-order multipole to any n th-order aberration coefficient. The report also derives the multipole strengths for the three techniques for introducing multipole components into a system: namely, 1) pure multipole fields, 2) non-uniform fields, and 3) contoured entrance or exit boundaries of magnets.

The notation used in this report is identical to TRANSPORT notation except for the following:

Replace x' and y' in the report by θ and φ respectively to convert to TRANSPORT notation.

REPORT SUBMITTED TO THE THIRD INTERNATIONAL MAGNET SYMPOSIUM
HELD IN HAMBURG, GERMANY-May 1970

by

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Summary

By extrapolating the systematics of the general first- and second-order theory of beam transport optics (1,2,3) to include higher order multipole terms, it has been possible to evolve a simple, step by step, procedure for the design of high resolving power static-magnetic beam transport systems. The choice of the appropriate dipole and quadrupole elements for a given system may be determined once the resolving power, solid angle, momentum range and detector system of the instrument have been specified. The partial derivative of any nth-order aberration coefficient with respect to an nth-order multipole component located anywhere in the system has been derived. From this "coupling coefficient", the strength and the optimum location of multipole element(s) to correct or modify a given aberration or group of aberrations is uniquely determined.

I. Introduction

Within the last two decades, significant advances have been made in the understanding of charged particle optics. Perhaps the first major contribution was the development of the theory of the Alternating Gradient Synchrotron (A.G.S.) by Courant, Livingston, and Snyder(4) which led to the first-order matrix algebra formulation of beam-transport optics. Subsequent to this a second-order matrix algebra was developed by Brown, Belbeoch, and Bounin(5); followed by the development at SLAC of the digital computer program called TRANSPORT(6) that is widely used today in many laboratories for solving first- and second-order static-magnetic beam transport problems. In principle, the second-order matrix formalism may be extended to any order, but in practice this approach has proved to be too cumbersome. Thus beyond second-order it has been more efficient to use computer ray-tracing programs which integrate the basic differential equation of motion of the charged particles through the known or assumed magnetic fields. The fundamental difficulty with ray-tracing has been the required computational time to complete a design involving the minimization of many higher-order aberrations.

In this report, we will outline a systematic procedure for the design of high-resolution systems based upon the extrapolation of the first- and second-order theory (1,2,3) to include higher-order multipole components. A general equation has been derived for the coupling coefficient of an nth-order multipole to any given nth-order aberration coefficient. As will be shown later, these coupling coefficients are a function only of the characteristic first-order trajectories (matrix elements) introduced and defined in References 1 and 2.

Given this information, a systematic procedure for designing high resolution beam transport systems is as follows:

- 1) Find a satisfactory first-order solution to the problem using TRANSPORT or its equivalent.
- 2) Calculate and make the necessary corrections to the second-order aberrations by introducing sextupole components into the system. The "best" locations and strengths of the sextupole components required may be selected via the coupling coefficients for the aberrations to be minimized.
- 3) Calculate and make the necessary corrections (via ray-tracing) to the third-order aberrations by introducing octupole components into the system. (Note that an nth-order multipole couples with terms of order n or higher but not with terms of order lower than n. Thus an octupole component will not disturb the first- and second-order solutions already found from steps 1 and 2.)
- 4) Repeat the above procedure up to the multipole order desired or needed to achieve the design objectives.

If the design requires a solution to nth-order and m multipoles at each order are necessary to minimize the aberrations, the number of computer runs previously needed to complete a design was at least $(n+m)^2$. Having a knowledge of the coupling coefficients, after the first-order design has been selected, now (in principle) reduces the number of computer runs required to n. Since ray-tracing is very time consuming, this is indeed a significant saving.

II. Theory*

The following results are applicable to static-magnetic charged particle optical systems possessing median plane symmetry. As in Ref. 1, we shall use a right-handed curvilinear coordinate system (x,y,t) where x and y are the transverse coordinates. x is the outward normal distance in the median plane away from the central trajectory, y is the perpendicular distance from the median plane, t is the distance along the central trajectory, and $h=h(t)$ is the curvature of the central trajectory.

The existence of the median plane requires that the scalar potential Φ be an odd function of y , i.e., $\Phi(x,y,t) = -\Phi(x,-y,t)$. The most general form of Φ may therefore, be expressed as follows:

$$\Phi(x,y,t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{2m+1,n} \frac{x^n}{n!} \frac{y^{2m+1}}{(2m+1)!} \quad (1)$$

where the coefficients $A_{2m+1,n}$ are functions of t .

In this coordinate system, the differential line element dT is given by

$$dT^2 = dx^2 + dy^2 + (1+hx)^2 dt^2$$

The Laplace equation has the form

$$\nabla^2 \Phi = \frac{1}{(1+hx)} \frac{\partial}{\partial x} \left[(1+hx) \frac{\partial \Phi}{\partial x} \right] + \frac{\partial^2 \Phi}{\partial y^2} + \frac{1}{(1+hx)} \frac{\partial}{\partial t} \left[\frac{1}{(1+hx)} \frac{\partial \Phi}{\partial t} \right] = 0 \quad (2)$$

Substitution of (1) into (2) gives the following recursion formula for the coefficients:

$$\begin{aligned} -A_{2m+3,n} &= A''_{2m+1,n} + nhA''_{2m+1,n-1} - nh'A'_{2m+1,n-1} + A_{2m+1,n+2} \quad (3) \\ &+ (3n+1)hA_{2m+1,n+1} + n(3n-1)h^2A_{2m+1,n} + n(n-1)h^3A_{2m+1,n-1} \\ &+ 3nhA_{2m+3,n-1} + 3n(n-1)h^2A_{2m+3,n-2} + n(n-1)(n-2)h^3A_{2m+3,n-3} \end{aligned}$$

where prime means $\frac{d}{dt}$, and where it is understood that all coefficients A with one or more negative subscripts are zero. This recursion formula expresses all

* The notation used in this report follows that used in Ref. 1 unless otherwise indicated.

the coefficients in terms of the midplane field $B_y(x, 0, t)$:

where

$$A_{1,n} = \left(\frac{\partial^n B_y}{\partial x^n} \right)_{\substack{x=0 \\ y=0}} = \text{functions of } t. \quad (4)$$

Since φ is an odd function of y , on the median plane we have $B_x = B_t = 0$. The normal (in x direction) derivatives of B_y on the reference curve defines B_y over the entire median plane, hence the magnetic field \vec{B} over the whole space. The components of the field are expressed in terms of φ explicitly by $\vec{B} = \nabla\varphi$ or

$$\begin{aligned} B_x &= \frac{\partial\varphi}{\partial x} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{2m+1,n+1} \frac{x^n}{n!} \frac{y^{2m+1}}{(2m+1)!} \\ B_y &= \frac{\partial\varphi}{\partial y} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{2m+1,n} \frac{x^n}{n!} \frac{y^{2m}}{(2m)!} \\ B_t &= \frac{1}{(1+hx)} \frac{\partial\varphi}{\partial t} = \frac{1}{(1+hx)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A'_{2m+1,n} \frac{x^n}{n!} \frac{y^{2m+1}}{(2m+1)!} \end{aligned} \quad (5)$$

The expression for the magnetic field on the midplane is

$$B_y(x, 0, t) = \sum_{n=0}^{\infty} A_{1,n} \frac{x^n}{n!} \quad (6)$$

At this point we deviate from the notation and formalism of Ref. 1 and introduce $K_n(t)$, the multipole strength per unit length; and S_n , the total multipole strength of a static-magnetic field.

We rewrite equation (6) as

$$B_y(x, 0, t) = B\rho \sum_{n=0}^{\infty} K_n(t) x^n \quad (7)$$

Where $B\rho = \frac{B}{h} = \frac{P_0}{e}$ is the magnetic rigidity of a particle of momentum P_0 and charge e along the central trajectory; from which

$$K_n(t) = \left(\frac{1}{B\rho} \right) \left(\frac{1}{n!} \right) \left(A_{1,n}(t) \right) = \left(\frac{1}{B\rho} \right) \left(\frac{1}{n!} \right) \left(\frac{\partial^n B_y}{\partial x^n} \right)_{x=y=0} \quad (8)$$

We define S_n as

$$S_n = \int_0^L K_n(t) dt \quad (9)$$

S_n so defined is the strength of the n th-order multipole component of a field over the interval of integration.

