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Transport of Anisotropic or Low-Intensity Turbulence

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by

Francis H. Harlow

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TRANSPORT OF ANISOTROPIC OR LOW-INTENSITY TURBULENCE

by

Francis H. Harlow

ABSTRACT

This report describes some extensions to the theory of turbulence transport, generalizing the theory to apply to flows with regions of weak turbulence or with strong departures from isotropy.

I. INTRODUCTION

Generalized transport equations have been proposed[1-3] for the theoretical investigation of turbulence in transient fluid flow problems. The validity of these equations is greatest for flows in which the turbulence is strong (the inertial range) and nearly isotropic; however, significant modifications are necessary for the low intensity (viscous) range and for regions of strong creation in which the turbulence is not isotropic.

In this report, detailed consideration is given to these and related matters, using the nomenclature of References 1 and 2, which we assume are available to the reader. For convenience, the transport equations are summarized from Reference 1.

$$\frac{\partial q}{\partial t} + u_k \frac{\partial q}{\partial x_k} = \sigma e_{jk} \frac{\partial u_j}{\partial x_k} + \frac{\theta}{\gamma} \frac{\partial}{\partial x_k}\left(\sigma \frac{\partial \varphi}{\partial x_k}\right) + \frac{\partial}{\partial x_k}\left[(\nu + \alpha\sigma) \frac{\partial q}{\partial x_k}\right] - 2\nu D, \tag{1}$$

$$\frac{\partial D}{\partial t} + u_k \frac{\partial D}{\partial x_k} = \frac{a\Delta'\sigma}{s^2} e_{jk} \frac{\partial u_j}{\partial x_k} - \frac{2\nu\Delta' D}{s^2} + \frac{a_2\Delta}{s^2} \frac{\partial}{\partial x_k}\left(\sigma \frac{\partial q}{\partial x_k}\right) + \frac{\partial}{\partial x_k}\left[(\nu + a_3\sigma) \frac{\partial D}{\partial x_k}\right] + a_4 \frac{\partial}{\partial x_k}\left(\frac{\Delta\sigma}{s^2} \frac{\partial \varphi}{\partial x_k}\right), \tag{2}$$

with

$$D = \frac{\Delta q}{s^2}$$

$$q = \frac{1}{2\gamma}\left(\frac{\sigma}{s}\right)^2$$

$$\Delta = \beta(1 + \delta\xi)$$

$$\Delta' = \beta'(1 + \delta'\xi). \tag{3}$$

The meanings of these quantities are as follows: q is the specific turbulence energy, D is a dissipation function, σ is the eddy viscosity, s is a turbulence scale function, φ is the pressure, e_{jk} is the rate of strain, u_j is the mean fluid velocity, $\xi \equiv \sigma/\nu$, and various universal constants are also present, as explained in Refs. 1 through 3.

The boundary conditions at a rigid wall are that $q = 0$, $\sigma = 0$, and $D = 0$. To see that this last is required, suppose that D does not vanish at a rigid wall. Then, from the definition of D, Eq. (3), s must vanish there. But in the D equation, near the wall,

$$\frac{\partial D}{\partial t} = \frac{1}{s^2}(\sigma S - 2\nu\Delta' D) + \ldots ,$$

where terms irrelevant for the discussion have been dropped. Here S is that part of the creation term that is finite at the wall, so that $\sigma S \to 0$ there. Accordingly, if D does not vanish at the wall, then $\partial D/\partial t \to -\infty$, and the value of D decays infinitely fast to zero.

II. LOW INTENSITY FLUX MODIFICATIONS

When the turbulence intensity is small, we must expect diffusive effects to modify significantly the transport of such imbedded properties as heat, solute, and momentum. To see what is involved, consider the density, Q, of some scalar quantity, for which there is a mean part, $\bar{Q}$, and a fluctuating part, Q'. The turbulence flux of this is $\overline{u_j' Q'}$, for which we have introduced the approximation

$$\overline{u_j' Q'} = - \text{constant} \times \sigma \frac{\partial \bar{Q}}{\partial x_j} . \tag{4}$$

The constant is universal, dimensionless, and of magnitude near unity.

Several heuristic justifications for this approximation have already been discussed. The following alternative is useful as a basis for generalization. We assume that a turbulence "eddy" moves in steps of length s', and that at each step the probable value of Q in it differs from the local mean by the same amount as the local mean differs at the two ends of the step. Thus

$$Q'(x_j) = \bar{Q}(x_j - s'\ell_j) - \bar{Q}(x_j), \tag{5}$$

where ℓ_j is a unit vector in the direction of motion. Then, to lowest order,

$$Q'(x_j) = -s'\ell_j \frac{\partial \bar{Q}}{\partial x_j} ,$$

and

$$\overline{u_k' Q'} = - \overline{u_k' s' \ell_j} \frac{\partial \bar{Q}}{\partial x_j}$$

$$= - \text{constant} \times \sigma \frac{\partial \bar{Q}}{\partial x_k} .$$

The fallacy of this derivation becomes apparent in the limit as the diffusion coefficient, λ, for Q becomes large. We then would expect the value of Q in the eddy to follow closely that of the surrounding mean, so that $Q' \to 0$ and $\overline{u_j' Q'} \to 0$. The values of σ and $\partial\bar{Q}/\partial x_j$, however, are independent of λ, so that while the left side of Eq. (4) goes to zero, the right side remains finite. The problem, therefore, is to modify the heuristic model so as to incorporate this tendency for Q' to be decreased by its diffusion into the adjacent mean.

To accomplish this, consider in some detail the diffusion of Q from an eddy during its motion from an initial point of equilibrium to a nearby point. En route,

$$\frac{dQ}{dt} = \frac{3\lambda}{R^2} (Q_0 - Q) ,$$

in which R is the eddy radius and Q_0 is the adjacent mean. Indeed, to the appropriate order of approximation,

$$Q_0 = u_k' \frac{\partial \bar{Q}}{\partial x_k} t ,$$

in which u_k' is the translational velocity relative to the mean, and Q has been (irrelevantly) scaled to vanish at t = 0.

