# Transposed Poisson structures on Block Lie algebras and superalgebra\& $\underbrace{}$ 

Ivan Kaygorodov ${ }^{a, b, c}$ \& Mykola Khrypchenko ${ }^{d}$<br>${ }^{a}$ Centro de Matemática e Aplicações, Universidade da Beira Interior, Portugal<br>${ }^{b}$ Saint Petersburg University, Russia<br>${ }^{c}$ Moscow Center for Fundamental and Applied Mathematics, Russia<br>${ }^{d}$ Departamento de Matemática, Universidade Federal de Santa Catarina, Brazil<br>E-mail addresses:<br>Ivan Kaygorodov (kaygorodov.ivan@gmail.com)<br>Mykola Khrypchenko (nskhripchenko@gmail.com)


#### Abstract

We describe transposed Poisson algebra structures on Block Lie algebras $\mathcal{B}(q)$ and Block Lie superalgebras $\mathcal{S}(q)$, where $q$ is an arbitrary complex number. Specifically, we show that the transposed Poisson structures on $\mathcal{B}(q)$ are trivial whenever $q \notin \mathbb{Z}$, and for each $q \in \mathbb{Z}$ there is only one (up to an isomorphism) non-trivial transposed Poisson structure on $\mathcal{B}(q)$. The superalgebra $\mathcal{S}(q)$ admits only trivial transposed Poisson superalgebra structures for $q \neq$ 0 and two non-isomorphic non-trivial transposed Poisson superalgebra structures for $q=0$. As a consequence, new Lie algebras and superalgebras that admit non-trivial Hom-Lie algebra structures are found.


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## Introduction

The origin of Poisson algebras lies in the Poisson geometry of the 1970s, and since then these algebras have shown their importance in several areas of mathematics and physics, such as Poisson manifolds, algebraic geometry, operads, quantization theory, quantum groups, and classical and quantum mechanics. The study of all possible Poisson algebra structures with fixed Lie or associative part is a popular topic in the theory of Poisson algebras [1, 12, 14, 28]. Recently, Bai, Bai, Guo, and Wu [2] have introduced a dual notion of the Poisson algebra, called transposed Poisson algebra, by exchanging the roles of the two binary operations in the Leibniz rule defining the Poisson algebra. They have shown that a transposed Poisson algebra defined this way not only shares common properties of a Poisson algebra, including the closedness under tensor products and the Koszul self-duality as an operad, but also admits a rich class of identities. More significantly, a transposed Poisson algebra naturally arises from a Novikov-Poisson algebra by taking the commutator Lie algebra of the Novikov algebra. Thanks to [3], any unital transposed Poisson algebra is a particular case of a "contact bracket" algebra [18] and a quasi-Poisson algebra [4]. Later, in a recent paper by Ferreira, Kaygorodov, and Lopatkin a relation between $\frac{1}{2}$-derivations of Lie algebras and transposed Poisson algebras have been established [9]. These ideas were used to describe all transposed Poisson structures on Witt and Virasoro algebras in [9]; on twisted Heisenberg-Virasoro, Schrodinger-Virasoro and extended Schrodinger-Virasoro algebras in [29]; on oscillator algebras in 3]. Fehlberg Júnior and Kaygorodov gave a way to construct new transposed Poisson algebras by the Kantor product of their multiplications in [8]. The Hom- and BiHom-versions of transposed Poisson algebras have been considered in [15, 17]. A list of actual open questions on transposed Poisson algebras is given in [3].

Block Lie algebras is a class of simple infinite dimensional Lie algebras introduced by Block in 1958 [5]. Since then, several generalizations of these algebras have appeared [7,19, 27]. Block Lie algebras, their generalizations and related algebras are still under an active investigation [6, 20 23, 25]. Thus, Đoković and Zhao introduced a generalization of Block algebras and described their derivations, isomorphisms and second cohomology in [7]. Xia, You and Zhou [26] defined, for a fixed complex number $q$, the Block algebra $\mathcal{B}(q)$ as a Lie algebra with a basis $\left\{L_{m, i} \mid m, i \in \mathbb{Z}\right\}$ and the following multiplication table

$$
\left[L_{m, i}, L_{n, j}\right]=(n(i+q)-m(j+q)) L_{m+n, i+j}, \quad i, j, m, n \in \mathbb{Z}
$$

They proved that, for distinct integers $q_{1}$ and $q_{2}$, the algebras $\mathcal{B}\left(q_{1}\right)$ and $\mathcal{B}\left(q_{2}\right)$ are nonisomorphic, and calculated the automorphism group and the algebra of derivations of $\mathcal{B}(q)$ for an arbitrary $q \in \mathbb{C}$. They also showed that the second scalar cohomology group of $\mathcal{B}(q)$ is one-dimensional and found the unique non-trivial central extension of $\mathcal{B}(q)$. Liu, Guo, Xiangqian and Zhao [16] described all biderivations (and, as an application, all commuting maps) on Block algebras $\mathcal{B}(q)$. $U(h)$-free modules over the Block algebra $\mathcal{B}(q)$ have been studied by Guo, Wang and Liu in [11]. Xia, You and Zhou introduced a superanalog of Block Lie algebras in [24].

In the present paper, we give a full description of transposed Poisson algebra structures on Block Lie algebras $\mathcal{B}(q)$ defined in [26] and transposed Poisson superalgebra structures on Block Lie superalgebras $\mathcal{S}(q)$ defined in [24]. More precisely, we prove in Theorem 2.14 that transposed Poisson algebra structures on $\mathcal{B}(q)$ are trivial for $q \notin \mathbb{Z}$, and there is only nontrivial such structure for $q \in \mathbb{Z}$. We also show in Theorem 3.16 that $\mathcal{S}(q)$ admits only trivial transposed Poisson superalgebra structures for $q \neq 0$, but in the case $q=0$ there are two non-trivial such structures.

## 1. Preliminaries

1.1. Transposed Poisson algebras. All the algebras below will be over $\mathbb{C}$ and all the linear maps will be $\mathbb{C}$-linear, unless otherwise stated.

Definition 1.1. Let $\mathfrak{L}$ be a vector space equipped with two nonzero bilinear operations $\cdot$ and $[\cdot, \cdot]$. The triple $(\mathfrak{L}, \cdot,[\cdot, \cdot])$ is called a transposed Poisson algebra if $(\mathfrak{L}, \cdot)$ is a commutative associative algebra and $(\mathfrak{L},[\cdot, \cdot])$ is a Lie algebra that satisfies the following compatibility condition

$$
\begin{equation*}
2 z \cdot[x, y]=[z \cdot x, y]+[x, z \cdot y] \tag{1}
\end{equation*}
$$

Transposed Poisson algebras were first introduced in a paper by Bai, Bai, Guo, and Wu [2].
Definition 1.2. Let $(\mathfrak{L},[\cdot, \cdot])$ be a Lie algebra. A transposed Poisson algebra structure on $(\mathfrak{L},[\cdot, \cdot])$ is a commutative associative multiplication $\cdot$ on $\mathfrak{L}$ which makes $(\mathfrak{L}, \cdot,[\cdot, \cdot])$ a transposed Poisson algebra.
Definition 1.3. Let $(\mathfrak{L},[\cdot, \cdot])$ be an algebra and $\varphi: \mathfrak{L} \rightarrow \mathfrak{L}$ a linear map. Then $\varphi$ is a $\frac{1}{2}$ derivation if it satisfies

$$
\begin{equation*}
\varphi([x, y])=\frac{1}{2}([\varphi(x), y]+[x, \varphi(y)]) . \tag{2}
\end{equation*}
$$

Observe that $\frac{1}{2}$-derivations are a particular case of $\delta$-derivations introduced by Filippov in 10$]$ (see also [13,30 and references therein).

Definitions 1.1 and 1.3 immediately imply the following key Lemma.
Lemma 1.4. Let $(\mathfrak{L}, \cdot,[\cdot, \cdot])$ be a transposed Poisson algebra and $z \in \mathfrak{L}$. Then the left multiplication $L_{z}$ of $(\mathfrak{L}, \cdot)$ is a $\frac{1}{2}$-derivation of $(\mathfrak{L},[\cdot, \cdot])$.
The basic example of a $\frac{1}{2}$-derivation is the multiplication by a field element. Such $\frac{1}{2}$ derivations will be called trivial.
Theorem 1.5. Let $\mathfrak{L}$ be a Lie algebra without non-trivial $\frac{1}{2}$-derivations. Then all transposed Poisson algebra structures on $\mathfrak{L}$ are trivial.

Let us recall the definition of Hom-structures on Lie algebras.
Definition 1.6. Let $(\mathfrak{L},[\cdot, \cdot])$ be a Lie algebra and $\varphi$ be a linear map. Then $(\mathfrak{L},[\cdot, \cdot], \varphi)$ is a Hom-Lie structure on $(\mathfrak{L},[\cdot, \cdot])$ if

$$
[\varphi(x),[y, z]]+[\varphi(y),[z, y]]+[\varphi(z),[x, y]]=0
$$

1.2. Transposed Poisson superalgebras. One naturally defines a transposed Poisson superalgebra as a superization of the notion of a transposed Poisson algebra.

Definition 1.7. Let $\mathfrak{L}=\mathfrak{L}_{0} \oplus \mathfrak{L}_{1}$ be a $\mathbb{Z}_{2}$-graded vector space equipped with two nonzero bilinear super-operations • and $[\cdot, \cdot]$. The triple $(\mathfrak{L}, \cdot,[\cdot, \cdot])$ is called a transposed Poisson superalgebra if $(\mathfrak{L}, \cdot)$ is a supercommutative associative superalgebra and $(\mathfrak{L},[\cdot, \cdot])$ is a Lie superalgebra that satisfies the following compatibility condition

$$
\begin{equation*}
2 z \cdot[x, y]=[z \cdot x, y]+(-1)^{|x||z|}[x, z \cdot y], x, y, z \in \mathfrak{L}_{0} \cup \mathfrak{L}_{1} . \tag{3}
\end{equation*}
$$

Definition 1.8. Let $(\mathfrak{L},[\cdot, \cdot])$ be a Lie superalgebra. A transposed Poisson superalgebra structure on $(\mathfrak{L},[\cdot, \cdot])$ is a supercommutative associative multiplication $\cdot$ on $\mathfrak{L}$ which makes $(\mathfrak{L}, \cdot,[\cdot, \cdot])$ a transposed Poisson superalgebra.

Definition 1.9. Let $(\mathfrak{L},[\cdot, \cdot])$ be a superalgebra and $\varphi$ a homogeneous linear map $\mathfrak{L} \rightarrow \mathfrak{L}$. Then $\varphi$ is called a $\frac{1}{2}$-superderivation if it satisfies

$$
\varphi([x, y])=\frac{1}{2}\left([\varphi(x), y]+(-1)^{|\varphi||x|}[x, \varphi(y)]\right), \quad x, y \in \mathfrak{L}_{0} \cup \mathfrak{L}_{1} .
$$

Lemma 1.10. Let $(\mathfrak{L}, \cdot,[\cdot, \cdot])$ be a transposed Poisson superalgebra and $z \in \mathfrak{L}_{0} \cup \mathfrak{L}_{1}$. Then the left multiplication $L_{z}$ of $(\mathfrak{L}, \cdot)$ is a $\frac{1}{2}$-superderivation of $(\mathfrak{L},[\cdot, \cdot])$ and $\left|L_{z}\right|=|z|$.

Let $\cdot$ be a transposed Poisson (super)algebra structure on a Lie (super)algebra ( $\mathfrak{L},[\cdot, \cdot]$ ). Then any automorphism $\phi$ of $(\mathfrak{L},[\cdot, \cdot])$ induces another transposed Poisson (super)algebra structure * on $(\mathfrak{L},[\cdot, \cdot])$ given by

$$
x * y=\phi\left(\phi^{-1}(x) \cdot \phi^{-1}(x)\right), \quad x, y \in \mathfrak{L} .
$$

Clearly, $\phi$ is an isomorphism of transposed Poisson (super)algebras ( $\mathfrak{L}, \cdot,[\cdot, \cdot]$ ) and $(\mathfrak{L}, *,[\cdot, \cdot])$.

