Journal of Statistical Physics, Vol. 31, No. 2, 1983

Transport Properties of the One-Dimensional Stochastic Lorentz Model: I. Velocity Autocorrelation Function

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Received November 16, 1982

Point scatterers are placed on the real line such that the distances between scatterers are independent identically distributed random variables (stationary renewal process). For a fixed configuration of scatterers a particle performs the following random walk: The particle starts at the point x with velocity v, $|v| = 1$. In between scatterers the particle moves freely. At a scatterer the particle is either transmitted or reflected, both with probability $1/2$. For given initial conditions of the particle the velocity autocorrelation function is a random variable on the scatterer configurations. If this variable is averaged over the distribution of scatterers, it decays not faster than $t^{-3/2}$.

KEY WORDS: Long time tails; stochastic Lorentz model; transfer matrix method.

1. INTRODUCTION

In 1905 Lorentz[1] introduced a model that is presently known as the Lorentz gas to describe the conduction of electrons in a metal. In this model one considers a mechanical point particle moving among randomly distributed hard spherical scatterers. The scatterers are infinitely heavy and therefore static. The point particle is specularly reflected at a scatterer and moves freely otherwise.

In the past decades the Lorentz gas has regained a remarkable popularity in kinetic theory as a relatively simple model for testing theories and

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0022-4715/83/0500-0231$03.00/0

conjectures. For example, Van Leeuwen and Weyland[2] studied the divergences in the formal density expansion of the self-diffusion coefficient (directly related to the conductivity in Lorentz's original interpretation) and showed how a resummation of most divergent terms gives rise to contributions that are nonanalytic in the density. Ernst and Weyland[3] showed that the velocity autocorrelation function of the point particle, the time integral of which is directly related to the self-diffusion coefficient, decays for large times as $t^{-(d+2)/2}$, where d is the dimensionality of the system. Hauge[4] and McKean[5] studied the solution of the Lorentz–Boltzmann equation, the kinetic equation describing the system at low scatterer density. Recently Keyes and Mercer[6] and Götze, Leutheuser, and Yip[7] proposed theories to describe the self-diffusion coefficient and the velocity autocorrelation function at higher density.

In computer experiments the diffusion coefficient and velocity autocorrelation function of the Lorentz gas have been studied by various groups of people.[8–11] Data are available up to 20 mean collision times at various densities of scatterers. Up to this time it is found that, after an exponential initial decay, the velocity autocorrelation function decays rather slowly and deviates strongly from the exponential decay predicted by the Lorentz–Boltzmann equation. In two dimensions at not too high scatterer density the data are fully consistent with the t^{-2} power law decay predicted by Ernst and Weyland,[3] although the prefactor of this long time tail seems to agree with their prediction over a very small density range only.[11]

Alder and Alley[10,11] also made a molecular dynamics study of an interesting modification of the Lorentz gas, replacing the deterministic reflection law by a stochastic one. They found a qualitatively similar behavior of the velocity autocorrelation function, which, for low scatterer density, would follow again from the theory of Ernst and Weyland. This indicates that the slow decay of the velocity autocorrelation function may be studied in stochastic models just as well as in deterministic ones.

In the present paper we want to study the long time behavior of the velocity autocorrelation function for a particular one-dimensional stochastic Lorentz model which was suggested to us by J. L. Lebowitz. (A preliminary account of our results has been given in Ref. 12.)

In Section 2 the model will be defined and we will state our main result. In Section 3 we compute a suitable form of the Laplace transform of the velocity autocorrelation function. In Section 4 we discuss an approximate expression for the velocity autocorrelation function. Similar approximations have been used before[11] and they are useful in higher dimensions where no rigorous arguments are known. In Section 5 we develop a transfer matrix method which we use in the Sections 6 and 7 to show that the approximate solution is asymptotically exact for long times.

2. THE ONE-DIMENSIONAL STOCHASTIC LORENTZ MODEL

The model consists of a single particle which moves at constant speed among scatterers placed randomly on a line. At encounters with a scatterer the light particle is reflected or transmitted with equal probability.

The probability measure for the scatterer configurations is defined as follows: Let $Q = (\ldots, q_{-1}, q_0, q_1, \ldots)$ be a locally finite configuration of scatterers on $\mathbb{R}$ (i.e., in each bounded interval a configuration contains only a finite number of scatterers), and let $\mathscr{X}$ denote the space of all locally finite configurations. For $Q \in \mathscr{X}$ the labeling is chosen such that

$$q_0 \leqslant 0, \qquad q_1 > 0, \qquad q_j < q_{j+1}. \tag{2.1}$$

The probability of coincidence $\{q_j = q_{j+1}\}$ will be zero. Let

$$\xi_j = q_{j+1} - q_j \tag{2.2}$$

Then $Q \in \mathscr{X}$ is specified by $q_1, \{\xi_j \,|\, j \in \mathbb{Z}\}$. We define a translation-invariant probability measure on $\mathscr{X}$. Let $\xi_j > 0$, $j \in \mathbb{Z}$, be independent, identically distributed random variables with a distribution given by a probability measure μ on $(0, \infty)$ which has a finite sixth moment. To achieve translation invariance the joint distribution of q_1 and ξ_0 has to be

$$\begin{array}{ll} \dfrac{1}{\langle \xi \rangle} dq_1\, \mu(d\xi_0) & \text{for} \quad 0 \leqslant q_1 \leqslant \xi_0 \\ 0 & \text{otherwise} \end{array} \tag{2.3}$$

Here, $\langle f(\xi) \rangle$ stands for $\int \mu(d\xi) f(\xi)$. This prescription actually defines a translation-invariant probability measure on $\mathscr{X}$. Expectation with respect to this measure is denoted by $\langle \cdot \rangle$.

For a fixed configuration $Q \in \mathscr{X}$ the stochastic motion of the particle can be described in the following way: Let $x(t) \in \mathbb{R}$ denote the position and $v(t) \in \{-1, 1\}$ the velocity of the particle at time t. The particle starts at $x(0) = x$ with velocity $v(0) = v$. Then $x(t) = x + v(0)t$ for $0 \leqslant t < t_0$, where t_0 is the time the particle first reaches a scatterer. At this scatterer the particle is transmitted with probability $1/2$ and reflected with probability $1/2$, i.e., $v(t_0) = \pm 1$ with probability $1/2$. Then $x(t) = x + v(0)t_0 + v(t_0)$ $(t - t_0)$ for $t_0 \leqslant t < t_1$, where t_1 is the time the particle reaches a second scatterer, etc. The expectation for this stochastic motion is denoted by $E^{(Q)}_{(x,v)}$. For a general initial distribution of the particle one has to average over (x, v) in the usual way.

We want to study the long time behavior of the velocity autocorrelation function

$$E^{(Q)}_{(x,v)}(v(t)v(0)) \tag{2.4}$$

still considered as a random variable on $\mathscr{X}$, as well as its expectation value with respect to some initial distribution and with respect to $\langle \cdot \rangle$. We do this through the Laplace transform

$$\int_0^\infty dt\, e^{-zt} E_{(x,v)}^{(Q)}(v(t)v(0)) \equiv F(z, Q \mid x, v) \tag{2.5}$$

z real and positive, $z > 0$, and $v = v(0)$.

The main result of our investigation is as follows:

Theorem. For given ϵ, $0 < \epsilon < 1/2$,

$$\langle F(z, Q \mid x, v)\rangle = \frac{\langle\xi\rangle^2}{2\tau} + (2z\tau)^{1/2}\frac{\langle\xi^2\rangle - \langle\xi\rangle^2}{4\tau} + O\left((z\tau)^{1-\epsilon}\right) \tag{2.6}$$

for $z \to 0$.

Here τ is the mean free time between collisions, which is equal to $\langle\xi\rangle$ with our choice $|v| = 1$ for the speed of the particle. The form of (2.6) is independent of this choice. As usual $f(z) = g(z) + O(z^\alpha)$ for $z \to 0$ means that $z^{-\alpha}(f(z) - g(z))$ remains bounded as $z \to 0$.