The solution is easily shown to be

$$Q' \equiv Q - Q_0 = \frac{R^2}{3\lambda} u_k' \frac{\partial \bar{Q}}{\partial x_k} \left(e^{-\frac{3\lambda t}{R^2}} - 1 \right) ,$$

so that

$$\overline{u_j' Q'} = \overline{u_j' u_k'} \frac{R^2}{3\lambda} \frac{\partial \bar{Q}}{\partial x_k} \left(e^{-\frac{3\lambda t}{R^2}} - 1 \right) .$$

But

$$\overline{u_j' u_k'} = \frac{2}{3} q \, \delta_{jk} + \text{higher order terms},$$

so that

$$\overline{u_j' Q'} = \frac{2R^2 q}{9\lambda} \frac{\partial \bar{Q}}{\partial x_j} \left(e^{-\frac{3\lambda t}{R^2}} - 1 \right)$$

$$= \frac{R^2 \sigma^2}{9\gamma s^2 \lambda} \frac{\partial \bar{Q}}{\partial x_j} \left(e^{-\frac{3\lambda t}{R^2}} - 1 \right) ;$$

From Ref. 2, Eq. (24),

$$\frac{R^2}{9s^2} = \frac{1}{2\beta} ,$$

and our result becomes

$$\overline{u_j' Q'} = \frac{\sigma^2}{2\beta\gamma\lambda} \frac{\partial \bar{Q}}{\partial x_j} \left(e^{-\frac{3\lambda t}{R^2}} - 1 \right) . \tag{6}$$

To complete the derivation, it is only necessary to determine t, the elapsed time for the motion. This must be related closely to the ratio of eddy size to mean fluctuating velocity, which, in turn, is measured by σ/s. Thus, we take

$$t \approx \frac{s^2}{\sigma} ,$$

leading to the exponential term in Eq. (6) in the form

$$\exp\left(- \text{constant} \times \frac{3\lambda s^2}{R^2\sigma}\right) ,$$

or

$$\exp\left(- \text{constant} \times \frac{2\beta\lambda}{3\sigma}\right) ,$$

in which the constant is of order unity.

Thus, Eq. (6) becomes

$$\overline{u_j'Q'} = \frac{\tau_Q\sigma^2}{\beta\gamma\lambda}\left(1 - e^{-\frac{\epsilon_Q\beta\lambda}{\sigma}}\right)\frac{\partial\bar{Q}}{\partial x_j} , \qquad (7)$$

in which τ_Q and ϵ_Q are constants with magnitudes of order unity, which are universal for each identification of Q, but may differ among the various possible identifications. (The derivation suggests $\tau_Q < 1$ and $\epsilon_Q < 1$, which, however, will require specific investigation of detailed circumstances to substantiate.)

Note that as $\sigma/\lambda \to 0$

$$\overline{u_j'Q'} \to - \frac{\tau_Q\sigma^2}{\beta\gamma\lambda}\frac{\partial\bar{Q}}{\partial x_j} , \qquad (8)$$

while as $\sigma/\lambda \to \infty$,

$$\overline{u_j'Q'} \to - \frac{\epsilon_Q\tau_Q\sigma}{\gamma}\frac{\partial\bar{Q}}{\partial x_j} . \qquad (9)$$

In this latter limit, the transport flux is independent of λ and reduces to Eq. (4) as expected. We shall see that the former limit takes a form that is crucial to the existence of critical conditions for turbulence onset and also is necessary to achieve agreement with a variety of weak-turbulence data.

If Q is a diffusive vector (such as velocity), then we expect a similar equation to hold for the flux tensor, modified only by an additive term to make the contraction correct. Thus

$$\overline{u_j'Q_k'} = \frac{1}{3}\,\overline{u_\ell'Q_\ell'}\,\delta_{jk} - \frac{\tau_Q\sigma^2}{\beta\gamma\lambda}\left(1 - e^{-\frac{\epsilon_Q\beta\lambda}{\sigma}}\right)\frac{\partial\bar{Q}_k}{\partial x_j} . \qquad (10)$$

If, indeed, $Q_k \equiv u_k$, the proper symmetry requirements lead to

$$\overline{u_j'u_k'} = \frac{2}{3}\,\bar{q}\,\delta_{jk} - \frac{\tau_u\sigma\xi}{\beta\gamma}\left(1 - e^{-\frac{\epsilon_u\beta}{\xi}}\right)e_{jk} , \qquad (11)$$

where $\bar{q} \equiv \frac{1}{2}\,\overline{u_j'u_j'}$ and $\xi \equiv \sigma/\nu$. This form differs from the proposal in Ref. 2, Eq. (8), but agreement with that proposal in the limit as $\xi \to \infty$ means that $\epsilon_u\tau_u = \gamma$. Note that we also expect $\epsilon_u\beta\delta \approx 1$, in order to place the transition intensity from viscous to inertial at the same level as that indicated for the Δ transition.

In case the scalar Q refers to temperature, then

$$\overline{u_j'T'} = - \frac{\tau_T pn\xi\sigma}{\beta\gamma}\left(1 - e^{-\frac{\epsilon_T\beta}{\xi pn}}\right)\frac{\partial\bar{T}}{\partial x_j} , \qquad (12)$$

in which pn is the molecular Prandtl number. Accordingly, the <u>turbulence</u> Prandtl number, pn_T, is given by

$$pn_T = \left(\frac{\tau_u}{\tau_T pn}\right)\left(\frac{1 - e^{-\frac{\epsilon_u\beta}{\xi}}}{1 - e^{-\frac{\epsilon_T\beta}{\xi pn}}}\right) , \qquad (13)$$

which varies with turbulence intensity if $\epsilon_u pn \neq \epsilon_T$. For $\xi \to 0$,

$$pn_T \to \frac{\tau_u}{\tau_T pn} ,$$

while for $\xi \to \infty$,

$$pn_T \to \frac{\epsilon_u\tau_u}{\epsilon_T\tau_T} .$$

Since experiments show that pn_T decreases with ξ, (see Schlichting,[4] p. 499), we therefore expect $\epsilon_u pn < \epsilon_T$. Indeed, the data discussed by Schlichting suggest that $pn_T = 1.0$ for $\xi = 0$, and $pn_T \approx 0.5$ for $\xi \to \infty$, giving, therefore

$$\tau_u \approx \tau_T pn \quad , \tag{14}$$

and

$$2\,\epsilon_u \tau_u \approx \epsilon_T \tau_T \; . \tag{15}$$

As noted before, we also have

$$\epsilon_u \tau_u = \gamma \; , \tag{16}$$

and

$$\epsilon_u \, \beta\delta \approx 1. \tag{17}$$

The solution of these for equations for ϵ_u, ϵ_T, τ_u and τ_T, however, gives coefficient values that depend upon pn, a conflict with universality that needs further investigation for resolution. Perhaps it is the interpretation of the experimental data that needs revision.