## 2. Transposed Poisson structures on Block Lie algebras

Definition 2.1. Let $q$ be a fixed complex number. The Block Lie algebra $\mathcal{B}(q)$ is the complex Lie algebra with a basis $\left\{L_{m, i} \mid m, i \in \mathbb{Z}\right\}$, where

$$
\begin{equation*}
\left[L_{m, i}, L_{n, j}\right]=(n(i+q)-m(j+q)) L_{m+n, i+j} \tag{4}
\end{equation*}
$$

for all $i, j, m, n \in \mathbb{Z}$.
Observe that $\mathcal{B}(q)=\bigoplus_{m, i \in \mathbb{Z}} \mathcal{B}(q)_{m, i}$ is a $\mathbb{Z} \times \mathbb{Z}$-grading, where $\mathcal{B}(q)_{m, i}=\mathbb{C} L_{m, i}$.
2.1. $\frac{1}{2}$-derivations of $\mathcal{B}(q)$. Let $\varphi$ be a linear map $\mathcal{B}(q) \rightarrow \mathcal{B}(q)$. Then the $\mathbb{Z} \times \mathbb{Z}$-grading of $\mathcal{B}(q)$ induces the decomposition

$$
\varphi=\sum_{r, s \in \mathbb{Z}} \varphi_{r, s}
$$

where $\varphi_{r, s}$ is a linear map $\mathcal{B}(q) \rightarrow \mathcal{B}(q)$ such that $\varphi_{r, s}\left(\mathcal{B}(q)_{m, i}\right) \subseteq \mathcal{B}(q)_{m+r, i+s}$ for all $m, i \in \mathbb{Z}$. Observe that $\varphi$ is a $\frac{1}{2}$-derivation of $\mathcal{B}(q)$ if and only if $\varphi_{r, s}$ is a $\frac{1}{2}$-derivation of $\mathcal{B}(q)$ for all $r, s \in \mathbb{Z}$. We write

$$
\begin{equation*}
\varphi_{r, s}\left(L_{m, i}\right)=d_{r, s}(m, i) L_{m+r, i+s} \tag{5}
\end{equation*}
$$

where $d_{r, s}(m, i) \in \mathbb{C}$.
Lemma 2.2. Let $\varphi_{r, s}: \mathcal{B}(q) \rightarrow \mathcal{B}(q)$ be a linear map satisfying (5). Then $\varphi_{r, s}$ is a $\frac{1}{2}$-derivation of $\mathcal{B}(q)$ if and only if

$$
\begin{align*}
2(n(i+q)-m(j+q)) d_{r, s}(m+n, i+j)= & (n(i+s+q)-(m+r)(j+q)) d_{r, s}(m, i) \\
& +((n+r)(i+q)-m(j+s+q)) d_{r, s}(n, j) . \tag{6}
\end{align*}
$$

Proof. It is obvious that $\varphi_{r, s}$ is a $\frac{1}{2}$-derivation of $\mathcal{B}(q)$ if and only if (2) holds on the basis $\left\{L_{m, i} \mid m, i \in \mathbb{Z}\right\}$. Using (4) and (5), for all $i, j, m, n \in \mathbb{Z}$ we have

$$
\begin{aligned}
2 \varphi_{r, s}\left(\left[L_{m, i}, L_{n, j}\right]\right) & =2 \varphi_{r, s}\left((n(i+q)-m(j+q)) L_{m+n, i+j}\right) \\
& =2(n(i+q)-m(j+q)) d_{r, s}(m+n, i+j) L_{m+n+r, i+j+s}
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
{\left[\varphi_{r, s}\left(L_{m, i}\right), L_{n, j}\right]+\left[L_{m, i}, \varphi_{r, s}\left(L_{n, j}\right)\right]=} & {\left[d_{r, s}(m, i) L_{m+r, i+s}, L_{n, j}\right]+\left[L_{m, i}, d_{r, s}(n, j) L_{n+r, j+s}\right] } \\
= & (n(i+s+q)-(m+r)(j+q)) d_{r, s}(m, i) L_{m+n+r, i+j+s} \\
& +((n+r)(i+q)-m(j+s+q)) d_{r, s}(n, j) L_{m+n+r, i+j+s} .
\end{aligned}
$$

Thus, (2) is equivalent to (6).
2.1.1. The case $q \neq 0$.

Lemma 2.3. Let $\varphi$ be a $\frac{1}{2}$-derivation of $\mathcal{B}(q)$ and $r \neq 0$. Then $\varphi_{r, s}=0$.
Proof. Taking $n=j=0$ in (6), we obtain

$$
-2 m q \cdot d_{r, s}(m, i)=-(m+r) q \cdot d_{r, s}(m, i)+(r(i+q)-m(s+q)) d_{r, s}(0,0)
$$

so

$$
\begin{equation*}
(r-m) q \cdot d_{r, s}(m, i)=(r(i+q)-m(s+q)) d_{r, s}(0,0) \tag{7}
\end{equation*}
$$

Choosing $m=r \neq 0$ and $i \neq s$ in (7) we get $d_{r, s}(0,0)=0$. Therefore,

$$
\begin{equation*}
d_{r, s}(m, i)=0, \text { if } m \neq r, \tag{8}
\end{equation*}
$$

by (7). Now, taking $m=-n$ and $j=-i$ in (6), we have

$$
\begin{equation*}
2 n q \cdot d_{r, s}(0,0)=(2 n q+n s+r i-r q) d_{r, s}(-n, i)+(2 n q+n s+r i+r q) d_{r, s}(n,-i) . \tag{9}
\end{equation*}
$$

Choosing, moreover, $n=-r \neq 0$ (in particular, $n \neq r$ ), we obtain from (8) and (9) that $0=(-3 q-s+i) d_{r, s}(r, i)$, whence

$$
\begin{equation*}
d_{r, s}(r, i)=0, \text { if } i \neq 3 q+s \tag{10}
\end{equation*}
$$

In view of (8) and (10), it remains to prove that $d_{r, s}(r, 3 q+s)=0$. Take $m=r, i=3 q+s$ and $n \notin\{0, r\}$ in (6). Then $0=(n(4 q+2 s)-2 r(j+q)) d_{r, s}(r, 3 q+s)$ by (8), so choosing $j \neq \frac{n}{r}(2 q+s)-q$, we get $d_{r, s}(r, 3 q+s)=0$.
Lemma 2.4. Let $\varphi$ be a $\frac{1}{2}$-derivation of $\mathcal{B}(q)$ and $s \neq 0$. If $s \neq q$, then $\varphi_{0, s}=0$. Moreover, $d_{0, q}(m, i)=0$, unless $(m, i)=(0,-2 q)$.

Proof. Write (6) with $r=0$ :

$$
\begin{align*}
2(n(i+q)-m(j+q)) d_{0, s}(m+n, i+j)= & (n(i+s+q)-m(j+q)) d_{0, s}(m, i) \\
& +(n(i+q)-m(j+s+q)) d_{0, s}(n, j) . \tag{11}
\end{align*}
$$

Then putting $n=j=0$ and $m \neq 0$, we obtain

$$
\begin{equation*}
d_{0, s}(m, i)=\left(1+s q^{-1}\right) d_{0, s}(0,0), \text { if } m \neq 0 \tag{12}
\end{equation*}
$$

We will now prove that $d_{0, s}(0,0)=0$.
Case 1. $q \in \mathbb{Z}$. Take $m=-n \neq 0$ and $j=-i$ in (11) and then apply (12):

$$
2 q \cdot d_{0, s}(0,0)=(2 q+s)\left(d_{0, s}(-n, i)+d_{0, s}(n,-i)\right)=2(2 q+s)\left(1+s q^{-1}\right) d_{0, s}(0,0)
$$

Assuming $d_{0, s}(0,0) \neq 0$, we obtain $q=(2 q+s)\left(1+s q^{-1}\right)$, whence $q^{2}+3 s q+s^{2}=0$. It follows that $q=\frac{-3 \pm \sqrt{5}}{2} s$, which contradicts $q \in \mathbb{Z}$.

Case 2. $q \notin \mathbb{Z}$. Observe that $1+s q^{-1} \neq 0$ in this case. Taking $i=j=0$ in (11), we have:

$$
2(n-m) q \cdot d_{0, s}(m+n, 0)=(n s+(n-m) q) d_{0, s}(m, 0)+((n-m) q-m s) d_{0, s}(n, 0)
$$

Choose $m, n, m \pm n \neq 0$. Then thanks to (12) and $1+s q^{-1} \neq 0$ :

$$
2(n-m) q \cdot d_{0, s}(0,0)=(n s+(n-m) q) d_{0, s}(0,0)+((n-m) q-m s) d_{0, s}(0,0)
$$

Assuming $d_{0, s}(0,0) \neq 0$, we come to $(n-m) s=0$. So, $n-m=0$, a contradiction.
Thus, $d_{0, s}(0,0)=0$, and hence by (12)

$$
\begin{equation*}
d_{0, s}(m, i)=0, \text { if } m \neq 0 \tag{13}
\end{equation*}
$$

It remains to analyze $d_{0, s}(0, i)$. To this end, put $m=-n \neq 0$ and $j=0$ in (11) and use (13):

$$
2(2 q+i) d_{0, s}(0, i)=(2 q+i+s)\left(d_{0, s}(-n, i)+d_{0, s}(n, 0)\right)=0
$$

Therefore,

$$
\begin{equation*}
d_{0, s}(0, i)=0, \text { if } i \neq-2 q . \tag{14}
\end{equation*}
$$

Combining (13) and (14), we conclude that $d_{0, s}(m, i)=0$ for $(m, i) \neq(0,-2 q)$.
Assume now that $s \neq q$. Take $m=0, n \neq 0$ and $i=-2 q$ in (11). We have

$$
-2 q \cdot d_{0, s}(n, j-2 q)=(s-q) d_{0, s}(0,-2 q)-q \cdot d_{0, s}(n, j),
$$

whence $(s-q) d_{0, s}(0,-2 q)=0$ by (13). Since $s \neq q$, then $d_{0, s}(0,-2 q)=0$.

Lemma 2.5. Let $q \in \mathbb{Z}$. Then the linear map $\alpha: \mathcal{B}(q) \rightarrow \mathcal{B}(q)$ such that

$$
\alpha\left(L_{m, i}\right)= \begin{cases}0, & (m, i) \neq(0,-2 q)  \tag{15}\\ L_{0,-q}, & (m, i)=(0,-2 q)\end{cases}
$$

is a $\frac{1}{2}$-derivation of $\mathcal{B}(q)$.
Proof. Observe that $\alpha=\alpha_{0, q}$. In view of Lemma 2.2 we need to check (6) for $(r, s)=(0, q)$ and

$$
d_{0, q}(m, i)= \begin{cases}0, & (m, i) \neq(0,-2 q)  \tag{16}\\ 1, & (m, i)=(0,-2 q)\end{cases}
$$

Case 1. $(m, i),(n, j),(m+n, i+j) \neq(0,-2 q)$. Then both sides of (6) are zero.
Case 2. $(m, i)=(0,-2 q)$. Then (6) becomes

$$
-2 n q \cdot d_{0, q}(n, j-2 q)=-n q \cdot d_{0, q}(n, j)
$$

If $n=0$, then it is trivially satisfied, otherwise both sides are zero by (16).
Case 3. $(n, j)=(0,-2 q)$. Then (6) becomes

$$
2 m q \cdot d_{r, s}(m, i-2 q)=m q \cdot d_{r, s}(m, i)
$$

so this case is similar to Case 2.
Case 4. $(m+n, i+j)=(0,-2 q)$. Then (6) becomes

$$
0=n q \cdot d_{0, q}(-n, i)+n q \cdot d_{0, q}(n,-i-2 q),
$$

and again this holds by (16).
Summarizing the results of Lemmas 2.3-2.5, we have the following.
Corollary 2.6. Let $\varphi$ be a $\frac{1}{2}$-derivation of $\mathcal{B}(q)$. If $q \notin \mathbb{Z}$, then $\varphi_{r, s}=0$ for all $(r, s) \neq(0,0)$. If $q \in \mathbb{Z}$, then $\varphi_{r, s}=0$ for all $(r, s) \notin\{(0,0),(0, q)\}$ and $\varphi_{0, q} \in\langle\alpha\rangle$.
Lemma 2.7. Let $\varphi$ be a $\frac{1}{2}$-derivation of $\mathcal{B}(q)$ satisfying (5). Then $\varphi_{0,0} \in\langle\mathrm{id}\rangle$.
Proof. Write (6) for $r=s=0$ :

$$
\begin{equation*}
2(n(i+q)-m(j+q)) d_{0,0}(m+n, i+j)=(n(i+q)-m(j+q))\left(d_{0,0}(m, i)+d_{0,0}(n, j)\right) \tag{17}
\end{equation*}
$$

Taking $n=j=0$ and $m \neq 0$ in (6), we obtain

$$
2 d_{0,0}(m, i)=d_{0,0}(m, i)+d_{0,0}(0,0)
$$

whence

$$
\begin{equation*}
d_{0,0}(m, i)=d_{0,0}(0,0), \text { if } m \neq 0 \tag{18}
\end{equation*}
$$

Now, put $m=-n \neq 0$ and $j=0$ in (17) and use (18):

$$
2(2 q+i) d_{0,0}(0, i)=(2 q+i)\left(d_{0,0}(-n, i)+d_{0,0}(n, 0)\right)=2(2 q+i) d_{0,0}(0,0)
$$

Hence,

$$
\begin{equation*}
d_{0,0}(0, i)=d_{0,0}(0,0), \text { if } i \neq-2 q . \tag{19}
\end{equation*}
$$

Finally, substitute $(m, i)=(0,-2 q)$ in (17). Then

$$
2 n \cdot d_{0,0}(n, j-2 q)=n\left(d_{0,0}(0,-2 q)+d_{0,0}(n, j)\right)
$$

So, for $n \neq 0$ we deduce from (18) that $2 d_{0,0}(0,0)=d_{0,0}(0,-2 q)+d_{0,0}(0,0)$, whence $d_{0,0}(0,-2 q)=d_{0,0}(0,0)$. Combining this with (18) and (19), we conclude that $\varphi_{0,0}=$ $d_{0,0}(0,0) \mathrm{id}$.
Proposition 2.8. Let $q \neq 0$. Then

$$
\Delta(\mathcal{B}(q))= \begin{cases}\langle\mathrm{id}\rangle, & q \notin \mathbb{Z}, \\ \langle\mathrm{id}, \alpha\rangle, & q \in \mathbb{Z} \backslash\{0\} .\end{cases}
$$