The bound obtained in this theorem is too weak to prove an asymptotic power law decay of the velocity autocorrelation function. Especially we cannot exclude the superposition of oscillations. On the other hand, the velocity autocorrelation function cannot be bounded as $ct^{-(3+\delta)/2}$ with any $\delta > 0$.

For our model the frequency-dependent diffusion coefficient for a fixed configuration Q is defined by

$$D(z, Q) = \lim_{L\to\infty} \frac{1}{2L}\int_{-L}^{L} dx\, H(z, Q \mid x) \tag{2.7}$$

where iz is the complex frequency and $H(z, Q \mid x) = \sum_{v=\pm 1} F(z, Q \mid x, v)$. For its fluctuations we have obtained the following bound[13]:

$$\begin{aligned}
&\langle D(z, Q)^2\rangle - \langle D(z, Q)\rangle^2 \\
&= \lim_{L\to\infty} \frac{1}{4L^2}\int_{-L}^{L} dx \int_{-L}^{L} dy \left(\langle H(z, Q \mid x) H(z, Q \mid y)\rangle - \langle D(z, Q)\rangle^2\right) \\
&\leqslant \lim_{L\to\infty} \frac{1}{2L}\int_{-L}^{L} dx \left(\langle H(z, Q \mid x)^2\rangle - \langle H(z, Q \mid x)\rangle^2\right) \\
&= 0(\sqrt{z})
\end{aligned} \tag{2.8}$$

Therefore in the static limit, $z \to 0$, $D(z, Q) \to \langle\xi\rangle^2/2\tau$ for almost all configurations.

3. TRANSFORMATION OF $F(z, Q \mid 0, 1)$

For notational simplicity we assume that the particle starts at the origin with velocity one. Let us denote by t_j the time of the $(j+1)$th collision, $j = 0, 1, \ldots$; by y_j the label of the scatterer with which the $(j+1)$th collision takes place, i.e., $x(t_j) = q_{y_j}, j = 0, 1, \ldots$; and let

$$\begin{aligned} v_j &= y_j - y_{j-1} = v(t) \qquad \text{for} \quad t_{j-1} \leqslant t < t_j \\ &= \text{velocity of the particle between scatterers } y_{j-1} \text{ and } y_j, \qquad j = 1, 2, \ldots \end{aligned}$$

$$\begin{aligned} \eta_j &= \begin{cases} \xi_{y_{j-1}} & \text{if} \quad v_j = 1 \\ \xi_{y_j} & \text{if} \quad v_j = -1 \end{cases} \\ &= \text{length of the interval between scatterers } y_{j-1} \text{ and } y_j, \qquad j = 1, 2, \ldots \end{aligned}$$

Since the speed of the particle is one,

$$t_j = q_1 + \sum_{i=1}^{j} \eta_i \tag{3.1}$$

We consider $(y_0, y_1, \ldots)$ as a path of the symmetric random walk. We use γ to denote such a path. Let $\mathfrak{R}_n(l)$ be the set of all paths of length n starting at l and let $\mathfrak{R}_n(k \mid l)$ be the set of all paths of length n starting at l and ending at k.

Let $\chi[a, b]$ denote the characteristic function of the interval $[a, b]$. Then

$$E_{(0,1)}^{(Q)}(v(t)) = \chi[0, q_1](t) + \sum_{n=1}^{\infty} \sum_{\gamma \in \mathfrak{R}_n(1)} 2^{-n} \chi[t_{n-1}, t_n](t) v_n \tag{3.2}$$

and its Laplace transform is given by

$$\begin{aligned} F(z, Q \mid 0, 1) &= \int_0^{\infty} dt\, e^{-zt} E_{(0,1)}^{(Q)}(v(t)) \\ &= \frac{1 - e^{-zq_1}}{z} + \sum_{n=1}^{\infty} \sum_{\gamma \in \mathfrak{R}_n(1)} 2^{-n} \prod e^{-zq_1} \prod_{i=1}^{n-1} e^{-z\eta_i} \frac{1 - e^{-z\eta_n}}{z} v_n \end{aligned} \tag{3.3}$$

It will turn out to be useful to separate (3.3) into the contributions of sets consisting of all those paths which end at a given scatterer. For this purpose we define

$$P_z(k \mid l) = \sum_{n=0}^{\infty} \sum_{\gamma \in \mathfrak{R}_n(k \mid l)} 2^{-n} \prod_{i=1}^{n} e^{-z\eta_i} \tag{3.4}$$

$P_z(k \mid l)$ is the Laplace transform of the probability density that a particle starting at scatterer l at time zero is at scatterer k at time t. Note that if $\eta_i = 1$ for all i, then $P_z(k \mid l)$ is just the Laplace transform of the transition probability of the symmetric random walk. Using (3.4) and the definition of

v_j we obtain

$$F(z, Q\,|\,0, 1) = \frac{1}{z}(1 - e^{-zq_1}) + \frac{1}{2}\sum_{k=-\infty}^{\infty} \frac{1 - e^{-z\xi_k}}{z} e^{-zq_1}[P_z(k\,|\,1) - P_z(k+1\,|\,1)] \tag{3.5}$$

The first term comes from the initial motion from the origin to q_1. The kth term in the sum collects all paths such that $q_k \leqslant x(t) \leqslant q_{k+1}$. The positive contribution comes from all those paths which cross the final interval ξ_k from the left to the right and the negative contribution comes from those crossing from the right to the left.

It is instructive to express $P_z(k\,|\,l)$ as matrix elements of an infinite-dimensional matrix. Let A be a matrix with matrix elements

$$A_{jj} = 0$$

$$A_{jj+1} = A_{j+1j} = \tfrac{1}{2}e^{-z\xi_j}$$

$$A_{ij} = 0 \qquad \text{for} \quad |i-j| > 1$$

A is a positive, symmetric, tridiagonal random matrix. The matrix elements A_{ij} represent the Laplace transforms of the direct (without visiting outside scatterers) transition probabilities between scatterers i and j. In terms of A we have

$$P_z(k\,|\,l) = \sum_{n=0}^{\infty} (A^n)_{kl} = \left((1-A)^{-1}\right)_{kl}$$

In this form we may think of $P_z(k\,|\,l)$ either as the thermal pair correlation function of a harmonic chain with random couplings or as the Green's function of a tight-binding Hamiltonian with off-diagonal disorder.

4. AN APPROXIMATION

In this section we construct an approximation to $F(z, Q\,|\,0, 1)$ valid for small z. As will be proved later on, this approximation is exact up to order $\sqrt{z}$ inclusive.

Before doing so we consider a particular case which illustrates the idea of the approximation in its simplest form. Let us fix a scatterer configuration Q' in such a way that $\xi_j = a > 0$ for $j \neq 0$ and $\xi_0 = 2a$, $q_1 = a$. Then in computing the velocity autocorrelation function $E_{(0,1)}^{(Q')}(v(t))$, the contribution of all those paths for which at time t the distance to the last scatterer seen is less than a cancel exactly. The only contribution left comes from those paths for which either $v(t) = 1$ and $0 \leqslant x(t) \leqslant a$ or $v(t) = -1$ and

$-a \leqslant x(t) \leqslant 0$. Since away from the origin the particle performs a symmetric random walk, $E_{(0,1)}^{(Q')}(v(t))$ equals

$$\begin{aligned} &\text{Prob}\{\text{even returns to the origin at time } t\} \\ &\quad -\text{Prob}\{\text{odd returns to the origin at time } t\} \end{aligned} \tag{4.1}$$

where Prob refers to the symmetric random walk. The odd returns carry a minus sign, since then the velocity at time t is directed opposite to the velocity at time zero. The Laplace transform of (4.1) can be computed explicitly. One obtains

$$F(z, Q' \mid 0, 1) = (1/z)(1 - e^{-za})\left[2 - (1 - e^{-2za})^{1/2}\right]^{-1}$$

which has the same small $-z$ behavior as $\langle F(z, Q \mid 0, 1)\rangle$ of the Theorem. In particular, $E_{(0,1)}^{(Q')}(v(t)) \sim -t^{-3/2}$ for large times. We see that the slow decay of the velocity autocorrelation function arises from a mismatch around the origin. (The obvious generalization of this model to higher dimensions can also be treated analytically.[14] One obtains a long time tail of the velocity autocorrelation function as $t^{-(d+2)/2}$ in d dimensions.)