Multipole Strengths for Pure Multipole Fields

Consider the scalar potential of an n th-order $[2(n+1)\text{pole}]$ pure multipole element:

$$\varphi = \frac{B_o r^{n+1}}{(n+1)a^n} \left[\sin(n+1) \theta \right] \quad (10)$$

where

$$x = r \cos\theta \quad \text{and} \quad y = r \sin\theta$$

B_o is the field at the pole and a is the radial distance to the pole from the central trajectory.

Expanding φ as a function of x and y and differentiating, we have

$$B_y = \frac{\partial\varphi}{\partial y} = \frac{B_o}{a^n} \left[x^n + \dots \dots \dots \right]$$

From which

$$K_n = \left(\frac{B_o}{a^n} \right) \left(\frac{1}{B\rho} \right)$$

and

$$S_n = \left(\frac{B_o}{a^n} \right) \left(\frac{L}{B\rho} \right) \quad (11)$$

Where L is the length of the multipole element.

For a dipole $n=0$ and the dipole strength is

$$S_0 = \frac{L}{\rho} = \alpha \text{ (The angle of bend of the central trajectory)}$$

For a quadrupole $n=1$ and

$$S_1 = \left(\frac{B_0}{a} \right) \left(\frac{L}{B\rho} \right)$$

For a sextupole $n=2$ and

$$S_2 = \left(\frac{B_0}{a^2} \right) \left(\frac{L}{B\rho} \right)$$

etc. for higher-order multipoles.

Multipole Strengths for a Non-Uniform Field Expansion

From the midplane field expansion of a non-uniform magnetic field

$$B_y(x, o, t) = B_y(o, o, t) \left[1 - nhx + \beta(hx)^2 + \gamma(hx)^3 + \dots \right] \quad (12)$$

the multipole strength factors are:

$$K_0 = h, \quad K_1 = -nh^2, \quad K_2 = \beta h^3, \quad \text{etc.}$$

and S_n evaluated over the length L of the central trajectory is:

$$S_0 = hL = \alpha \text{ as before,}$$

$$S_1 = -nh^2L, \quad \text{and} \quad S_2 = \beta h^3L, \quad \text{etc.}$$

Multipole Strengths for a Contoured Entrance or Exit Boundary of a Magnet

A third method of introducing multipole components is via a curved entrance or exit boundary of a magnet. To calculate the multipole strengths in this case, we integrate equation (7), holding x constant, as follows:

$$\int_0^L B_y(x, o, t) dt = B\rho \sum x^n \int_0^L K_n(t) dt = B\rho \sum S_n x^n \quad (13)$$

To relate this to the field boundary, we assume B_y to be a constant inside the effective field boundary and zero outside (i.e., we ignore the finite extent of the fringing field). In this sharp-cutoff approximation, the field boundary $Z = Z(x)$ is:

$$Z = \frac{1}{B_y} \int_0^L B_y(x, 0, t) dt = \frac{1}{h} \sum S_n x^n = -x \tan \beta + \frac{S_2}{h} x^2 + \dots \quad (14)$$

where $h = \frac{1}{\rho}$ and β is the angle of rotation of the entrance or exit face of the magnet at $x=0$. A positive β implies radial (x) defocusing and transverse (y) focusing. We note that:

$$S_1 = -h \tan \beta = \text{The "quadrupole strength"}$$

The radius of curvature of the boundary is related to the sextupole strength as follows:

$$\frac{1}{R} = \frac{Z''}{(1+Z'^2)^{3/2}} = \frac{2 S_2}{h \sec^3 \beta}$$

or

$$S_2 = \frac{h \sec^3 \beta}{2R} = \text{The "sextupole strength"}$$

From equation (13), we note that a positive multipole component of the field increases the $\int B dt$ for a positive x; thus a positive sextupole is represented by a concave surface of the entrance or exit boundary.

The Description of the Trajectories as a Taylor's Expansion

The deviation of an arbitrary trajectory from the central trajectory is described by expressing x and y as functions of t . The expressions will also contain x_0, y_0, x'_0, y'_0 and δ , where the subscript 0 indicates that the quantity is evaluated at $t=0$. The prime (') denotes the derivative with respect to t , and $\delta = \frac{\Delta P}{P_0}$ is the fractional momentum deviation of the ray from that of the central trajectory. These five initial boundary values will have the value zero for the central trajectory itself. x and y are expressed as a five-fold Taylor expansion using these initial boundary values. The expansions are written:

$$\begin{aligned}
 x(t) &= \sum (x|x_0^\kappa y_0^\lambda x_0'^\mu y_0'^\nu \delta^\chi) x_0^\kappa y_0^\lambda x_0'^\mu y_0'^\nu \delta^\chi \\
 y(t) &= \sum (y|x_0^\kappa y_0^\lambda x_0'^\mu y_0'^\nu \delta^\chi) x_0^\kappa y_0^\lambda x_0'^\mu y_0'^\nu \delta^\chi
 \end{aligned}
 \tag{15}$$

Here, the parentheses are symbols for the Taylor coefficients; the first part of the symbol identifies the coordinate represented by the expansion, and the second indicates the term in question. These coefficients are functions of t to be determined. The symbol \sum indicates summation over zero and all positive integer values of the exponents $\kappa, \lambda, \mu, \nu, \chi$; . The constant term is zero, and the terms that would indicate a coupling between the coordinates x and y are also zero; this results from the midplane symmetry. Thus we have

$$\begin{aligned}
 (x|1) &= (y|1) = 0 \\
 (x|y_0) &= (y|x_0) = 0 \\
 (x|y_0') &= (y|x_0') = 0
 \end{aligned}
 \tag{16}$$

Here, the first line is a consequence of choosing the central trajectory as the reference axis, while the second and third lines follow directly from considerations of median plane symmetry.

Since they will appear often in the formalism, it is convenient to introduce the following abbreviations for the first-order Taylor coefficients:

$$\begin{aligned}
 (x|x_0) &= c_x(t) & (x|x_0') &= s_x(t) & (x|\delta) &= d_x(t) \\
 (y|y_0) &= c_y(t) & (y|y_0') &= s_y(t)
 \end{aligned}
 \tag{17}$$

When the transverse position of an arbitrary trajectory at position t is written as a first-order Taylor's expansion as a function of the initial boundary conditions, the above five quantities are just the coefficients appearing in the expansion for the transverse coordinates x and y as follows:

$$x(t) = c_x(t) x_0 + s_x(t) x_0' + d_x(t) \delta + \text{higher-order terms}$$

and

$$y(t) = c_y(t) y_0 + s_y(t) y_0' + \text{higher-order terms.}$$

III. Solution of the Equations of Motion

The general differential equation of motion of a charged particle in a static-magnetic field valid to all orders in x and y and their derivatives as derived in Ref. 1, equation (5) is:

$$\begin{aligned} & \hat{x} \left\{ [x'' - h(1+hx)] - \frac{x'}{(T')^2} [x'x'' + y'y'' + (1+hx)(hx'+h'x)] \right\} \\ & + \hat{y} \left\{ y'' - \frac{y'}{(T')^2} [x'x'' + y'y'' + (1+hx)(hx'+h'x)] \right\} \\ & + \hat{t} \left\{ (2hx'+h'x) - \frac{(1+hx)}{(T')^2} [x'x'' + y'y'' + (1+hx)(hx'+h'x)] \right\} \\ & = \frac{e}{P} T' (\vec{T}' \times \vec{B}) = \frac{e}{P} T' \left\{ \hat{x}[y'B_t - (1+hx)B_y] + \hat{y}[(1+hx)B_x - x'B_t] \right. \\ & \qquad \qquad \qquad \left. + \hat{t}[x'B_y - y'B_x] \right\} \end{aligned} \tag{18}$$

If this equation is solved to n th-order for the Taylor's coefficients of equation (15), it will be observed that the result has the remarkably simple form:

$$\begin{aligned} (x_i | x_o^\kappa y_o^\lambda x_o^\mu y_o^\nu \delta^\chi) &= \frac{t^n}{n!} \left[\frac{n!}{\kappa! \lambda! \mu! \nu! \chi!} \right] \int_0^t G_i(t, \tau) c_x^\kappa(\tau) c_y^\lambda(\tau) s_x^\mu(\tau) s_y^\nu(\tau) d_x^\chi(\tau) K_n(\tau) d\tau \\ &+ \text{Terms containing } K_0, \dots, K_{n-1} \end{aligned} \tag{19}$$

where the variable of integration is τ and $n = (\kappa + \lambda + \mu + \nu + \chi)$.