III. INCLUSION OF THE TRANSPORT OF Q

Several extensions are required to calculate the transport of Q. First, there must be a transport equation for Q itself. This may be a simple diffusion equation, or it may be a complicated Liouville equation for a distribution function. Second, the mean momentum equation requires an interaction term, such as a mean buoyancy term if Q is heat, or a mean drag term if Q is dust density. Third, the two turbulence transport equations require creation terms describing such interactions as the creation of turbulence by buoyancy or flow past grains, and the creation of $\mathcal{D}$ by similar processes.

For the transport of Q, we assume, for example, an equation of the type

$$\frac{\partial Q}{\partial t} + u_j \frac{\partial Q}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\lambda \frac{\partial Q}{\partial x_j}\right) + S \; ,$$

in which S is an internal source such as that of heat from a chemical reaction. Averaging, we find

$$\frac{\partial \bar{Q}}{\partial t} + \bar{u}_j \frac{\partial \bar{Q}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\lambda \frac{\partial \bar{Q}}{\partial x_j} - \overline{u_j' Q'}\right) + \bar{S} \; , \tag{18}$$

which, together with Eq. (7), constitutes the required equation.

The equation for mean momentum is similar. We start from the Boussinesq approximation

$$\frac{\partial u_j}{\partial t} + u_k \frac{\partial u_j}{\partial x_k} + \frac{\partial \varphi}{\partial x_j} = -c g_j (Q - Q_0) + g_j + \frac{\partial}{\partial x_k}\left(\nu \frac{\partial u_j}{\partial x_k}\right),$$

in which g_j is the body acceleration, φ is the ratio of pressure to (constant) density, and c is a coefficient of density change accompanying the variations of Q from the reference value, Q_0. (Thus if Q is temperature, then c is the bulk expansion coefficient; if Q is solute concentration, then c relates this to the corresponding density changes. If Q is density of suspended particles, then an additional term may be required to describe drag, which is a function of the difference between fluid and dust velocities; but for now this possibility is ignored.)

Averaging the momentum equation, we obtain

$$\frac{\partial \bar{u}_j}{\partial t} + \bar{u}_k \frac{\partial \bar{u}_j}{\partial x_k} + \frac{\partial \bar{\varphi}}{\partial x_j} = c g_j (\bar{Q} - Q_0) + g_j + \frac{\partial}{\partial x_k}\left(\nu \frac{\partial \bar{u}_j}{\partial x_k} - \overline{u_j' u_k'}\right), \tag{19}$$

which, together with Eq. (11), constitutes the appropriate result.

Following Ref. 2, we may derive the first of the two turbulence transport equations, that of the turbulence energy per unit mass, q. An intermediate step is

$$\frac{\partial \bar{q}}{\partial t} + \bar{u}_k \frac{\partial \bar{q}}{\partial x_k} = -\overline{u_j' u_k'} \frac{\partial \bar{u}_j}{\partial x_k} - 2\nu\mathcal{D} + \frac{\partial}{\partial x_k}\left(\nu \frac{\partial \bar{q}}{\partial x_k} - \overline{u_k' q'} - \overline{u_k' \varphi'}\right) - c g_j \overline{u_j' Q'} \; , \tag{20}$$

where

$$\mathcal{D} \equiv \frac{1}{2}\overline{\left(\frac{\partial u_j'}{\partial x_k}\right)^2} = \frac{\Delta \bar{q}}{s^2} \; .$$

Into this are to be inserted the flux expressions, Eqs. (7) and (11), with ν being the diffusion coefficient for q and φ.

In similar fashion, the second turbulence transport equation, for $\mathcal{D}$, follows the development in Ref. 1, with the result

$$\frac{\partial \mathcal{D}}{\partial t} + \bar{u}_k \frac{\partial \mathcal{D}}{\partial x_k} = - \frac{a\Delta'}{s^2} \overline{u'_j u'_k} \frac{\partial \bar{u}_j}{\partial x_k} - \frac{\Delta}{s^2} \frac{\partial}{\partial x_k} (\overline{u'_k q'}) - \frac{\partial}{\partial x_k} (\overline{u'_k \mathcal{D}'}) - \frac{\partial}{\partial x_k} \left(\frac{\Delta}{s^2} \overline{u'_k \varphi'} \right) + \nu \frac{\partial^2 \mathcal{D}}{\partial x_k^2} - \frac{2\nu\Delta'\mathcal{D}}{s^2} - \frac{cg_j\Delta'}{s^2} \overline{u'_j Q'} . \qquad (21)$$

Again, the approximations, Eqs. (7) and (11), are to be used, with the additional condition that the diffusion coefficient for $\mathcal{D}$ be ν.

The three occurrences of Δ' in Eq. (21) are implied to be the same, but may, in fact, differ slightly among themselves. Use of Δ' in the first and last terms on the right side is a modification from the proposal in Ref. 1; it enables $a \approx 1.0$ (actually $a \lesssim 1.0$) and also $\tau_2 \approx \tau_1$ [of Eq. (27), below], (actually, $\tau_2 \lesssim \tau_1$). It also enables meaningful results to be obtained for all levels of turbulence with fixed values of a and τ_2, which turns out to be impossible if Δ is used in those terms, rather than Δ'.

To transform to usable notation, consistent with previous usage, we summarize these various flux approximations as derived from Eqs. (7) and (11). We assume for now that all of the ϵ_Q values are the same, namely

$$\epsilon_Q \equiv \frac{\epsilon}{\beta\delta} . \qquad (22)$$

Then

$$\overline{u'_j u'_k} = \frac{2}{3} \bar{q}\, \delta_{jk} - \frac{\sigma\delta\xi}{\epsilon} \left(1 - e^{-\frac{\epsilon}{\delta\xi}} \right) e_{jk} , \qquad (23)$$

$$\overline{u'_k q'} = - \frac{\alpha'\sigma\delta\xi}{\epsilon} \left(1 - e^{-\frac{\epsilon}{\delta\xi}} \right) \frac{\partial \bar{q}}{\partial x_k} , \qquad (24)$$

[When this is used in Eq. (20), $\alpha' = \alpha$; in Eq. (21), $\alpha' = a_2$.]

$$\overline{u'_k \varphi'} = - \frac{\theta'\sigma\delta\xi}{\epsilon} \left(1 - e^{-\frac{\epsilon}{\delta\xi}} \right) \frac{\partial \bar{\varphi}}{\partial x_k} , \qquad (25)$$

[When this is used in Eq. (20), $\theta' = \theta/\gamma$; in Eq. (21), $\theta' = a_4$.]

$$\overline{u'_k \mathcal{D}'} = - \frac{a_3\sigma\delta\xi}{\epsilon} \left(1 - e^{-\frac{\epsilon}{\delta\xi}} \right) \frac{\partial \mathcal{D}}{\partial x_k} , \qquad (26)$$

and

$$\overline{u'_k Q'} = - \frac{\tau_Q \sigma^2 \delta}{\lambda\epsilon} \left(1 - e^{-\frac{\lambda\epsilon}{\delta\sigma}} \right) \frac{\partial \bar{Q}}{\partial x_k} . \qquad (27)$$

[If Q is identified as temperature, then when this is used in Eq. (20), $\tau_Q = \tau_1$ while in Eq. (21), $\tau_Q = \tau_2$.]