In the spirit of this example we may hope that for small z only the precise length of the initial and final interval counts. For the motion in between, the different interval lengths should approximately average out and so the motion may be approximated by a random walk of step size $\langle \xi \rangle$. With these assumptions we approximate $F(z, Q \mid 0, 1)$ for small z by

$$\begin{aligned} \tilde{F}(z, Q \mid 0, 1) &= (1/z)(1 - e^{-zq_1}) \\ &\quad + \frac{1}{2} \sum_{k=-\infty}^{\infty} \frac{1 - e^{-z\xi_k}}{z} e^{-zq_1}\left[\bar{P}_z(k \mid 1) - \bar{P}_z(k+1 \mid 1)\right] \end{aligned} \tag{4.2}$$

$\bar{P}_z(k \mid l)$ is given by (3.4) with $\langle \xi \rangle$ substituted for η_i. $\bar{P}_z(k \mid l)$ is known explicitly,

$$\bar{P}_z(k \mid l) = \frac{1}{\beta} \alpha^{|k-l|} \tag{4.3}$$

where $\beta = (1 - e^{-2z\langle \xi \rangle})^{1/2}$ and $\alpha = [1 - (1 - e^{-2z\langle \xi \rangle})^{1/2}]e^{z\langle \xi \rangle}$. The small z behavior is $\beta = (2\langle \xi \rangle z)^{1/2} + O(z)$ and $\alpha = 1 - (2\langle \xi \rangle z)^{1/2} + O(z)$.

Since $\tilde{F}(z, Q \mid 0, 1)$ as given in (4.2) is a sum of independent variables (up to the correlation between q_1 and ξ_0) its various properties are easily investigated. In particular, the mean of $\tilde{F}$ behaves for small z as the mean of F in the theorem. Further, it is worth noticing that, in calculating the average of $\tilde{F}$, in general the terms containing $\bar{P}_z(k \mid 1)$ and $-\bar{P}_z((k-1)+1 \mid 1)$ cancel each other, since the distribution of ξ_k does not depend on k. The only nonvanishing contributions just come from correlations between the initial interval q_1 and the final interval ξ_k. Similar correlations between initial and final track of the light particle are at the heart of the calculations by Ernst and Weyland[3,12] for the Lorentz gas in higher dimensions.

5. A TRANSFER MATRIX FOR $P_z(k\,|\,0)$

Equation (3.4) expresses $P_z(k\,|\,0)$ as a sum over paths. Let γ be a path of the random walk of length n such that $\gamma(0) = 0$ and $\gamma(n) = k$, $k \geqslant 0$. Then let $2n_j(\gamma)$ be the number of crossings through the interval $[j, j+1]$ of the path γ in the case that either $j < -1$ or $j \geqslant k$ and let $2n_j(\gamma) + 1$ be the number of crossings through the interval $[j, j+1]$ of the path γ in the case that $0 \leqslant j \leqslant k-1$. By definition $n_j = 0, 1, 2, \ldots$ with the restriction that $n_j = 0$, if $n_i = 0$ with either $j < i \leqslant -1$ or $k \leqslant i < j$. The crucial observation is that the random weight $\prod \exp[-z\eta_i]$ in (3.4) depends on γ only through the $\{n_j\,|\,j \in \mathbb{Z}\}$. One obtains

$$\prod_{i=1}^{n} \exp(-z\eta_i) = \exp\left(-2z \sum_{j=-\infty}^{\infty} \xi_j n_j - z \sum_{j=0}^{k-1} \xi_j\right) \tag{5.1}$$

To express the sum over paths as a sum over $\{n_j\,|\,j \in \mathbb{Z}\}$ one has to compute the number of paths corresponding to a specified set of $\{n_j\}$. We will do this by first assuming that $n_{-N-1} = 0 = n_{k+M}$. The general case is then obtained by the limiting procedure $M, N \to \infty$.

One way to solve this combinatorial problem is the following: At each site one specifies a finite sequence of the symbols L, R (e.g., $RRLLR$) such that at site j there are

$$\begin{array}{rl} n_j - 1 \text{ times } L & \text{for all } j \\ n_j \text{ times } R & \text{if either } \; j \leqslant -1 \;\text{ or }\; j \geqslant k \\ n_j + 1 \text{ times } R & \text{if } \; 0 \leqslant j \leqslant k-1 \end{array}$$

For given sequences of L, R symbols a path of the random walker is constructed: The random walker starts at the origin. If the first symbol at zero is L the walker goes to the left and if it is R the walker goes to the right. If the walker returns for the nth time to the origin he uses the $(n+1)$th symbol of the sequence at the origin to continue his path. The same prescription applies to all other sites. If in addition we require that the last symbols in the sequences for sites $j < k$ are R and for sites $j' > k$ are L, then there is a one-to-one correspondence between sequences of L, R-symbols and paths of the random walk which start at 0, end at k, and have a number of crossings as specified by the set $\{n_j\,|\,j \in \mathbb{Z}\}$, $n_{-N-1} = 0$

$= n_{k+M+1}$. To count the number of these paths, one has to count the number of allowed L, R sequences. These are

$$\prod_{j=-N}^{-2} \binom{n_j + n_{j+1} - 1}{n_j} \prod_{j=-1}^{k-1} \binom{n_j + n_{j+1}}{n_j} \prod_{j=k}^{k+M-1} \binom{n_j + n_{j+1} - 1}{n_{j+1}} \tag{5.2}$$

As usual,

$$\binom{n-1}{n} = \delta_{n0} \tag{5.3}$$

With this convention the restriction on the n_j's mentioned before can be lifted, because all sequences not satisfying this restriction yield vanishing contributions.

Using (5.1) and (5.2) $P_z(k \mid 0)$ is given by the sum

$$\begin{aligned} P_z(k \mid 0) = {} & \lim_{M\to\infty} \lim_{N\to\infty} \sum_{n_{-N}=0}^{\infty} \cdots \sum_{n_{k+M-1}=0}^{\infty} \\ & \times \left(\frac{1}{2}\right)^{2n_{-N}} \exp(-z\xi_{-N} 2n_{-N}) \binom{n_{-N} + n_{-N+1} - 1}{n_{-N}} \cdots \\ & \times \binom{n_{-2} + n_{-1} - 1}{n_{-2}} \left(\frac{1}{2}\right)^{2n_{-1}} \exp(-z\xi_{-1} 2n_{-1}) \binom{n_{-1} + n_0}{n_0} \\ & \times \left(\frac{1}{2}\right)^{2n_0 + 1} \exp\left[-z\xi(2n_0 + 1)\right] \cdots \\ & \times \left(\frac{1}{2}\right)^{2n_{k-1}} \exp\left[-z\xi_{k-1}(2n_{k-1} + 1)\right] \\ & \times \binom{n_{k-1} + n_k}{n_k} \left(\frac{1}{2}\right)^{2n_k} \exp(-z\xi_k 2n_k) \binom{n_k + n_{k+1} - 1}{n_{k+1}} \cdots \\ & \times \binom{n_{k+M-2} + n_{k+M-1} - 1}{n_{k+M-1}} \left(\frac{1}{2}\right)^{2n_{k+M-1}} \exp(-z\xi_{k+M-1} 2n_{k+M-1}) \end{aligned} \tag{5.4}$$

