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Transport of Turbulence Energy Decay Rate

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Francis H. Harlow and Paul I. Nakayama

ABSTRACT

The second of two previously-proposed turbulence transport
equations is here derived so as to determine the variation
of several of the universal functions. The derivation also
extends the usefulness of the equations, strengthens their
validity, and clarifies the interpretations of the turbu-
lence fields variables.

I. INTRODUCTION

In a recent paper,[1] hereinafter referred to as
I, we proposed a coupled pair of transport equa-
tions to describe the transient dynamics of a tur-
bulent fluid. The mean flow of the fluid is simul-
taneously described by the Navier-Stokes equations
with an eddy viscosity function. The equations are
written in universal, invariant form, expressing
the Eulerian derivatives of the turbulence functions
in terms of convective and diffusive fluxes, and in
terms of creation and decay processes. The turbu-
lence functions for the effective eddy viscosity,
σ, and the size-scale function, s, are postulated
to relate to the specific turbulence energy, $\bar{q}$,
through the Prandtl-Wieghardt formula

$$\bar{q} = \frac{1}{2\gamma}\left(\frac{\sigma}{s}\right)^2 , \tag{1}$$

in which γ is a universal constant of magnitude
near unity.

One of the transport equations was derived
directly from the Navier-Stokes equations, utiliz-
ing some appropriate moment approximations. The
second equation was proposed on purely heuristic
grounds, and was shown to be valid in several ap-
plications. The purpose of this report is to de-
rive the second equation in the same manner as the
first, and thereby to determine more rigorously

the behavior of several universal functions that
have not previously been specified, and to streng-
then considerably the validity of the theory.

The first of the two equations expresses the
transport of turbulence energy. In the nomencla-
ture of I, this can be written [see Eq. (I-18)],

$$\frac{\partial \bar{q}}{\partial t} + \bar{u}_k \frac{\partial \bar{q}}{\partial x_k} = \sigma \bar{e}_{jk} \frac{\partial \bar{u}_j}{\partial x_k} + \frac{\theta}{\gamma}\frac{\partial}{\partial x_k}\left(\sigma \frac{\partial \bar{\phi}}{\partial x_k}\right)$$
$$+ \frac{\partial}{\partial x_k}\left[(\nu + \alpha\sigma)\frac{\partial \bar{q}}{\partial x_k}\right] - 2\nu\mathfrak{D} \tag{2}$$

A basic assumption of this derivation is that

$$\frac{1}{2}\overline{\left(\frac{\partial u'_j}{\partial x_k}\right)^2} = \mathfrak{D} \equiv \frac{\Delta \bar{q}}{s^2} \tag{3}$$

in which $\Delta \equiv \beta(1 + \delta\xi)$, β is a constant whose value
is 5.0, δ is a constant of order 0.01, and $\xi \equiv \sigma/\nu$.
The other universal constants in Eq. (2), γ, θ, and
α, have magnitudes expected to be near unity.

The second of the two transport equations was
proposed in the form [see Eq. (I-29)]

$$\frac{\partial \Lambda}{\partial t} + \bar{u}_k \frac{\partial \Lambda}{\partial x_k} = 3\sigma^2 s^2 \frac{\partial}{\partial x_k}\left[(\nu + \psi\sigma)\frac{\partial s}{\partial x_k}\right]$$
$$+ 3\sigma^3 sF(\xi) - 3\sigma s^5 \left(\frac{\partial \bar{u}_j}{\partial x_k}\right)^2 g(\xi), \tag{4}$$

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in which Λ is the Loitsiansky function, $\sigma^2 s^3$, and ψ is a universal constant of magnitude near unity. $F(\xi)$ and $g(\xi)$ are universal functions for which the detailed behavior was unknown, but which were determined empirically over a rather restricted range of values of ξ.

II. DERIVATION OF DECAY-RATE TRANSPORT EQUATION

It is now evident that Λ is not an appropriate variable for which to derive the second transport equation. (We could, however, convert the results of this paper to an equation for Λ, to correspond with the previously proposed equation.) Instead, it is ϑ for which the second transport equation should be derived, since it is this energy decay term that serves primarily to introduce the scale function, s.

