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TRANSVERSAL INFINITESIMAL AUTOMORPHISMS FOR HARMONIC KÄHLER FOLIATIONS

Dedicated to Professor Morio Obata on his sixtieth birthday

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Summary. In this paper we consider a harmonic Kähler foliation $\mathscr{F}$ and study the infinitesimal automorphisms of $\mathscr{F}$ which are either transversally holomorphic or transversally Killing. A special study is made for the case of foliations with constant transversal scalar curvature.

1. Introduction. Let $\mathscr{F}$ be a transversally oriented foliation on a compact oriented manifold M. It is given by an exact sequence of vector bundles

$$0 \to L \to TM \xrightarrow{\pi} Q \to 0 ,$$

where L is the tangent bundle and Q the normal bundle of $\mathscr{F}$. We have an associated exact sequence of Lie algebras

$$0 \to \Gamma L \to V(\mathscr{F}) \xrightarrow{\pi} \Gamma Q^L \to 0 ,$$

where $V(\mathscr{F})$ denotes the algebra of infinitesimal automorphisms of $\mathscr{F}$, and ΓQ^L the portion of ΓQ invariant under the action of L by Lie derivatives [KT 2], [MO]. The foliation is assumed to be transversally Kähler. By a *Kähler foliation* $\mathscr{F}$ we mean a foliation satisfying the following conditions: (i) $\mathscr{F}$ is Riemannian, with a bundle-like metric g_M on M inducing the holonomy invariant metric g_Q on $Q = L^\perp$, (ii) there is a holonomy invariant almost complex structure $J: Q \to Q$, where $\dim Q = q = 2n$ (real dimension), with respect to which g_Q is Hermitian, i.e. $g_Q(J\mu, J\nu) = g_Q(\mu, \nu)$ for $\mu, \nu \in \Gamma Q$, and (iii) if ∇ denotes the unique metric and torsionfree connection in Q, then ∇ is almost complex, i.e. $\nabla J = 0$. Note that $\Phi(\mu, \nu) = g_Q(\mu, J\nu)$ defines a basic 2-form Φ, which is closed as a consequence of $\nabla g_Q = 0$ and $\nabla J = 0$.

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Associated to ∇ are transversal curvature data, in particular the transversal Ricci curvature tensor $S_\nabla = \mathrm{Ric}_\nabla$, the associated Ricci operator $\rho_\nabla: Q \to Q$, the transversal scalar curvature c_∇, and the transversal Jocabi operator $J_\nabla = \Delta - \rho_\nabla: \Gamma Q \to \Gamma Q$. In this paper we study geometric properties of infinitesimal automorphisms $Y \in V(\mathscr{F})$. For $Y \in V(\mathscr{F})$ the transversal part $\pi(Y)$ is denoted by $\bar{Y}$, and the transversal Lie derivative operator $\Theta(Y): \Gamma Q \to \Gamma Q$ is defined as in [KT 2]. In view of the variational meaning of J_∇ [KT 2], it is natural to assume $\mathscr{F}$ to be *harmonic*, i.e. all leaves of $\mathscr{F}$ are minimal submanifolds of (M, g_M).

THEOREM A. *Let $\mathscr{F}$ be a harmonic Kähler foliation on a closed orientable manifold M, and Y an infinitesimal automorphism of $\mathscr{F}$. Then the following properties are equivalent:*

(i) *$\bar{Y}$ is transversally holomorphic, i.e. $\Theta(Y)J = 0$,*

(ii) *$\bar{Y}$ is a transversal Jacobi field, i.e. $J_\nabla \bar{Y} = 0$.*

Combining Theorem A with the results of [KTT] yields the following consequence.

THEOREM B. *Let $\mathscr{F}$ be a harmonic Kähler foliation on a closed orientable manifold M, and Y an infinitesimal automorphism of $\mathscr{F}$. Then the following properties are equivalent:*

(i) *$\bar{Y}$ is transversally Killing, i.e. $\Theta(Y)g_Q = 0$,*

(ii) *$\bar{Y}$ is a transversally divergencefree Jacobi field,*

(iii) *$\bar{Y}$ is transversally holomorphic and transversally divergencefree.*

For the point foliation these are results of Bochner [B] and Yano [Y]. The next result generalizes a theorem of Bochner [B] to the foliation context. The corresponding result for Riemannian foliations was proved in [KT 2].

THEOREM C. *Let $\mathscr{F}$ be a harmonic Kähler foliation on a closed orientable manifold M with transversal Ricci operator $\rho_\nabla \leqq 0$. Then every transversally holomorphic infinitesimal automorphism $Y \in V(\mathscr{F})$ satisfies $\nabla \bar{Y} = 0$. If $\rho_\nabla < 0$ for at least one point $x \in M$, then every $Y \in V(\mathscr{F})$ with transversally holomorphic $\bar{Y}$ satisfies $Y \in \Gamma L$.*

Note that the commutative diagram

$$\begin{array}{ccc} V(\mathscr{F}) & \longrightarrow & \Gamma Q^L \\ \cup & & \cup \\ K(\mathscr{F}) & \longrightarrow & \bar{K}(\mathscr{F}) \end{array}$$

describes the projection of the transversally metric infinitesimal automorphisms $K(\mathscr{F})$ to the Lie algebra $\bar{K}(\mathscr{F})$ of transversal Killing fields.