This equation can be understood in terms of a transfer matrix formulation. We define three matrices T, S, and R with matrix elements

$$T(m,n) = \left(\frac{1}{2}\right)^{n+m} \Phi(2zm) \binom{m+n-1}{n} \tag{5.5}$$

$$S(m,n) = \left(\frac{1}{2}\right)^{n+m+1} \Phi(z(2m+1)) \binom{m+n}{n} \tag{5.6}$$

$$R(m,n) = \left(\frac{1}{2}\right)^{n+m} \binom{m+n}{n} \tag{5.7}$$

with

$$\Phi(nz) = \langle e^{-zn\xi} \rangle = \int \mu(d\xi) e^{-zn\xi} \tag{5.8}$$

$m, n = 0, 1, 2, \ldots$. We think of these matrices as acting on either $l_1(\mathbb{N})$ or $l_2(\mathbb{N})$. Note that if $\psi_0(n) = \delta_{0n}$, then $(T\psi_0)(m) = T(m, 0) = (\frac{1}{2})^m \Phi(2zm)$. Therefore for $k \geqslant 0$

$$\langle P_z(k\,|\,0) \rangle = \lim_{N\to\infty} \lim_{M\to\infty} \langle T^N \psi_0 \,|\, RS^k T^M \psi_0 \rangle \tag{5.9}$$

Here $\langle \cdot \,|\, \cdot \rangle$ denotes the scalar product in $l_2(\mathbb{N})$.

We will investigate first

$$\lim_{N\to\infty} T^N \psi_0$$

In order to do so it is useful to keep in mind the probabilistic interpretation of the matrix T. Let $j > k$. Then $T(m_j, n_{j+1})$ is the Laplace transform of the conditional probability for the interval ξ_j being traversed $2m_j$ times during time t conditioned on the interval ξ_{j+1} being traversed $2n_{j+1}$ times. The matrix S may be interpreted in a similar way.

6. BOUNDS ON THE INVARIANT VECTOR OF T

Let l_1 be the space of sequences with norm

$$\|\phi\| = \sum_{n=0}^{\infty} |\phi(n)| \tag{6.1}$$

Let $l_1^\perp$ be the subspace of vectors with first component zero. Since

$$(T\phi)(0) = \phi(0) \tag{6.2}$$

$l_1^\perp$ is mapped into itself under T.

Proposition 1. T is a strict contraction in $l_1^\perp$, satisfying

$$\|T\phi\| \leqslant \Phi(2z)\|\phi\| \tag{6.3}$$

for $\phi \in l_l^\perp$.

Proof. Let $\phi(0) = 0$. Then, since Φ is decreasing,

$$\begin{aligned} \|T\phi\| &\leqslant \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} 2^{-(n+m)} \Phi(2zn) \binom{n+m-1}{m} |\phi(m)| \\ &\leqslant \Phi(2z) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} 2^{-(n+m)} \binom{n+m-1}{m} |\phi(m)| \\ &= \Phi(2z) \sum_{m=1}^{\infty} |\phi(m)| = \Phi(2z)\|\phi\| \end{aligned} \tag{6.4}$$

For the last equality we used the series expansion

$$\sum_{n=0}^{\infty}\binom{n+m}{m}x^{n}=(1-x)^{-(m+1)}, \qquad |x|<1 \quad \blacksquare \tag{6.5}$$

Proposition 2. There is a unique vector $\psi \in l_1$ with $\psi(0) = 1$ which is invariant under T. $T^N\psi_0$ is a pointwise increasing sequence in N and

$$\lim_{N\to\infty} T^N\psi_0 = \psi = T\psi \tag{6.6}$$

Furthermore,

$$\|\psi\| \leqslant \frac{1}{1-\Phi(2z)} \tag{6.7a}$$

$$0 \leqslant \psi(n) \leqslant \Phi(2zn) \tag{6.7b}$$

Proof. Let

$$\phi(n) = 2^{-n}\Phi(2zn), \qquad n>0, \quad \phi(0)=0$$

Then $T\psi_0 = \psi_0 + \phi$, $T^2\psi = \psi_0 + \phi + T\phi$, etc. Hence, by contraction,

$$\lim_{N\to\infty} T^N\psi_0 = \psi = \psi_0 + \frac{1}{1-T}\phi$$

Now $\|\phi\| \leqslant \Phi(2z)$ and therefore, by (6.3),

$$\|\psi\| \leqslant \sum_{n=0}^{\infty} \Phi(2z)^n,$$

proving (6.7a).

Since $\Phi(2nz)$ is decreasing in z, we conclude from (6.6) that $\psi(n)$ is also decreasing in z. Introducing explicitly the z dependence, $[T(z)^N\psi_0](n) \leqslant [T(0)^N\psi_0](n)$, which implies $\psi(n) \leqslant \lim_{N\to\infty}[T(0)^N\psi_0] = 1$. Inserting this bound in $\psi = T\psi$ results in (6.7b). $\blacksquare$

$\psi(n)$ has the following probabilistic interpretation. Let $\mathfrak{R}(n)$ be the set of all paths of arbitrary length which start from 0, end at 0, always stay to the right of 0, and return exactly n times to the origin. Then, cf. (3.4),

$$\psi(n) = \sum_{\gamma\in\mathfrak{R}(n)} 2^n\Big\langle \prod_i e^{-z\eta_i}\Big\rangle \tag{6.8}$$

So $\psi(n)$ is the Laplace transform of the probability of the nth return to the origin at time t with reflecting boundary conditions at the origin. (This boundary condition is responsible for the factor 2^n.)

In the random walk approximation, as used already in Section 4,

$$\psi(n) \sim \alpha^n \tag{6.9}$$

cf. (4.3). This suggests breaking up $\{0, 1, 2, \ldots\}$ into two parts.

(i) $$0 \leqslant n \leqslant \bar{n}$$

where $\bar{n}$ is the largest integer such that

$$\bar{n} < z^{-(1/2)-\epsilon} \tag{6.10}$$

For $n \leqslant \bar{n}$, nz is a small quantity. So by expanding $\Phi(nz)/\alpha^n$ in powers of z one may obtain approximations to the invariant vector ψ with (6.9) as zeroth-order approximation. Our interest in relatively fine details forces us to expand through fifth order, which leads to rather extensive calculations.

(ii) $$n > \bar{n}$$

In this case $\alpha^{\bar{n}} \sim e^{-z^{-\epsilon}}$. The contributions to ψ of paths returning very often to the origin are extremely small and can be estimated not to contribute to the z-dependent velocity autocorrelation function in the orders we are interested in.

Let P be the projection onto the subspace of vectors ψ such that $\psi(n) = 0$ for $n > \bar{n}$. (Note that P depends on z.)

We want to construct upper and lower bounds for the invariant vector in the form

$$(P\psi_-)(n) - \Delta\psi(n) \leqslant \psi(n) \leqslant (P\psi_+)(n) + \Delta\psi(n) \tag{6.11}$$

where $\|\Delta\psi(n)\| \leqslant ce^{-z^{-\epsilon}}$.

We first construct the ψ_- and ψ_+ in the form of an expansion of $\psi_\pm(n)$ in powers of n. In principle this expansion can be carried through any order. We expand through order n^5, since this is the case that will be needed later on.

We define

$$x = 1 - (1 - e^{-2z\langle\xi\rangle})^{1/2} \tag{6.12}$$

x equals $\alpha e^{-z\langle\xi\rangle}$ and it satisfies

$$x = \frac{2}{2-x}\,\frac{1}{2e^{2z\langle\xi\rangle}} \tag{6.13}$$

The form of the bound on ψ is specified in the following lemma.