The starting point for the present derivation is Eq. (I-6), describing the dynamics of the fluctuating part of the flow:

$$\frac{\partial u'_j}{\partial t} + u'_k \frac{\partial \bar{u}_j}{\partial x_k} + \bar{u}_k \frac{\partial u'_j}{\partial x_k} + u'_k \frac{\partial u'_j}{\partial x_k} = -\frac{\partial \varphi'}{\partial x_j}$$
$$+ \frac{\partial}{\partial x_k}\left(\nu \frac{\partial u'_j}{\partial x_k} + \overline{u'_j u'_k}\right). \quad (5)$$

Differentiating this with respect to x_ℓ, multiplying by $\partial u'/\partial x_\ell$, and taking the ensemble average yields

$$\frac{\partial}{\partial t}\overline{\left[\frac{1}{2}\left(\frac{\partial u'_j}{\partial x_\ell}\right)^2\right]} + \overline{\frac{\partial u'_j}{\partial x_\ell}\frac{\partial}{\partial x_\ell}\left[u'_k \frac{\partial \bar{u}_j}{\partial x_k} + \bar{u}_k \frac{\partial u'_j}{\partial x_k} + u'_k \frac{\partial u'_j}{\partial x_k}\right]}$$
$$= -\overline{\frac{\partial u'_j}{\partial x_\ell}\frac{\partial^2 \varphi'}{\partial x_j \partial x_\ell}} + \nu \overline{\frac{\partial u'_j}{\partial x_\ell}\frac{\partial^2}{\partial x_k^2}\left(\frac{\partial u'_j}{\partial x_\ell}\right)}. \quad (6)$$

This, then, is the exact transport equation for ϑ. It is the result that we make tractable by the introduction of appropriate moment approximations, thereby obtaining the required second equation for turbulence transport.

Equation (6) can be rewritten in the form

$$\frac{\partial}{\partial t}\overline{\left[\frac{1}{2}\left(\frac{\partial u'_j}{\partial x_\ell}\right)^2\right]} =$$

$$- \frac{\partial^2 \bar{u}_j}{\partial x_\ell \partial x_k}\overline{\left[u'_k \frac{\partial u'_j}{\partial x_\ell}\right]} \quad (7a)$$

$$- \frac{\partial \bar{u}_j}{\partial x_k}\overline{\left[\frac{\partial u'_j}{\partial x_\ell}\frac{\partial u'_k}{\partial x_\ell}\right]} \quad (7b)$$

$$- \frac{\partial \bar{u}_k}{\partial x_\ell}\overline{\left[\frac{\partial u'_j}{\partial x_\ell}\frac{\partial u'_j}{\partial x_k}\right]} \quad (7c)$$

$$- \bar{u}_k \frac{\partial}{\partial x_k}\overline{\left[\frac{1}{2}\left(\frac{\partial u'_j}{\partial x_\ell}\right)^2\right]} \quad (7d)$$

$$- \overline{\frac{\partial u'_j}{\partial x_\ell}\frac{\partial u'_k}{\partial x_\ell}\frac{\partial u'_j}{\partial x_k}} \quad (7e)$$

$$- \overline{u'_k \frac{\partial}{\partial x_k}\left[\frac{1}{2}\left(\frac{\partial u'_j}{\partial x_\ell}\right)^2\right]} \quad (7f)$$

$$- \overline{\frac{\partial u'_j}{\partial x_\ell}\frac{\partial^2 \varphi'}{\partial x_j \partial x_\ell}} \quad (7g)$$

$$+ \nu \overline{\frac{\partial u'_j}{\partial x_\ell}\frac{\partial^2}{\partial x_k^2}\left(\frac{\partial u'_j}{\partial x_\ell}\right)}. \quad (7h)$$

The terms are labeled separately for reference to the manner in which each is approximated below.

Consistent with the procedure in I, we approximate the bracketed factor in Term (7a) by the following

$$\overline{\left[u'_k \frac{\partial u'_j}{\partial x_\ell}\right]} = -\text{constant} \times \sigma \frac{\partial^2 \bar{u}_j}{\partial x_k \partial x_\ell} + A_j \delta_{k\ell}$$

$$+ B_k \delta_{j\ell} + C_\ell \delta_{jk} + D\epsilon_{jk\ell}, \quad (8)$$

in which A_j, B_k, C_ℓ, and D are undetermined functions and $\epsilon_{jk\ell}$ is the Levi-Civita tensor. These undetermined functions can be found by examining the appropriate contractions of Eq. (8), and by multiplying through by $\epsilon_{jk\ell}$ and solving. The only possibly contributing function, however, is A_j, for which we find

$$A_j = \frac{1}{6} \frac{\partial \overline{q}}{\partial x_j} , \tag{9}$$

but this gives a higher-order coupling to the mean field that we now consider to be negligible, and thus ignore.