Similarly the commutative diagram

$$\begin{array}{ccc} V(\mathscr{F}) & \longrightarrow & \Gamma Q^L \\ \cup & & \cup \\ H(\mathscr{F}) & \longrightarrow & \bar{H}(\mathscr{F}) \end{array}$$

describes the Lie algebra $\bar{H}(\mathscr{F})$ of transversal holomorphic fields. Then we prove the following theorem.

THEOREM D. *Let $\mathscr{F}$ be a harmonic Kähler foliation on a closed orientable manifold M with constant transversal scalar curvature c_∇. Then the Lie algebra $\bar{H}(\mathscr{F})$ of transversally holomorphic fields splits into the direct sum of the abelian Lie algebra of parallel transversally holomorphic fields and the Lie algebra of the transversally holomorphic fields which are annihilated by all basic harmonic 1-forms. The latter Lie algebra is the complexification of the real Lie subalgebra of transversally Killing fields annihilated by basic harmonic 1-forms.*

For a point foliation this is a result of Lichnerowicz [L].

The key elements for the proofs presented below are the transversal divergence theorem of [KTT], valid for harmonic foliations and the de Rham-Hodge theory for basic forms of a Riemannian foliations as developed in [EH] and [KT 5]. Further applications can be made in the spirit of [TT].

2. Transversally holomorphic infinitesimal automorphisms. Let R_∇ be the curvature associated to the unique metric and torsionfree connection ∇ in the normal bundle of the Riemannian foliation $\mathscr{F}$. Let similarly S_∇ be the Ricci curvature. For a Kähler foliation we have then the following identities:

$$R_\nabla(\mu, \nu)\circ J = J\circ R_\nabla(\mu, \nu) \,, \tag{2.1}$$

$$R_\nabla(J\mu, J\nu) = R_\nabla(\mu, \nu) \,, \tag{2.2}$$

$$S_\nabla(J\mu, J\nu) = S_\nabla(\mu, \nu) \,, \tag{2.3}$$

$$R_\nabla(\lambda, \mu)\nu + R_\nabla(\mu, \nu)\lambda + R_\nabla(\nu, \lambda)\mu = 0 \,. \tag{2.4}$$

In fact, (2.1) follows from $\nabla J = 0$. (2.2) and (2.3) follow from $g_Q(J\mu, J\nu) = g_Q(\mu, \nu)$. Finally (2.4) is a consequence of the Jacobi identity. The proofs are similar to the usual ones in Kähler geometry [KN].

An infinitesimal automorphism $Y \in V(\mathscr{F})$ gives rise to a transversally holomorphic field $\bar{Y} = \pi(Y)$ if and only if

$$\Theta(Y)J = 0 \,, \tag{2.5}$$

where for $Z \in \Gamma L^{\perp}$ by definition

$$(\Theta(Y)J)(Z) = \Theta(Y)(JZ) - J(\Theta(Y)Z) .$$

But this expression equals $\pi[Y, JZ] - J\pi[Y, Z]$, which yields the formula

$$(\Theta(Y)J)(Z) = -\nabla_{JZ}\bar{Y} + J\nabla_{Z}\bar{Y} , \tag{2.6}$$

so that (2.5) holds if and only if

$$\nabla_{JZ}\bar{Y} = J\nabla_{Z}\bar{Y} \quad \text{for all} \quad Z \in \Gamma L^{\perp} . \tag{2.7}$$

In the sequel it will be convenient to use the following orthonormal frame on M. For $x \in M$ let $\{e_A\}$, $A = 1, \cdots, m = \dim M$ be an oriented orthonormal basis of T_xM with e_i, $i = 1, \cdots, p = \dim L$ in L_x and e_α, $e_{\alpha+n} = Je_\alpha$ in $L_x^{\perp}$ for $\alpha = p+1, \cdots, p+n$ ($\mathscr{F}$ is of codimension $q = 2n$ on M^m, $m = p + 2n$). The transversal Kähler property of $\mathscr{F}$ allows then to extend e_α, Je_α to local vector fields E_α, $JE_\alpha \in \Gamma L^{\perp}$ such that for $\alpha, \beta = p+1, \cdots, p+n$

$$(\nabla_{E_\alpha}E_\beta)_x = 0 , \quad (\nabla_{E_\alpha}JE_\beta)_x = 0 , \quad (\nabla_{JE_\alpha}E_\beta)_x = 0 , \quad (\nabla_{JE_\alpha}JE_\beta)_x = 0 . \tag{2.8}$$

As a consequence of torsionfreeness [KT 1, 1.5]

$$[E_\alpha, E_\beta]_x , \quad [E_\alpha, JE_\beta]_x , \quad [JE_\alpha, JE_\beta]_x \in L_x . \tag{2.9}$$

The E_α, JE_α can be chosen as (local) infinitesimal automorphisms of $\mathscr{F}$, so that

$$\nabla_X E_\alpha = \pi[X, E_\alpha] = 0 \quad \text{for} \quad X \in \Gamma L . \tag{2.10}$$

We can complete E_α, JE_α by the Gram-Schmidt process to a moving local frame by adding $E_i \in \Gamma L$ with $(E_i)_x = e_i$, $i = 1, \cdots, p$.