Proof. A consequence of Corollary 2.6 and Lemmas 2.5 and 2.7
2.1.2. The case $q=0$. In this case (6) reduces to

$$
\begin{align*}
2(n i-m j) d_{r, s}(m+n, i+j)= & (n(i+s)-(m+r) j) d_{r, s}(m, i) \\
& +((n+r) i-m(j+s)) d_{r, s}(n, j) \tag{20}
\end{align*}
$$

Lemma 2.9. Let $\varphi$ be a $\frac{1}{2}$-derivation of $\mathcal{B}(0)$. If $(r, s) \neq(0,0)$, then $\varphi_{r, s}=0$.
Proof. Put $m=j=0$ in (20):

$$
\begin{equation*}
2 n i \cdot d_{r, s}(n, i)=n(i+s) d_{r, s}(0, i)+i(n+r) d_{r, s}(n, 0) \tag{21}
\end{equation*}
$$

In particular, substituting $n=0, i \neq 0$ and $n \neq 0, i=0$ in (21), we see that

$$
\begin{equation*}
d_{r, s}(0,0)=0, \text { if }(r, s) \neq(0,0) \tag{22}
\end{equation*}
$$

On the other hand, taking $m=-n$ and $j=0$ in (20), we have

$$
\begin{equation*}
2 n i \cdot d_{r, s}(0, i)=n(i+s) d_{r, s}(-n, i)+(n(i+s)+r i) d_{r, s}(n, 0) \tag{23}
\end{equation*}
$$

Observe that (23) with $i=0$ together with (22) give

$$
\begin{equation*}
d_{r, s}(-n, 0)=-d_{r, s}(n, 0), \text { if } s \neq 0 \tag{24}
\end{equation*}
$$

Furthermore, if $i=-j$ and $m=0$, then (20) becomes

$$
\begin{equation*}
2 n i \cdot d_{r, s}(n, 0)=(n(i+s)+r i) d_{r, s}(0, i)+i(n+r) d_{r, s}(n,-i) \tag{25}
\end{equation*}
$$

As above, this implies

$$
\begin{equation*}
d_{r, s}(0,-i)=-d_{r, s}(0, i), \text { if } r \neq 0 \tag{26}
\end{equation*}
$$

Case 1. $r \neq 0$ and $s \neq 0$. Then $n=-r$ in (21) and (25) together with (24) give

$$
\begin{align*}
2 i \cdot d_{r, s}(-r, i) & =(i+s) d_{r, s}(0, i)  \tag{27}\\
2 i \cdot d_{r, s}(r, 0) & =-s \cdot d_{r, s}(0, i) \tag{28}
\end{align*}
$$

On the other hand, $n=r$ in (23) gives

$$
\begin{equation*}
2 i \cdot d_{r, s}(0, i)=(i+s) d_{r, s}(-r, i)+(2 i+s) d_{r, s}(r, 0) \tag{29}
\end{equation*}
$$

Multiplying (29) by $2 i$ and using (27) and (28), we get

$$
4 i^{2} \cdot d_{r, s}(0, i)=(i+s)^{2} d_{r, s}(0, i)-s(2 i+s) d_{r, s}(0, i)=i^{2} \cdot d_{r, s}(0, i)
$$

Combining this with (22), we conclude that

$$
\begin{equation*}
d_{r, s}(0, i)=0, \text { if } r \neq 0 \text { and } s \neq 0 \tag{30}
\end{equation*}
$$

Similarly, $i=-s$ in (21) and (23) together with (26) yield

$$
\begin{align*}
2 n \cdot d_{r, s}(n,-s) & =(n+r) d_{r, s}(n, 0),  \tag{31}\\
2 n \cdot d_{r, s}(0, s) & =-r \cdot d_{r, s}(n, 0) \tag{32}
\end{align*}
$$

Now, $i=s$ in (25) gives

$$
\begin{equation*}
2 n \cdot d_{r, s}(n, 0)=(2 n+r) d_{r, s}(0, s)+(n+r) d_{r, s}(n,-s) \tag{33}
\end{equation*}
$$

Multiplying (33) by $2 n$ and using (31) and (32), we come to

$$
4 n^{2} \cdot d_{r, s}(n, 0)=-r(2 n+r) d_{r, s}(n, 0)+(n+r)^{2} d_{r, s}(n, 0)=n^{2} d_{r, s}(n, 0)
$$

So, together with (22) this results in

$$
\begin{equation*}
d_{r, s}(n, 0)=0, \text { if } r \neq 0 \text { and } s \neq 0 \tag{34}
\end{equation*}
$$

It follows from (21), (30) and (34) that

$$
\begin{equation*}
d_{r, s}(n, i)=0, \text { if } r \neq 0, s \neq 0, n \neq 0 \text { and } i \neq 0 \tag{35}
\end{equation*}
$$

Combining (30), (34) and (35), we obtain

$$
\begin{equation*}
d_{r, s}(n, i)=0, \text { if } r \neq 0 \text { and } s \neq 0 \tag{36}
\end{equation*}
$$

Case 2. $r \neq 0$ and $s=0$. Substituting $n=-r$ into (21) and (23), we have

$$
\begin{align*}
2 i \cdot d_{r, 0}(-r, i) & =i \cdot d_{r, 0}(0, i)  \tag{37}\\
2 i \cdot d_{r, 0}(0, i) & =i \cdot d_{r, 0}(r, i) \tag{38}
\end{align*}
$$

On the other hand, the substitution $n=r$ in (211) and (23) leads to

$$
\begin{align*}
2 i \cdot d_{r, 0}(r, i) & =i \cdot d_{r, 0}(0, i)+2 i \cdot d_{r, 0}(r, 0)  \tag{39}\\
2 i \cdot d_{r, 0}(0, i) & =i \cdot d_{r, 0}(-r, i)+2 i \cdot d_{r, 0}(r, 0) \tag{40}
\end{align*}
$$

It follows from (37) and (40) that $3 i \cdot d_{r, 0}(0, i)=4 i \cdot d_{r, 0}(r, 0)$. On the other hand, (38) and (39) imply that $3 i \cdot d_{r, 0}(0, i)=2 i \cdot d_{r, 0}(r, 0)$. This, together with (22) results in

$$
\begin{equation*}
d_{r, 0}(0, i)=0, \text { if } r \neq 0 \tag{41}
\end{equation*}
$$

Hence, (21), (23) and (25) take the following form

$$
\begin{equation*}
2 n i \cdot d_{r, 0}(n, i)=i(n+r) d_{r, 0}(n, 0) \tag{42}
\end{equation*}
$$

$$
\begin{align*}
0 & =n i \cdot d_{r, 0}(-n, i)+i(n+r) d_{r, 0}(n, 0),  \tag{43}\\
2 n i \cdot d_{r, 0}(n, 0) & =i(n+r) d_{r, 0}(n,-i) . \tag{44}
\end{align*}
$$

Replacing $i$ by $-i$ in (44) and combining this with (42), we arrive at $i^{2}(r-n)(r+3 n) d_{r, 0}(n, 0)=$ 0 , so $d_{r, 0}(n, 0)=0$, whenever $n \notin\left\{r,-\frac{r}{3}\right\}$. Hence, $d_{r, 0}(n, i)=0$ for $n \notin\left\{r,-\frac{r}{3}\right\}$ by (41) and (42). However, if $n \in\left\{r,-\frac{r}{3}\right\}$, then $-n \notin\left\{r,-\frac{r}{3}\right\}$, so $d_{r, 0}(-n, i)=0$, yielding $d_{r, 0}(n, 0)=0$ thanks to (43). Thus,

$$
\begin{equation*}
d_{r, 0}(n, i)=0, \text { if } r \neq 0, \tag{45}
\end{equation*}
$$

completing the proof of this case.
Case 3. $r=0$ and $s \neq 0$. Taking $i=-s$ in (21) and (25), we obtain

$$
\begin{align*}
2 n \cdot d_{0, s}(n,-s) & =n \cdot d_{0, s}(n, 0)  \tag{46}\\
2 n \cdot d_{0, s}(n, 0) & =n \cdot d_{0, s}(n, s) . \tag{47}
\end{align*}
$$

On the other hand, substituting $i=s$ in (21) and (25), we get

$$
\begin{align*}
& 2 n \cdot d_{0, s}(n, s)=2 n \cdot d_{0, s}(0, s)+n \cdot d_{0, s}(n, 0)  \tag{48}\\
& 2 n \cdot d_{0, s}(n, 0)=2 n \cdot d_{0, s}(0, s)+n \cdot d_{0, s}(n,-s) \tag{49}
\end{align*}
$$

Observe that (46) and (49) imply that $3 n \cdot d_{0, s}(n, 0)=4 n \cdot d_{0, s}(0, s)$, while (47) and (48) yield $3 n \cdot d_{0, s}(n, 0)=2 n \cdot d_{0, s}(0, s)$. So, taking into account (22) as well, we conclude that

$$
\begin{equation*}
d_{0, s}(n, 0)=0, \text { if } s \neq 0 \tag{50}
\end{equation*}
$$

Thus, (21), (23) and (25) reduce to

$$
\begin{align*}
2 n i \cdot d_{0, s}(n, i) & =n(i+s) d_{0, s}(0, i)  \tag{51}\\
2 n i \cdot d_{0, s}(0, i) & =n(i+s) d_{0, s}(-n, i)  \tag{52}\\
0 & =n(i+s) d_{0, s}(0, i)+n i \cdot d_{0, s}(n,-i) . \tag{53}
\end{align*}
$$

Replacing $n$ by $-n$ in (52) and combining it with (51), we come to $n(s-i)(s+3 i) d_{0, s}(0, i)=0$, whence $d_{0, s}(0, i)=0$ for $i \notin\left\{s,-\frac{s}{3}\right\}$. Consequently, $d_{0, s}(n, i)=0$, if $i \notin\left\{s,-\frac{s}{3}\right\}$ in view of (50) and (51). Finally, if $i \in\left\{s,-\frac{s}{3}\right\}$, then $-i \notin\left\{s,-\frac{s}{3}\right\}$, so $d_{0, s}(n,-i)=0$. Hence $d_{0, s}(0, i)=0$ from (53), and we again have $d_{0, s}(n, i)=0$ thanks to (51). Resuming, we have proved

$$
\begin{equation*}
d_{0, s}(n, i)=0, \text { if } s \neq 0 \tag{54}
\end{equation*}
$$

The result now follows from (36), (45) and (54).
Lemma 2.10. Let $\varphi$ be a $\frac{1}{2}$-derivation of $\mathcal{B}(0)$ satisfying (5). Then $d_{0,0}(m, i)=d_{0,0}\left(m^{\prime}, i^{\prime}\right)$ for all $(m, i),\left(m^{\prime}, i^{\prime}\right) \in \mathbb{Z} \times \mathbb{Z} \backslash\{(0,0)\}$.

Proof. Equality (20) written for $r=s=0$ becomes

$$
2(n i-m j) d_{0,0}(m+n, i+j)=(n i-m j)\left(d_{0,0}(m, i)+d_{0,0}(n, j)\right) .
$$

Hence,

$$
\begin{equation*}
2 d_{0,0}(m+n, i+j)=d_{0,0}(m, i)+d_{0,0}(n, j), \text { if } n i-m j \neq 0 \tag{55}
\end{equation*}
$$

In particular, taking $m=j=0$ in (55), we obtain

$$
\begin{equation*}
2 d_{0,0}(n, i)=d_{0,0}(n, 0)+d_{0,0}(0, i), \text { if } n i \neq 0 \tag{56}
\end{equation*}
$$

On the other hand, taking $m=-n$ and $j=0$ in (55), we get

$$
\begin{equation*}
2 d_{0,0}(0, i)=d_{0,0}(n, 0)+d_{0,0}(-n, i), \text { if } n i \neq 0 \tag{57}
\end{equation*}
$$

Similarly, the substitution $j=-i$ and $m=0$ into (55) leads to

$$
\begin{equation*}
2 d_{0,0}(n, 0)=d_{0,0}(0, i)+d_{0,0}(n,-i), \text { if } n i \neq 0 \tag{58}
\end{equation*}
$$

Now, combining (56) and (58) we get

$$
3 d_{0,0}(n, 0)=2 d_{0,0}(0, i)+d_{0,0}(0,-i), \text { if } n i \neq 0
$$

In particular, $d_{0,0}(n, 0)=d_{0,0}(-n, 0)$ for all $n$. Similarly, (56) and (57) yield

$$
3 d_{0,0}(0, i)=2 d_{0,0}(n, 0)+d_{0,0}(-n, 0)=3 d_{0,0}(n, 0)
$$

i.e.

$$
\begin{equation*}
d_{0,0}(n, 0)=d_{0,0}(0, i), \text { if } n i \neq 0 \tag{59}
\end{equation*}
$$

Then (56) and (59) imply

$$
d_{0,0}(n, i)=d_{0,0}(n, 0)=d_{0,0}(0, i)
$$

which is clearly a constant not depending on $(n, i)$ such that $n i \neq 0$.
Proposition 2.11. We have $\Delta(\mathcal{B}(0))=\langle\mathrm{id}, \alpha\rangle$, where $\alpha$ is as in Lemma 2.5.
Proof. It follows from Lemmas 2.9 and 2.10 that $\Delta(\mathcal{B}(0)) \subseteq\langle i d, \alpha\rangle$. Conversely, any element of $\langle\mathrm{id}, \alpha\rangle$ is a $\frac{1}{2}$-derivation of $\mathcal{B}(0)$ by Lemma 2.5.