The Terms (7b) and (7c) can both be found from contractions of the more general tensor

$$\overline{\frac{\partial u'_i}{\partial x_k} \frac{\partial u'_j}{\partial x_\ell}} .$$

This we assume can be decomposed into properly symmetric products of the available second-order tensors, δ_{ij}, e_{ij} and these same tensors with the various other indices. The scalar coefficients of these products can then be determined by examination of appropriate contractions, two of which must vanish because of the incompressibility condition, and one of which must reduce to the definition of $\mathscr{D}$ in Eq. (3). The result can be written

$$\text{Term (7b)} = \frac{a_1 \Delta \sigma}{s^2} e_{jk} \frac{\partial \overline{u}_j}{\partial x_k} , \tag{10}$$

$$\text{Term (7c)} = \frac{b_1 \Delta \sigma}{s^2} e_{jk} \frac{\partial \overline{u}_j}{\partial x_k} , \tag{11}$$

in which a_1 and b_1 are universal constants, and the magnitude of a_1 is expected to be near unity. (The derivation suggests nothing about the magnitude of b_1, but a value near unity is later shown to be reasonable.)

Reduction of Term (7d) is easily accomplished:

$$\text{Term (7d)} = -\overline{u}_k \frac{\partial}{\partial x_k}\left(\frac{\Delta \overline{q}}{2}\right) = -\overline{u}_k \frac{\partial \mathscr{D}}{\partial x_k} . \tag{12}$$

Term (7e) is treated in a manner analogous to the assumption in Eq. (3):

$$\text{Term (7e)} = -\frac{\Delta}{s^2} \overline{u'_j u'_k \frac{\partial u'_j}{\partial x_k}}$$

$$= -\frac{\Delta}{s^2} \overline{u'_k \frac{\partial q'}{\partial x_k}}$$

$$= \frac{a_2 \Delta}{s^2} \frac{\partial}{\partial x_k}\left(\sigma \frac{\partial \overline{q}}{\partial x_k}\right) . \tag{13}$$

In similar fashion,

$$\text{Term (7f)} = -\frac{\partial}{\partial x_k}\left[\left(\frac{\Delta}{2}\right)' \overline{u'_k q'}\right]$$

$$= a_3 \frac{\partial}{\partial x_k}\left(\sigma \frac{\partial \mathscr{D}}{\partial x_k}\right); \tag{14}$$

and

$$\text{Term (7g)} = -\frac{\partial}{\partial x_j}\left(\overline{\frac{\partial u'_j}{\partial x_\ell} \frac{\partial \varphi'}{\partial x_\ell}}\right)$$

$$= a_4 \frac{\partial}{\partial x_j}\left(\frac{\Delta \sigma}{s^2} \frac{\partial \overline{\varphi}}{\partial x_j}\right) . \tag{15}$$

In these expressions, a_2, a_3, and a_4 are again universal constants with magnitudes near unity.

Finally, Term (7h) is reduced through a two-step process like those used above.

$$\text{Term (7h)} = \nu \frac{\partial}{\partial x_k}\left(\overline{\frac{\partial u'_j}{\partial x_\ell} \frac{\partial}{\partial x_k} \frac{\partial u'_j}{\partial x_\ell}}\right) - \nu \overline{\left(\frac{\partial}{\partial x_k} \frac{\partial u'_j}{\partial x_\ell}\right)^2}$$

$$= \nu \frac{\partial^2}{\partial x_k^2}\left[\overline{\frac{1}{2}\left(\frac{\partial u'_j}{\partial x_\ell}\right)^2}\right] - \frac{\nu \Delta'}{s^2} \overline{\left(\frac{\partial u'_j}{\partial x_\ell}\right)^2}$$

$$= \nu \frac{\partial^2}{\partial x_k^2}\left(\frac{\Delta \overline{q}}{s^2}\right) - \frac{2\nu \Delta \Delta' \overline{q}}{s^4} ,$$

$$= \nu \frac{\partial^2 \mathscr{D}}{\partial x_k^2} - \frac{2\nu \Delta' \mathscr{D}}{s^2} , \tag{16}$$

in which $\Delta' \equiv \beta'(1 + \delta\xi)$, and β' is a universal constant with magnitude near that of β.