In terms of such a moving frame the transversal Ricci operator $\rho_\nabla : Q \to Q$ is given by

$$\rho_\nabla = \sum_{\alpha=p+1}^{p+n} J \circ R_\nabla(E_\alpha, JE_\alpha) . \tag{2.11}$$

In fact, let $\mu \in \Gamma Q$. Then by (2.4), (2.1)

$$\begin{aligned} R_\nabla(E_\alpha, JE_\alpha)\mu &= -R_\nabla(JE_\alpha, E_\alpha)\mu = R_\nabla(E_\alpha, \mu)JE_\alpha + R_\nabla(\mu, JE_\alpha)E_\alpha \\ &= JR_\nabla(E_\alpha, \mu)E_\alpha - JR_\nabla(\mu, JE_\alpha)JE_\alpha \\ &= -J(R_\nabla(\mu, E_\alpha)E_\alpha + R_\nabla(\mu, JE_\alpha)JE_\alpha) \end{aligned}$$

or

$$JR_\nabla(E_\alpha, JE_\alpha)\mu = R_\nabla(\mu, E_\alpha)E_\alpha + R_\nabla(\mu, JE_\alpha)JE_\alpha .$$

It follows that

$$\rho_\nabla\mu = \sum_{\alpha=p+1}^{p+n}(R_\nabla(\mu, E_\alpha)E_\alpha + R_\nabla(\mu, JE_\alpha)JE_\alpha)$$

is given by (2.11).

PROOF OF THEOREM A. To establish (i)$\Rightarrow$(ii) we calculate for $Y \in V(\mathscr{F})$ with transversally holomorphic $\bar{Y}$

$$(\Delta \bar{Y})_x = (d_\nabla^* d_\nabla \bar{Y})_x = -\sum_{A=1}^{m} (\nabla_{e_A}(d_\nabla \bar{Y}))(e_A) = -\sum_{A=1}^{m} (\nabla_{e_A}\nabla_{E_A}\bar{Y} - \nabla_{\nabla^M_{e_A}E_A}\bar{Y}) .$$

By (2.8) we have for $\alpha = p+1, \cdots, p+n$

$$\nabla_{e_\alpha} E_\alpha = \pi(\nabla^M_{e_\alpha} E_\alpha) = 0$$

and therefore $\nabla^M_{e_\alpha} E_\alpha \in L_x$. Since $\bar{Y}$ is projectable, i.e. $\Theta(X)\bar{Y} = 0$ or equivalently $\nabla_X \bar{Y} = 0$ for every $X \in \Gamma L$, the term $\nabla_{\nabla^M_{e_\alpha}E_\alpha}\bar{Y}$ vanishes. Similarly for the corresponding terms involving JE_α. On the other hand, for a projectable $\bar{Y}$

$$\sum_{i=1}^{p} \nabla_{\nabla^M_{e_i}E_i}\bar{Y} = \nabla_{\tau_x}\bar{Y}$$

where $\tau_x = \sum_{i=1}^p \pi(\nabla^M_{e_i}E_i)$ is the mean curvature vector field of $\mathscr{F}$. Since $\mathscr{F}$ is assumed to be harmonic, τ vanishes, and these contributions also disappear. It follows that

$$(\Delta \bar{Y})_x = -\sum_{i=1}^{p} \nabla_{e_i}\nabla_{E_i}\bar{Y} - \sum_{\alpha=p+1}^{p+n} \nabla_{e_\alpha}\nabla_{E_\alpha}\bar{Y} - \sum_{\alpha=p+1}^{p+n} \nabla_{Je_\alpha}\nabla_{JE_\alpha}\bar{Y} .$$

The first sum disappears since $\bar{Y}$ is projectable. Since $\bar{Y}$ is holomorphic, by (2.7) the terms in the third sum equal

$$\nabla_{Je_\alpha}\nabla_{JE_\alpha}\bar{Y} = \nabla_{Je_\alpha}(J\nabla_{E_\alpha}\bar{Y}) = J(\nabla_{Je_\alpha}\nabla_{E_\alpha}\bar{Y}) .$$

But

$$R_\nabla(Je_\alpha, e_\alpha)\bar{Y}_x = \nabla_{Je_\alpha}\nabla_{E_\alpha}\bar{Y} - \nabla_{e_\alpha}\nabla_{JE_\alpha}\bar{Y} - \nabla_{[JE_\alpha, E_\alpha]_x}\bar{Y}$$

where the last term vanishes by (2.9). It follows that

$$(2.12)\qquad \begin{aligned}(\Delta \bar{Y})_x &= -\sum_\alpha \nabla_{e_\alpha}\nabla_{E_\alpha}\bar{Y} - \sum_\alpha J(\nabla_{e_\alpha}\nabla_{JE_\alpha}\bar{Y} + R_\nabla(Je_\alpha, e_\alpha)\bar{Y}_x)\\ &= \sum_\alpha JR_\nabla(e_\alpha, Je_\alpha)\bar{Y}_x ,\end{aligned}$$

where we have again used (2.7). By (2.11), (2.12) we find then $(J_\nabla \bar{Y})_x = (\Delta \bar{Y})_x - \rho_\nabla \bar{Y}_x = 0$ and $\bar{Y}$ is indeed a transversal Jacobi field.