Lemma 1. Let

$$\psi_{+(-)}(n) = x^n\left\{1 + a_1 zn + a_2 z^{3/2} n^2 + a_3 z^{5/2} n^3 + a_4 z^3 n^4 + \alpha_{+(-)} z^{4-\epsilon} n^5\right\}$$

with z-dependent coefficients $a_1, \ldots, a_4$ which tend to a finite limit as $z \to 0$ and with z-independent α_+ (α_-). These coefficients can be chosen in such a way that for z small enough

$$(PT\psi_+)(n) \leqslant (P\psi_+)(n) \tag{6.14}$$

and

$$(PT\psi_-)(n) \geqslant (P\psi_-)(n) \tag{6.15}$$

Proof. We make the ansatz

$$\phi(n) = x^n\left(1 + \sum_{j=1}^{5} b_j n^j\right) \tag{6.16}$$

with z-dependent coefficients b_j. Then

$$(T\phi)(n) = \Phi(2zn)e^{2zn\langle\xi\rangle}x^n\left(1 + \sum_{j=1}^{5} c_j n^j\right)$$

using the identity

$$\sum_{m=0}^{\infty}\binom{n+m=1}{m}y^m m^j = \left(y\frac{d}{dy}\right)^j(1-y)^{-n} \qquad \text{for} \quad j \geqslant 0$$

The coefficient c_j is a linear combination of $b_j, b_{j+1}, \ldots, b_5$. We only need that

$$c_j = \left(\frac{x}{2-x}\right)^j b_j + \sum_{i=j+1}^{5} c_{ji}b_i \tag{6.17}$$

where the c_{ji} have a finite limit as $z \to 0$. For $n < z^{-(1/2)-\epsilon}$, we want to show that by a suitable choice of the b_j's

$$(T\phi)(n) \leqslant \phi(n) \tag{6.18}$$

Equivalently,

$$\begin{aligned}&\left\{1 - \left[e^{2z\langle\xi\rangle n}\Phi(2zn)\right]^{-1}\right\}\left(1 + \sum_{j=1}^{5} b_j n^j\right)\\ &\qquad \leqslant \sum_{j=1}^{5}\left\{\left(1 - \left(\frac{x}{2-x}\right)^j\right)b_j - \sum_{i=j+1}^{5} c_{ji}b_i\right\}n^j\end{aligned} \tag{6.19}$$

We expand $1 - [e^{2z\langle\xi\rangle n}\Phi(2zn)]^{-1}$ in powers of zn. Recalling that we required the sixth moment of $\mu(d\xi)$ to exist and that $nz \leqslant z^{(1/2)-\epsilon}$, we obtain

$$\left[1 - \left(e^{2z\langle\xi\rangle n}\Phi(2zn)\right]^{-1} = \sum_{j=1}^{5} d_j(nz)^j + O(n^6z^6) \tag{6.20}$$

The coefficients d_j are independent of z. Since $x \to 1$ as $z \to 0$,

$$\left[1 - \left(\frac{x}{2-x}\right)^j\right] = z^{1/2}c_{jj} \tag{6.21}$$

where c_{jj} has a finite strictly positive limit as $z \to 0$. We insert (6.20) and

(6.21) in (6.19) and equate the coefficients of n^j, $1 \leqslant j \leqslant 4$, in (6.19). For $j = 1$

$$b_1 = z^{-1/2} c_{11}^{-1} \left(\sum_{i=2}^{5} c_{1i} b_i \right) \tag{6.22}$$

For $j = 2$

$$b_2 = c_{22}^{-1} \left(z^{3/2} d_2 + z^{-1/2} \sum_{i=3}^{5} c_{2i} b_i \right) \tag{6.23}$$

We insert (6.23) in (6.22). Then

$$b_1 = c'_{11} + z^{-1} \left(\sum_{j=3}^{5} c'_{1j} b_j \right) \tag{6.24}$$

with new coefficients which have a finite limit as $z \to 0$. Proceeding in this fashion up to $j = 4$, one obtains

$$\begin{aligned} b_1 &= z a_1 + z^{-2} c'_{15} b_5 \\ b_2 &= z^{3/2} a_2 + z^{-3/2} c'_{25} b_5 \\ b_3 &= z^{5/2} a_3 + z^{-1} c'_{35} b_5 \\ b_4 &= z^3 a_4 + z^{-1/2} c'_{45} b_5 \end{aligned} \tag{6.25}$$

with coefficients having a finite limit as $z \to 0$. Inserting (6.25) in (6.19) disregarding the terms proportional to b_5, the dominant contribution for $z \to 0$ and $n < z^{-1/2-\epsilon}$ is

$$d_2 a_4 n^6 z^5 \leqslant n^5 \sqrt{z}\, b_5 \tag{6.26}$$

Choosing $b_5 = z^{4-\epsilon} \alpha_+$, with α_+ independent of z, (6.26) can be satisfied for all $n < z^{-(1/2)-\epsilon}$. With this choice of b_5 in each equation of (6.25) the first term dominates the second one.

For the lower bound only the sign of (6.26) is reversed,

$$n^5 \sqrt{z}\, b_5 \leqslant d_2 a_4 n^6 z^5 \tag{6.27}$$

(6.27) is satisfied with $b_5 = \alpha_- z^{4-\epsilon}$, $\alpha_- = -\alpha_+$. ■

With the aid of this Lemma the proof of (6.11) is provided in the following proposition.

Proposition 3. Let ψ_+ and ψ_- be as in Lemma 1. Then, for z small enough, there exists a vector $\Delta\psi$ such that

$$\|\Delta\psi\| \leqslant c e^{-z^{-\epsilon}/2} \tag{6.28}$$

and such that

$$\begin{aligned} \psi_+(n) &\leqslant (P\psi_+)(n) + (\Delta\psi)(n) \\ \psi_-(n) &\geqslant (P\psi_-)(n) - (\Delta\psi)(n) \end{aligned} \tag{6.29}$$

Proof. From Lemma 1 $(PT\psi_+)(n) \leqslant (P\psi_+)(n)$. Then

$$\begin{aligned} (TP\psi_+)(n) &= (PT\psi_+)(n) + \left[(1-P)T\psi_+\right](n) - \left[T(1-P)\psi_+\right](n) \\ &\leqslant (P\psi_+)(n) + \left[(1-P)T\psi_+\right](n) - \left[T(1-P)\psi_+\right](n) \end{aligned} \tag{6.30}$$

We act with T on both sides of (6.30). Since T is positive, the inequality is preserved. We obtain the term $TP\psi_+$ which is estimated by (6.30). Iterating this procedure, one has

$$\begin{aligned} (T^N P\psi_+)(n) \leqslant (P\psi_+)(n) &+ \sum_{j=0}^{N-1} \left[T^j(1-P)T\psi_+\right](n) \\ &- \sum_{j=0}^{N-1} \left(T^{j+1}(1-P)\psi_+\right)(n) \end{aligned} \tag{6.31}$$

Since $\psi_+(0) = 1$, by Propositions 1 and 2,

$$\lim_{N\to\infty} T^N P\psi_+ = \psi \tag{6.32}$$

Therefore

$$\psi(n) \leqslant (P\psi_+)(n) + \tilde{\psi}_+(n)$$

with

$$\tilde{\psi}_+ = \sum_{j=0}^{\infty} \left[T^j(1-P)T\psi_+ - T^{j+1}(1-P)\psi_+\right] \tag{6.33}$$

By Proposition 1

$$\|\tilde{\psi}_+\| \leqslant \frac{1}{1-\Phi(2z)} \left(\|(1-P)T\psi_+\| + \|(1-P)\psi_+\|\right) \tag{6.34}$$

A typical term in this norm is $\phi(n) = x^n n^j$ with $j = 0, 1, \ldots, 5$. Then

$$\|(1-P)\phi\| = \sum_{n=\bar{n}+1}^{\infty} x^n n^j = x^{\bar{n}} \sum_{n=1}^{\infty} x^n (n+\bar{n})^j \tag{6.35}$$

Since $x \approx 1 - (2\langle\xi\rangle z)^{1/2}$,

$$x^{\bar{n}} \leqslant c e^{-z^{-\epsilon}} \tag{6.36}$$

for z small enough. The sum grows as an inverse power depending on j. $\|(1-P)T\phi\|$ is estimated similarly.