IV. TURBULENCE DECAY THROUGH THE INERTIAL-VISCOUS TRANSITION

Reference 1 clarified the meaning of the scale function, s, showing it to be proportional to the integral scale and related to the microscale, λ, by the equation

$$\frac{s^2}{\Delta} = \frac{\lambda^2}{\beta} .$$

Thus the derivation in Ref. 2 is in error when it compares ds^2/dt with the experimental results for the microscale during low intensity decay. Instead, we may derive the decay result

$$\frac{\partial \lambda^2}{\partial t} = \beta \frac{\partial}{\partial t} \left(\frac{s^2}{\Delta} \right) = 2\beta\nu \left(\frac{\Delta'}{\Delta} - 1 \right) .$$

Thus, to agree with the experimental results, Δ' cannot be directly proportional to Δ, as suggested in Ref. 1. Indeed, with $\Delta' = \beta'(1 + \delta'\xi)$, the low and high intensity limits can be achieved if $\beta = 5$ (as before), $\beta' = 7$, and $\delta' = 10\delta/7$. This, then, gives $d\lambda^2/dt = 4\nu$ as $\xi \to 0$ and $d^2\lambda/dt \to 10\nu$ as $\xi \to \infty$, in agreement with the requirements discussed in Ref. 2. Actually, there is evidence to support the contention that the Δ and Δ' functions should vary more rapidly from β (or β') to $\beta\delta\xi$ (or $\beta'\delta'\xi$) than is indicated by the expressions

$$\Delta = \beta(1 + \delta\xi) ,$$

$$\Delta' = \beta'(1 + \delta'\xi) .$$

Experimentally, the transition in $d\lambda^2/dt$ from 4 to 10ν is quite abrupt, near $\xi = 7$, rather than as gradual as the above functional forms predict. In addition, the transitions predicted by the modified flux approximations are also relatively sudden, suggesting the same for Δ and Δ' by analogy. Finally, it appears that the laminar instability regime discussed in Section VII requires this abrupt transition in order that the small-ξ solutions may hold over an appreciable range of Rayleigh or Reynolds (or similar) number. Thus, for example, in the Bénard problem, independence of the solution from Prandtl-number variations requires the turbulence to be in the viscous range, which in turn allows prediction of a critical Rayleigh number. Persistence of this viscous range up to the Rayleigh number for sudden Nusselt number transition ($Ra \approx 4 \times 10^4$) then requires also the abruptness of the various functional transitions near $\xi = 7$, the viscous-inertial boundary.

From a heuristic, physical viewpoint, the transition may be related to a basic change in flow details, closely analogous to the onset of separated flow that occurs about a cylinder at Reynolds number of order 5. (The turbulence Reynolds number, ξ, has magnitude of about 7 at the viscous-inertial transition, so that the correspondence in Reynolds number magnitudes is quite close.) Thus the abruptness of transition may correspond to the suddenness of the onset of flow separations within the field of eddies. Unfortunately, there has not yet been developed a technique for deriving this, to obtain more realistic expressions for Δ and Δ', so that only empirical relationships can now be proposed.

The transition value for ξ, incidentally, must be closely associated with having $\delta\xi \approx 1$. In Ref. 2, Eq. (24), the heuristic derivations predict

$$\delta = \frac{9}{8\sqrt{2\beta\gamma}} \quad ,$$

which was there thought to be too large. In contrast, I now think that this value may be nearly correct.

There is another error regarding the interpretation of s found in Ref. 2. Equation (17), there, proposes a relationship among q, s, and σ, which is said to disagree with that proposed by Hinze.[5] Actually, the two proposals are equivalent for high intensity turbulence, as can be seen by a careful examination of the meaning of each. Thus, while we write

$$q = \frac{1}{2\gamma}\left(\frac{\sigma}{s}\right)^2 = \frac{1}{2\gamma}\left(\frac{\sigma}{\lambda}\right)^2\left(\frac{1}{1+\delta\xi}\right) \rightarrow \frac{\sigma\nu}{2\gamma\delta\lambda^2}$$

for large ξ, Hinze puts (in our nomenclature)

$$\sigma = \text{const} \times \frac{2q\lambda^2}{\nu} \; ,$$

which is equivalent to our proposal.

V. A REMARK ON EQUILIBRIUM FLOWS

In the momentum equation, we have

$$\frac{\partial u_j}{dt} = g_j - \frac{\partial}{\partial x_j}\left(\varphi + \frac{2}{3} q\right) + \ldots$$

plus other terms not entering the discussion of this section. We now consider the case of a turbulent fluid with no mean velocity, for which, then

$$\frac{\partial}{\partial x_j}\left(\bar{\varphi} + \frac{2}{3}\bar{q}\right) = \bar{g}_j .$$

If we suppose, for the moment, that g_j has a fluctuating part <u>correlated with the fluid turbulence fluctuations</u>, then derivation of the q equation proceeds as follows

$$\frac{\partial \bar{q}}{\partial t} \equiv \overline{u_j' \frac{\partial u_j'}{\partial t}} = \overline{g_j' u_j'} - \overline{u_j' \frac{\partial \varphi'}{\partial x_j}} + \ldots$$

If, in addition, we make the (ridiculous) assumption that such a g_j could be derived from a potential, i.e.,

$$g_j \equiv -\frac{\partial V}{\partial x_j} \; ,$$

then

$$\frac{\partial \bar{q}}{\partial t} = -\frac{\partial\, \overline{u_j' V'}}{\partial x_j} - \frac{\partial\, \overline{u_j' \varphi'}}{\partial x_j} + \ldots \; ,$$

and our approximation procedure produces, for intense turbulence (in which $\delta\xi \gg 1$),

$$\frac{\partial \bar{q}}{\partial t} = \frac{\partial}{\partial x_j}\left(\frac{\theta}{\gamma}\,\sigma\,\frac{\partial \bar{\varphi}}{\partial x_j} + K\,\sigma\,\frac{\partial \bar{V}}{\partial x_j}\right) + \ldots$$

$$= \frac{\partial}{\partial x_j}\left(\frac{\theta}{\gamma}\,\sigma\,\frac{\partial \bar{\varphi}}{\partial x_j} - K\,\sigma\,\overline{g_j}\right) + \ldots$$