The x_i have the following meaning:

$$x_1 = x(t) \quad x_2 = x'(t) \quad x_3 = y(t) \quad x_4 = y'(t)$$

$c_x, c_y, s_x, s_y,$ and d_x are defined by equation (17) and in general are functions of the variable of integration τ over the interval of integration. K_n is defined by equation (8) and in general is also a function of τ .

The G_i 's are Green's functions where:

$$\begin{aligned}
 G_1(t, \tau) &= (x(t)|x'(\tau)) = s_x(t)c_x(\tau) - c_x(t)s_x(\tau) \\
 G_2(t, \tau) &= (x'(t)|x'(\tau)) = s'_x(t)c_x(\tau) - c'_x(t)s_x(\tau) \\
 G_3(t, \tau) &= (y(t)|y'(\tau)) = s_y(t)c_y(\tau) - c_y(t)s_y(\tau) \\
 G_4(t, \tau) &= (y'(t)|y'(\tau)) = s'_y(t)c_y(\tau) - c'_y(t)s_y(\tau)
 \end{aligned} \tag{20}$$

Note that the G_i 's are just first-order Taylor's coefficients measured from the location (τ) of the multipole component to the end of the system (t).

Thus we see that the coupling coefficient to an n th-order multipole is a function only of the first-order matrix elements c_x, c_y, s_x, s_y, d_x and their derivatives with respect to t .

From median-plane symmetry considerations, the allowed aberrations are those with y and/or y' appearing an even number of times in the Taylor coefficient. For example $(x|x_0^2), (x|y_0 y_0')$ and $(y|y_0^2 y_0')$ are allowed aberrations; whereas $(x|y_0), (x|x_0^2 y_0')$ or $(y|y_0^2)$ are not allowed and are therefore equal to zero.

The minus sign is used when y and/or y' appear 0, 4, 8, 12 \dots times and the plus sign is used when y and/or y' appear 2, 6, 10 \dots times. For example for the coefficients $(x|x_0^2)$ and $(y|y_0^3)$, the minus sign is applicable; whereas for the coefficients $(x|y_0^2)$ and $(y'|y_0^3 y_0'^2)$ the plus sign is applicable.

Equation (19) is derived by observing in the pattern of the solution of the differential equation that an n th-order aberration term containing the n th-order multipole strength factor K_n cannot include multipole strength factors of lower order than n ; or stated physically, an n th-order multipole cannot couple to aberrations (terms) of order lower than n . This fact allows the recursion formula equation (3) to be reduced to the simple form

$$A_{2m+3, n} = -A_{2m+1, n+2} \tag{21}$$

in so far as it applies to the derivation of n th-order terms containing only K_n . As a consequence, the scalar potential for deriving these terms assumes the simplified form

$$\Phi(x, y, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^m A_{1, 2m+n} \frac{x^n}{n!} \frac{y^{2m+1}}{(2m+1)!} \tag{22}$$

From which, it follows that

$$B_x(x, y, t) = \left(\frac{\Gamma_0}{e}\right) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^m K_{2m+n+1} \frac{(2m+n+1)!}{n!(2m+1)!} x^n y^{2m+1}$$

and

$$B_y(x, y, t) = \left(\frac{P_0}{e}\right) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^m K_{2m+n} \frac{(2m+n)!}{n! 2m!} x^n y^{2m} \quad (23)$$

For terms containing only K_n , the basic differential equations assume the form:

$$\begin{aligned} x'' + \dots &= -\frac{e}{P} B_y \\ y'' + \dots &= \frac{e}{P} B_x \end{aligned} \quad (24)$$

Substituting the Taylor's expansion of equation (15) and solving for the n th-order terms using a conventional Green's function solution (see Ref. 1) yields equation (19) above.

IV. Interpretation and Use of Equation (19)

For most practical cases of interest, K_n will be a constant over the interval of integration. In this event we may define the coupling coefficient of an n th-order multipole to an n th-order aberration as the partial derivative of equation (19) with respect to the K_n in question as follows:

$$\frac{\partial(x_i | x_o^{\kappa} y_o^{\lambda} x_o^{\mu} y_o^{\nu} \delta X)}{\partial K_n} = \pm \left[\frac{n!}{\kappa! \lambda! \mu! \nu! X!} \right] \int_0^L G_i c_x^{\kappa} c_y^{\lambda} s_x^{\mu} s_y^{\nu} d_x X d\tau \quad (25)$$

where now the interval of integration is over the multipole length L represented by K_n . For a distributed multipole component (such as in a non-uniform field bending magnet), equation (25) is used.

In many cases where a curved entrance or exit pole contour is used or a short multipole magnet is used such that the characteristic first-order functions c_x , c_y , s_x , s_y and d_x are essentially constants over the interval of integration (the length of one multipole), then the coupling coefficient is best defined as the partial derivative of equation (19) with respect to S_n as follows:

$$\frac{\partial(x_i | x_o^{\kappa} y_o^{\lambda} x_o^{\mu} y_o^{\nu} \delta^{\chi})}{\partial S_n} = \pm \left[\frac{n!}{\kappa! \lambda! \mu! \nu! \chi!} \right] G_i c_x^{\kappa} c_y^{\lambda} s_x^{\mu} s_y^{\nu} d_x^{\chi} \quad (26)$$

Examples

Assume a situation where the end of the system is a point-to-point image or the origin (i.e., $s_x(t)=0$), then using equation 26, the coupling coefficients of a sextupole of strength S_2 to various second order aberration coefficients are:

$$\frac{\partial(x | x_o' \delta)}{\partial S_2} = c_x(t) s_x^2 d_x$$

$$\frac{\partial(x | x_o'^2)}{\partial S_2} = c_x(t) s_x^3$$

$$\frac{\partial(x | y_o y_o')}{\partial S_2} = - 2 c_x(t) c_y s_x s_y \quad (27)$$

etc. Where the Green's function used in these examples is

$$G_1 = s_x(t) c_x - c_x(t) s_x = - c_x(t) s_x \quad (\text{since } s_x(t) = 0 \text{ for point-to-point imaging})$$

The aberration and $c_x(t)$ are evaluated at the end of the system. $c_x(t)$ is equal to the magnification M_x in the examples given. The remaining coefficients c_y , s_x , s_y and d_x are evaluated at the location of the sextupole S_2 . The above results are in agreement with Table VII of Ref. 1.

To illustrate a more complex example, consider the fourth-order aberration coefficient $(y | y_o^2 y_o' \delta)$ and assume parallel-to-point imaging in the y coordinate (i.e., $c_y(t) = 0$). The appropriate Green's function is:

$$G_3 = s_y(t) c_y - c_y(t) s_y = s_y(t) c_y$$

and the coupling coefficient to a fourth-order multipole of strength S_4 is:

$$\frac{\partial(y | y_o^2 y_o' \delta)}{\partial S_4} = - \left(\frac{4!}{2!} \right) s_y(t) c_y^3 s_y d_x \quad (28)$$

where again the aberration coefficient $(y|y_o^2 y_o' \delta)$ and $s_y(t)$ are evaluated at the end of the optical system and c_y , s_y , and d_x are evaluated at the location of the fourth-order multipole S_4 .

V. A Systematic Procedure for Designing High Resolution Systems

First-Order Considerations

In many respects, the determination of a satisfactory first-order magnetic-optical design is more difficult to achieve than is the subsequent higher-order design. This is true not only because the basic equipment configuration is dominated by first-order optical considerations but also because the choice of the first-order optics affects the magnitude of all higher-order aberrations and the ease with which these aberrations may be minimized by introducing multipole components into the design.

The dominating design parameters that must be clearly specified in order to evolve a first-order design are the momentum resolving power; the spatial resolution of the particle detector system to be used (this determines the momentum dispersion required); the required phase space acceptance (the solid angle, the source size, and the momentum range) of the instrument, and the first-order imaging requirements in both the x and y coordinates.

Given the above specifications (assuming they are self-consistent), the optical mode and physical configuration of the instrument may be determined. Often, more than one theoretical solution exists; in which case the choice is usually resolved by practical or economic considerations. In other cases, no solution is evident and the basic specifications must be modified accordingly. In any event, the following equations and discussion are applicable to the solution of the problem.