We may thus join Propositions 2.8 and 2.11 to get the following theorem.
Theorem 2.12. For all $q \in \mathbb{C}$ we have

$$
\Delta(\mathcal{B}(q))= \begin{cases}\langle\mathrm{id}\rangle, & q \notin \mathbb{Z} \\ \langle\mathrm{id}, \alpha\rangle, & q \in \mathbb{Z}\end{cases}
$$

Filippov proved that each nonzero $\delta$-derivation $(\delta \neq 0,1)$ of a Lie algebra, gives a non-trivial Hom-Lie algebra structure [10, Theorem 1]. Hence, by Theorem [2.12, we have the following corollary.

Corollary 2.13. The Lie algebra $\mathcal{B}(q)_{q \in \mathbb{Z}}$ admits a non-trivial Hom-Lie algebra structure.
2.2. Transposed Poisson structures on $\mathcal{B}(q)$. Using Theorem 2.12 we can describe all the transposed Poisson structures on $(\mathcal{B}(q),[\cdot, \cdot])$.

Theorem 2.14. If $q \notin \mathbb{Z}$, then all the transposed Poisson structures on $(\mathcal{B}(q),[\cdot, \cdot])$ are trivial. If $q \in \mathbb{Z}$, then, up to an isomorphism, there is only one non-trivial transposed Poisson structure - on $(\mathcal{B}(q),[\cdot, \cdot])$ given by

$$
\begin{equation*}
L_{0,-2 q} \cdot L_{0,-2 q}=L_{0,-q} . \tag{60}
\end{equation*}
$$

Proof. Let $(\mathcal{B}(q), \cdot,[\cdot, \cdot])$ be a transposed Poisson algebra, so that $(\mathcal{B}(q), \cdot)$ is commutative and (1) holds. For any $(m, i) \in \mathbb{Z} \times \mathbb{Z}$ denote by $\varphi^{m, i}$ the left multiplication by $L_{m, i}$ in ( $\mathcal{B}(q), \cdot)$, so that

$$
\begin{equation*}
L_{m, i} \cdot L_{n, j}=\varphi^{m, i}\left(L_{n, j}\right) \tag{61}
\end{equation*}
$$

Since $(\mathcal{B}(q), \cdot)$ is commutative, we also have

$$
\begin{equation*}
L_{m, i} \cdot L_{n, j}=L_{n, j} \cdot L_{m, i}=\varphi^{n, j}\left(L_{m, i}\right) . \tag{62}
\end{equation*}
$$

By (1) we have $\varphi^{m, i} \in \Delta(\mathcal{B}(q))$. If $q \notin \mathbb{Z}$, then it follows from Theorem 2.12 that $\varphi^{m, i}=a^{m, i} \mathrm{id}$ for some $a^{m, i} \in \mathbb{C}$. It is then an immediate consequence of (61) and (62) that $a^{m, i}=0$ for all $(m, i) \in \mathbb{Z} \times \mathbb{Z}$. Thus, the product • is trivial for $q \notin \mathbb{Z}$.

So, we will focus on the case $q \in \mathbb{Z}$. Write, using Theorem 2.12,

$$
\begin{equation*}
\varphi^{m, i}=a^{m, i} \mathrm{id}+b^{m, i} \alpha \tag{63}
\end{equation*}
$$

where $a^{m, i}, b^{m, i} \in \mathbb{C}$ and $\alpha$ is given by (15). On the one hand, by (15), (61) and (63),

$$
L_{m, i} \cdot L_{n, j}=a^{m, i} L_{n, j}+b^{m, i} \alpha\left(L_{n, j}\right)= \begin{cases}a^{m, i} L_{n, j}, & (n, j) \neq(0,-2 q)  \tag{64}\\ a^{m, i} L_{0,-2 q}+b^{m, i} L_{0,-q}, & (n, j)=(0,-2 q)\end{cases}
$$

On the other hand, by (15), (62) and (63),

$$
L_{m, i} \cdot L_{n, j}=a^{n, j} L_{m, i}+b^{n, j} \alpha\left(L_{m, i}\right)= \begin{cases}a^{n, j} L_{m, i}, & (m, i) \neq(0,-2 q)  \tag{65}\\ a^{n, j} L_{0,-2 q}+b^{n, j} L_{0,-q}, & (m, i)=(0,-2 q)\end{cases}
$$

Case 1. $(m, i),(n, j) \neq(0,-2 q)$. Then $a^{m, i} L_{n, j}=a^{n, j} L_{m, i}$ by (64) and (65), so taking $(m, i) \neq(n, j)$ we conclude that $a^{m, i}=a^{n, j}=0$. Thus, $L_{m, i} \cdot L_{n, j}=0$.

Case 2. $(m, i)=(0,-2 q),(n, j) \neq(0,-2 q)$. Then $a^{0,-2 q} L_{n, j}=a^{n, j} L_{0,-2 q}+b^{n, j} L_{0,-q}$ by (64) and (65). Choosing $(n, j) \neq(0,-q)$, we obtain $a^{0,-2 q}=0$, so $L_{m, i} \cdot L_{n, j}=0$.

Case 3. $(m, i) \neq(0,-2 q),(n, j)=(0,-2 q)$. This case is symmetric to Case 2, so again $L_{m, i} \cdot L_{n, j}=0$.

Case 4. $(m, i)=(n, j)=(0,-2 q)$. Then $L_{m, i} \cdot L_{n, j}=a^{0,-2 q} L_{0,-2 q}+b^{0,-2 q} L_{0,-q}=b^{0,-2 q} L_{0,-q}$, because $a^{0,-2 q}=0$, as proved in Case 2 .

Thus, the product in $(\mathcal{B}(q), \cdot)$ is of the form

$$
\begin{equation*}
L_{0,-2 q} \cdot L_{0,-2 q}=c L_{0,-q} \tag{66}
\end{equation*}
$$

where $c \in \mathbb{C}$. Assume that $c \neq 0$ (otherwise the transposed Poisson structure is trivial). Observe that $L_{0,-q} \in Z(\mathcal{B}(q))$, where $Z(\mathcal{B}(q))$ is the center of the Lie algebra $(\mathcal{B}(q),[\cdot, \cdot])$. Indeed,

$$
\left[L_{m, i}, L_{0,-q}\right]=(0 \cdot(i+q)-m(-q+q)) L_{m+n, i+j}=0
$$

for all $(m, i) \in \mathbb{Z} \times \mathbb{Z}$. Hence the linear map $\phi$ such that $\phi\left(L_{m, i}\right)=L_{m, i}$ for $(m, i) \neq(0,-q)$ and $\phi\left(L_{0,-q}\right)=k L_{0,-q}$ is an automorphism of $(\mathcal{B}(q),[\cdot, \cdot])$ for any $k \in \mathbb{C}^{*}$. If $q \neq 0$, then taking $k=c^{-1}$, we obtain an isomorphic transposed Poisson structure $*$ on $(\mathcal{B}(q),[\cdot, \cdot])$ in which the only non-zero product is

$$
L_{0,-2 q} * L_{0,-2 q}=\phi\left(L_{0,-2 q}\right) * \phi\left(L_{0,-2 q}\right)=\phi\left(L_{0,-2 q} \cdot L_{0,-2 q}\right)=\phi\left(c L_{0,-q}\right)=c \cdot c^{-1} L_{0,-q}=L_{0,-q},
$$

so, up to an isomorphism, we may consider $c=1$ in (661). If $q=0$, then taking $k=c$, we obtain an isomorphic transposed Poisson structure $*$ on $(\mathcal{B}(q),[\cdot, \cdot])$ in which the only non-zero product is

$$
L_{0,0} * L_{0,0}=c^{-1} \phi\left(L_{0,0}\right) * c^{-1} \phi\left(L_{0,0}\right)=c^{-2} \phi\left(L_{0,0} \cdot L_{0,0}\right)=c^{-2} \phi\left(c L_{0,0}\right)=c^{-2} \cdot c^{2} L_{0,0}=L_{0,0},
$$

so again, up to an isomorphism, it suffices to take $c=1$ in (66).
Conversely, consider the commutative algebra structure on $\mathcal{B}(q)$ given by (60). It is clearly associative. To prove (1), observe that $\mathcal{B}(q) \cdot \mathcal{B}(q) \subseteq Z(\mathcal{B}(q))$. Hence, the right-hand side of (1) is always zero. In fact, the left-hand side of (1) is zero as well, because $[\mathcal{B}(q), \mathcal{B}(q)] \subseteq \operatorname{Ann}(\mathcal{B}(q))$, where $\operatorname{Ann}(\mathcal{B}(q))$ is the annihilator of $(\mathcal{B}(q), \cdot)$. For, assuming $\left[L_{m, i}, L_{n, j}\right] \in\left\langle L_{0,-2 q}\right\rangle$ we obtain from (4) that $m+n=0$ and $i+j=-2 q$. But then

$$
n(i+q)-m(j+q)=n(i+q)+n(-i-2 q+q)=n(i+q)-n(i+q)=0
$$

so $\left[L_{m, i}, L_{n, j}\right]=0$. Thus, $\left[L_{m, i}, L_{n, j}\right] \in \operatorname{Ann}(\mathcal{B}(q))$ for all $(m, i),(n, j) \in \mathbb{Z} \times \mathbb{Z}$, as needed.

## 3. Transposed Poisson structures on Block Lie superalgebras

We are going to study the Lie superalgebra $\mathcal{S}(q)$ whose definition comes from [24] with the only difference that $q$ will be an arbitrary complex number and the set of indices of the basis elements will be the whole $\mathbb{Z} \times \mathbb{Z}$. Thus, $\mathcal{S}(q)_{0}=\mathcal{B}(q)$ with basis $\left\{L_{m, i} \mid m, i \in \mathbb{Z}\right\}$ and multiplication given by (4). Now, $\mathcal{S}(q)_{1}$ is spanned by $\left\{G_{m, i} \mid m, i \in \mathbb{Z}\right\}$, where

$$
\begin{align*}
{\left[L_{m, i}, G_{n, j}\right] } & =\left(n(i+q)-m\left(j+\frac{q}{2}\right)\right) G_{m+n, i+j},  \tag{67}\\
{\left[G_{m, i}, G_{n, j}\right] } & =2 q L_{m+n, i+j} . \tag{68}
\end{align*}
$$

3.1. Even $\frac{1}{2}$-derivations of $\mathcal{S}(q)$. In this subsection we consider only even linear maps $\varphi$ : $\mathcal{S}(q) \rightarrow \mathcal{S}(q)$, i.e. those which satisfy $\varphi\left(\mathcal{S}(q)_{i}\right) \subseteq \mathcal{S}(q)_{i}$ for $i \in\{0,1\}$. We thus have $|\varphi|=0$, so $\varphi$ is a $\frac{1}{2}$-superderivation of $\mathcal{S}(q)$ if and only if $\varphi$ is a usual $\frac{1}{2}$-derivation of $\mathcal{S}(q)$. We now write

$$
\varphi=\sum_{r, s \in \mathbb{Z}} \varphi_{r, s}
$$

where

$$
\begin{array}{r}
\varphi_{r, s}\left(L_{m, i}\right)=d_{r, s}^{0}(m, i) L_{m+r, i+s} \\
\varphi_{r, s}\left(G_{m, i}\right)=d_{r, s}^{1}(m, i) G_{m+r, i+s} \tag{70}
\end{array}
$$

for some $d_{r, s}^{i}(m, i) \in \mathbb{C}, i=0,1$. Since

$$
\left[\left(\mathcal{S}(q)_{k}\right)_{m, i},\left(\mathcal{S}(q)_{l}\right)_{n, j}\right] \subseteq\left(\mathcal{S}(q)_{k+l}\right)_{m+n, i+j}
$$

for all $k, l \in \mathbb{Z}_{2}$ and $m, n, i, j \in \mathbb{Z}$ by (67) and (68), then $\varphi \in \Delta^{0}(\mathcal{S}(q))$ if and only if $\varphi_{r, s} \in$ $\Delta^{0}(\mathcal{S}(q))$ for all $r, s \in \mathbb{Z}$.