To prove (ii)$\Rightarrow$(i) we consider conversely $Y \in V(\mathscr{F})$ such that $J_\nabla \bar{Y} = 0$. The desired result follows then clearly from the following general formula

$$(2.13)\qquad \langle \Theta(Y)J, \Theta(Y)J\rangle = 2\langle J_\nabla \bar{Y}, \bar{Y}\rangle ,$$

where the left hand side denotes the global scalar product in the space of endomorphisms of Q, while the right hand side denotes the global scalar product in ΓQ. It remains to prove (2.13).

First we evaluate for $x \in M$

$$
\begin{aligned}
&(\Theta(Y)J, \Theta(Y)J)_x \\
&\quad = \sum_{\alpha=p+1}^{p+n} g_Q((\Theta(Y)J)(e_\alpha), (\Theta(Y)J)(e_\alpha)) + \sum_{\alpha=p+1}^{p+n} g_Q((\Theta(Y)J)(Je_\alpha), (\Theta(Y)J)(Je_\alpha)) \\
&\quad = \sum_\alpha g_Q(\nabla_{Je_\alpha}\bar{Y} - J\nabla_{e_\alpha}\bar{Y}, \nabla_{Je_\alpha}\bar{Y} - J\nabla_{e_\alpha}\bar{Y}) \\
&\qquad + \sum_\alpha g_Q(\nabla_{e_\alpha}\bar{Y} + J\nabla_{Je_\alpha}\bar{Y}, \nabla_{e_\alpha}\bar{Y} + J\nabla_{Je_\alpha}\bar{Y}) .
\end{aligned}
$$

The second sum equals

$$
\sum_\alpha g_Q(J\nabla_{e_\alpha}\bar{Y} - \nabla_{Je_\alpha}\bar{Y}, J\nabla_{e_\alpha}\bar{Y} - \nabla_{Je_\alpha}\bar{Y})
$$

and thus equals the first sum. It follows that

$$
\begin{aligned}
&(\Theta(Y)J, \Theta(Y)J)_x \\
&\quad = 2\sum_\alpha g_Q(\nabla_{Je_\alpha}\bar{Y} - J\nabla_{e_\alpha}\bar{Y}, \nabla_{Je_\alpha}\bar{Y}) + 2\sum_\alpha g_Q(J\nabla_{Je_\alpha}\bar{Y} + \nabla_{e_\alpha}\bar{Y}, \nabla_{e_\alpha}\bar{Y}) \\
&\quad = 2\sum_\alpha Je_\alpha g_Q(\nabla_{JE_\alpha}\bar{Y} - J\nabla_{E_\alpha}\bar{Y}, \bar{Y}) - 2\sum_\alpha g_Q(\nabla_{Je_\alpha}\nabla_{JE_\alpha}\bar{Y} - \nabla_{Je_\alpha}J(\nabla_{E_\alpha}\bar{Y}), \bar{Y}) \\
&\qquad + 2\sum_\alpha e_\alpha g_Q(J\nabla_{JE_\alpha}\bar{Y} + \nabla_{E_\alpha}\bar{Y}, \bar{Y}) - 2\sum_\alpha g_Q(\nabla_{e_\alpha}J(\nabla_{JE_\alpha}\bar{Y}) + \nabla_{e_\alpha}\nabla_{E_\alpha}\bar{Y}, \bar{Y}) \\
&\quad = 2(\mathrm{div}_B Z)_x + 2g_Q(\Delta\bar{Y}, \bar{Y})_x + 2g_Q(\sum_\alpha JR_\nabla(JE_\alpha, E_\alpha)\bar{Y}, \bar{Y})_x ,
\end{aligned}
$$

where $Z \in \Gamma Q^L$ is the g_Q-dual of $\lambda \in \Omega_B^1(\mathscr{F})$ defined by

$$
\lambda(X) = g_Q(\nabla_X\bar{Y} + J\nabla_{JX}\bar{Y}, \bar{Y}) \quad \text{for} \quad X \in \Gamma Q ,
$$

and the transversal divergence $\mathrm{div}_B Z$ of Z is defined as the unique scalar satisfying $\Theta(Z)\nu = \mathrm{div}_B Z \cdot \nu$, ν being the transversal volume form defined by g_Q [KTT]. The third term is by (2.11) equal to $2g_Q(-\rho_\nabla\bar{Y}, \bar{Y})_x$. It follows that

$$
(\Theta(Y)J, \Theta(Y)J)_x = 2(\mathrm{div}_B Z)_x + 2g_Q(J_\nabla\bar{Y}, \bar{Y})_x . \tag{2.14}
$$

By the transversal divergence theorem of [KTT], the integral of $\mathrm{div}_B Z$ over M vanishes for harmonic $\mathscr{F}$. Thus we obtain (2.13) by integrating (2.14) over M. □

PROOF OF THEOREM B. This is an immediate consequence of the characterization in [KTT] of transversal Killing fields as transversally divergencefree Jacobi fields, together with Theorem A. □