For the case of no velocity, then,

$$\frac{\partial \bar{q}}{\partial t} = -\frac{\partial}{\partial x_j}\left[\frac{2\theta}{3\gamma}\,\sigma\,\frac{\partial \bar{\varphi}}{\partial x_j} + \left(K - \frac{\theta}{\gamma}\right)\sigma\,\overline{g_j}\right] + \ldots$$

Now, we do not expect the g_j term to contribute to the energy equation. (For the Bénard problem, for example, this contribution is quickly observed to be nonsense.) Its absence, however, requires $K = \theta/\gamma$, which would be reasonable except for the meaningless basis on which the K term was derived. The implication, then, is that $K = 0$ and that the term $\theta\sigma g_j/\gamma$ also should not have appeared. Apparently the presence of this latter term arises from an impropriety of the assumption $\overline{u_j'\varphi'} = -\frac{\theta}{\gamma}\,\sigma\,\frac{\partial \bar{\varphi}}{\partial x_j}$. The $\bar{\varphi}$ that appears in this assumption should relate only to that part of the pressure that is capable of fluctuations correlated to the turbulence field, and not to the part that is related to maintaining equilibrium with the constant body acceleration.

VI. STEADY-STATE SOLUTIONS

Strength of creation is characterized by a molecular Reynolds number, Rayleigh number, or other similar parameter; we call Rg some appropriate measure of this, as defined more specifically below. The geometry may be that of a pipe, the space between plates, or other configuration, in general characterized by a dimension, h. If a nontrivial steady-state solution exists, then this is characterized by mean or central values of σ and s, which, in dimensionless form, are denoted by ξ_s ($\equiv \sigma_s/\nu$) and z_s ($\equiv s_s/h$), where subscript s refers to steady state.

In addition, there may be other significant parameters such as the Prandtl number, but their effects are ignored for the present qualitative discussion. (In the Bénard problem, for example, the influence of Prandtl-number variations comes naturally from the analysis.)

In general, we expect that the two transport equations will allow the unique determination of ξ_s and z_s, each separately a function of Rg. Figure 1 illustrates a hypothetical example. For $Rg < A$, the only steady-state solution is $\xi_s = 0$, the steady laminar flow.

For $A < Rg < B$, the steady-state solution corresponds to equilibrium laminar instability fluctuations. This is the viscous range in which the flux terms are proportional to ξ^2. In the Bénard problem, for example, it would correspond to the regime of uniform, steady convective cells. For cylindrical Couette flow, it would also represent the steady-cell configuration.

For $Rg > B$, the flow is fully turbulent and the variations of ξ_s and z_s with Rg are abruptly somewhat different. The transition at B is also manifested in such functionals as the surface drag in its variation with Reynolds number, or the Nusselt number in its variation with Rayleigh number.

In addition to this steady-state flow, however, there is the more delicate consideration of stability as an initial-value problem. Usually the question of stability is treated only in the limit of infinitesimal (linear) perturbations. More generally, initial-value stability can be represented as an additional curve in the ξ-Rg plot. Figures 2 through 4 show some possibilities for this.

In each figure, the dashed line shows the variation of ξ_c, a critical initial value such that if the initial value ξ_0 exceeds ξ_c, then the turbulence grows to the steady-state value ξ_s, but if $\xi_0 < \xi_c$, then ξ decays to zero.

<u>Infinitesimal</u> stability theory in effect examines the behavior of ξ for values just above the abscissa. In Fig. 2 is a case in which the analysis would predict stability for <u>all</u> values of Rg. (Plane Couette flow is an example.) Hasen[6] has examined such a flow from quite a different theoretical point of view, deriving the ξ_c variation with Reynolds number in the form

$$(\mathrm{Ampl})_c \approx \frac{sU}{h\,\mathrm{Re}_1^{\frac{2}{3}}} \times \text{constant},$$

where $\mathrm{Re}_1 = \frac{Us^2}{h\nu}$ is assumed large and $(\mathrm{Ampl})_c = \sigma_c/s$. Thus

$$\xi_c \approx \frac{s^2 U}{h\nu\,\mathrm{Re}_1^{\frac{2}{3}}} \times \text{constant},$$

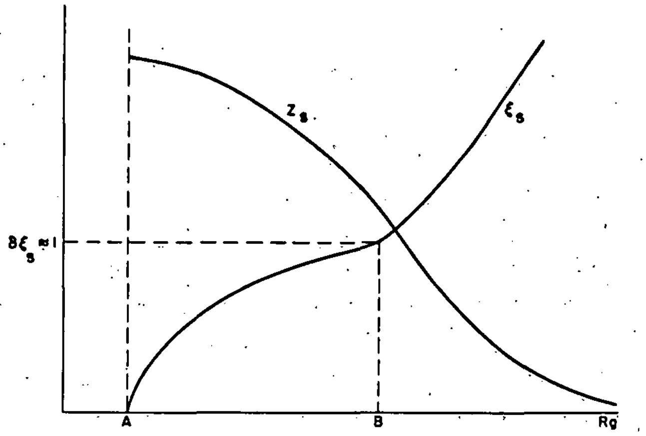

Fig. 1. Hypothetical variations of steady-state values of z and ξ.

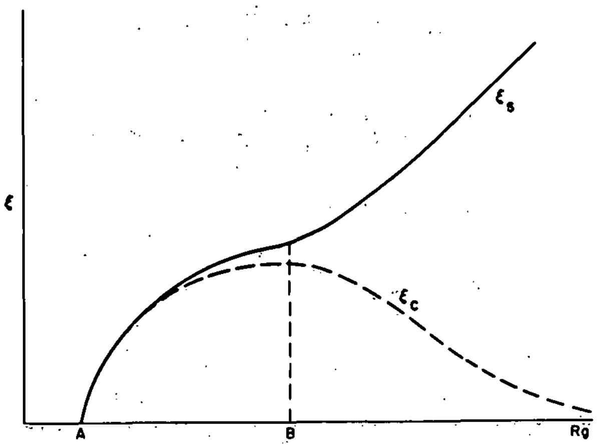

Fig. 2. An example with no critical value of Rg.