Lemma 3.1. Let $\varphi_{r, s}: \mathcal{S}(q) \rightarrow \mathcal{S}(q)$ be a linear map satisfying (69) and (70). Then $\varphi_{r, s} \in$ $\Delta^{0}(\mathcal{S}(q))$ if and only if $\left.\varphi_{r, s}\right|_{\mathcal{S}(q)_{0}} \in \Delta(\mathcal{B}(q))$ and

$$
\begin{align*}
2\left(n(i+q)-m\left(j+\frac{q}{2}\right)\right) d_{r, s}^{1}(m+n, i+j)= & \left(n(i+s+q)-(m+r)\left(j+\frac{q}{2}\right)\right) d_{r, s}^{0}(m, i) \\
& +\left((n+r)(i+q)-m\left(j+s+\frac{q}{2}\right)\right) d_{r, s}^{1}(n, j) \tag{71}
\end{align*}
$$

$$
\begin{equation*}
2 q d_{r, s}^{0}(m+n, i+j)=q\left(d_{r, s}^{1}(m, i)+d_{r, s}^{1}(n, j)\right) \tag{72}
\end{equation*}
$$

Proof. By (67) we see that $\varphi_{r, s} \in \Delta^{0}(\mathcal{S}(q))$ if and only if

$$
\begin{align*}
2\left(n(i+q)-m\left(j+\frac{q}{2}\right)\right) \varphi_{r, s}\left(G_{m+n, i+j}\right) & =\left[\varphi_{r, s}\left(L_{m, i}\right), G_{n, j}\right]+\left[L_{m, i}, \varphi_{r, s}\left(G_{n, j}\right)\right]  \tag{73}\\
4 q \varphi_{r, s}\left(L_{m+n, i+j}\right) & =\left[\varphi_{r, s}\left(G_{m, i}\right), G_{n, j}\right]+\left[G_{m, i}, \varphi_{r, s}\left(G_{n, j}\right)\right] . \tag{74}
\end{align*}
$$

In view of (69) and (70) the left-hand side of (73) equals

$$
2\left(n(i+q)-m\left(j+\frac{q}{2}\right)\right) d_{r, s}^{1}(m+n, i+j) G_{m+n+r, i+j+s}
$$

while the right-hand side of (73) is

$$
\begin{aligned}
& {\left[d_{r, s}^{0}(m, i) L_{m+r, i+s}, G_{n, j}\right]+\left[L_{m, i}, d_{r, s}^{1}(n, j) G_{n+r, j+s}\right]} \\
& \quad=\left(n(i+s+q)-(m+r)\left(j+\frac{q}{2}\right)\right) d_{r, s}^{0}(m, i) G_{m+n+r, i+j+s} \\
& \quad+\left((n+r)(i+q)-m\left(j+s+\frac{q}{2}\right)\right) d_{r, s}^{1}(n, j) G_{m+n+r, i+j+s}
\end{aligned}
$$

Thus, we come to (71). Now, the left-hand side of (74) is

$$
4 q d_{r, s}^{0}(m+n, i+j) L_{m+n+r, i+j+s}
$$

while the right-hand side of (74) equals

$$
\begin{aligned}
& {\left[d_{r, s}^{1}(m, i) G_{m+r, i+s}, G_{n, j}\right]+\left[G_{m, i}, d_{r, s}^{1}(n, j) G_{n+r, j+s}\right]} \\
& \quad=2 q d_{r, s}^{1}(m, i) L_{m+n+r, i+j+s}+2 q d_{r, s}^{1}(n, j) L_{m+n+r, i+j+s}
\end{aligned}
$$

whence (72).

We are going to specify the result of Lemma 3.1 taking into account the description of $\Delta(\mathcal{B}(q))$ from Theorem 2.12.

Lemma 3.2. Let $\varphi_{r, s}: \mathcal{S}(q) \rightarrow \mathcal{S}(q)$ be a linear map satisfying (69) and (70) and such that $\left.\varphi_{r, s}\right|_{\mathcal{S}(q)_{0}} \in \Delta(\mathcal{B}(q))$.

If $q \notin \mathbb{Z}$ or $q \in \mathbb{Z}$ and $(r, s) \notin\{(0,0),(0, q)\}$, then $\varphi_{r, s} \in \Delta^{0}(\mathcal{S}(q))$ if and only if

$$
\begin{align*}
2\left(n(i+q)-m\left(j+\frac{q}{2}\right)\right) d_{r, s}^{1}(m+n, i+j) & =\left((n+r)(i+q)-m\left(j+s+\frac{q}{2}\right)\right) d_{r, s}^{1}(n, j)  \tag{75}\\
q\left(d_{r, s}^{1}(m, i)+d_{r, s}^{1}(n, j)\right) & =0 \tag{76}
\end{align*}
$$

If $q \in \mathbb{Z} \backslash\{0\}$, then $\varphi_{0,0} \in \Delta^{0}(\mathcal{S}(q))$ if and only if

$$
\begin{gather*}
\left(n(i+q)-m\left(j+\frac{q}{2}\right)\right)\left(2 d_{0,0}^{1}(m+n, i+j)-d_{0,0}^{1}(n, j)-d_{0,0}^{0}(0,0)\right)=0,  \tag{77}\\
d_{0,0}^{1}(m, i)+d_{0,0}^{1}(n, j)=2 d_{0,0}^{0}(0,0) . \tag{78}
\end{gather*}
$$

If $q \in \mathbb{Z} \backslash\{0\}$, then $\varphi_{0, q} \in \Delta^{0}(\mathcal{S}(q))$ if and only if

$$
\begin{align*}
& 2\left(n(i+q)-m\left(j+\frac{q}{2}\right)\right) d_{0, q}^{1}(m+n, i+j)=\left(n(i+q)-m\left(j+\frac{3 q}{2}\right)\right) d_{0, q}^{1}(n, j) \\
& \text { for }(m, i) \neq(0,-2 q), \\
& 2 n d_{0, q}^{1}(n, j-2 q)= n d_{0, q}^{1}(n, j), \\
& d_{0, q}^{1}(m, i)+d_{0, q}^{1}(n, j)=0 \text { for }(m+n, i+j) \neq(0,-2 q),  \tag{79}\\
& d_{0, q}^{1}(-n,-j-2 q)+d_{0, q}^{1}(n, j)=2 d_{0, q}^{0}(0,-2 q), \tag{80}
\end{align*}
$$

If $q=0$, then $\varphi_{0,0} \in \Delta^{0}(\mathcal{S}(q))$ if and only if

$$
\begin{equation*}
(n i-m j)\left(2 d_{0,0}^{1}(m+n, i+j)-d_{0,0}^{0}(m, i)-d_{0,0}^{1}(n, j)\right)=0, \tag{81}
\end{equation*}
$$

where $d_{0,0}^{0}(m, i)=d_{0,0}^{0}\left(m^{\prime}, i^{\prime}\right)$ for all $(m, i),\left(m^{\prime}, i^{\prime}\right) \in \mathbb{Z} \times \mathbb{Z} \backslash\{(0,0)\}$.
Now we will consider each case from Lemma 3.2 separately.
Lemma 3.3. Let $\varphi$ be a $\frac{1}{2}$-derivation of $\mathcal{S}(q)$. If $q \notin \mathbb{Z}$ and $(r, s) \neq(0,0)$, or $q \in \mathbb{Z}$ and $(r, s) \notin\{(0,0),(0, q)\}$, then $\varphi_{r, s}=0$.
Proof. The map $\varphi_{r, s}$ acts as in (69) and (70), and we know by Theorem (2.12 that $d_{r, s}^{0}=0$, so it remains to prove that $d_{r, s}^{1}=0$.

Case 1. $q \neq 0$. Then $d_{r, s}^{1}(m, i)+d_{r, s}^{1}(n, j)=0$ by (76). Taking $n=m$ and $j=i$, we get $d_{r, s}^{1}(m, i)=0$ for all $(m, i) \in \mathbb{Z} \times \mathbb{Z}$.

Case 2. $q=0$. Then

$$
\begin{equation*}
2(n i-m j) d_{r, s}^{1}(m+n, i+j)=((n+r) i-m(j+s)) d_{r, s}^{1}(n, j) \tag{82}
\end{equation*}
$$

by (75). Taking $m=j=0$ and $i \neq 0$ in (82), we get

$$
\begin{equation*}
2 n d_{r, s}^{1}(n, i)=(n+r) d_{r, s}^{1}(n, 0) \tag{83}
\end{equation*}
$$

In particular, $d_{r, s}^{1}(n, i)$ does not depend on $i \neq 0$. On the other hand, $m=0$ and $j=-i \neq 0$ in (82) yield

$$
\begin{equation*}
2 n d_{r, s}^{1}(n, 0)=(n+r) d_{r, s}^{1}(n,-i)=(n+r) d_{r, s}^{1}(n, i) \tag{84}
\end{equation*}
$$

Case 2.1. $n \notin\left\{r,-\frac{r}{3}\right\}$. Then

$$
\left|\begin{array}{cc}
2 n & -(n+r) \\
n+r & -2 n
\end{array}\right|=(n+r)^{2}-4 n^{2}=(r-n)(r+3 n) \neq 0
$$

Hence, the linear system (83) and (84) has only the trivial solution $d_{r, s}^{1}(n, i)=d_{r, s}^{1}(n, 0)=0$.
Case 2.2. $n=r$. It follows from (82) that

$$
\begin{equation*}
2(r i-m j) d_{r, s}^{1}(m+r, i+j)=(2 r i-m(j+s)) d_{r, s}^{1}(r, j) . \tag{85}
\end{equation*}
$$

If $r \neq 0$, then choosing $m \notin\left\{0,-\frac{4 r}{3}\right\}, i \neq \frac{m(j+s)}{2 r}$ in (85) and using the result of Case 2.1, we get $d_{r, s}^{1}(r, j)=0$. If $r=0$, then choosing $m \neq 0$ in (85) we similarly get $d_{0, s}^{1}(0, j)=0$ whenever $j \neq-s$. Finally, choosing $m=-n \neq 0$ and $j=-i-s$ in (82) we obtain $s d_{0, s}^{1}(0,-s)=0$. Since $(r, s) \neq(0,0)$ by assumption, then $s \neq 0$, so $d_{0, s}^{1}(0,-s)=0$.

Case 2.3. $n=-\frac{r}{3}$. It follows from (82) that

$$
\begin{equation*}
-2\left(\frac{r i}{3}+m j\right) d_{r, s}^{1}\left(m-\frac{r}{3}, i+j\right)=\left(\frac{2 r i}{3}-m(j+s)\right) d_{r, s}^{1}\left(-\frac{r}{3}, j\right) \tag{86}
\end{equation*}
$$

We may assume that $r \neq 0$, since $r=0$ was considered in Case 2.2. Then choosing $m \notin\left\{0, \frac{4 r}{3}\right\}$, $i \neq \frac{3 m(j+s)}{2 r}$ in (86) and using the result of Case 2.1, we get $d_{r, s}^{1}\left(-\frac{r}{3}, j\right)=0$.
Lemma 3.4. Let $\varphi$ be a $\frac{1}{2}$-derivation of $\mathcal{S}(q)$. If $q \in \mathbb{Z} \backslash\{0\}$, then $\varphi_{0, q}=0$.
Proof. Taking $(m, i)=(n, j) \neq(0,-q)$ in (79), we obtain $d_{0, q}^{1}(m, i)=0$ for $(m, i) \neq(0,-q)$. In particular, $d_{0, q}^{1}(0,0)=0$. But then substituting $(m, i)=(0,-q)$ and $(n, j)=(0,0)$ in the same (79), we come to $d_{0, q}^{1}(0,-q)=0$. Thus, $d_{0, q}^{1}=0$, which implies $d_{0, q}^{0}(0,-2 q)=0$ by (80). Since we already know by Proposition 2.8 that $d_{0, q}^{0}(m, i)=0$ for all $(m, i) \neq(0,-2 q)$, then $d_{0, q}^{0}=0$.
Lemma 3.5. Let $\varphi$ be a $\frac{1}{2}$-derivation of $\mathcal{S}(q)$. If $q \neq 0$, then $d_{0,0}^{1}(m, i)=d_{0,0}^{0}(0,0)$ for all $m, i \in \mathbb{Z}$.

Proof. The substitution $m=i=n=j=0$ in (78) gives $d_{0,0}^{1}(0,0)=d_{0,0}^{0}(0,0)$. Now, substituting only $n=j=0$ in (78), we obtain $d_{0,0}^{1}(m, i)=2 d_{0,0}^{0}(0,0)-d_{0,0}^{1}(0,0)=d_{0,0}^{0}(0,0)$. Observe that (77) is then trivially satisfied.
Lemma 3.6. Let $\varphi$ be a $\frac{1}{2}$-derivation of $\mathcal{S}(0)$. Then $d_{0,0}^{1}(m, i)=d_{0,0}^{0}\left(m^{\prime}, i^{\prime}\right)$ for all $(m, i),\left(m^{\prime}, i^{\prime}\right) \in$ $\mathbb{Z} \times \mathbb{Z} \backslash\{(0,0)\}$.

Proof. Substituting $m=j=0$ and $n, i \neq 0$ in (81) we get

$$
\begin{equation*}
2 d_{0,0}^{1}(n, i)-d_{0,0}^{0}(0, i)-d_{0,0}^{1}(n, 0)=0, \text { if } n i \neq 0 \tag{87}
\end{equation*}
$$

On the other hand, the substitution $n=i=0$ and $m, j \neq 0$ in (81) gives

$$
\begin{equation*}
2 d_{0,0}^{1}(m, j)-d_{0,0}^{0}(m, 0)-d_{0,0}^{1}(0, j)=0, \text { if } m j \neq 0 \tag{88}
\end{equation*}
$$

Since $d_{0,0}^{0}(0, i)=d_{0,0}^{0}(m, 0)$ for all $m, i \neq 0$ by Lemma 2.10, it follows from (87) and (88) (with $m=n$ and $i=j$ ) that

$$
\begin{equation*}
d_{0,0}^{1}(n, 0)=d_{0,0}^{1}(0, i), \text { if } n i \neq 0 \tag{89}
\end{equation*}
$$

Now take $m=-n \neq 0, i \neq 0$ and $j=0$ in (81):

$$
2 d_{0,0}^{1}(0, i)-d_{0,0}^{0}(-n, i)-d_{0,0}^{1}(n, 0)=0 .
$$

In view of (89) we conclude that

$$
\begin{equation*}
d_{0,0}^{1}(0, i)=d_{0,0}^{1}(n, 0)=d_{0,0}^{0}(-n, i) . \tag{90}
\end{equation*}
$$

Since $d_{0,0}^{0}(-n, i)=d_{0,0}^{0}(0, i)$ by Lemma 2.10, it follows from (87) and (90) that $d_{0,0}^{1}(n, i)=$ $d_{0,0}^{0}(0, i)$ for all $n, i \neq 0$. Combining this with (89) and (90), we obtain the desired result.