PROOF OF THEOREM C. Let $\bar{Y}$ be transversally holomorphic. Then $\bar{Y}$ is a transversal Jacobi field by Theorem A. By [KT 2, (4.3)] we have then the identity

$$-\frac{1}{2}\Delta g_Q(\bar{Y}, \bar{Y}) = g_Q(\nabla \bar{Y}, \nabla \bar{Y}) - g_Q(\rho_\nabla \bar{Y}, \bar{Y}) . \tag{2.15}$$

From here on the proof is exactly as in [KT 2]. Namely $\rho_\nabla \leqq 0$ yields immediately $g_Q(\nabla \bar{Y}, \nabla \bar{Y}) = 0$ and hence $\nabla \bar{Y} = 0$. If $\rho_\nabla < 0$ for at least one $x \in M$, then one concludes in addition $\bar{Y}_x = 0$ and hence $\bar{Y} = 0$. □

EXAMPLE 2.16 (Complex hyperbolic space CH^n, see [KN, p. 282]). This is the quotient space of the anti de Sitter space H_1^{2n+1} under the canonical circle action. Here

$$H_1^{2n+1} = \{z = (z_0, z_1, \cdots, z_n) \in C^{n+1} | (z, z) = -1\}$$

with the Hermitian form $(z, w) = -z_0\bar{w}_0 + \sum_{k=1}^n z_k\bar{w}_k$. The signature of the induced metric on H_1^{2n+1} is $(1, 2n)$, and the sectional curvature is constant -1. The canonical S^1-action on H_1^{2n+1} defines a fibration

$$H_1^{2n+1} \to CH^n = SU(1, n)/S(U(1) \times U(n)) ,$$

which gives rise naturally to a harmonic (and totally geodesic) Kähler foliation $\mathscr{F}$ on H_1^{2n+1}. The transversal holomorphic sectional curvature is constant -4. The assumptions of Theorem C are satisfied, and it follows that every $Y \in V(\mathscr{F})$ with transversally holomorphic $\bar{Y}$ satisfies $Y \in \Gamma L$. In this case $\rho_\nabla < 0$ and $\mathscr{F}$ is a (strictly) stable harmonic foliation of vanishing index and nullity (see [KT 2] for the second variation formula).

3. Dual basic 1-forms. For use below it is convenient to consider the basic 1-form associated to $\bar{Y}$ by g_Q-duality. Recall that the basic forms are given by

$$\Omega_B^{\cdot}(\mathscr{F}) = \{\omega \in \Omega^{\cdot}(M) | i(X)\omega = 0, \Theta(X)\omega = 0 \text{ for all } X \in \Gamma L\} .$$

The exterior differential d restricts to $d_B\colon \Omega_B^{\cdot} \to \Omega_B^{\cdot+1}$. The adjoint δ_B with respect to the induced scalar product $\langle\ ,\ \rangle_B$ on $\Omega_B^{\cdot}$ yield then the basic Laplacian $\Delta_B = \delta_B d_B + d_B \delta_B$. In the harmonic case, δ_B is locally given by the usual formula (in the general case, the mean curvature produces correction terms). Similarly, in the harmonic case the Weitzenböck formula on basic 1-forms is the usual $\Delta_B\omega = -\mathrm{trace}\,\nabla^2\omega + \rho_\nabla(\omega)$ (while in the general case the mean curvature produces correction terms).

Let ω be the basic 1-form associated to $\bar{Y}$ by g_Q-duality. It then follows that the Jacobi condition $J_\nabla \bar{Y} = 0$ is equivalent to

$$\Delta_B\omega = 2\rho_\nabla(\omega) , \tag{3.1}$$

where $(\rho_\nabla(\omega))(\mu) = \omega(\rho_\nabla(\mu))$ for $\mu \in \Gamma Q$. By Theorem A it follows that this identity characterizes transversally holomorphic infinitesimal automorphisms, while by Theorem B transversally Killing fields have additionally to satisfy $\delta_B\omega = -\mathrm{div}_B \bar{Y} = 0$.

We will need the identity

$$2\delta S_{\nabla} = -d_B c_{\nabla} \tag{3.2}$$

for the transversal Ricci curvature S_{∇} and transversal scalar curvature c_{∇}. It involves the differential operator $\delta\colon \Gamma S^2Q^* \to \Gamma Q^*$ defined in [KTT]. It is proved by contracting the second Bianchi identity for the transversal curvature R_{∇} as in the case of a foliation by points. We further need the following identity for $\omega \in \Omega_B^1(\mathscr{F})$ and the transversal Kähler form $\Phi \in \Omega_B^2(\mathscr{F})$:

$$\begin{aligned} &\delta_B(\omega \wedge \Phi)(\mu, \nu) - (\delta_B \omega)\cdot \Phi(\mu, \nu) \\ &\quad = d_B(J\omega)(\mu, \nu) - (d_B\omega)(J\mu, \nu) - (d_B\omega)(\mu, J\nu) \end{aligned} \tag{3.3}$$

for $\mu, \nu \in \Gamma Q$. To verify (3.3), let E_a be one of the vector fields $E_{p+1}, \cdots, E_{p+n}$; $E_{p+n+1} = JE_{p+1}, \cdots, E_{p+2n} = JE_{p+n}$. Then at a point $x \in M$ we observe for $a, b, c = p+1, \cdots, p+2n$