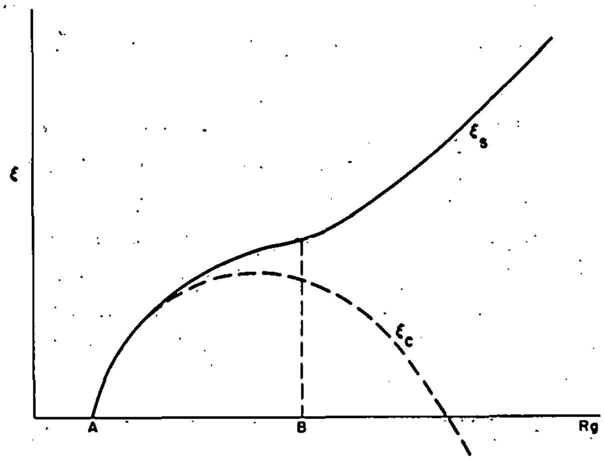

Fig. 3. An example with a critical value of Rg.

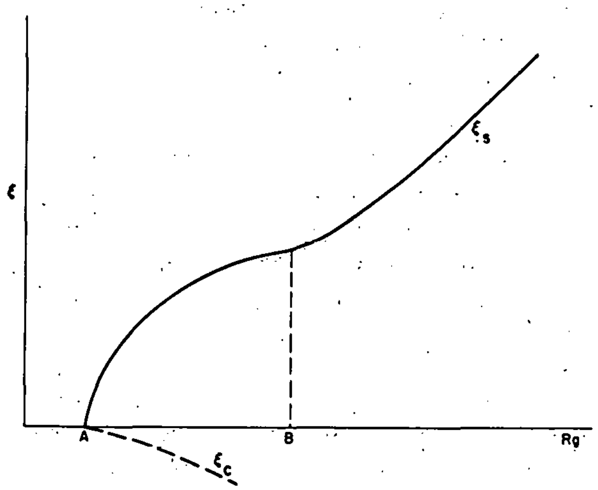

Fig. 4. The degeneration of critical Rg.

or

$$\xi_c \approx \mathrm{Re}_1^{\frac{1}{3}} \times \text{constant} ,$$

or

$$\xi_c \approx \text{constant} \times \left(\frac{\delta}{h}\right)^{\frac{2}{3}} \mathrm{Re}^{\frac{1}{3}} ,$$

where Re = Uh/ν.

Figure 3 is an example in which a critical value of Rg would be predicted by the analysis. The meaning here is that for sufficiently small perturbations, there is a limiting Rg below which the perturbations will always damp; but for finite-amplitude perturbations, the critical value of Rg for transition to turbulence decreases with ξ_0. Many physical examples can be cited in which the critical conditions for transition to turbulence depend in this fashion upon the initial "noise" level. Indeed, the example in Fig. 2 also shows this.

Figure 4 differs in that the linear theory would show laminar instability for Rg > A, and <u>any</u> perturbation would grow to ξ_s as long as Rg > A. An example is that of wake flow behind a circular cylinder, characterized by a Reynolds number, Rg ≡ Re. For Re < A (≈ 40), the flow is steady. For A ≤ Re < B, the regular von Karman vortex street occurs. For Re > B, the flow becomes more nearly truly turbulent. (Thus the regular two-dimensional vortex street becomes unstable to three-dimensional perturbations, and the flow becomes "irregular.")

It should be remarked that the ξ_c line varies somewhat with Z_0, but the qualitative conclusions remain, altered only in the detailed quantitative values. Linear stability theory, for example, usually looks for the critical value of Rg by minimizing its variations as a function of Z_0. The dashed ξ_c lines in Figs. 2 through 4 are meant to denote the minimum ξ_c as a function of Z_0 for each Rg.

If we standardize the meaning of Rg in such a way as to have only a linear dissipative coefficient in the denominator, then it appears that many of the curves become closely similar for various types of phenomena. Thus, for shear flows, Rg = Re, the Reynolds number; for the Bénard problem, Rg = $\sqrt{Ra}$, where Ra is the Rayleigh number; for cylindrical Couette flow, Rg = $\sqrt{T}$, where T is the Taylor number. Circumstances with behavior like that in Fig. 4 (flow past a cylinder, Bénard convection, narrow-gap cylindrical Couette flow, etc.) then all exhibit critical values of Rg at A $\approx$ 41, and we may suppose that this is indicative of a nearly universal magnitude for all types of flows (Figs. 2 and 3, also).

The point B appears also to be universal in this system of measuring Rg, occurring at B $\approx$ 200. [There is, for example, a flexure in Nusselt number variation in the Bénard problem at Ra $\approx (200)^2$.] This is the point at which laminar fluctuations (cells, regular vortices, etc.) begin to degenerate into full turbulence. Here, too, is the transition from the viscous range of turbulence to the inertial.

VII. INTERMITTENCY

The interpretation of intermittency in turbulence has an important bearing on the assumption that weak turbulence in our transport equations does, in fact, represent the regular laminar instability state for those circumstances in which such should occur.

Consider first a region of intense turbulence adjacent to a laminar region. If there is shear between the two regions, the Kelvin-Helmholtz instability may amplify large scale perturbations, producing large eddies with scales independent of the turbulence scale. Accordingly, the interface between the regions becomes irregular, and intermittency can be observed as the fluid sweeps by. This is an example in which the large scale distortions of the flow are not at all included in the turbulence spectrum, and are calculated only by the mean-flow equations. (Of course, the presence of the turbulence introduces eddy viscosity which can affect profoundly the growth rate of the large scale disturbance, and thereby contribute to determination of maximum-growth-rate wave length.) Thus, the observed intermittency is manifested as a succession of periods of very rapid fluctuations, separated by pauses of calm, the rapid fluctuations corresponding to the small turbulence scale and the periods of calm to the large scale instability.

Consider now a succession of experiments in which the turbulence is progressively weaker. In general, its scale is correspondingly progressively larger. For sufficiently weak turbulence ($\delta\xi \approx 1$), the scale approaches that of the perturbing influence, so that the mean-flow instability is amplified with a scale spectrum nearly the same as that of the turbulence itself. In this case, the two phenomena are no longer separable, and it is reasonable to combine the effects into a net flow that can be considered all turbulence (in our turbulence transport approach) <u>or</u> all mean flow (in the approaches that solve the full equations of unstable laminar flow). In such cases, intermittency has no meaning.

Thus, in the framework of our interpretation, two distinctly different processes in regions of intense turbulence merge into one when the turbulence is weak. In the latter case the scale distribution has converged into one for which a single scale function is appropriate, and the results can be represented either by the low intensity transport equations or by solutions in detail of the full laminar instability problem.