1) First-Order Resolving Power

A general equation for the first-order resolving power has been derived in References (1,2, and 3). For point-to-point imaging the first-order momentum resolving power R_1 is defined as the ratio of the momentum dispersion at the image plane to the total image size. Thus if $2x_o$ is the total source size then from Reference 1 we have:

$$R_1 = \frac{P}{\Delta P} = \left| \frac{d_x(t)}{2x_o c_x(t)} \right| = \frac{1}{2x_o} \left| \int_0^t s_x(\tau) h(\tau) d\tau \right| \quad (29)$$

Note that $h(\tau)d\tau = d\alpha$ is the differential angle of bend of the central trajectory of the optical system.

Equation (29) may be expressed in a number of useful forms. If we consider a particle originating at the source with $x_0 = 0$ and $\delta = \frac{\Delta P}{P_0} = 0$ and lying in the midplane (i.e., a monoenergetic point source), the first-order equation of its trajectory is

$$x(\tau) = s_x(\tau)x'_0 \quad (30)$$

We may then rewrite equation (29) as follows:

$$R_1 = \frac{1}{2x_0 x'_0} \left| \int_0^t x(\tau)h(\tau)d\tau \right| = \frac{(l-l_0)}{2x_0 x'_0} \quad (31)$$

where $(l-l_0)$ is the path length difference between the trajectory described by equation (30) and the central trajectory. Or we may also write equation (31) in the form

$$R_1 = \frac{1}{2x_0 x'_0} \left| \int_0^t \frac{B x(\tau)d\tau}{B\rho} \right| = \left(\frac{1}{2x_0 x'_0} \right) \left(\frac{1}{B\rho} \right) \left| \int_0^t B dA \right| \quad (32)$$

where $\int B dA$ is the magnetic flux enclosed between the central trajectory and the trajectory described by equation (30), and $B\rho$ is the magnetic rigidity of the central trajectory. Please note, however, that if the trajectory of equation (30) crosses the central trajectory or the sign of B changes, this changes the sign of the integration. From equation (32) we may define resolving power as the magnetic flux enclosed per unit phase space area ($2x_0 x'_0$), per unit momentum ($B\rho$) of the central ray.

In any given design, one or more of the above equations may be used as a guide toward achieving the required resolving power. One of the design decisions that must be made is the appropriate choice of the dipole magnet parameters (width and length) to achieve the required $\int B dA$. From first-order considerations, this choice is dominated primarily by practical and economic factors. However, a study of the nature of the origin of aberrations (see for example Ref. 1) suggests that it is advisable to keep the amplitude of s_x small. In order to simultaneously satisfy this requirement and meet the required resolving power R_1 , we see from equation (29) that the total angle of bend α of the central trajectory should be chosen as large as is practical. Also, in general, the focal plane angle tends to be more normal to the optic axis for larger α - a property usually desired in most designs.

2) Dispersion

From Reference 1, 2, or 3; for point-to-point imaging ($s_x(t)=0$) the dispersion at the image plane is

$$d_x(t) = -c_x(t) \int_0^t s_x(\tau)h(\tau)d\tau \quad (33)$$

where $c_x(t)$ is the magnification at the image plane.

The dispersion and hence the magnification in the design of a spectrometer is dominated almost entirely by a compromise between the spatial resolution of the particle detectors used at the image plane and the momentum range to be covered by the instrument; or in the case of a momentum defining (analyzing) system, by the acceptable momentum-defining slit spacings.

3) The Selection of the Optical Mode

By optical mode, we mean the type of imaging (e.g., point-to-point or parallel-to-point, etc.) required at the image plane in both the x and y coordinates, and the number of intermediate images imposed between the source and image planes. The imaging requirements at the image plane are usually dominated by the physics to be performed by the instrument and the nature of the particle detectors used. However often (especially at low energies) the imaging in the y plane may be unimportant as far as the physics requirements are concerned which in turn provides some additional flexibility in the optics design.

A study of the coupling coefficients to the aberration coefficients (equation 19) shows the not surprising result, that multipoles located at intermediate images in a system do not couple to aberrations in the plane in which the intermediate image occurs. Hence it often proves beneficial to intentionally create an intermediate image in the y plane of an optical system so as to achieve some degree of "orthogonality" in the minimizing of x and y aberrations.

The considerations of 1), 2), and 3) above are the determining factors in the selection of the first-order solution of a system design.

The optical mode and dispersion of the system are determined to a great extent by the choice of the quadrupole components chosen to achieve the first-order imaging although it is clear that the dipole elements also influence the first-order imaging to a greater or lesser extent depending upon the total angle of bend of the system.

4) Aberrations and their Correction

A study of the source of second- and higher-order aberrations (see for example Ref. 1) suggests that it is advisable to maintain the characteristic

first-order functions c_x , s_x , d_x and c_y , s_y and their derivatives as small as is feasible through the magnetic elements of a system when choosing the first-order design. This procedure will tend to reduce the initial size of the aberrations and hence simplify the problem of minimizing them by the addition of multipole components to the system design.

The procedure for minimizing aberrations has already been outlined in the Introduction and as such will not be repeated here. The "key" to the minimization procedure is the coupling coefficient given by the integral expression in equation (19). The "best" location for the correcting multipole is where the coupling coefficient has its maximum value.

The preferred method of introducing the multipole components, i.e., via pure multipoles, contoured entrance or exit boundaries, or non-uniform fields is a combination of practical and economical considerations and, of course, personal taste and experience. All three methods have been used with pure multipoles dominating the situation for higher energy physics and the other two methods dominating medium and low-energy physics applications. All three techniques should be considered in any given design situation to be certain that an important economic or practical advantage has not been ignored.

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THE EFFECT OF BEAM LINE MAGNET MISALIGNMENTS

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The misalignment of a magnet in a beam line will cause an alteration of the beam envelope at any later point in that beam line. The position of a misaligned magnet may be described in terms of six coordinates, three translational and three rotational. The effect of a misalignment on a single particle trajectory is derived to first order, including bilinear terms. A bilinear effect is one which affects the beam line focusing characteristics, but not the central ray, such as the effect of rotating a quadrupole about its axis. The effect on the beam envelope is calculated, both for a known magnet displacement and for an uncertain magnet position. The formalism has been included in the computer program TRANSPORT.¹

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I. Introduction

The effects of magnet misalignments are an important consideration at every stage of beam line design, installation, and operation. The selection of the optical mode, determination of surveying accuracy requirements, and the choice of correcting elements are all dependent on misalignment information.

Two types of misalignment information are typically needed. To assess the general effect of misalignments in the design stage, one needs to know the change in beam position and beam line transmission characteristics due to uncertainties in the position of each magnetic element in each separate coordinate. Secondly, to provide for correcting elements, one needs to know the effect on the beam of specific misalignments.

In the following we derive a method of determining the effect of magnet misalignments on a particle beam. We first define a reference system in which to express misalignments. Then we determine the effect of a misalignment on individual particle trajectories. Finally we express the effect on the beam envelope which describes the ensemble of particles comprising the beam.

II. Particle Trajectory Coordinates

To specify the position and direction of a particle at any instant in time, we employ a coordinate system defined with respect to the beam line reference trajectory.² The z axis is taken to point along the reference trajectory; the x axis points to the left, and the y axis points up. The position and direction of

the particle trajectory can then be given by a vector with six components

$$X = \begin{pmatrix} x \\ x' \\ y \\ y' \\ l \\ \delta \end{pmatrix} \quad (1)$$

The quantities x , x' , y , and y' are respectively the horizontal displacement and slope, and vertical displacement and slope of the ray with respect to the central reference trajectory. l is the longitudinal separation of the ray from a ray which enters the beam line at the same time as the given ray and travels along the central trajectory. δ is the fractional momentum deviation of the particle from the design momentum of the beam line.

When a charged particle passes through a perfectly aligned magnet, the transformation may be described to first order by the matrix equation²

$$X(1) = R X(0) \quad (2)$$

The sets of six coordinates $X(0)$ and $X(1)$ give the particle position and direction at the entrance and exit faces of the magnet respectively.

When a magnet is misaligned, the central trajectory of the magnet is no longer continuous with the central trajectory of

the beam line (see figure 1 below). In particular, at both the entrance and exit faces, the reference coordinate system external to the magnet no longer coincides with the reference coordinate system internal to the magnet (see figure 2 below). The misaligned and aligned reference coordinate systems are related by a translation of origin plus a rotation of axes.