$$\begin{aligned} \delta_B(\omega\wedge\Phi)_x(e_a, e_b) &= -\sum_c \nabla_{e_c}(\omega\wedge\Phi)(E_c, E_a, E_b) \\ &= -\sum_c \nabla_{e_c}[\omega(E_c)\cdot\Phi(E_a, E_b) - \omega(E_a)\cdot\Phi(E_c, E_b) + \omega(E_b)\cdot\Phi(E_c, E_a)] \\ &= -\sum_c \nabla_{e_c}[\omega(E_c)\cdot g_Q(E_a, JE_b) - \omega(E_a)\cdot g_Q(E_c, JE_b) + \omega(E_b)\cdot g_Q(E_c, JE_a)] \\ &= -\sum_c [(\nabla_{e_c}\omega)(e_c)\cdot g_Q(e_a, Je_b) - (\nabla_{e_c}\omega)(e_a)\cdot g_Q(e_c, Je_b) + (\nabla_{e_c}\omega)(e_b)\cdot g_Q(e_c, Je_a)] \\ &= (\delta_B\omega)\cdot\Phi(e_a, e_b) + (\nabla_{Je_b}\omega)(e_a) - (\nabla_{Je_a}\omega)(e_b)\,. \end{aligned}$$

But by direct calculation

$$d_B(J\omega)(e_a, e_b) - (d_B\omega)(Je_a, e_b) - (d_B\omega)(e_a, Je_b) = (\nabla_{Je_b}\omega)(e_a) - (\nabla_{Je_a}\omega)(e_b)\,,$$

which completes the proof of (3.3).

We further need the characterization

$$\nabla_{JZ}\omega = -J\nabla_Z\omega \quad \text{for all} \quad Z \in \Gamma L^{\perp} \tag{3.4}$$

for $\omega \in \Omega_B^1(\mathscr{F})$ associated to $\bar{Y} \in \bar{H}(\mathscr{F})$ by g_Q-duality. This is simply the dual version of (2.7). Finally we need for $\bar{Y} \in \bar{H}(\mathscr{F})$ the J-invariance of $d_B\omega$, i.e.

$$(d_B\omega)(JZ, W) + (d_B\omega)(Z, JW) = 0 \quad \text{for all} \quad Z, W \in \Gamma L^{\perp}\,. \tag{3.5}$$

This is a consequence of (3.4).

4. Proof of theorem D. For the sake of simplicity we identify from now on $\bar{H}(\mathscr{F})$ with the corresponding space of dual basic 1-forms, and similarly for $\bar{K}(\mathscr{F})$. Consider the de Rham-Hodge direct sum decomposition of $\Omega_B^1(\mathscr{F})$ [EH], [KT 5], which represents ω in the form

(4.1) $$\omega = d_B\alpha + \delta_B\beta + \pi_B\omega ,$$

where $\pi_B: \Omega_B^1 \to \mathscr{H}_B^1$ is the orthogonal projection onto the basic harmonic 1-forms $\mathscr{H}_B^1$. Define $\zeta = d_B\alpha$, $\xi = \delta_B\beta + \pi_B\omega$. Then

(4.2) $$\omega = \xi + \zeta \quad \text{with} \quad \delta_B\xi = 0 , \quad d_B\zeta = 0 .$$

We first prove that for $\omega \in \bar{H}(\mathscr{F})$

(4.3) $$\delta_B(\xi \wedge \Phi) = 0 \quad \text{and} \quad d_B(J\xi) = 0 .$$

By (3.5) we have

$$(d_B\omega)(JE_a, E_b) + (d_B\omega)(E_a, JE_b) = 0 ,$$

and since $d_B\zeta = 0$, it follows that

$$(d_B\xi)(JE_a, E_b) + (d_B\xi)(E_a, JE_b) = 0 .$$

From (3.3) and $\delta_B\xi = 0$ it follows further that

$$\delta_B(\xi \wedge \Phi) = d_B(J\xi) .$$

Since the left and right hand sides are in orthogonal spaces, they both vanish.

Next we prove for the operator $\delta^*: \Gamma Q^* \to \Gamma S^2 Q^*$ adjoint to δ (see [KTT]), and $\omega \in \bar{H}(\mathscr{F})$, the identity

(4.4) $$(\delta^*\xi)(JZ, W) - (\delta^*\xi)(Z, JW) = 0 \quad \text{for all} \quad Z, W \in \Gamma L^\perp .$$

In fact, at each $x \in M$, by (4.3), (3.3)

$$\begin{aligned} 0 &= \delta_B(\xi \wedge \Phi)(e_a, e_b) + d_B(J\xi)(e_a, e_b) \\ &= (\nabla_{Je_b}\xi)(e_a) - (\nabla_{Je_a}\xi)(e_b) + (\nabla_{e_a}J\xi)(e_b) - (\nabla_{e_b}J\xi)(e_a) \\ &= 2[(\delta^*\xi)(e_a, Je_b) - (\delta^*\xi)(Je_a, e_b)] \end{aligned}$$

which yields (4.4).