VIII. NONISOTROPIC TURBULENCE

We assume that the full burden of the nonisotropy can be placed upon σ and q, but that s remains a scalar function. This assumption appears not to be strictly valid, as there is much evidence to suggest the existence of elongated or otherwise distorted eddies. As a first approximation to the study of nonisotropic turbulence, however, the assumption leads to a considerably simplified formulation worth testing in some representative applications.

The basic task is to find an approximation for the

Reynolds stress, which is taken to have the form

$$\overline{u_i'u_j'} = 2q_{ij} - \tfrac{1}{2}\left(\sigma_{ik}e_{kj} + \sigma_{jk}e_{ki}\right) + \tfrac{1}{3}\,\sigma_{\ell k}e_{\ell k}\delta_{ij}. \quad (28)$$

The last term is introduced in such a way that $q \equiv \frac{1}{2}\,\overline{u_i'u_i'} \equiv q_{ii}$ does not depend upon the rate of strain; the form also is chosen to ensure symmetry. We further assume that

$$q_{ij} = \frac{1}{6\gamma s^2}\,\sigma_{ik}\sigma_{jk}. \quad (29)$$

In this formulation, the tensor eddy viscosity, σ_{ij}, is assumed to be symmetric, so that the tensor eddy energy is correspondingly symmetric. (In some kinds of problems, the two tensors are everywhere diagonal, and may even have only two differing components. The Bénard problem is one in which this convenient situation occurs.) Equations (28) and (29) reduce to our previous formulations for isotropic turbulence, for which $\sigma_{ij} \equiv \sigma\delta_{ij}$. (Note that for isotropic turbulence the σ for each direction must be the same as the total σ, which also relates to the effect that occurs in each individual direction, rather than to a summation of effects. Thus, $\sigma_{ij}\sigma_{ij} = 3\sigma$.)

For high intensity turbulence, the flux approximation for a scalar likewise can be generalized to show the effects of this nonisotropy model. We write

$$\overline{u_j'Q'} = -\,\text{constant} \times \sigma_{jk}\,\frac{\partial\bar{Q}}{\partial x_k}. \quad (30)$$

For low intensity turbulence, modifications like those described in the first sections of this report are needed, but such are not proposed here.

To find the energy transport equations, insert the flux approximations into Eq. (7) of Ref. 2. We choose an example with both shear and buoyancy creation, the last with coefficient of volume expansion T_0^{-1}.

$$\left(\frac{\partial}{\partial t} + \bar{u}_k\frac{\partial}{\partial x_k}\right)\left[2q_{ij} - \tfrac{1}{2}\left(\sigma_{i\ell}e_{\ell j} + \sigma_{j\ell}e_{\ell i}\right) + \tfrac{1}{3}\,\sigma_{\ell n}e_{\ell n}\delta_{ij}\right] \quad \text{(a)}$$

$$= -\frac{\partial\bar{u}_j}{\partial x_k}\left[2q_{ik} - \tfrac{1}{2}\left(\sigma_{i\ell}e_{\ell k} + \sigma_{k\ell}e_{\ell i}\right) + \tfrac{1}{3}\,\sigma_{\ell m}e_{\ell m}\delta_{ik}\right] \quad \text{(b)}$$

$$-\frac{\partial\bar{u}_i}{\partial x_k}\left[2q_{jk} - \tfrac{1}{2}\left(\sigma_{j\ell}e_{\ell k} + \sigma_{k\ell}e_{\ell j}\right) + \tfrac{1}{3}\,\sigma_{\ell m}e_{\ell m}\delta_{jk}\right] \quad \text{(c)}$$

$$-\frac{\partial}{\partial x_k}\,\overline{u_k'\left[2q_{ij}' - \tfrac{1}{2}\left(\sigma_{i\ell}e_{\ell j}' + \sigma_{j\ell}e_{\ell i}'\right) + \tfrac{1}{3}\,\sigma_{\ell m}e_{\ell m}'\delta_{ij}\right]} \quad \text{(d)}$$

$$-\overline{u_i'\frac{\partial\varphi'}{\partial x_j}} - \overline{u_j'\frac{\partial\varphi'}{\partial x_i}} \quad \text{(e)}$$

$$+\nu\left(\frac{\partial^2\overline{u_i'u_j'}}{\partial x_k^2} - 2\,\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}\right) \quad \text{(f)}$$

$$-\frac{1}{T_0}\left(g_j\,\overline{u_i'T'} + g_i\,\overline{u_j'T'}\right). \quad \text{(g)}$$

$$(31)$$

Note that shear effects create turbulence for the energy component along the direction of flow, while buoyancy effects produce turbulence along the direction of the temperature gradient and gravity. The dominant coupling among the directions comes from the triple-correlation term (31-d) and from the pressure terms (31-e). Two of the terms can be transformed immediately. Thus, for the second part of (31-f) we write

$$-2\nu\,\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} \rightarrow -\frac{4\nu\Delta}{s^2}\,q_{ij}, \quad (32)$$

from which the rate-of-strain terms have been dropped [just as one might also drop them in (31-a)], and in which

$$\Delta = \beta\left[1 + \frac{\delta}{\nu}\left(\frac{\sigma_{ij}\sigma_{ij}}{3}\right)^{\frac{1}{2}}\right]. \quad (33)$$

Also

$$(31\text{-g})_{ij} = \frac{\tau_1}{2T_0}\left(g_j\sigma_{ik} + g_i\sigma_{jk}\right)\frac{\partial T}{\partial x_k}. \quad (34)$$

In addition, the first part of (31-d) becomes

$$-2\frac{\partial}{\partial x_k}\left(\overline{u_k'q_{ij}'}\right) \rightarrow 2\alpha\frac{\partial}{\partial x_k}\left(\sigma_{kn}\frac{\partial q_{ij}}{\partial x_n}\right). \quad (35)$$