We continue to use $X(0)$ and $X(1)$ to denote respectively the entrance and exit face ray coordinates in the aligned coordinate systems. We use a subscript f to denote the ray coordinates $X_f(0)$ and $X_f(1)$ expressed in the misaligned reference coordinate systems. To first order the ray coordinates in the misaligned coordinate systems may be expressed in terms of those in the aligned coordinate systems by an affine transformation

$$X_f(0) = S_0 X(0) - D_0 \quad (3)$$

$$X_f(1) = S_1 X(1) - D_1 \quad (4)$$

The symbols S_0 and S_1 represent six by six matrices, whose form will be derived below. The two six-vectors D_0 and D_1 are translations in the six dimensional space of particle coordinates. The three vectors \bar{D}_0 and \bar{D}_1 formed from the displacement coordinates (x , y , and z) of D_0 and D_1 give the displacement due to the misalignment of the origins of the reference coordinate systems. These two three-vectors are shown in figure 1.

III. Magnet Misalignment Coordinates

The alignment of a rigid magnet has six degrees of freedom, three translational and three rotational. These are conveniently represented by the six quantities

$$m = \begin{pmatrix} \delta x \\ \theta_x \\ \delta y \\ \theta_y \\ \delta z \\ \theta_z \end{pmatrix} \quad (5)$$

where δx , δy , δz are the displacements in the x, y, and z directions, and θ_x , θ_y , and θ_z are the rotations about the x, y, and z axes respectively. The origin of the xyz coordinate system, called the pivot, is the point about which the misalignments are measured. If the pivot point is located at some point on the reference trajectory, the x, y, and z axes of the alignment coordinate system are taken to coincide with the x, y, and z axes of the beam line reference coordinate system.

The misalignments form a mathematical group, which is the Euclidean group in three dimensions. This group is non-commutative and the order in which the misalignments are imposed is important if terms of higher order than linear are included. In practice, however, misalignment values are sufficiently small so that a first-order approach is justified. For these reasons, we consider

only those terms which are of first order in the misalignment parameters.

IV. Transformation of Particle Trajectory Coordinates

We now temporarily delete the indices 0 and 1 indicating the entrance and exit magnet faces respectively, and consider the effect of a misalignment at a single magnet face. Later we will combine the results from the two faces to obtain the net effect of a misalignment.

When the components of the misalignment vector m are small, we may expand the matrix S and the centroid displacement D in the misalignment parameters. Retaining only first-order terms we have

$$\begin{aligned} D &= Am \\ S &= I + Bm \end{aligned} \tag{6}$$

The six by six matrix A represents a transformation from the misalignment parameters to the particle coordinates. I is the identity matrix, and B represents a set of six matrices, one for each of the misalignment parameters. A single six by six matrix Bm is obtained by multiplying each of the six matrices by its corresponding misalignment parameter and summing the results. In terms of the misalignment parameters, the particle coordinates in the misaligned reference coordinate system now take the form

$$X_f = X - Am + BXm \tag{7}$$

To derive the forms of the matrices A and B, we consider separately the effect of each of the various misalignment parameters on each of the ray coordinates. First we derive the effect on the ray coordinates of the various misalignments as expressed in the coordinate system of the aligned magnet face. Then we will express the misalignment of the magnet face in terms of the misalignment parameters about the pivot point.

A rigid translation of the magnet face will change the x, y, and z coordinates of a ray by the amount of the displacement. The z translation will also introduce a short drift distance (positive or negative length) at the magnet face, and will contribute to B via the transformation matrix of that drift space.

To determine the effect of a rotational misalignment we form from the ray angles x' ($= dx/dz$) and y' ($= dy/dz$) and the number 1 ($= dz/dz$), a three-vector $(x', y', 1)$ giving the ray direction. We let $\bar{\theta}_x$, $\bar{\theta}_y$, $\bar{\theta}_z$ be the three rotational components of the misalignment vector. Then, including only first-order effects, this three-vector is transformed as

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}_f = \begin{pmatrix} 1 & \bar{\theta}_z & -\bar{\theta}_y \\ -\bar{\theta}_z & 1 & \bar{\theta}_x \\ \bar{\theta}_y & -\bar{\theta}_x & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} \quad (8)$$

In the misaligned coordinate system the ray angles become

$$\begin{aligned} x'_f &= x' - \bar{\theta}_y + \bar{\theta}_z y' \\ y'_f &= y' + \bar{\theta}_x - \bar{\theta}_z x' \end{aligned} \quad (9)$$

Thus coordinate rotations about the aligned magnet face x and y axes only shift the ray angles. A rotation about the z axis mixes x' and y'.

If we let \bar{m} represent the misalignment parameters relative to the aligned magnet face coordinate system, and \bar{A} and \bar{B} be the corresponding matrices, then equation (6) holds using the barred quantities. Using the results derived above, the matrices \bar{A} and \bar{B} are now given by

$$\bar{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (10)$$

$$\bar{B}_{125} = \bar{B}_{345} = 1$$

$$\bar{B}_{136} = \bar{B}_{246} = -\bar{B}_{316} = -\bar{B}_{426} = -1 \quad (11)$$

The first two indices for B correspond to the ray coordinates and the third corresponds to the misalignment parameters. All other elements of \bar{B} are zero.

In order to express the quantities \bar{m} in terms of m , the misalignment parameters at the pivot point, we need two items. First is the orthogonal matrix O giving the three translational coordinates at the magnet face in terms of those at the pivot point

$$X_f = O X_p \quad (12)$$

Also needed is the three-vector P which gives the position of the origin of the aligned magnet face coordinate system, in the coordinate system of the pivot.

We now define two three-vectors which give the translational m_x and rotational m_θ parts of the misalignment vector m . We also do the same for \bar{m} . Then the contribution of m_x to \bar{m}_x is given by equation (11), so that

$$\bar{m}_x = O m_x \quad (13)$$

The contribution of m_x to \bar{m}_θ is zero, since parallel translations do not affect angles.

The displacement of a point due to a rotation about the pivot is given by the vector product of the rotation vector and the position vector of the point. Therefore the displacements of the magnet face \bar{m}_x due to a rotation at the pivot are given by

$$\bar{m}_x = O \left(m_\theta \times P \right) \quad (14)$$

The orthogonal transformation indicated by the matrix O gives the misalignment parameters in the magnet face coordinate

system. Finally the transformation of rotational misalignment parameters is again given by equation (11), so that

$$\bar{m}_\theta = O m_\theta \quad (15)$$

V. Evaluation of the Relevant Matrices

We choose the pivot to be the origin of the aligned magnet entrance face coordinate system. Therefore we have

$$\begin{aligned} A_o &= \bar{A} \\ B_o &= \bar{B} \end{aligned} \quad (16)$$

For the exit face, the matrix O transforms from the aligned entrance face coordinate system to the aligned exit face coordinate system. The vector P gives the position of the origin of the aligned exit face coordinate system in the aligned entrance face coordinate system. In figure 1 it is the vector which reaches from A to B.

For the exit face of a bending magnet we therefore have

$$O = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} \quad (17)$$

$$P = \begin{pmatrix} -\rho(1 - \cos\alpha) \\ 0 \\ \rho\sin\alpha \end{pmatrix} \quad (18)$$

where ρ is the radius of curvature of the central trajectory and α is the total bend angle. We then derive for the matrices A_1 and B_1

$$A_1 = \begin{pmatrix} \cos\alpha & 0 & 0 & \sin\alpha & \sin\alpha & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\sin\alpha & 1 & 0 & 0 & -\rho(1-\cos\alpha) \\ 0 & -\cos\alpha & 0 & 0 & 0 & -\sin\alpha \\ -\sin\alpha & 0 & 0 & -\rho(1-\cos\alpha) & \cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (19)$$

$$B(1)_{121} = B(1)_{341} = -\sin\alpha$$

$$B(1)_{124} = B(1)_{344} = -\rho(1 - \cos\alpha)$$

$$B(1)_{125} = B(1)_{345} = \cos\alpha$$

$$B(1)_{132} = B(1)_{242} = -B(1)_{312} = -B(1)_{422} = \sin\alpha$$

$$B(1)_{136} = B(1)_{246} = -B(1)_{316} = -B(1)_{426} = -\cos\alpha \quad (20)$$

All other elements of $B(1)$ are zero.

To calculate A_1 and B_1 for a quadrupole, we take the limit $\alpha \rightarrow 0$ with $\alpha\rho = L$, the length of the magnet being held

fixed. Then we have

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -L & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (21)$$

and $B_1 = \bar{B}$ as given above.