Combining these results and dropping bars, we obtain

$$\frac{\partial \mathscr{D}}{\partial t} + u_k \frac{\partial \mathscr{D}}{\partial x_k}$$

$$= \frac{a \Delta \sigma}{s^2} e_{jk} \frac{\partial u_j}{\partial x_k} + \frac{a_2 \Delta}{s^2} \frac{\partial}{\partial x_k}\left(\sigma \frac{\partial q}{\partial x_k}\right)$$

$$+ a_3 \frac{\partial}{\partial x_k}\left(\sigma \frac{\partial \mathscr{D}}{\partial x_k}\right) + a_4 \frac{\partial}{\partial x_k}\left(\frac{\Delta \sigma}{s^2} \frac{\partial \varphi}{\partial x_k}\right)$$

$$+ \nu \frac{\partial^2 \mathscr{D}}{\partial x_k^2} - \frac{2\nu \Delta' \mathscr{D}}{s^2}, \tag{17}$$

where $a \equiv a_1 + b_1$.

This, then, is our principal result, the proposed second transport equation to be coupled with the energy equation of I.

III. DISCUSSION

In what follows, we examine the properties of Eq. (17) and compare it with the previously proposed form, Eq. (4). The first and last terms on the right side of Eq. (17) are of primary significance for the comparison. Ignoring for a moment the other terms, we combine Eq. (17) with Eq. (2) to show that

$$\left(\frac{\partial}{\partial t} + u_k \frac{\partial}{\partial x_k}\right)(\sigma^2 s^n)$$

$$= \frac{\gamma \sigma s^{n+2}}{2 + \delta\xi}\left\{8 + 2n - (4 + 2n) a\right.$$

$$+ \left[8 + 3n - (4 + 2n) a\right]\delta\xi\Big\}\, e_{jk}\frac{\partial u_j}{\partial x_k}$$

$$- \nu\beta\sigma^2 s^{n-2}\left(\frac{1 + \delta\xi}{2 + \delta\xi}\right)\left\{8 + 2n - (4 + 2n)\frac{\beta'}{\beta}\right.$$

$$+ \left[8 + 3n - (4 + 2n)\frac{\beta'}{\beta}\right]\delta\xi\Big\}$$

$$+ \text{diffusion and pressure terms.} \qquad (18)$$

As discussed in I, σ is a decay invariant when ξ is small, indicating that $\beta' = 2\beta$. The question arises as to whether this relationship is consistent with other properties that the equations should have. To partially answer this, we compare the creation and decay terms in Eq. (18) with the corresponding terms in Eq. (4). With $n = 3$, this leads to the identifications

$$\xi F(\xi) = \frac{\beta(1 + \delta\xi)}{3(2 + \delta\xi)}\left[10\frac{\beta'}{\beta} - 14 + \left(10\frac{\beta'}{\beta} - 17\right)\delta\xi\right], \qquad (19)$$

$$g(\xi) = \frac{\gamma}{3(2 + \delta\xi)}\left[10 a - 14 + (10 a - 17)\delta\xi\right]. \qquad (20)$$

With $\beta' = 2\beta$, we see that for $\delta\xi \ll 1$,

$$\xi F(\xi) \rightarrow \beta,$$

$$g(\xi) \rightarrow \frac{\gamma}{3}(5 a - 7),$$

while for $\delta\xi \gg 1$,

$$F(\xi) \rightarrow \beta\delta,$$

$$g(\xi) \rightarrow \frac{\gamma}{3}(10 a - 17).$$

In the low intensity limit, $\xi F(\xi)$ thus has the value 5.0, which is exactly the value deduced empirically in I. For intense turbulence, $F(\xi)$ is of order of magnitude δ. If $a = 2$, then $g(\xi) \equiv \gamma$, but

this leads to σ being a creation invariant for all values of ξ. It thus appears that $1.7 < a < 2.0$.