Now we establish that for $\omega \in \bar{H}(\mathscr{F})$

(4.5) $$\delta_B(\rho_\nabla\xi) = 0 .$$

At $x \in M$ we have

$$\begin{aligned} \delta_B(\rho_\nabla\xi)_x &= -\sum_a \nabla_{e_a}((\rho_\nabla\xi)(E_a)) = -\sum_a \nabla_{e_a}(\xi(\rho_\nabla E_a)) \\ &= -\sum_a (\nabla_{e_a}\xi)(\rho_\nabla e_a) - \xi(\sum_a \nabla_{e_a}(\rho_\nabla E_a)) \\ &= -\sum_{a,b} S_\nabla(e_a, e_b)\cdot(\nabla_{e_a}\xi)(e_b) - \sum_{a,b} \xi(e_b)\cdot g_Q(\nabla_{e_a}(\rho_\nabla E_a), e_b) . \end{aligned}$$

For the second sum we find

$$\sum_a g_Q(\nabla_{e_a}(\rho_\nabla E_a), e_b) = \sum_a (\nabla_{e_a}S_\nabla)(e_a, e_b) ,$$

and therefore by the definition of δS_∇

$$\sum_b \xi(e_b)\cdot\sum_a (\nabla_{e_a} S_\nabla)(e_a, e_b) = -\sum_b \xi(e_b)\cdot(\delta S_\nabla)(e_b) .$$

Using (3.2) and the constancy of the transversal scalar curvature c_∇, we find that this term vanishes. Using the symmetry of S_∇, we find therefore

$$\begin{aligned}\delta_B(\rho_\nabla\xi)_x &= -\sum_{a,b} S_\nabla(e_a, e_b)\left[\frac{1}{2}((\nabla_{e_a}\xi)(e_b) + (\nabla_{e_b}\xi)(e_a))\right]\\ &= -\sum_{a,b} S_\nabla(e_a, e_b)\cdot(\delta^*\xi)(e_a, e_b) .\end{aligned}$$

Using (2.3) and (4.4) establishes now (4.5).

We want to show that for $\omega \in \bar{H}(\mathscr{F})$, ζ and ξ in (4.2) are both also in $\bar{H}(\mathscr{F})$. By Theorem A and (3.1) we have $\Delta_B\omega = 2\rho_\nabla(\omega)$, i.e. $\Delta_B(\zeta + \xi) = 2\rho_\nabla(\zeta + \xi)$, or equivalently

$$\Delta_B\zeta - 2\rho_\nabla\zeta = -(\Delta_B\xi - 2\rho_\nabla\xi) .$$

Applying δ_B to both sides, and using (4.5) and $\delta_B\xi = 0$, we have

$$\delta_B(\Delta_B\zeta - 2\rho_\nabla\zeta) = 0 . \tag{4.6}$$

Thus

$$\langle J_\nabla\zeta, \zeta\rangle = \langle\Delta_B\zeta - 2\rho_\nabla\zeta, d_B\alpha\rangle = 0 ,$$

and it follows from (2.13) that $\zeta \in \bar{H}(\mathscr{F})$. Therefore also $\xi = \omega - \zeta \in \bar{H}(\mathscr{F})$. Note that in fact $\xi \in \bar{K}(\mathscr{F})$ by Theorem B, since $\delta_B\xi = 0$.

Next we want to show that $\zeta = J\eta$ for some $\eta \in \bar{K}(\mathscr{F})$. Necessarily we have

$$\eta = -J\zeta \tag{4.7}$$

and with this definition we now show that indeed $\eta \in \bar{K}(\mathscr{F})$. Clearly $\eta \in \bar{H}(\mathscr{F})$, so that by Theorem B it remains to show that

$$\delta_B\eta = \delta_B(-J\zeta) = 0 . \tag{4.8}$$

We observe for $x \in M$

$$\begin{aligned}(\delta_B\eta)_x &= -(\delta_B J\zeta)_x = \sum_a (\nabla_{e_a}(J\zeta))(e_a) = \sum_a (J\nabla_{e_a}\zeta)(e_a) = \sum_a (\nabla_{e_a}\zeta)(Je_a)\\ &= \sum_a (\nabla_{e_a} d_B\alpha)(Je_a) = \sum_{a,b} g_Q(Je_a, e_b)\cdot(\nabla d_B\alpha)(e_a, e_b) .\end{aligned}$$

Now $g_Q(Je_a, e_b) = -g_Q(e_a, Je_b)$, while $(\nabla d_B\alpha)(e_a, e_b) = (\nabla d_B\alpha)(e_b, e_a)$. From this (4.8) follows.

The previous results show that the decomposition (4.2) represents $\bar{H}(\mathscr{F})$ as the (not necessarily direct) sum

$$\bar{H}(\mathscr{F}) = \bar{K}(\mathscr{F}) + J\bar{K}(\mathscr{F}) . \tag{4.9}$$

The canonical homomorphism of $\bar{K}(\mathscr{F})\otimes C$ onto $\bar{H}(\mathscr{F})$ has the kernel, the ideal of elements $\xi\otimes 1 + \xi'\otimes\sqrt{-1}$ with $\xi, \xi' \in \bar{K}(\mathscr{F})$ such that

$$\xi + J\xi' = 0 . \tag{4.10}$$

But $\xi = -J\xi'$ implies by (4.3)

$$d_B\xi = -d_B J\xi' = 0 .$$

Thus $\Delta_B\xi = 0$, and hence by (4.10) also $\Delta_B\xi' = \Delta_B J\xi = J\Delta_B\xi = 0$.