VI. Effect on the Beam Envelope

To first order, the coordinates at the misaligned magnet exit face are related to those at the misaligned entrance face by a transfer matrix, so that

$$X_f(1) = R X_f(0) \quad (22)$$

or

$$X(1) - A_1 m + B_1 X(1) m = R \left[X(0) - A_0 m + B_0 X(0) m \right] \quad (23)$$

If we solve for $X(1)$ and discard all terms in m of order higher than first, we then derive

$$X(1) = R X(0) + \left[A_1 - R A_0 \right] m + \left[R B_0 - B_1 R \right] X(0) m \quad (24)$$

For later use we define two new matrices F and G given by

$$\begin{aligned} F &= A_1 - R A_0 \\ G &= R B_0 - B_1 R \end{aligned} \quad (25)$$

so that

$$X(1) = RX(0) + Fm + GX(0)m \quad (26)$$

An ensemble of particles in a beam line is often represented as a six-dimensional ellipsoid. The equation of this ellipsoid may be written in matrix form as follows:

$$X^T \sigma^{-1} X = 1 \quad (27)$$

where X^T is the transpose of the coordinate vector X , and σ is a real, positive definite, symmetric matrix. The square roots of the diagonal terms of the sigma matrix are a measure of the beam size in each coordinate. If the centroid of this ellipsoid does not fall on the central trajectory, then one needs to specify this centroid position also. The sigma matrix then gives the beam dimensions as measured about the centroid.

The beam envelope entering a misaligned magnet may be described in terms of the position in the aligned coordinate system of the beam centroid and the sigma matrix. For a known misalignment m , the centroid is transformed as in equation (25). The sigma matrix is transformed by

$$\begin{aligned} \sigma(1) = R\sigma(0)R^T + G\sigma(0)mR^T + R\sigma(0)m^T G^T \\ + G\sigma(0)mm^T G^T \end{aligned} \quad (28)$$

where the superscript T indicates a transpose.

For an uncertainty in position we define a covariance

matrix $\langle mm^T \rangle$, measuring the distribution of possible magnet positions. The sigma matrix, which represents the beam envelope entering the magnet may contain contributions from both the original beam and from the uncertainty in positions of previous magnets. We assume there is no correlation of errors of positioning between any two magnets. The beam centroid is unaffected by an uncertainty in position. The transformed sigma matrix becomes

$$\sigma(1) = R\sigma(0)R^T + F\langle mm^T \rangle F^T + G\sigma(0)\langle mm^T \rangle G^T \quad (29)$$

If the original sigma matrix is zero, then the resultant sigma matrix represents the uncertainty in the beam centroid upon leaving the magnet. If the original sigma matrix encloses a region of phase space, then the resultant sigma matrix represents the envelope of possible particle trajectories, including both the undisturbed sigma matrix and the effects of the misalignment.

VII. Implementation

This model for misalignments has been implemented in the computer program TRANSPORT.¹ An arbitrary misalignment m may be imposed on any magnet or section of the beam line. Misalignments may also be nested. The effect of all misalignments may then be added into the sigma matrix and thereby be traced through the system. Alternatively, the effects of separate components of the misalignment vector on individual magnets

may be stored in a table. This table is traced through the beam line and may be compared with the unperturbed sigma matrix at any later point. Details of implementation are described in the TRANSPORT manual.

Figure Captions

- Figure 1** Perfectly aligned and misaligned bending magnets.
With the misaligned magnet the beam line reference trajectory is no longer continuous with that inside the magnet. The displacements of the origins of the entrance and exit face reference coordinate systems are shown as \bar{D}_0 and \bar{D}_1 respectively.
- Figure 2** Magnet entrance and exit face coordinate systems.
The misalignment causes both a translation and a rotation of the reference coordinate system.

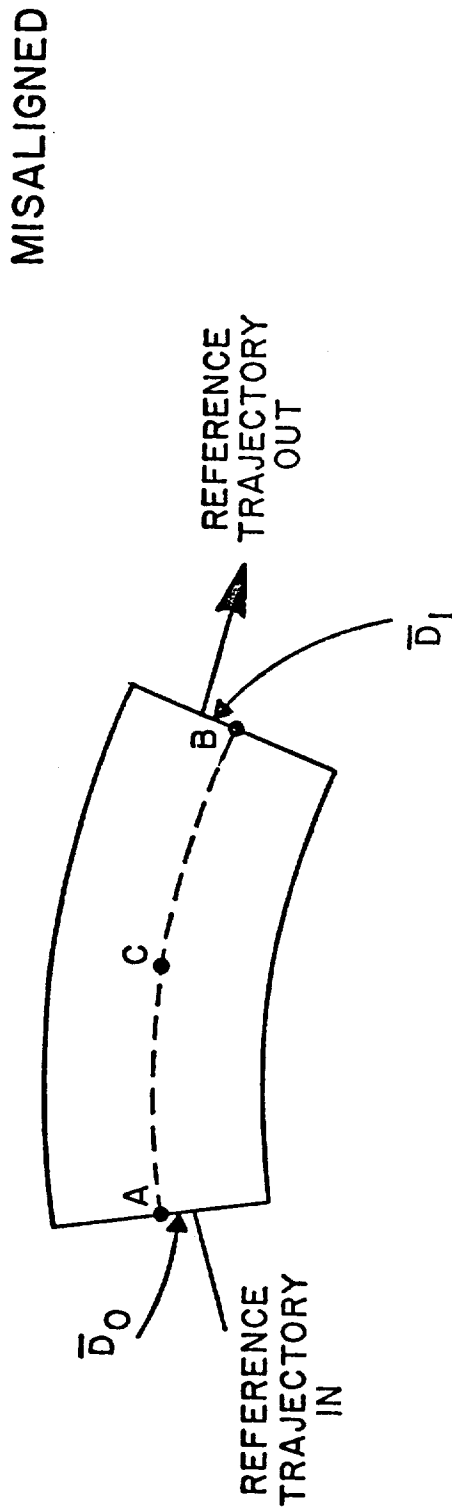
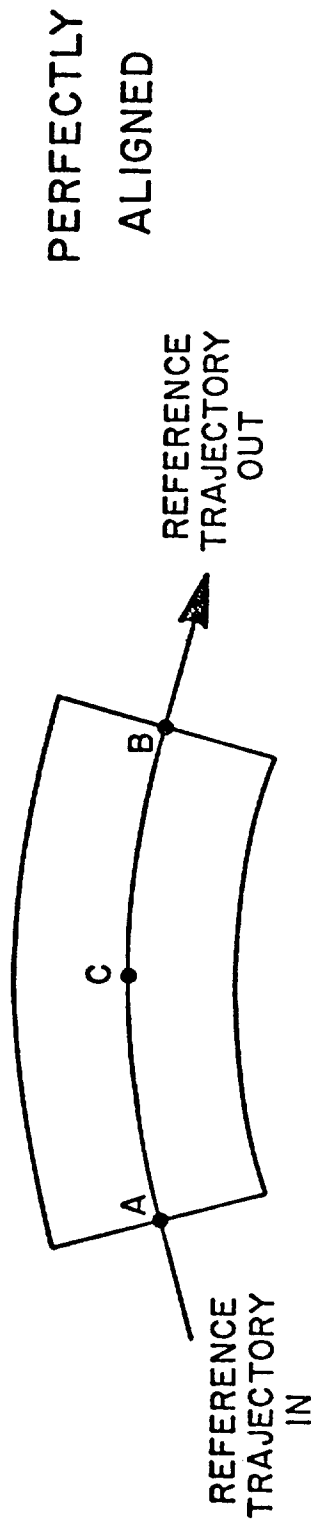
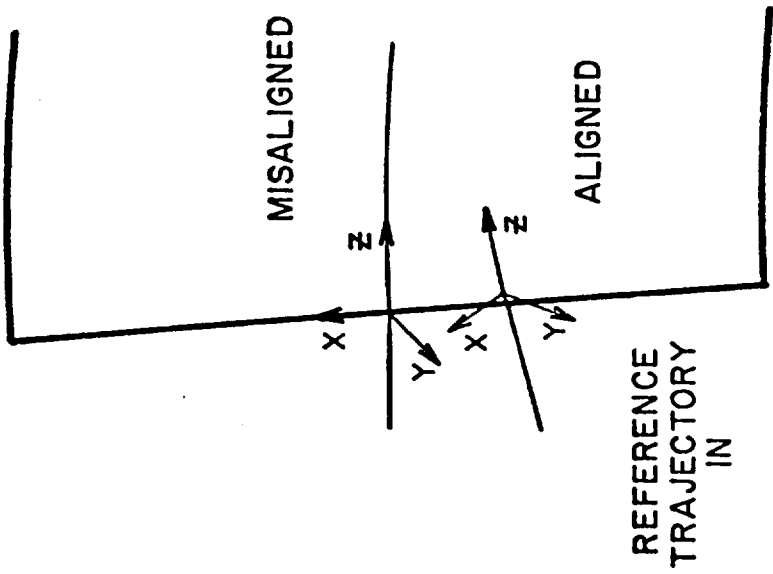
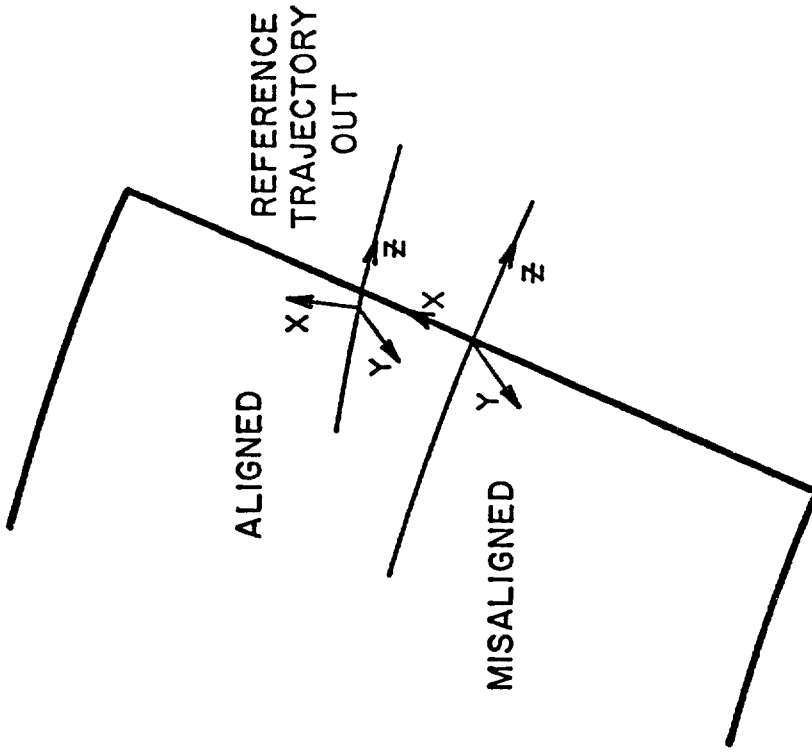