By the same method one obtains

$$\psi(n) \geqslant (P\psi_-)(n) - \tilde{\psi}_-(n) \tag{6.37}$$

with

$$\tilde{\psi}_-(n) = \sum_{j=0}^{\infty} \left[T^{j+1}(1-P)\psi_- - T^j(1-P)T\psi_- \right] \tag{6.38}$$

Finally $\Delta\psi$ is chosen as $(\Delta\psi)(n) = |\tilde{\psi}_+(n)| + |\tilde{\psi}_-(n)|$. ∎

7. THE AVERAGE $\langle F(z, Q\,|\,0,1)\rangle$ FOR SMALL z

Having introduced the transfer matrices T, S, and R and having derived upper and lower bounds on the invariant vector of T, we are now in a position to prove the theorem. Starting from (3.5) we average $F(z, Q\,|\,0,1)$ with the aid of (2.3) first over q_1 and then over the ξ_j's. The result reads

$$\begin{aligned} \langle F(z, Q\,|\,0,1)\rangle &= \frac{1}{z\langle\xi\rangle}\langle \xi_0 + (1/z)(e^{-z\xi_0} - 1)\rangle \\ &\quad + \frac{1}{2z^2\langle\xi\rangle}\sum_{k=-\infty}^{\infty}\langle (1 - e^{-z\xi_0} - e^{-z\xi_k} + e^{-z\xi_0}e^{-z\xi_k}) \\ &\qquad \times \left[P_z(k\,|\,1) - P_z(k+1\,|\,1)\right]\rangle \end{aligned} \tag{7.1}$$

Since for $z > 0$ the sum is absolutely convergent, we may freely rearrange the various terms. The sum $\sum_k [P_z(k\,|\,1) - P_z(k+1\,|\,1)]$ vanishes pointwise. Therefore the sum $\sum_k \langle (1 - e^{-z\xi_0})[P_z(k\,|\,1) - P_z(k+1\,|\,1)]\rangle$ vanishes. For the sum $\sum_k \langle e^{-z\xi_k}[P_z(k\,|\,1) - P_z(k+1\,|\,1)]\rangle$ we use the shift invariance of the average to relabel the ξ_j's such that ξ_k has always the new label zero. Then by the previous argument this sum also vanishes. For the remaining sum one uses again the shift invariance of the average to obtain

$$\begin{aligned} \langle F(z, Q\,|\,0,1)\rangle &= \frac{1}{z\langle\xi\rangle}\langle \xi + (1/z)(e^{-z\xi} - 1)\rangle \\ &\quad + \frac{1}{2z^2\langle\xi\rangle}\sum_{k=-\infty}^{\infty}\langle (e^{-z\xi_k} - e^{-z\xi_{k-1}})e^{-z\xi_{-1}}P_z(k\,|\,0)\rangle \end{aligned} \tag{7.2}$$

Finally, by reflection symmetry,

$$\begin{aligned} &\langle F(z, Q\,|\,0,1)\rangle \\ &\quad = \frac{1}{z\langle\xi\rangle}\langle \xi + (1/z)(e^{-z\xi_{-1}})\rangle \\ &\quad\quad + \frac{1}{4z^2\langle\xi\rangle}\sum_{k=-\infty}^{\infty}\langle (e^{-z\xi_0} - e^{-z\xi_{-1}})(e^{-z\xi_{k-1}} - e^{-z\xi_k})P_z(k\,|\,0)\rangle \end{aligned} \tag{7.3}$$

As a next step $\langle F(z, Q\,|\,0,1)\rangle$ is broken up into four terms:

$$
\begin{aligned}
F_1(z) &= \frac{1}{z\langle\xi\rangle}\langle \xi + (1/z)(e^{-z} - 1)\rangle. \\
F_2(z) &= \frac{1}{2z^2\langle\xi\rangle}\sum_{k=z}^{\infty}\langle (e^{-z\xi_0} - e^{-z\xi_{-1}})(e^{-z\xi_{k-1}} - e^{-z\xi_k})P_z(k\,|\,0)\rangle \\
F_3(z) &= \frac{-1}{4z^2\langle\xi\rangle}\langle (e^{-z\xi_0} - e^{-z\xi_{-1}})^2 P_z(0\,|\,0)\rangle \\
F_4(z) &= \frac{1}{2\langle\xi\rangle z^2}\langle (e^{-z\xi_0} - e^{-z\xi_{-1}})(e^{-z\xi_0} - e^{-z\xi_1})P_z(1\,|\,0)\rangle
\end{aligned}
\tag{7.4}
$$

We discuss the small-z behavior of each term separately. The results for F_1 through F_4 are expressed in Lemmas 2 to 5, respectively.

Lemma 2.

$$
F_1(z) = \frac{\langle\xi^2\rangle}{2\langle\xi\rangle} + O(z) \tag{7.5}
$$

Lemma 3.

$$
F_2(z) = O(z^{1-6\epsilon}) \tag{7.6}
$$

Proof. Let

$$
\tilde{S}(m,n) = \left(\frac{1}{2}\right)^{m+n+1}\frac{\Phi(2zn)\Phi(z(2m+2)) - \Phi(z(2n+1))\Phi(z(2m+1))}{\Phi(2zn)} \times \binom{n+m}{n} \tag{7.7}
$$

Then for $k \geqslant 2$

$$
\begin{aligned}
&\langle (e^{-z\xi_0} - e^{-z\xi_{-1}})(e^{-z\xi_{k-1}} - e^{-z\xi_k})P_z(k\,|\,0)\rangle \\
&\quad = \langle \tilde{S}\psi\,|\,RS^{k-2}\tilde{S}\psi\rangle
\end{aligned}
\tag{7.8}
$$

$$
(P'\phi)(n) = \begin{cases} \phi(n) & \text{for} \quad n \leqslant \bar{n} \\ x^n\phi(0) & \text{for} \quad n > \bar{n} \end{cases}
$$

Then we decompose (7.8) into

$$
\begin{aligned}
\langle \tilde{S}\psi\,|\,RS^{k-2}\tilde{S}\psi\rangle &= \langle P\tilde{S}P'\psi\,|\,R(PSP)^{k-2}P\tilde{S}P'\psi\rangle \\
&\quad + \langle P\tilde{S}(1-P')\psi\,|\,R(PSP)^{k-2}P\tilde{S}P'\psi\rangle \\
&\quad + \langle (1-P)\tilde{S}\psi\,|\,R(PSP)^{k-2}P\tilde{S}P'\psi\rangle \\
&\quad + \cdots + \langle \tilde{S}\psi\,|\,RS^{k-2}\tilde{S}(1-P')\psi\rangle
\end{aligned}
\tag{7.9}
$$

We first estimate the terms containing $(1-P)$ and $(1-P')$. Since $|(\tilde{S}\psi)(n)| \leqslant (S\psi)(n)$, these are bounded by

$$\{\langle PS(1-P')\psi \mid R(PSP)^{k-2}PSP'\psi\rangle + \cdots + \langle S\psi \mid RS^{k-2}S(1-P')\psi\rangle\} \tag{7.10}$$

We have for $n \leqslant \bar{n}$

$$\begin{aligned}(SP\psi)(n) \leqslant (S\psi)(n) &= \frac{\Phi(z(2n+1))}{\Phi(z(2n+2))}(T\psi)(n+1) \\ &= \frac{\Phi(z(2n+1))}{\Phi(z(2n+2))}\psi(n+1) \leqslant e^{z\langle\xi\rangle}\psi(n+1) \\ &\leqslant e^{z\langle\xi\rangle}\frac{\psi_+(n+1)}{\psi_-(n)}\psi(n) \leqslant \left(1-a_1'\sqrt{z}\right)\psi(n)\end{aligned} \tag{7.11}$$

with strictly positive constant a_1', where we used Proposition 3 and Lemma 1 in the last step. The second inequality, which is valid for all n, implies

$$\|(1-P)S\psi\| \leqslant e^{z\langle\xi\rangle}\|(1-P)\psi\| \tag{7.12}$$

Furthermore, from the definition of S,

$$\|S\phi\| \leqslant \Phi(z)\|\phi\| \tag{7.13}$$

Let $\chi(n) = x^n$. Since $x^n \leqslant (T\chi)(n)$, $(P'\psi)(n) \leqslant \psi(n)$, which implies firstly that

$$\|(1-P')\psi\| \leqslant \|(1-P)\psi\| \tag{7.14}$$

and secondly that (7.11) and (7.12) are still valid with P replaced by P'. By (6.7b)

$$|\langle\psi \mid R\phi\rangle| \leqslant 2e^{z\langle\xi\rangle}\|S\phi\| \tag{7.15}$$