The predicted behaviors of $F(\xi)$ and $g(\xi)$ are qualitatively as expected from previous empirical evidence and quantitatively in agreement in the low-intensity limit, but somewhat higher than previously proposed for the high-intensity limit. We are now inclined to believe that the presently derived values are correct, and we are further investigating this matter with detailed numerical solutions of the equations.

In summary, the proposed pair of turbulence transport equations now becomes

$$\frac{\partial q}{\partial t} + u_k \frac{\partial q}{\partial x_k} = \sigma e_{jk}\frac{\partial u_j}{\partial x_k} + \frac{\theta}{\gamma}\frac{\partial}{\partial x_k}\left(\sigma\frac{\partial\varphi}{\partial x_k}\right)$$

$$+ \frac{\partial}{\partial x_k}\left[(\nu + \alpha\sigma)\frac{\partial q}{\partial x_k}\right] - 2\nu\Omega; \qquad (21)$$

$$\frac{\partial\Omega}{\partial t} + u_k \frac{\partial\Omega}{\partial x_k} = \frac{a\Delta\sigma}{s^2}e_{jk}\frac{\partial u_j}{\partial x_k} - \frac{2\nu\Delta'\Omega}{s^2}$$

$$+ \frac{a_2\Delta}{s^2}\frac{\partial}{\partial x_k}\left(\sigma\frac{\partial q}{\partial x_k}\right) + \frac{\partial}{\partial x_k}\left[(\nu + a_3\sigma)\frac{\partial\Omega}{\partial x_k}\right]$$

$$+ a_4 \frac{\partial}{\partial x_k}\left(\frac{\Delta\sigma}{s^2}\frac{\partial\varphi}{\partial x_k}\right) \qquad (22)$$

with

$$\Omega = \frac{\Delta q}{s^2}, \qquad (23)$$

$$q = \frac{1}{2\gamma}\left(\frac{v}{s}\right)^2, \qquad (24)$$

$$\Delta = \beta(1 + \delta\xi). \qquad (25)$$

The initial and boundary conditions for these equations, together with their solutions for a variety of applications, are being prepared for publication. Successful applications to such problems as free-vortex-layer transition, the turbulent Bénard problem, intermittency, boundary layer turbulence, and wall-induced decay support the validity of the equations, and allow determination of the universal constants they contain. The decay-rate transport equation is further supported by the derivation by Chou[2] of a "vorticity decay" equation that closely resembles our Eq. (22) when

the latter is specialized to his circumstances. Chou's equation was shown to apply to a channel flow problem;[3] in general, however, the more extensive form given by our Eq. (22) would be required. Our derivations are also supported by the work of Schubauer and Tchen,[4] who derive flux approximations like ours from a postulated Boltzmann equation for the turbulence field.

Finally, the derivations clarify the meaning of the scale function, s, showing that this is proportional to the turbulence integral scale for all levels of turbulence intensity, but equal to the microscale in the limit of weak turbulence. [This follows from the use of s in the Loitsiansky function, which is known to be properly estimated with the integral scale, rather than the microscale. It is also dictated by the requirement for intense turbulence that the decay term in Eq. (3) be proportional to $q^{\frac{3}{2}}$/(integral scale).]
From Eq. (3)

$$\frac{\Delta}{s^2} = \frac{\beta}{\lambda^2}$$

where λ is the Taylor microscale, so that

$$s = \lambda\sqrt{1 + \delta\xi} \ . \tag{26}$$

As discussed in I, ξ is the turbulence Reynolds number based on s, the integral scale. Thus, the result in Eq. (26) is in agreement with that of Hinze,[5] who shows that the ratio of integral scale to microscale is proportional to the turbulence Reynolds number based on microscale and, accordingly, to the square root of the Reynolds number based on the integral scale.

REFERENCES

1. F. H. Harlow and P. I. Nakayama, Phys. Fluids 10, 2323 (1967).

2. P. Y. Chou, Quart.Appl. Math. 3, 38 (1945).

3. P. Y. Chou, Quart.Appl. Math. 3, 198 (1945).

4. G. B. Schubauer and C. M. Tchen, "Turbulent Flow," Section B, pp. 75-195, of Turbulent Flows and Heat Transfer, (C. C. Lin, ed.), Vol. 5 of High Speed Aerodynamics and Jet Propulsion, Princeton University Press (1959).

5. J. O. Hinze, Turbulence, McGraw-Hill Book Co. Inc., New York (1959), pp. 185, 186.