Lemma 3.7. The linear maps $\alpha, \beta: \mathcal{S}(0) \rightarrow \mathcal{S}(0)$ such that

$$
\begin{align*}
& \alpha\left(L_{m, i}\right)=\left\{\begin{array}{ll}
0, & (m, i) \neq(0,0), \\
L_{0,0}, & (m, i)=(0,0),
\end{array} \alpha\left(G_{m, i}\right)=0,\right.  \tag{91}\\
& \beta\left(G_{m, i}\right)=\left\{\begin{array}{ll}
0, & (m, i) \neq(0,0), \\
G_{0,0}, & (m, i)=(0,0),
\end{array} \quad \beta\left(L_{m, i}\right)=0\right. \tag{92}
\end{align*}
$$

are $\frac{1}{2}$-derivations of $\mathcal{S}(0)$.
Proof. Let us first prove that $\alpha \in \Delta^{0}(\mathcal{S}(0))$. We have $\alpha=\alpha_{0,0}$, where

$$
d_{0,0}^{0}(m, i)= \begin{cases}0, & (m, i) \neq(0,0)  \tag{93}\\ 1, & (m, i)=(0,0)\end{cases}
$$

and $d_{0,0}^{1}=0$. By Lemma 2.5 we know that $\left.\varphi_{0,0}\right|_{\mathcal{S}()_{0}} \in \Delta(\mathcal{B}(0))$. It remains to check (81) which reduces to $(n i-m j) d_{0,0}^{0}(m, i)=0$. The latter is trivial by (93)).

Now we will prove that $\beta \in \Delta^{0}(\mathcal{S}(q))$. Again, $\beta=\beta_{0,0}$, but now

$$
d_{0,0}^{1}(m, i)= \begin{cases}0, & (m, i) \neq(0,0)  \tag{94}\\ 1, & (m, i)=(0,0)\end{cases}
$$

and $d_{0,0}^{0}=0$. The equality (81) that we need to verify takes the form

$$
\begin{equation*}
(n i-m j)\left(2 d_{0,0}^{1}(m+n, i+j)-d_{0,0}^{1}(n, j)\right)=0 . \tag{95}
\end{equation*}
$$

Consider the following cases.

Case 1. $(n, j),(m+n, i+j) \neq(0,0)$. Then both sides of (95) are zero because $d_{0,0}^{1}(m+n, i+$ $j)=d_{0,0}^{1}(n, j)=0$ by (94).

Case 2. $(n, j)=(0,0)$ or $(m+n, i+j)=(0,0)$. Then $n i-m j=0$, so again both sides of (95) are zero.

Proposition 3.8. For all $q \in \mathbb{C}$ we have

$$
\Delta^{0}(\mathcal{S}(q))= \begin{cases}\langle\mathrm{id}\rangle, & q \neq 0 \\ \langle\mathrm{id}, \alpha, \beta\rangle, & q=0\end{cases}
$$

Proof. This follows from Lemmas 3.3 3.7,
3.2. Odd $\frac{1}{2}$-derivations of $\mathcal{S}(q)$. Recall that a linear map $\varphi: \mathcal{S}(q) \rightarrow \mathcal{S}(q)$ is odd, if $\varphi\left(\mathcal{S}(q)_{i}\right) \subseteq \mathcal{S}(q)_{1-i}$ for $i \in\{0,1\}$. In this case $|\varphi|=1$, and $\varphi$ is a $\frac{1}{2}$-superderivation of $\mathcal{S}(q)$ if and only if

$$
\begin{align*}
& \varphi([x, y])=\frac{1}{2}([\varphi(x), y]+[x, \varphi(y)]), \quad x \in \mathcal{S}(q)_{0}  \tag{96}\\
& \varphi([x, y])=\frac{1}{2}([\varphi(x), y]-[x, \varphi(y)]), x \in \mathcal{S}(q)_{1} \tag{97}
\end{align*}
$$

Denote by $\Delta^{1}(\mathcal{S}(q))$ the space of odd $\frac{1}{2}$-superderivations of $\mathcal{S}(q)$. As usual, for any $\varphi \in$ $\Delta^{1}(\mathcal{S}(q))$, we write

$$
\varphi=\sum_{r, s \in \mathbb{Z}} \varphi_{r, s}
$$

where

$$
\begin{array}{r}
\varphi_{r, s}\left(L_{m, i}\right)=d_{r, s}^{0}(m, i) G_{m+r, i+s} \\
\varphi_{r, s}\left(G_{m, i}\right)=d_{r, s}^{1}(m, i) L_{m+r, i+s} \tag{99}
\end{array}
$$

for some $d_{r, s}^{i}(m, i) \in \mathbb{C}, i=0,1$. We have $\varphi \in \Delta^{1}(\mathcal{S}(q))$ if and only if $\varphi_{r, s} \in \Delta^{1}(\mathcal{S}(q))$ for all $r, s \in \mathbb{Z}$.

Lemma 3.9. Let $\varphi_{r, s}: \mathcal{S}(q) \rightarrow \mathcal{S}(q)$ be a linear map satisfying (98) and (99). Then $\varphi_{r, s} \in$ $\Delta^{0}(\mathcal{S}(q))$ if and only if

$$
\begin{align*}
2(n(i+q)-m(j+q)) d_{r, s}^{0}(m+n, i+j)= & d_{r, s}^{0}(m, i)\left(n\left(i+s+\frac{q}{2}\right)-(m+r)(j+q)\right) \\
& +d_{r, s}^{0}(n, j)\left((n+r)(i+q)-m\left(j+s+\frac{q}{2}\right)\right), \tag{100}
\end{align*}
$$

$$
\begin{align*}
2\left(n(i+q)-m\left(j+\frac{q}{2}\right)\right) d_{r, s}^{1}(m+n, i+j)= & 2 q \cdot d_{r, s}^{0}(m, i) \\
& +((n+r)(i+q)-m(j+s+q)) d_{r, s}^{1}(n, j), \tag{101}
\end{align*}
$$

$$
\begin{align*}
4 q \cdot d_{r, s}^{0}(m+n, i+j)= & \left(n(i+s+q)-(m+r)\left(j+\frac{q}{2}\right)\right) d_{r, s}^{1}(m, i) \\
& +\left(m(j+s+q)-(n+r)\left(i+\frac{q}{2}\right)\right) d_{r, s}^{1}(n, j) \tag{102}
\end{align*}
$$

Proof. Fix $r, s \in \mathbb{Z}$. Writing (96) for $\varphi=\varphi_{r, s}$ with $x=L_{m, i}$ and $y=L_{n, j}$, we have by (98)

$$
\begin{aligned}
2 \varphi_{r, s}\left(\left[L_{m, i}, L_{n, j}\right]\right) & =\left[\varphi_{r, s}\left(L_{m, i}\right), L_{n, j}\right]+\left[L_{m, i}, \varphi_{r, s}\left(L_{n, j}\right)\right] \\
& =d_{r, s}^{0}(m, i)\left[G_{m+r, i+s}, L_{n, j}\right]+d_{r, s}^{0}(n, j)\left[L_{m, i}, G_{n+r, j+s}\right]
\end{aligned}
$$

Then thanks to (44), (67) and (98) we get (100). Similarly, $\varphi=\varphi_{r, s}, x=L_{m, i}$ and $y=G_{n, j}$ in (96) result in

$$
\begin{aligned}
2 \varphi_{r, s}\left(\left[L_{m, i}, G_{n, j}\right]\right) & =\left[\varphi_{r, s}\left(L_{m, i}\right), G_{n, j}\right]+\left[L_{m, i}, \varphi_{r, s}\left(G_{n, j}\right)\right] \\
& =d_{r, s}^{0}(m, i)\left[G_{m+r, i+s}, G_{n, j}\right]+d_{r, s}^{1}(n, j)\left[L_{m, i}, L_{n+r, j+s}\right]
\end{aligned}
$$

So, using (4), (67) and (98) we come to (101). Now take $\varphi=\varphi_{r, s}, x=G_{m, i}$ and $y=G_{n, j}$ in (97) and use (99):

$$
\begin{aligned}
2 \varphi_{r, s}\left(\left[G_{m, i}, G_{n, j}\right]\right) & =\left[\varphi_{r, s}\left(G_{m, i}\right), G_{n, j}\right]-\left[G_{m, i}, \varphi_{r, s}\left(G_{n, j}\right)\right] \\
& =d_{r, s}^{1}(m, i)\left[L_{m+r, i+s}, G_{n, j}\right]-d_{r, s}^{1}(n, j)\left[G_{m, i}, L_{n+r, j+s}\right]
\end{aligned}
$$

Applying (67), (68) and (98), we arrive at (102).
Let $\varphi_{r, s} \in \Delta^{1}(\mathcal{S}(q))$ and $\psi_{m, i}$ be the left multiplication by $G_{m, i}$ in $\mathcal{S}(q)$. Then $\psi_{m, i}$ is an odd superderivation of $\mathcal{S}(q)$, so that the supercommutator $\left[\varphi_{r, s}, \psi_{m, i}\right]=\varphi_{r, s} \circ \psi_{m, i}+\psi_{m, i} \circ \varphi_{r, s}$ is an even $\frac{1}{2}$-superderivation of $\mathcal{S}(q)$, whose description was given in Proposition 3.8. So, on the one hand,

$$
\begin{align*}
& {\left[\varphi_{r, s}, \psi_{m, i}\right]\left(L_{n, j}\right)=\varphi_{r, s}\left(\left[G_{m, i}, L_{n, j}\right]\right)+\left[G_{m, i}, \varphi_{r, s}\left(L_{n, j}\right)\right]} \\
& =\left(n\left(i+\frac{q}{2}\right)-m(j+q)\right) \varphi_{r, s}\left(G_{m+n, i+j}\right)+\left[G_{m, i}, d_{r, s}^{0}(n, j) G_{n+r, j+s}\right] \\
& =\left(n\left(i+\frac{q}{2}\right)-m(j+q)\right) d_{r, s}^{1}(m+n, i+j) L_{m+n+r, i+j+s}+2 q \cdot d_{r, s}^{0}(n, j) L_{m+n+r, i+j+s} \\
& =\left(\left(n\left(i+\frac{q}{2}\right)-m(j+q)\right) d_{r, s}^{1}(m+n, i+j)+2 q \cdot d_{r, s}^{0}(n, j)\right) L_{m+n+r, i+j+s},  \tag{103}\\
& {\left[\varphi_{r, s}, \psi_{m, i}\right]\left(G_{n, j}\right)=\varphi_{r, s}\left(\left[G_{m, i}, G_{n, j}\right]\right)+\left[G_{m, i}, \varphi_{r, s}\left(G_{n, j}\right)\right]} \\
& =2 q \cdot \varphi_{r, s}\left(L_{m+n, i+j}\right)+\left[G_{m, i}, d_{r, s}^{1}(n, j) L_{n+r, j+s}\right] \\
& =2 q \cdot d_{r, s}^{0}(m+n, i+j) G_{m+n+r, i+j+s}-\left(m(j+s+q)-(n+r)\left(i+\frac{q}{2}\right)\right) d_{r, s}^{1}(n, j) G_{m+n+r, i+j+s} \\
& =\left(2 q \cdot d_{r, s}^{0}(m+n, i+j)-\left(m(j+s+q)-(n+r)\left(i+\frac{q}{2}\right)\right) d_{r, s}^{1}(n, j)\right) G_{m+n+r, i+j+s} . \tag{104}
\end{align*}
$$