Using (7.11) to (7.15) the various terms of (7.10) can be estimated. We have

$$\begin{aligned}\langle\psi \mid RS^k(1-P')\psi\rangle &\leqslant 2e^{z\langle\xi\rangle}\|S^{k+1}(1-P')\psi\| \\ &\leqslant 2e^{z\langle\xi\rangle}\Phi(z)^{k+1}\|(1-P)\psi\| \leqslant 2e^{z\langle\xi\rangle}\Phi(z)^{k+1}e^{-z^{-\epsilon}/2}\end{aligned} \tag{7.16}$$

by Proposition 3. For $j = 1, \ldots, k$ we have

$$\begin{aligned}\langle\psi \mid RS^{k-j}(1-P')(SP)^{j-1}SP'\psi\rangle &\leqslant \left(1-a_1'\sqrt{z}\right)^{j-1}\langle\psi \mid RS^{k-j}(1-P)S\psi\rangle \\ &\leqslant \left(1-a_1'\sqrt{z}\right)^{j-1}2e^{2z\langle\xi\rangle}\Phi(z)^{k-j}e^{-z^{-\epsilon}/2}\end{aligned} \tag{7.17}$$

Finally,

$$\langle\psi\,|(1-P')R(PS)^k P'\psi\rangle \leqslant \left(1-a_1'\sqrt{z}\right)^k \langle\psi\,|(1-P')R\psi\rangle$$
$$\leqslant \left(1-a_1'\sqrt{z}\right)^k \Phi(z) 2e^{2z\langle\xi\rangle}e^{-z^{-\epsilon}/2} \quad (7.18)$$

Therefore the sum of all terms in $F_2(z)$ of the form (7.10) is bounded by

$$ce^{-z^{-\epsilon}/2}\sum_{k=2}^{\infty}\sum_{j=0}^{k}\left(1-a'\sqrt{z}\right)^j \phi(z)^{k-j}$$

which is exponentially small. So we are left with the main contribution

$$\sum_{k=2}^{\infty}\langle P\tilde{S}P'\psi\,|\,R(PSP)^{k-2}P\tilde{S}P'\psi\rangle \quad (7.19)$$

For $m,n \leqslant z^{-(1/2)-\epsilon}$ one has

$$\Phi(2zn)^{-1}\{\Phi(2zm)\Phi(z(2n+2)) - \Phi(z(2m+1))\Phi(z(2n+1))\}$$
$$= c_1 z^2(n+1-m) + O(z^{2-2\epsilon}) \quad (7.20)$$

with $c_1 > 0$. Then for $n \leqslant \bar{n}$

$$|(P\tilde{S}P'\psi)(n)| \leqslant c_1 z^2\left|\sum_{m=0}^{\infty}\left(\frac{1}{2}\right)^{m+n+1}\binom{n+m}{m}P'\psi(m)(n+1-m)\right|$$
$$+ c_2 z^{2-2\epsilon}\sum_{m=0}^{\infty}\left(\frac{1}{2}\right)^{m+n+1}\binom{n+m}{m}(P'\psi_+)(m)$$
$$\leqslant c_1 z^2\left|\sum_{m=0}^{\infty}\left(\frac{1}{2}\right)^{m+n+1}\binom{n+m}{m}x^m(n+1-m)\right|$$
$$+ c_3 z^2 z^{(1/2)-2\epsilon}\sum_{m=0}^{\bar{n}}\left(\frac{1}{2}\right)^{m+n+1}\binom{n+m}{m}x^m z^{-(1/2)-\epsilon}$$
$$+ c_4 z^{2-2\epsilon}(P'\psi_+)(n)$$
$$\leqslant cz^{2-3\epsilon}(P'\psi_+)(n) \quad (7.21)$$

In the third step we used Lemma 1 which bounds the higher terms in ψ by $z^{(1/2)-2\epsilon}$ and

$$\sum_{m=0}^{\infty} m\left(\frac{x}{2}\right)^m\binom{n+m}{m} = \left(x\frac{d}{dx}\right)\left(\frac{2}{2-x}\right)^{n+1} = (n+1)\left(\frac{2}{2-x}\right)^{n+1}$$
$$= (n+1)\left[x + 0(z)\right]^{n+1}$$

By (7.11)

$$\langle P\tilde{S}P'\psi R(PSP)^{k-2}P\tilde{S}P'\psi\rangle \leqslant cz^{4-6\epsilon}\left(1-a_1'\sqrt{z}\right)^{k-2}\langle P'\psi_+ \mid RP'\psi_+\rangle \tag{7.22}$$

$\langle P'\psi_+ \mid RP'\psi_+\rangle$ grows as $z^{-1/2}$ for small z.

Summing over k we obtain

$$|F_2(z)| \leqslant cz^{-2}z^{4-6\epsilon}z^{-1/2}\sum_{k=2}^{\infty}\left(1-a_1'\sqrt{z}\right)^{k-2} \leqslant c'z^{1-6\epsilon} \quad \blacksquare \tag{7.23}$$

Let

$$\psi_0(n) = x^n\left\{1 + a_1zn + a_2z^{3/2}n^2 + a_3z^{5/2}n^3 + a_4z^3n^4\right\} \tag{7.24}$$

with coefficients as in Lemma 1 and let

$$\begin{aligned}&\left[\Phi(z2n)\Phi(z(2m+2)) - 2\Phi(z(2n+1))\Phi(z(2m+1))\right.\\&\qquad\left.+\Phi(z(2n+2))\Phi(z2m)\right]/z^2\Phi(2zm)\Phi(2zn)\\&\quad= b_{00} + z(b_{10}n + b_{01}m + b_1) + z^2\left(b_{20}m^2 + b_{11}nm + b_{02}n^2\right)\\&\qquad+O(z^3n^3 + z^3n^2m + z^3nm^2 + z^3m^3) + O(z^2n + z^2m)\end{aligned} \tag{7.25}$$

for $m, n < z^{-(1/2)-\epsilon}$. The explicit values of the b_{ij} are given in the Appendix.

Lemma 4.

$$\begin{aligned}F_3(z) = &-\frac{1}{4\langle\xi\rangle}\sum_{m,n=0}^{\infty}\binom{n+m}{m}2^{-(m+n)}\psi_0^{(n)}\psi_0^{(m)}\\&\times\left\{b_{00} + z(b_{10}n + b_{01}m + b_1) + z^2\left(b_{20}m^2 + b_{11}nm + b_{02}n^2\right)\right\}\\&+O(z^{1-\epsilon})\end{aligned} \tag{7.26}$$

Proof. We have

$$\begin{aligned}F_3(z) = &-\frac{1}{4\langle\xi\rangle z^2}\sum_{m,n=0}^{\infty}\psi(m)\psi(n)\binom{n+m}{m}2^{-(m+n)}\frac{1}{\Phi(2zm)\Phi(2zn)}\\&\times\left[\Phi(z2n)\Phi(z(2m+2)) - 2\Phi(z(2n+1))\Phi(z(2m+1))\right.\\&\left.+\Phi(z(2n+2))\Phi(z2m)\right]\end{aligned} \tag{7.27}$$