figure 1



MAGNET ENTRANCE FACE
COORDINATE SYSTEM



MAGNET EXIT FACE
COORDINATE SYSTEM

figure 2

FIRST-ORDER PARAMETER OPTIMIZATION AND COVARIANCE

Section V

To optimize the selected parameters, TRANSPORT uses the method of non-linear least squares, differential correction, a good description of which is found in Ref. (7) below pages, 390-393. A useful by-product of this method is the covariance matrix C, printed by the program at the successful conclusion of any run involving parameter fitting. In many applications C may be used to estimate tolerances on the fitted parameters subject to the specified tolerances (i.e., the standard deviations) of the constraints.

The covariance matrix C is symmetric. This admits a geometrical interpretation as an ellipsoid, and is printed in the same suggestive format as is the beam ellipsoid σ , except that in this case the dimension is equal to N, the number of parameters varied. The center of the ellipsoid is at

$$\vec{\lambda}^0 = (\lambda_1^0, \lambda_2^0 \dots \lambda_N^0) , \quad (1)$$

the N values found by TRANSPORT to be the best estimate of the varied parameters.

The equation of the covariance ellipsoid is

$$(\vec{\lambda} - \vec{\lambda}^0) C^{-1} (\vec{\lambda} - \vec{\lambda}^0)^T \leq 1 \quad (2)$$

where $(C_{11})^{1/2}$, the first diagonal element printed, measures the maximum extent of the ellipsoid along the λ_1 axis (the first varied parameter) in the same sense that $(\sigma_{11})^{1/2}$ measures the maximum extent of the beam ellipsoid along the x axis. The off diagonal elements are normalized so that they are ≤ 1 in magnitude, in analogy with the r_{ij} of the beam matrix, and can be interpreted as measures of the orientation of the covariance ellipsoid.

(7) SOLMITZ, Analysis of Experiments in Particle Physics, Ann. Rev. Nuc. Sci., Vol 14, 1964.

K. Halbach, "A Program for Inversion of System Analysis and its Application to the Design of Magnets", Second International Conference on Magnet Technology, Oxford, 1967.

The best estimates (or optimized values) of the varied parameters λ are precisely those that minimize the quantity:

$$\chi^2 = \sum_{K=1}^M \left[\frac{\xi_K - f_K(\lambda_1 \dots \lambda_N)}{S_{KK}} \right]^2 \quad (3)$$

where:

M = number of constraints

f_K = a function selected by the code digits (i,j) on the constraint definition (Type Code 10.). For example, i = -1, j = 1 means that the transform matrix element R_{11} is to be constrained.

ξ_K = the desired value of the selected function.

S_{KK} = the desired accuracy of fit (i.e., the standard deviation).

In our notation this minimum is expressed by:

$$\chi^2_{\min} = \sum_{K=1}^M \left[\frac{\xi_K - f_K(\lambda_1^0 \dots \lambda_N^0)}{S_{KK}} \right]^2 \quad (4)$$

χ^2_{\min} is printed at the successful conclusion of any run involving parameter fitting. Whether or not the optimization (χ^2_{\min} , $\vec{\lambda}^0$) is 'acceptable' depends on each application and must be evaluated by the user. Values of $\chi^2_{\min} \sim (M-N)$ are sometimes (but not necessarily) regarded as 'good'. In particular if $M=N$, then an exact solution, $\chi^2_{\min} = 0$ should be found by TRANSPORT.

If the resulting fit is acceptable, then the following interpretation may be put on the covariance matrix C : Let the parameters be changed to values $\vec{\lambda}$ near the optimum $\vec{\lambda}^0$, such that they stay within the ellipsoid defined by:

$$(\vec{\lambda} - \vec{\lambda}^0) C^{-1} (\vec{\lambda} - \vec{\lambda}^0)^T \leq 1$$

Then the resulting deviation of the specified constrained quantities is bounded by:

$$\sum_{K=1}^M \left[\frac{\xi_K - f_K(\lambda)}{S_{KK}} \right]^2 \leq \chi^2_{\min} + 1 \quad (5)$$

This interpretation is strictly true if the constrained functions f_K are linear in the parameters $\vec{\lambda}$. In the non-linear case, it is an approximation valid only in some neighborhood of $\vec{\lambda}^0$.

Example:

On the following page is an example of a TRANSPORT data deck and the resulting covariance fit of a first-order run. We have ask for a point-to-point image in both the x and y planes by varying the fields of the quadrupole triplet. The following definitions and solution are applicable:

$$B_1 = \lambda_1, B_2 = \lambda_2, B_1^0 = \lambda_1^0, B_2^0 = \lambda_2^0, f_1 = R_{12}, f_2 = R_{34}, \xi_1 = 0, \xi_2 = 0$$

$$S_{11} = 0.005, S_{22} = 0.005 \text{ and } N=2.$$

From the data deck and the TRANSPORT printout* shown on the following page, we learn that:

$$\frac{\chi^2_{\min}}{N} = 0.8$$

$$B_1^0 = -7.4096 \text{ Kg}, B_2^0 = 6.1577 \text{ Kg}$$

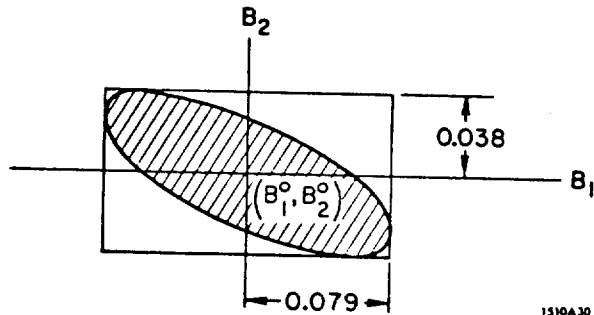
$$\sqrt{C_{11}} = 0.079$$

$$r_{12} = -0.883 \quad \sqrt{C_{22}} = 0.038$$

} Covariance matrix information

The ellipsoid $(B - B^0) C^{-1} (B - B^0)^T \leq 1.0$ can then be constructed as

shown:



1510A30

* Note: the printout format is:

COVARIANCE (FIT χ^2_{\min}/N)

$$\begin{matrix} \sqrt{C_{11}} \\ r_{12} & \sqrt{C_{22}} \\ \vdots & \vdots \\ r_{1n} & \dots & r_{n,n-1} & \sqrt{C_{NN}} \end{matrix}$$

Example:

```

1.  'USE TRIPLET TO FORM POINT-TO-POINT IMAGE AT TARGET'
2.  0
3.  13. 2. ;
4.  15. 1. 'I1' ; 15. 8. 'FT' ;
5.  1. 0. 0. 0. 0. 0. 0. 1.05 ;
6.  3.0 12. ;
7.  5.02 1.5 -8.0 4.0 'Q1' ;
8.  3.0 0.5 ;
9.  5.01 3.0 7.0 4.0 'Q2' ;
10. 3.0 0.5 ;
11. 5.02 1.5 -8. 4.0 'Q1' ;
12. 3.0 7.0 ;
13. 10. -1. 2. 0. .005 'FIT1' ;
14. 10. -3. 4. 0. .005 'FIT2' ;
15. SFNTINEL
16. /*
    
```

} Data Deck for TRANSPORT Run

\swarrow (X^2_{\min})
 *COVARIANCE (FIT 0.8)
 0.079 }
 -0.883 0.038 } Covariance Matrix
 USE TRIPLET TO FORM POINT-TO-POINT IMAGE AT TARGET

```

*BEAM*      1.000000      1.05 GEV
*DRIFT*     3.0      12.0000 FT      B10
*QUAD*     5.00      1.50000 FT      -7.4096 KG      4.000 IN ( -3.208 FT )
*DRIFT*     3.0      0.5000 FT      B20
*QUAD*     5.00      3.00000 FT      6.1577 KG      4.000 IN ( 2.673 FT )
*DRIFT*     3.0      0.5000 FT      B10
*QUAD*     5.00      1.50000 FT      -7.4096 KG      4.000 IN ( -3.208 FT )
*DRIFT*     3.0      7.0000 FT      ξ1
*FIT*      10.0      -1. 2.  0.0 / 0.005  { Constraint on R12 }
          LABEL = FIT1
          -0.000
          ξ2
*FIT*      10.0      -3. 4.  0.0 / 0.005  { Constraint on R34 }
          LABEL = FIT2
          -0.000
*LENGTH*   26.0000 FT
    
```

Interpretation:

So long as B_1 and B_2 fall inside the shaded area (this is the tolerance requirement), then the ellipsoid representing the corresponding deviations of the matrix elements R_{12} and R_{34} is using Eq. (5):

$$\left(\frac{0 - R_{12}}{0.005}\right)^2 + \left(\frac{0 - R_{34}}{0.005}\right)^2 \leq \chi^2_{\min} + 1.0 = 2.6$$

or

$$R_{12}^2 + R_{34}^2 \leq 2.6 (0.005)^2$$

Note that it is not enough to prescribe tolerances $|\Delta B_1| < .079$ and $|\Delta B_2| < .038$, since there is unshaded area inside the rectangle defined by these values. The strongly tilted covariance ellipse (i.e., $|r_{12}| \sim 1$) suggests that the triplet power supplies should be designed so that any drift in magnetic field B_1 causes a compensating drift in the magnetic field B_2 so as to stay inside the shaded area shown in Figure 1.

TABLE 1

Summary of TRANSPORT type codes.

PHYSICAL ELEMENT	TYPE CODE	2nd ENTRY	3rd ENTRY	4th ENTRY	5th ENTRY	6th ENTRY	7th ENTRY	8th ENTRY	9th ENTRY
BEAM	1.vvvvvv0	x (cm)	θ (mr)	y (cm)	ϕ (mr)	z (cm)	δ (percent)	P_0	
r.m.s. ADDITION TO BEAM ENVELOPE	1.vvvvvv00	Δx (cm)	$\Delta \theta$ (mr)	Δy (cm)	$\Delta \phi$ (mr)	Δz (cm)	$\Delta \delta$ (percent)	ΔP	0
POLE FACE ROTATION	2.v	ANGLE OF ROTATION (degrees)							
DRIFT	3.v	LENGTH (metres)							
BENDING MAGNET	4.vvv	LENGTH (metres)	FIELD (KG)	FIELD GRADIENT (n-value)					
QUADRUPOLE	5.vv0	LENGTH (metres)	FIELD (KG)	HALF-APERTURE (cm)					
TRANSFORM 1 UPDATE	6.0	0.0	1.0						
TRANSFORM 2 UPDATE	6.0	0.0	2.0						
BEAM CENTROID SHIFT	7.0	SHIFT (x) (cm)	SHIFT (θ) (mr)	SHIFT (y) (cm)	SHIFT (ϕ) (mr)	SHIFT (z) (cm)	SHIFT (δ percent)		
ALIGNMENT TOLERANCE	8.vvvvvv0	DISPLACEMENT (x) (cm)	ROTATION (θ) (mr)	DISPLACEMENT (y) (cm)	ROTATION (ϕ) (mr)	DISPLACEMENT (z) (cm)	ROTATION (α) (mr)		
REPEAT CONTROL	9.0	NUMBER OF REPEATS		DESIRED VALUE OF (I,J) MATRIX ELEMENTS	ACCURACY OF FIT				
FITTING CONSTRAINTS	10.0	$\pm I$	J						
Note: +I is used for fitting a beam (σ) matrix element. -I is used for fitting an R1 matrix element. - (I + 20) is used for fitting an R2 matrix element.									
ACCELERATOR	11.0	LENGTH (metres)	E (energy gain) (GeV)	ϕ (phase lag) (degrees)	(WAVELENGTH) (cm)				
BEAM (Rotated Ellipse)	12.0	THE FIFTEEN CORRELATIONS AMONG THE SIX ELEMENTS (This entry must be preceded by a type code 1.0 entry.)							
INPUT/OUTPUT OPTIONS	13.0	CONTROL CODE NUMBER			R(J,4)				J
ARBITRARY R MATRIX	14.vvvvvv0	R(J,1)			R(J,5)		R(J,6)		
UNITS CONTROL (Transport Dimensions)	15.0	CODE	UNIT SYMBOL	SCALE FACTOR (if required)					
QUADRATIC TERM OF BENDING FIELD	16.0v	1.0	$\epsilon(1) = \beta(\frac{1}{\rho_0})^2$	ρ_0 in units of transverse length (cm)					
MASS OF PARTICLES IN BEAM	16.0	3.0	M/m (dimensionless)	m = mass of electron					
HALF-APERTURE OF BENDING MAGNET IN x-PLANE	16.0	4.0	w/2 (cm)						
HALF-APERTURE OF BENDING MAGNET IN y-PLANE (gap)	16.0	5.0	g/2 (cm)						
LENGTH OF SYSTEM	16.0	6.0	L (metres)						
FRINGE FIELD CORRECTION COEFFICIENT	16.0	7.0	K_1 (dimensionless)						
FRINGE FIELD CORRECTION COEFFICIENT	16.0	8.0	K_2 (dimensionless)						
CURVATURE OF ENTRANCE FACE OF BENDING MAGNET	16.0v	12.0	(1/R ₁) (1/metres)						
CURVATURE OF EXIT FACE OF BENDING MAGNET	16.0v	13.0	(1/R ₂) (1/metres)						
FOCAL PLANE ROTATION	16.0	15.0	Angle of focal plane rotation (degrees). See type code 16.0 for details.						
INITIAL BEAM LINE x-COORDINATE	16.0	16.0	x_0						
INITIAL BEAM LINE y-COORDINATE	16.0	17.0	y_0						
INITIAL BEAM LINE z-COORDINATE	16.0	18.0	z_0						
INITIAL BEAM LINE HORIZONTAL ANGLE	16.0	19.0	θ_0						
INITIAL BEAM LINE VERTICAL ANGLE	16.0	20.0	ϕ_0						
SECOND-ORDER CALCULATIONS	17.0								
SEXTUPOLE	18.0v	LENGTH (metres)	FIELD (KG)	HALF-APERTURE (cm)					
SOLENOID	19.vv	LENGTH (metres)	FIELD (KG)						
BEAM ROTATION	20.v	ANGLE OF ROTATION (degrees)							
STRAY FIELD	21.0	See later section of report.							
Note: The v's following the type codes indicate the parameters which may be varied. See section under type code 10.0 for a detailed explanation of Vary Codes. The units are standard TRANSPORT units (as shown) unless changed via type code 15.0 entries.									