And on the other hand, if $q \neq 0$, then

$$
\begin{align*}
{\left[\varphi_{r, s}, \psi_{m, i}\right]\left(L_{n, j}\right) } & = \begin{cases}0, & (r+m, s+i) \neq(0,0) \\
c L_{n, j}, & (r+m, s+i)=(0,0)\end{cases}  \tag{105}\\
{\left[\varphi_{r, s}, \psi_{m, i}\right]\left(G_{n, j}\right) } & = \begin{cases}0, & (r+m, s+i) \neq(0,0), \\
c G_{n, j}, & (r+m, s+i)=(0,0)\end{cases} \tag{106}
\end{align*}
$$

for some constant $c \in \mathbb{C}$. And if $q=0$, then

$$
\begin{align*}
& {\left[\varphi_{r, s}, \psi_{m, i}\right]\left(L_{n, j}\right)= \begin{cases}0, & (r+m, s+i) \neq(0,0), \\
c_{1} L_{n, j}, & (r+m, s+i)=(0,0) \text { and }(n, j) \neq(0,0), \\
c_{2} L_{n, j}, & (r+m, s+i)=(0,0) \text { and }(n, j)=(0,0),\end{cases} }  \tag{107}\\
& {\left[\varphi_{r, s}, \psi_{m, i}\right]\left(G_{n, j}\right)= \begin{cases}0, & (r+m, s+i) \neq(0,0), \\
c_{1} G_{n, j}, & (r+m, s+i)=(0,0) \text { and }(n, j) \neq(0,0), \\
c_{3} G_{n, j}, & (r+m, s+i)=(0,0) \text { and }(n, j)=(0,0),\end{cases} } \tag{108}
\end{align*}
$$

for some constants $c_{1}, c_{2}, c_{3} \in \mathbb{C}$.
Lemma 3.10. Let $q \neq 0$ and $\varphi \in \Delta^{1}(\mathcal{S}(q))$. If $q \notin 2 \mathbb{Z}$, then $\varphi_{r, s}=0$. Otherwise, $\varphi_{r, s}=0$ for all $(r, s) \neq\left(0, \frac{q}{2}\right)$ and $\varphi_{0, \frac{q}{2}}\left(L_{m, i}\right)=0$ for all $m, i \in \mathbb{Z}, \varphi_{0, \frac{q}{2}}\left(G_{m, i}\right)=0$ for all $(m, i) \neq\left(0,-\frac{3 q}{2}\right)$.
Proof. By (103) (106) we have for all $(m, i) \neq(-r,-s)$

$$
\begin{array}{r}
\left(n\left(i+\frac{q}{2}\right)-m(j+q)\right) d_{r, s}^{1}(m+n, i+j)+2 q \cdot d_{r, s}^{0}(n, j)=0 \\
2 q \cdot d_{r, s}^{0}(m+n, i+j)-\left(m(j+s+q)-(n+r)\left(i+\frac{q}{2}\right)\right) d_{r, s}^{1}(n, j)=0 \tag{110}
\end{array}
$$

Taking $m=n=0$ and $i \neq-s$ in (109), we obtain

$$
d_{r, s}^{0}(0, j)=0 \text { for all }(r, s) \text { and for all } j .
$$

Then $n=0, m \neq 0$ and $j \neq-q$ in (109) give $d_{r, s}^{1}(m, i+j)=0$. Since any $k \in \mathbb{Z}$ can be written as $i+j$ with $i \neq-s$ and $j \neq-q$ (by choosing $j \notin\{-q, k+s\}$ ), we obtain

$$
d_{r, s}^{1}(m, k)=0 \text { for all }(r, s) \text { and for all }(m, k) \text { with } m \neq 0 .
$$

Hence, $m=0, n \neq 0$ and $i \neq-s$ in (109) yield

$$
d_{r, s}^{0}(n, j)=0 \text { for all }(r, s) \text { and for all }(n, j) \text { with } n \neq 0 .
$$

Thus,

$$
d_{r, s}^{0}=0 \text { for all }(r, s) .
$$

Now substitute $m=-n \neq 0$ and $i \neq-s$ into (109) to get $\left(i+j+\frac{3 q}{2}\right) d_{r, s}^{1}(0, i+j)=0$, so

$$
d_{r, s}^{1}(0, k)=0 \text { for all }(r, s) \text { and for all } k \neq-\frac{3 q}{2} .
$$

Using (110) with $n=0$ and $j=-\frac{3 q}{2}$, we come to $\left(m\left(s-\frac{q}{2}\right)-r\left(i+\frac{q}{2}\right)\right) d_{r, s}^{1}\left(0,-\frac{3 q}{2}\right)=0$. If $r \neq 0$, then choosing $i \notin\left\{\frac{m}{r}\left(s-\frac{q}{2}\right)-\frac{q}{2},-s\right\}$ we conclude that $d_{r, s}^{1}\left(0,-\frac{3 q}{2}\right)=0$. Otherwise, choosing $m \neq 0$ (so that $m \neq-r$ ), we see that $d_{0, s}^{1}\left(0,-\frac{3 q}{2}\right)=0$ unless $s=\frac{q}{2}$.
Lemma 3.11. Let $\varphi \in \Delta^{1}(\mathcal{S}(0))$. Then $\varphi_{r, s}=0$ for all $(r, s) \neq(0,0), \varphi_{0,0}\left(G_{m, i}\right)=0$ for all $(m, i) \neq(0,0), \varphi_{0,0}\left(L_{m, i}\right)=c G_{m, i}$ for all $(m, i) \neq(0,0)$ and some constant $c \in \mathbb{C}$.
Proof. By (103), (104), (107) and (108) we have for all $(m, i) \neq(-r,-s)$

$$
\begin{align*}
(n i-m j) d_{r, s}^{1}(m+n, i+j) & =0  \tag{111}\\
(m(j+s)-i(n+r)) d_{r, s}^{1}(n, j) & =0 \tag{112}
\end{align*}
$$

If $j \neq-s$, then choosing $m \notin\left\{\frac{i(n+r)}{j+s},-r\right\}$ in (112) we get $d_{r, s}^{1}(n, j)=0$. And if $n \neq-r$, then choosing $i \notin\left\{\frac{m(j+s)}{n+r},-s\right\}$ in (112) we again obtain $d_{r, s}^{1}(n, j)=0$. Thus,

$$
d_{r, s}^{1}(n, j)=0 \text { for all }(n, j) \neq(-r,-s)
$$

Let $(r, s) \neq(0,0)$. Then take $m=-r-n$ and $j=-s-i$ in (111). Since $n i-m j=$ $n i-(r+n)(s+i)=-r s-n s-r i$, we have two cases. If $r \neq 0$, then choosing $i \notin\left\{-\frac{n s}{r}-s,-s\right\}$, we conclude that $d_{r, s}^{1}(-r,-s)=0$. Otherwise, choosing $n \neq 0$ (so that $m=-n \neq-r$ ), we again come to $d_{r, s}^{1}(-r,-s)=0$. Thus,

$$
d_{r, s}^{1}=0 \text { for all }(r, s) \neq(0,0) .
$$

Regarding $d_{r, s}^{0}$, we see that it satisfies (100) with $q=0$, which has exactly the same form as (20). Applying the proofs of Lemmas 2.9 and 2.10 to $d_{r, s}^{0}$, we conclude that

$$
\begin{aligned}
d_{r, s}^{0} & =0 \text { for all }(r, s) \neq(0,0) \\
d_{0,0}^{0}(m, i) & =d_{0,0}^{0}\left(m^{\prime}, i^{\prime}\right) \text { for all }(m, i),\left(m^{\prime}, i^{\prime}\right) \in \mathbb{Z} \times \mathbb{Z} \backslash\{(0,0)\}
\end{aligned}
$$

Lemma 3.12. Let $q \in 2 \mathbb{Z}$. Then the linear map $\gamma: \mathcal{S}(q) \rightarrow \mathcal{S}(q)$ such that

$$
\gamma\left(L_{m, i}\right)=0, \gamma\left(G_{m, i}\right)= \begin{cases}0, & (m, i) \neq\left(0,-\frac{3 q}{2}\right)  \tag{113}\\ L_{0,-q}, & (m, i)=\left(0,-\frac{3 q}{2}\right)\end{cases}
$$

is an odd $\frac{1}{2}$-derivation of $\mathcal{S}(q)$.
Proof. Clearly, $\gamma=\gamma_{0, \frac{q}{2}}$. We are going to verify (100) (102) for $(r, s)=\left(0, \frac{q}{2}\right)$ and

$$
d_{0, \frac{q}{2}}^{0}=0, d_{0, \frac{q}{2}}^{1}(m, i)= \begin{cases}0, & (m, i) \neq\left(0,-\frac{3 q}{2}\right) \\ 1, & (m, i)=\left(0,-\frac{3 q}{2}\right)\end{cases}
$$

Obviously, (100) is trivially satisfied.
Case 1. $(m, i),(n, j),(m+n, i+j) \neq\left(0,-\frac{3 q}{2}\right)$. Then both sides of (101) and (102) are zero.

Case 2. $(m, i)=\left(0,-\frac{3 q}{2}\right)$. Then (101) and (102) become

$$
\begin{aligned}
-q n \cdot d_{0, \frac{q}{2}}^{1}\left(n, j-\frac{3 q}{2}\right) & =-\frac{q n}{2} \cdot d_{0, \frac{q}{2}}^{1}(n, j), \\
0 & =q n \cdot d_{0, \frac{q}{2}}^{1}(n, j) .
\end{aligned}
$$

Either $n=0$ or $d_{0, \frac{q}{2}}^{1}\left(n, j-\frac{3 q}{2}\right)=d_{0, \frac{q}{2}}^{1}(n, j)=0$, so both sides of these equalities are zero.
Case 3. $(n, j)=\left(0,-\frac{3 q}{2}\right)$. Then (101) and (102) become

$$
\begin{aligned}
2 q m \cdot d_{0, \frac{q}{2}}^{1}\left(m, i-\frac{3 q}{2}\right) & =0, \\
0 & =q m \cdot d_{0, \frac{q}{2}}^{1}(m, i) .
\end{aligned}
$$

Either $m=0$ or $d_{0, \frac{q}{2}}^{1}\left(m, i-\frac{3 q}{2}\right)=d_{0, \frac{q}{2}}^{1}(m, i)=0$, so again both sides are always zero.
Case 4. $(m+n, i+j)=\left(0,-\frac{3 q}{2}\right)$. Then Then (101) and (102) become

$$
\begin{aligned}
& 0=q n \cdot d_{0, \frac{q}{2}}^{1}(n, j), \\
& 0=\frac{q n}{2} \cdot d_{0, \frac{q}{2}}^{1}(-n, i)-\frac{q n}{2} \cdot d_{0, \frac{q}{2}}^{1}(n, j) .
\end{aligned}
$$

Either $n=0$ or $d_{0, \frac{q}{2}}^{1}(n, j)=d_{0, \frac{q}{2}}^{1}(-n, i)=0$, so once again both sides are zero.
Lemma 3.13. The linear maps $\delta, \varepsilon: \mathcal{S}(0) \rightarrow \mathcal{S}(0)$ such that

$$
\begin{align*}
& \delta\left(G_{m, i}\right)=0, \delta\left(L_{m, i}\right)= \begin{cases}0, & (m, i) \neq(0,0) \\
G_{0,0}, & (m, i)=(0,0)\end{cases}  \tag{114}\\
& \varepsilon\left(G_{m, i}\right)=0, \varepsilon\left(L_{m, i}\right)=G_{m, i}, \tag{115}
\end{align*}
$$

are odd $\frac{1}{2}$-derivations of $\mathcal{S}(0)$.
Proof. We first prove that $\delta \in \Delta^{1}(\mathcal{S}(0))$. Clearly, $\delta=\delta_{0,0}$, where

$$
d_{0,0}^{0}(m, i)= \begin{cases}0, & (m, i) \neq(0,0)  \tag{116}\\ 1, & (m, i)=(0,0)\end{cases}
$$

and $d_{0,0}^{1}=0$. Equalities (101) and (102) are trivially satisfied, while (100) reduces to

$$
\begin{equation*}
2(n i-m j) d_{0,0}^{0}(m+n, i+j)=(n i-m j)\left(d_{0,0}^{0}(m, i)+d_{0,0}^{0}(n, j)\right) . \tag{117}
\end{equation*}
$$

If $(0,0) \in\{(m, i),(n, j),(m+n, i+j)\}$, then $n i-m j=0$. Otherwise, $d_{0,0}^{0}(m+n, i+j)=$ $d_{0,0}^{0}(m, i)=d_{0,0}^{0}(n, j)=0$ by (116). Thus, $\delta \in \Delta^{1}(\mathcal{S}(0))$ by Lemma 3.9.

Now, let us prove that $\varepsilon \in \Delta^{1}(\mathcal{S}(0))$. Again, $\varepsilon=\varepsilon_{0,0}$, but now

$$
\begin{equation*}
d_{0,0}^{0}(m, i)=1 \text { for all }(m, i) \tag{118}
\end{equation*}
$$

and $d_{0,0}^{1}=0$. As above, (101) and (102) are trivial and (100) reduces to (117). In view of (118) the latter is also satisfied. So, $\varepsilon \in \Delta^{1}(\mathcal{S}(0))$ by Lemma 3.9,
Proposition 3.14. For all $q \in \mathbb{C}$ we have

$$
\Delta^{1}(\mathcal{S}(q))= \begin{cases}\{0\}, & q \notin 2 \mathbb{Z} \\ \langle\gamma\rangle, & q \in 2 \mathbb{Z} \backslash\{0\} \\ \langle\gamma, \delta, \varepsilon\rangle, & q=0\end{cases}
$$

Proof. A consequence of Lemmas 3.10 3.13.
Observe that $\gamma, \delta, \varepsilon$ from Proposition 3.14 are odd analogues of $\alpha, \beta$, id from Proposition 3.8.
Filippov proved that each nonzero $\delta$-derivation $(\delta \neq 0,1)$ of a Lie algebra gives a non-trivial Hom-Lie algebra structure [10, Theorem 1]. It is easy to see that a superanalog of his result is also true. Hence, by Proposition 3.14 we have the following corollary.