We restrict the sums up to $\bar{n}$. This causes an exponentially small error. We then use the expansion (7.25) and the bounds of Proposition 3. We extend the sum again up to ∞ causing another exponentially small error. Then we have a sum of terms of the form

$$\sum_{n,m=0}^{\infty} x^n x^m \binom{n+m}{m} 2^{-(m+n)} n^\alpha m^\beta \tag{7.28}$$

with integer α, $\beta = 0, 1, \dots$. The sum (7.28) diverges as $z^{-(\alpha+\beta+1)/2}$ as $z \to 0$. The error term in Proposition 3 is of the order $z^{4-\epsilon} n^5$. Its contribution behaves then as $z^{1-\epsilon}$. The errors from expanding the term in square brackets in (7.27) are of the order $z^3 n^3$ or $z^2 n$. Their contribution behaves then as z. ■

Let

$$\begin{aligned}
&\frac{1}{z^2 \Phi(2zk)\Phi(2zn)} \{\Phi(2zk)\Phi(2zn)\Phi(z(2m+3)) \\
&\qquad + \Phi(z(2k+1))\Phi(z(2m+1))\Phi(z(2n+1)) \\
&\qquad - \Phi(z(2k+1))\Phi(z(2m+2))\Phi(z2n) \\
&\qquad - \Phi(z2k)\Phi(z(2m+2))\Phi(z(2n+1))\} \\
&= c_{000} + z(c_{100}k + c_{010}m + c_{001}n + c_1) \\
&\quad + z^2 \left[\sum_{\substack{i,j,l=0 \\ i+j+l=2}}^{2} c_{ijl} k^i m^j n^l \right] + O(n^3 z^3 + m^3 z^3 + k^3 z^3) \\
&\quad + O(z^2 m + z^2 n + z^2 k)
\end{aligned} \tag{7.29}$$

for $k, m, n \leqslant z^{-(1/2)-\epsilon}$. The explicit values of the c_{ijl} are given in the Appendix.

Lemma 5.

$$\begin{aligned}
F_4(z) = \frac{1}{4\langle\xi\rangle} &\sum_{k,m,n=0}^{\infty} \psi_0(k)\psi_0(n) \binom{k+m}{k} 2^{-(k+m)} \\
&\times \binom{n+m}{m} 2^{-(m+n)} \Bigg[c_{000} + z(c_{100}k + c_{010}m + c_{001}n + c_1) \\
&\qquad + z^2 \left[\sum_{\substack{i,j,l=0 \\ i+j+l=2}}^{2} c_{ijl} k^i m^j n^l \right] \Bigg] + O(z^{1-\epsilon})
\end{aligned} \tag{7.30}$$

Proof. We have

$$F_4(z) = \frac{1}{4\langle\xi\rangle z^2} \sum_{k,m,n=0}^{\infty} \psi(k)\psi(n)\binom{k+m}{k}2^{-(k+m)}$$
$$\times \binom{n+m}{n}2^{-(m+n)} \frac{1}{\Phi(2zk)\Phi(2zn)}$$
$$\times \big[\Phi(2zk)\Phi(z(2m+3))\Phi(2zn)$$
$$+ \Phi(z(2k+1))\Phi(z(2m+1))\Phi(z(2n+1))$$
$$- \Phi(z(2k+1))\Phi(z(2m+2))\Phi(z(2n))$$
$$- \Phi(z2k)\Phi(z(2m+2))\Phi(z(2n+1))\big] \tag{7.31}$$

We restrict the sums up to $\bar{n}$. This causes an exponentially small error. We then use the expansion (7.29) and the bounds of Proposition 3. We extend the sums to ∞ causing another exponentially small error. Then we have a sum of terms of the form

$$\sum_{k,m,n} x^k x^n \binom{k+m}{k} 2^{-(k+m)} \binom{m+n}{n} 2^{-(m+n)} k^\alpha m^\beta n^\gamma \tag{7.32}$$

with integer $\alpha, \beta, \gamma = 0, 1 \ldots$. The sum diverges as $z^{-(\alpha+\beta+\gamma+1)/2}$ as $z \to 0$. As in the proof of Lemma 4 one checks that the error term is of order $z^{1-\epsilon}$. ■

We note that $F_3(z)$ and $F_4(z)$ diverge as $z^{-1/2}$ as $z \to 0$. However, the leading terms cancel exactly. The remainder consists of terms of the order $z^{j/2}$, $j = 0, 1, \ldots$. So $F_3(z) + F_4(z) = c_1 + c_2\sqrt{z} + O(z^{1-\epsilon})$. By explicit computation one checks that the coefficients c_1 and c_2 are as stated in the theorem. This concludes the proof of the theorem.

8. CONCLUSIONS

We found that the velocity autocorrelation function decays on the average as $t^{-3/2}$. A problem not answered by our result is, How long does one have to wait before the velocity autocorrelation function actually assumes its asymptotic behavior? One way to investigate this is by means of computer simulations of the model. Grassberger[15] has performed such simulations (in fact, he studies the more general case, where reflection and transmission probability at a scatterer differ from each other). For our model his results indicate that the asymptotic regime for the average velocity autocorrelation function is reached after 15 mean collision times.

A model similar to the stochastic Lorentz gas has been investigated in

Ref. 12. In this model one considers the situation where the time of flight between two scatterers is very short compared to the waiting time at a scatterer and therefore may be approximated to be zero. So, on randomly placed points on the line a particle performs a continuous time random walk. Since the transition probabilities for this process are known, quantities such as the mean and variance of the square displacement can be computed explicitly. With an appropriately defined velocity, one finds the same asymptotic behavior of the velocity autocorrelation function as for the stochastic Lorentz gas.

It is our intention to investigate the theory of the stochastic Lorentz model in two further papers. In the first of these we will investigate the fluctuations in the velocity autocorrelation function originating from the random distribution of scatterers, and in the second we will investigate the low-frequency, small-wave-number behavior of the Green's function for the moving particle.

ACKNOWLEDGMENT

We would like to thank N. Kuiper and J. L. Lebowitz for their most kind hospitality at IHES and Rutgers University, respectively, where parts of this work were completed.

APPENDIX

For the convenience of the reader we report the numerical values of the coefficients which are needed for the computation at the end of Section 7.

(i) The coefficients of the invariant vector (Lemma 1) in the limit $z \to 0$ are

$$a_1 = \frac{1}{8\langle\xi\rangle}$$

$$a_2 = \frac{1}{4(2\langle\xi\rangle)^{1/2}}$$

(ii) The coefficients in Lemma 4 are

$$b_{00} = 2\langle\xi^2\rangle - \langle\xi\rangle^2$$

$$b_1 = -2\langle\xi^3\rangle + 2\langle\xi\rangle\langle\xi^2\rangle$$

$$b_{10} = b_{01} = -2\langle\xi^3\rangle + 6\langle\xi^2\rangle\langle\xi\rangle - 4\langle\xi\rangle^3$$

$$b_{20} = b_{02} = 2\langle\xi^4\rangle - 8\langle\xi^3\rangle\langle\xi\rangle - 2\langle\xi^2\rangle\langle\xi^2\rangle + 16\langle\xi^2\rangle\langle\xi\rangle^2 - 8\langle\xi\rangle^4$$

$$b_{11} = -8\langle\xi^2\rangle\langle\xi^2\rangle + 16\langle\xi^2\rangle\langle\xi\rangle^2 - 8\langle\xi\rangle^4$$

(iii) The coefficients in Lemma 5 are

$$c_{000} = \langle \xi^2 \rangle - \langle \xi \rangle^2$$

$$c_1 = -2\langle \xi^3 \rangle + 3\langle \xi \rangle\langle \xi^2 \rangle - \langle \xi \rangle^3$$

$$c_{100} = c_{001} = 0$$

$$c_{010} = -2\langle \xi^3 \rangle + 4\langle \xi \rangle\langle \xi^2 \rangle - 2\langle \xi \rangle^3$$

$$c_{101} = -c_{110} = -c_{011} = 4(\langle \xi^2 \rangle - \langle \xi \rangle^2)^2$$

$$c_{020} = 2\langle \xi^4 \rangle - 4\langle \xi \rangle\langle \xi^3 \rangle + 2\langle \xi^2 \rangle\langle \xi \rangle^2$$

$$c_{200} = c_{002} = 0$$

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