To complete the proof of Theorem D, we show that the harmonic part $\pi_B\omega$ in (4.1) defines a parallel vector field. Let

$$J\xi = J(\delta_B\beta) + J(\pi_B\omega) \equiv \Psi + \varphi \in J\bar{K}(\mathscr{F})\subset\bar{H}(\mathscr{F}) . \tag{4.11}$$

Since $\Delta_B\varphi = \Delta_B(J\pi_B\omega) = J\Delta_B\pi_B\omega = 0$, it follows that $d_B\varphi = 0$ and $\delta_B\varphi = 0$. By (4.3) we have $d_B(J\xi) = 0$, hence also $d_B\Psi = 0$. Thus the decomposition (4.11) satisfies the same conditions as (4.2), and by the arguments made above to prove $\xi \in \bar{K}(\mathscr{F})$, we conclude also that $\varphi \in \bar{K}(\mathscr{F})$. This condition together with $d_B\varphi = 0$ implies now $\nabla\varphi = 0$. But then

$$J\nabla(\pi_B\omega) = \nabla(J\pi_B\omega) = \nabla\varphi = 0$$

and thus $\nabla\pi_B\omega = 0$, as claimed. The parallel transversal holomorphic fields form a abelian Lie algebra, since for two such fields $\bar{Y}$, $\bar{Y}'$ we have $[\bar{Y}, \bar{Y}'] = 0$ as a consequence of the torsionfreeness of ∇. This completes the proof of Theorem D.

5. Kähler-Einstein foliations. This is a special case of the situation discussed above, where $\rho_\nabla = c\cdot\mathrm{id}\colon Q \to Q$ for some constant $c = c_\nabla/q$. By (3.1) it follows that $\omega \in \bar{H}(\mathscr{F})$ is characterized by $\Delta_B\omega = 2c\cdot\omega$.

If $c = 0$ this shows that $\bar{H}(\mathscr{F}) \cong \mathscr{H}^1_B$. Thus in this case $\bar{K}(\mathscr{F}) \cong \bar{H}(\mathscr{F})$.

COROLLARY 5.1. *Let $\mathscr{F}$ be a harmonic Kähler foliation with zero transversal Ricci curvature. Then $\bar{H}(\mathscr{F}) \cong \bar{K}(\mathscr{F})$, and this Lie algebra is an abelian subalgebra of the algebra of all parallel transversal fields.*

In view of Theorem C, the only case of interest for $c \neq 0$ is $c > 0$. In this case $\Delta_B\omega = 2c\cdot\omega$ applied to (4.1) implies that $\pi_B\omega = 0$. The previous arguments imply then the following result (see Matsushima [MA] for the case of a point foliation).

COROLLARY 5.2. *Let $\mathscr{F}$ be a harmonic Kähler-Einstein foliation with $c > 0$. Then $\bar{H}(\mathscr{F}) \cong \bar{K}(\mathscr{F})\oplus J\bar{K}(\mathscr{F})$.*

EXAMPLE 5.3. *Let* $P^{2n+1} \xrightarrow{\pi} CP^n$ be a principal circle bundle with a connection form η. Let ξ be the vertical vector field characterized by

$\eta(\xi) = 1$. Let $\bar{g}$ be the Fubini-Study metric of constant holomorphic sectional curvature 4 on CP^n. Then $g = \pi^*\bar{g} + \eta\otimes\eta$ defines a Riemannian metric on P, for which ξ is easily verified to be a unit Killing vector field. The fibers of π are geodesics (see e.g. [KT 1, 3.20]), and define a harmonic Kähler-Einstein foliation. Thus Corollary 5.2 applies and

$$\bar{H}(\mathscr{F}) = \bar{K}(\mathscr{F}) \oplus J\bar{K}(\mathscr{F}) . \tag{5.4}$$

We can moreover estimate the nullity of $\mathscr{F}$ as follows:

$$\text{nullity of } \mathscr{F} \geqq 2[(n+1)^2 - 1] . \tag{5.5}$$

In fact, the nullity of $\mathscr{F}$ is the dimension of the space of Jacobi fields of $\mathscr{F}$, and hence it exceeds the dimension of space of transversal Jacobi automorphisms of $\mathscr{F}$. But by Theorem A, the latter space coincides with the space of transversally holomorphic automorphisms of $\mathscr{F}$. By (5.4), this space has twice the dimension of the space of transversal Killing automorphisms of $\mathscr{F}$. It follows that

$$\text{nullity of } \mathscr{F} \geqq 2\cdot\dim(\mathrm{Isom}_0(CP^n)) = 2\cdot\dim SU(n+1) = 2[(n+1)^2 - 1] .$$

In view of [KT 1, (3.20)] this inequality holds more generally for the foliation defined by any principal G-bundle over CP^n. We further note that for the special case of the Hopf fibration $S^{2n+1} \to CP^n$, the index of $\mathscr{F}$ is $\geqq 2n+1$ by [KT 4], while the (complex) dimension of the space of holomorphic foliations near $\mathscr{F}$ is precisely $(n+1)^2 - 1$ by [DK, p. 79].

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