Consider now the special case of buoyancy turbulence, in which the mean velocity vanishes. For this we may write

$$\frac{\partial q_{ij}}{\partial t} = \frac{\partial}{\partial x_k}\left[\left(\nu\delta_{kn} + \alpha\sigma_{kn}\right)\frac{\partial q_{ij}}{\partial x_n}\right] - \frac{2\nu\Delta}{s^2}\,q_{ij}$$

$$+\frac{\theta}{2\gamma}\frac{\partial}{\partial x_k}\left(\sigma_{ki}\frac{\partial\varphi}{\partial x_j} + \sigma_{kj}\frac{\partial\varphi}{\partial x_i}\right)$$

$$+ \frac{\tau_1}{2T_0} (g_j \sigma_{ik} + g_i \sigma_{jk}) \frac{\partial T}{\partial x_k}$$

$$+ \Phi_{ij} , \qquad (36)$$

in which Φ_{ij} is the residual from the pressure and triple-correlation terms, chosen in such a way that $\Phi_{ii} \equiv 0$. For the φ gradients in Eq. (36) we invoke the momentum equilibrium condition

$$\frac{\partial \varphi}{\partial x_j} = -2 \frac{\partial q_{jk}}{\partial x_k} , \qquad (37)$$

which is to be used in both Eqs. (36) and (39). Also, with

$$\mathcal{D} \equiv \frac{\Delta q_{ii}}{s^2} , \qquad (38)$$

we may write, consistent with the assumption of scalar s, a single equation for $\mathcal{D}$ in the form

$$\frac{\partial \mathcal{D}}{\partial t} = - \frac{2\nu\Delta'\mathcal{D}}{s^2} + \frac{a_2 \Delta}{s^2} \frac{\partial}{\partial x_k} \left(\sigma_{jk} \frac{\partial q_{ii}}{\partial x_j} \right)$$

$$+ \nu \frac{\partial^2 \mathcal{D}}{\partial x_k^2} + a_3 \frac{\partial}{\partial x_k} \left(\sigma_{jk} \frac{\partial \mathcal{D}}{\partial x_j} \right)$$

$$+ a_4 \frac{\partial}{\partial x_k} \left(\frac{\Delta \sigma_{jk}}{s^2} \frac{\partial \varphi}{\partial x_j} \right) + \frac{\tau_2 \Delta'}{T_0 s^2} \sigma_{jk} g_j \frac{\partial T}{\partial x_k} , \qquad (39)$$

which closely resembles the previous proposal and reduces to it for isotropic turbulence.

Thus, the crucial remaining question is how to approximate Φ_{ij}. For this, diligent investigation has not produced a satisfactory answer. Instead, we have been forced to construct Φ_{ij} on a heuristic basis, from a combination of all available tensors satisfying certain physical and mathematical requirements. The result is

$$\Phi_{ij} = - \frac{\omega}{s^2} \left(\sigma_{i\ell} q_{\ell j} - \frac{1}{3} \sigma_{\ell m} q_{\ell m} \delta_{ij} \right)$$

$$- \frac{\zeta}{2s^2} \sigma_{nm} \left(\frac{\partial \sigma_{n\ell}}{\partial x_i} \frac{\partial \sigma_{mj}}{\partial x_\ell} + \frac{\partial \sigma_{n\ell}}{\partial x_j} \frac{\partial \sigma_{mi}}{\partial x_\ell} \right)$$

$$+ \frac{\zeta}{3s^2} \delta_{ij} \sigma_{nm} \frac{\partial \sigma_{n\ell}}{\partial x_k} \frac{\partial \sigma_{mk}}{\partial x_\ell} , \qquad (40)$$

in which ω and ζ are universal, dimensionless, positive constants with magnitudes near unity. The guidelines for choosing this form are as follows.

1. We require $\Phi_{ii} \equiv 0$.
2. When the source terms to the q_{ij} equation are diagonal, then Φ_{ij} should be diagonal.
3. Part of Φ_{ij} should represent the conversion of q to φ as an eddy stops in a distance s, then reaccelerates in the other directions. This is the ω part.
4. Part of Φ_{ij} should represent the conversion of q to other directions when the component of q is decreasing in its own direction, as near a wall. This is the ζ part.
5. The conversion terms in the ζ part must be positive (or negative) definite, when the circumstances are appropriate. This can be most conveniently tested for circumstances in which Φ_{ij} is diagonal.
6. The correct dimensionality of the ζ term should not require inverse contracted tensors.
7. The ζ term should not remove energy from a direction as a result of gradients in that direction of the energy in another direction, nor as a result of gradients in another direction of energy in that direction.

The ω term depends upon the difference in σ-q products, rather than upon q differences, in order to relate the conversion timing for each direction to the time scale appropriate for that direction. Conditions 5 and 6 above dictate that the ζ term be formed of σ derivatives rather than q derivatives.

Note that the ω and ζ coupling terms produce quite different effects. Suppose that energy is created predominantly in only one direction; then the ω term tends to create isotropy by pressure coupling to the other directions, working effectively even in regions of homogeneity. In contrast, the ζ term, which is effective only in regions of inhomogeneity, describes especially the effect of a wall in converting the normal component of energy to the tangential directions. The ζ term can therefore contribute to increasing anisotropy, an essential feature for duplicating experimental results in a variety of circumstances.

(It should be noted, however, that conceivable circumstances may arise in which the ζ-term coupling is not appropriate, as for example, near the front of a turbulence diffusion wave. For such problems, the Φ_{ij} tensor may require significant modification.) The full effects of these equations for nonisotropic turbulence can be illustrated much more effectively through application to examples and comparisons with experiments. For this purpose we have studied extensively the example of the turbulence between horizontal plates heated from below (the Bénard problem), and report the details elsewhere.

IX. COMPARISON WITH THE THEORY OF ROTTA

J. Rotta,[7,8] has proposed a theoretical approach that resembles ours in several respects. The first of his two papers[7] concerns the energy equation, and in particular the effects of pressure-velocity correlations on the redistribution of energy among the various directions. His proposed form, which always enhances isotropy, is quite similar to the ω term in Eq. (40), differing principally in the rate coefficient. Rotta factors this outside of the energy difference, while Eq. (40) proposes that the transfer rate of energy from a direction should depend upon the turbulence strength in that direction. A more important difference between the two forms is in the ζ coupling terms, which Rotta omits entirely, and which may contribute to nonisotropy.

The energy dissipation term of Rotta is essentially identical to ours. It is likely, however, that neither is sufficiently abrupt in representing the transition from low to high intensity.

Rotta's first paper[7] shows application of the theory to a specific problem, but neglects the energy diffusion and scale transport that are certainly necessary for general applicability. His second paper[8] proposes corrections for these deficiencies.

The scale equation is derived by Rotta from a two-point-correlation-function equation. The technique resembles that of Chou,[9] but differs in several significant respects. It seems that our derivation, based directly upon the decay-rate definition of scale,[1] is likely to be more realistic in its representation of the processes of turbulence transport. The matter can be settled, however, only when rigorous derivations have been accomplished, or the results compared in detail with experiments.

In summary, it is apparent that Rotta contributed significantly to the foundations of this type of turbulence theory. Our approach and results do not seriously conflict with his; instead, they extend his ideas in a manner that should prove more widely applicable and somewhat more tractable, at the same time utilizing field variables with more direct physical significance to the engineer.

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