Corollary 3.15. $\mathcal{S}(q)_{q \in 2 \mathbb{Z}}$ admits non-trivial Hom-Lie superalgebra structures.
3.3. Transposed Poisson superalgebra structures on $\mathcal{S}(q)$. With the help of Proposition 3.14 we are now ready to describe the transposed Poisson superalgebra structures on $(\mathcal{S}(q),[\cdot, \cdot])$.
Theorem 3.16. If $q \neq 0$, then all the transposed Poisson superalgebra structures on $(\mathcal{S}(q),[\cdot, \cdot])$ are trivial. If $q=0$, then the non-trivial transposed Poisson superalgebra structures $(\mathcal{S}(q), \cdot,[\cdot, \cdot])$ on $(\mathcal{S}(q),[\cdot, \cdot])$ are, up to an isomorphism, of one of the following two forms

$$
\begin{align*}
& L_{0,0} \cdot L_{0,0}=L_{0,0}, L_{0,0} \cdot G_{0,0}=G_{0,0} \cdot L_{0,0}=G_{0,0}  \tag{119}\\
& L_{0,0} \cdot L_{0,0}=L_{0,0} \tag{120}
\end{align*}
$$

Proof. Let $(\mathcal{S}(q), \cdot,[\cdot, \cdot])$ be a transposed Poisson superalgebra, i.e. $(\mathcal{S}(q), \cdot)$ is supercommutative and (3) holds. Given $(m, i) \in \mathbb{Z} \times \mathbb{Z}$, denote by $\varphi^{m, i}$ and $\psi^{m, i}$ the left multiplications by $L_{m, i}$ and $G_{m, i}$, respectively, in $(\mathcal{S}(q), \cdot)$, i.e.

$$
\begin{align*}
& L_{m, i} \cdot L_{n, j}=\varphi^{m, i}\left(L_{n, j}\right), L_{m, i} \cdot G_{n, j}=\varphi^{m, i}\left(G_{n, j}\right),  \tag{121}\\
& G_{m, i} \cdot L_{n, j}=\psi^{m, i}\left(L_{n, j}\right), G_{m, i} \cdot G_{n, j}=\psi^{m, i}\left(G_{n, j}\right) \tag{122}
\end{align*}
$$

In view of supercommutativity of $(\mathcal{S}(q), \cdot)$, we have

$$
\begin{align*}
& L_{m, i} \cdot L_{n, j}=\varphi^{n, j}\left(L_{m, i}\right), L_{m, i} \cdot G_{n, j}=\psi^{n, j}\left(L_{m, i}\right),  \tag{123}\\
& G_{m, i} \cdot L_{n, j}=\varphi^{n, j}\left(G_{m, i}\right), G_{m, i} \cdot G_{n, j}=-\psi^{n, j}\left(G_{m, i}\right) \tag{124}
\end{align*}
$$

By (3) we have $\varphi^{m, i} \in \Delta^{0}(\mathcal{S}(q))$ and $\psi^{m, i} \in \Delta^{1}(\mathcal{S}(q))$.
Case 1. $q \notin 2 \mathbb{Z}$. Then $\varphi^{m, i}=a^{m, i} \mathrm{id}$ for some $a^{m, i} \in \mathbb{C}$ by Proposition 3.8 and $\psi^{m, i}=0$ by Proposition (3.14. It follows from (121) and (123)) that $a^{m, i}=0$ for all $(m, i) \in \mathbb{Z} \times \mathbb{Z}$. So, $\cdot$ is trivial whenever $q \notin 2 \mathbb{Z}$.

Case 2. $q \in 2 \mathbb{Z} \backslash\{0\}$. As above, $\varphi^{m, i}=a^{m, i} \mathrm{id}$ by Proposition 3.8, so again (121) and (123) imply $a^{m, i}=0$, whence $\varphi^{m, i}=0$. So, $L_{m, i} \cdot L_{n, j}=L_{m, i} \cdot G_{n, j}=0$. However,

$$
\begin{equation*}
\psi^{m, i}=b^{m, i} \gamma \tag{125}
\end{equation*}
$$

with $b^{m, i} \in \mathbb{C}$ and $\gamma$ given by (113). We immediately deduce from (113), (122) and (125) that $\psi^{m, i}\left(L_{n, j}\right)=0$, so $G_{m, i} \cdot L_{n, j}=0$. Regarding $G_{m, i} \cdot G_{n, j}$, on the one hand, by (113), (122) and (125)

$$
G_{m, i} \cdot G_{n, j}=b^{m, i} \gamma\left(G_{n, j}\right)= \begin{cases}0, & (n, j) \neq\left(0,-\frac{3 q}{2}\right), \\ b^{m, i} L_{0,-q}, & (n, j)=\left(0,-\frac{3 q}{2}\right) .\end{cases}
$$

On the other hand, by (113), (124) and (125)

$$
G_{m, i} \cdot G_{n, j}=-b^{n, j} \gamma\left(G_{m, i}\right)= \begin{cases}0, & (m, i) \neq\left(0,-\frac{3 q}{2}\right) \\ -b^{n, j} L_{0,-q}, & (m, i)=\left(0,-\frac{3 q}{2}\right)\end{cases}
$$

Thus, the product $G_{m, i} \cdot G_{n, j}$ is zero unless $(m, i)=(n, j)=\left(0,-\frac{3 q}{2}\right)$. But $G_{0,-\frac{3 q}{2}} \cdot G_{0,-\frac{3 q}{2}}$ is also zero, because $\left(\mathcal{S}(q)_{1}, \cdot\right)$ is anticommutative.

Case 3. $q=0$. Write $\varphi^{(m, i)}=a^{m, i} \mathrm{id}+b^{m, i} \alpha+c^{m, i} \beta$ and $\psi^{(m, i)}=p^{m, i} \gamma+q^{m, i} \delta+r^{m, i} \varepsilon$ in view of Propositions 3.8 and 3.14. On the one hand, by (91), (92), (113)-(115), (121) and (122)

$$
\begin{aligned}
& L_{m, i} \cdot L_{n, j}=a^{m, i} L_{n, j}+b^{m, i} \alpha\left(L_{n, j}\right)+c^{m, i} \beta\left(L_{n, j}\right)= \begin{cases}a^{m, i} L_{n, j}, & (n, j) \neq(0,0), \\
\left(a^{m, i}+b^{m, i}\right) L_{0,0}, & (n, j)=(0,0) .\end{cases} \\
& L_{m, i} \cdot G_{n, j}=a^{m, i} G_{n, j}+b^{m, i} \alpha\left(G_{n, j}\right)+c^{m, i} \beta\left(G_{n, j}\right)= \begin{cases}a^{m, i} G_{n, j}, & (n, j) \neq(0,0), \\
\left(a^{m, i}+c^{m, i}\right) G_{0,0}, & (n, j)=(0,0) .\end{cases} \\
& G_{m, i} \cdot G_{n, j}=p^{m, i} \gamma\left(G_{n, j}\right)+q^{m, i} \delta\left(G_{n, j}\right)+r^{m, i} \varepsilon\left(G_{n, j}\right)= \begin{cases}0, & (n, j) \neq(0,0), \\
p^{m, i} L_{0,0}, & (n, j)=(0,0) .\end{cases}
\end{aligned}
$$

On the other hand, by (91), (92), (113)-(115), (123) and (124)

$$
\begin{aligned}
& L_{m, i} \cdot L_{n, j}=a^{n, j} L_{m, i}+b^{n, j} \alpha\left(L_{m, i}\right)+c^{n, j} \beta\left(L_{m, i}\right)= \begin{cases}a^{n, j} L_{m, i}, & (m, i) \neq(0,0), \\
\left(a^{n, j}+b^{n, j}\right) L_{0,0}, & (m, i)=(0,0) .\end{cases} \\
& L_{m, i} \cdot G_{n, j}=p^{n, j} \gamma\left(L_{m, i}\right)+q^{n, j} \delta\left(L_{m, i}\right)+r^{n, j} \varepsilon\left(L_{m, i}\right)= \begin{cases}r^{n, j} G_{m, i}, & (m, i) \neq(0,0), \\
\left(q^{n, j}+r^{n, j}\right) G_{0,0}, & (m, i)=(0,0) .\end{cases} \\
& G_{m, i} \cdot G_{n, j}=-p^{n, j} \gamma\left(G_{m, i}\right)-q^{n, j} \delta\left(G_{m, i}\right)-r^{n, j} \varepsilon\left(G_{m, i}\right)= \begin{cases}0, & (m, i) \neq(0,0), \\
-p^{n, j} L_{0,0}, & (m, i)=(0,0) .\end{cases}
\end{aligned}
$$

Consider the product $L_{m, i} \cdot L_{n, j}$. If $(m, i),(n, j) \neq(0,0)$, then $a^{m, i} L_{n, j}=a^{n, j} L_{m, i}$, so taking $(m, i) \neq(n, j)$ we conclude that $a^{m, i}=a^{n, j}=0$. Thus, $L_{m, i} \cdot L_{n, j}=0$. If $(m, i)=(0,0)$, $(n, j) \neq(0,0)$, then $a^{0,0} L_{n, j}=\left(a^{n, j}+b^{n, j}\right) L_{0,0}=b^{n, j} L_{0,0}$. So, we obtain $a^{0,0}=b^{n, j}=0$, whence
$L_{m, i} \cdot L_{n, j}=0$. Similarly, $(m, i) \neq(0,0),(n, j)=(0,0)$ implies $L_{m, i} \cdot L_{n, j}=0$. Finally, if $(m, i)=(n, j)=(0,0)$, then $L_{m, i} \cdot L_{n, j}=\left(a^{0,0}+b^{0,0}\right) L_{0,0}=b^{0,0} L_{0,0}$, because $a^{0,0}=0$.

As to $L_{m, i} \cdot G_{n, j}$, the case $(m, i),(n, j) \neq(0,0)$ gives $L_{m, i} \cdot G_{n, j}=0$ and $r^{n, j}=0$, the case $(m, i)=(0,0),(n, j) \neq(0,0)$ gives $L_{m, i} \cdot G_{n, j}=0$ and $q^{n, j}=0$, the case $(m, i) \neq(0,0)$, $(n, j)=(0,0)$ gives $L_{m, i} \cdot G_{n, j}=0$ and $r^{0,0}=c^{m, i}=0$, the case $(m, i)=(n, j)=(0,0)$ gives $L_{m, i} \cdot G_{n, j}=c^{0,0} G_{0,0}=q^{0,0} G_{0,0}$.

Regarding $G_{m, i} \cdot G_{n, j}$, we see that it is zero unless $(m, i)=(n, j)=(0,0)$, in which case it must also be zero thanks to anti-commutativity of $\left(\mathcal{S}(0)_{1}, \cdot\right)$.

Thus, the only possible non-zero products in $(\mathcal{S}(0), \cdot)$ are of the form

$$
L_{0,0} \cdot L_{0,0}=c_{1} L_{0,0}, \quad L_{0,0} \cdot G_{0,0}=G_{0,0} \cdot L_{0,0}=c_{2} G_{0,0}
$$

Moreover, it follows from $L_{0,0} \cdot\left(L_{0,0} \cdot G_{0,0}\right)=\left(L_{0,0} \cdot L_{0,0}\right) \cdot G_{0,0}$ that $c_{2}^{2}=c_{1} c_{2}$.
Since $L_{0,0}, G_{0,0} \in Z(\mathcal{S}(0))$, then any linear map of the form $\phi\left(L_{m, i}\right)=L_{m, i}$ for $(m, i) \neq(0,0)$, $\phi\left(G_{n, j}\right)=G_{n, j}$ for $(n, j) \neq(0,0), \phi\left(L_{0,0}\right)=k_{11} L_{0,0}+k_{12} G_{0,0}$ and $\phi\left(G_{0,0}\right)=k_{21} L_{0,0}+k_{22} G_{0,0}$ with $k_{11} k_{22} \neq k_{12} k_{21}$ is an automorphism of $(\mathcal{S}(0),[\cdot, \cdot])$.

Let $c_{2} \neq 0$. Then $c_{1}=c_{2}$, and we obtain the following multiplication table

$$
\begin{equation*}
L_{0,0} \cdot L_{0,0}=c_{1} L_{0,0}, L_{0,0} \cdot G_{0,0}=G_{0,0} \cdot L_{0,0}=c_{1} G_{0,0} \tag{126}
\end{equation*}
$$

Applying $\phi$ with $k_{11}=k_{22}=c_{1}$ and $k_{12}=k_{21}=0$ to (126), we come to a transposed Poisson structure isomorphic to (119)

Let $c_{2}=0$. Then we obtain the following multiplication table

$$
\begin{equation*}
L_{0,0} \cdot L_{0,0}=c_{1} L_{0,0} . \tag{127}
\end{equation*}
$$

Applying $\phi$ with $k_{11}=c_{1}, k_{22}=1$ and $k_{12}=k_{21}=0$ to (127), we come to a transposed Poisson structure isomorphic to (120).

Conversely, each of the two associative and supercommutative multiplications (119) and (120) defines a transposed Poisson algebra structure on $\mathcal{S}(0)$, because $\mathcal{S}(0) \cdot \mathcal{S}(0) \subseteq\left\langle L_{0,0}, G_{0,0}\right\rangle \subseteq$ $Z(\mathcal{S}(0))$ and $[\mathcal{S}(0), \mathcal{S}(0)] \subseteq \operatorname{Ann}(\mathcal{S}(0))$. They are non-isomorphic because the dimensions of $S(0)^{2}$ are different under these two multiplications.

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