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Transversals for Families of Translates of a Two-Dimensional Convex Compact Set*

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Abstract. The following Theorem 1 gives an affirmative answer to Grünbaum's old question. Let F be a family of translates of a convex compact set $K \subset \mathbb{R}^2$. If every two elements of F have a common point, then there exist three points $A, B, C \in \mathbb{R}^2$ such that every element of F contains some of these points.

In the well-known survey paper [4] (see [7] and also Conjecture 6.2 in Chapter 2.1, p. 407, of [5]) Grünbaum posed the following question: for any family of translates of a convex compact set in a plane in which any two sets have nonempty intersection does there exist a 3-*transversal*, i.e., three points such that each set of the family contains at least one of them? There are some partial solutions: for unit disks [8], triangles [3], centrally symmetric sets [6], and sets of constant width [2]. In this paper we solve this problem.

Theorem 1. *For a family $F = \{K + x : x \in X\}$ of translates of a convex compact set K in $\mathbb{R}^2$ in which any two sets have a nonempty intersection there exists a* 3-*transversal*.

In [4] and [7] there is also some information concerning similar statements for $\mathbb{R}^n$, $n \geq 3$, e.g., an estimate on the order of a transversal for a family of translates of a convex compact set in which any two sets have a nonempty intersection. Further, from the statement of Theorem 2 it will be seen that this problem is closely related to the problem of partitioning a figure into smaller parts (Borsuk's problem) and covering a figure with smaller copies (Hadwiger's problem).

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Consider some auxiliary statements that will be used in the proof of Theorem 1. For any two sets A and B in $\mathbb{R}^n$ let $A+B=\{a+b \mid a \in A, b \in B\}$ denote the Minkowski's sum and $A-B=A+(-B)$.

The following is another formulation of Theorem 1:

Theorem 2. *If $X-X \subseteq K-K$ for a set $X \in \mathbb{R}^2$ and a convex compact set K, then X may be covered by three translates of $-K$.*

Remark. Since $K-K$ is centrally symmetric, the condition $X-X \subseteq K-K$ does not change when K is replaced by $-K$ and we construct a covering of X by three translates of K.

Lemma 1. *Theorems* 1 *and* 2 *are equivalent*.

Proof. We show that Theorem 2 implies Theorem 1. If $(K+x_1) \cap (K+x_2) \neq \emptyset$, then for any $x_1, x_2 \in X$ there exist points $p, y_1, y_2 \in K$ such that $p=x_1+y_1$ and $p=x_2+y_2$. Thus, $x_1-x_2=y_2-y_1$, so that $X-X \subseteq K-K$.

By Theorem 2 there exist points x_1, x_2, x_3 such that for any $x \in X$ there exists $i=1,2,3$ and $y \in K$ such that $x=x_i-y$, or, in other words, $x_i \in x+K$ for all $x \in X$ and some i. This is the statement of Theorem 1. ☐

The converse can be proved by the same reasoning in reverse order.

Definitions. Let $X \subset \mathbb{R}^n$ be a bounded set, let $K \subset \mathbb{R}^n$ be a convex compact set, and let $a \in S^{n-1}$; S^{n-1} is the unit sphere. The ratio $w(X,K,a)$ of distances between parallel supporting hyperplanes of X and K with the normal vector a is called the width of X relative to K in the direction of a. It may also be defined as the ratio of lengths of the images of X and K under the orthogonal projection onto the line spanned by a. It is clear that $w(X,K,a)=2w(X,K-K,a)$, $w(X-X,K,a)=2w(X,K,a)$, $w(X,K,a)=w(X-X,K-K,a)$, and that $w(X,K,a)$ is the distance between supporting hyperplanes of X with the normal vector a in the sense of the Banach metric $d_{\mathrm{B}}(x,y)$ defined by the unit ball $B_1=K-K$. A convex compact set X is called a body of constant width relative to K if $w(X,K,a)=\text{const}$.

The following simple lemma clarifies the geometric meaning of $X-X \subseteq K-K$.

Lemma 2. *The following statements are equivalent*:

(1) $X-X \subseteq K-K$.
(2) X *has diameter* diam $X \leq 1$ *in the Banach metric* $d_{\mathrm{B}}(x,y)$, $B_1=K-K$.
(3) $w(X,K,a) \leq 1$ *for each* a.

Proof. (1) ⇔ (2) Clearly.

(2) ⇒ (3) Using (1) ⇔ (2), we have $X-X \subseteq B_1$. Since $w(X,K,a)=w(X-X,B_1,a)$ it follows that $w(X,K,a) \leq 1$.

(3) $\Rightarrow$ (1) The sets B_1 and $X - X$ are both symmetric with the center $\mathbf{0}$; B_1 is convex and therefore is an intersection of centrally symmetric strips with center $\mathbf{0}$. Since $w(X, K, a) = w(X - X, B_1, a) \leq 1$, each such strip contains $\operatorname{conv}(X - X)$ and hence $\operatorname{conv}(X - X) \subseteq B_1$ and $X - X \subseteq B_1$. □

Since condition (3) of Lemma 2 will not change if we replace X by $\overline{\operatorname{conv}(X)}$, we may assume set X of Theorem 2 to be convex.

Lemma 3. *Let B be a two-dimensional Banach space, B_1 being the unit ball. If $X \subset B$ and* $\operatorname{diam}_B X \leq 1$, *then there exists a convex compact set F such that $X \subseteq F$ and $w(F, B, a) = \frac{1}{2}$ for all $a \in S^1$.*

Lemma 3 is well known (see p. 62 of [1]).

Now Theorem 2 may be deduced using Lemma 3 from the following special case:

Theorem 3. *If $X, K \subset \mathbb{R}^2$ are convex compact sets and $X - X = K - K$, then X may be covered by three translates of K.*

Remark. The condition $X - X = K - K$ means that X and K have the same width in every direction (see Lemma 2).

To prove Theorem 3, we need the next lemma:

Lemma 4. *Let $\triangle A_1 B_1 C_1$ be formed by the midpoints of the sides of $\triangle ABC$. If a line l does not intersect $\triangle A_1 B_1 C_1$ and is not parallel to any of its sides, then l together with the two lines containing sides of $\triangle ABC$, form a triangle of greater area than that of $\triangle ABC$.*

Proof of Lemma 4. We consider two essentially different cases (see Figs. 1 and 2):

Case 1: *line l does not intersect $\triangle ABC$ and A has the greatest distance from l*. Then, obviously, lines l, AB, and AC form a triangle of greater area than $\triangle ABC$.

Case 2: *line l intersects the sides AB and AC and the line BC in F, E, and D, respectively, and C lies between B and D*. By the assumption of Lemma 4, $AE < EC$. It is clear to see that the triangle symmetric to $\triangle DEC$ with respect to E contains $\triangle AFE$ and therefore $S_{\triangle AFE} < S_{\triangle DEC}$. Thus, $S_{\triangle BFD} > S_{\triangle ABC}$.

Other cases reduce to ones studied above by reordering vertices of $\triangle ABC$. □

We introduce some notation (see Fig. 3).

Let $a \in S^1$ and let X be a convex compact set. Denote by $l_+(a, X)$ and $l_-(a, X)$ the supporting lines of X perpendicular to a such that $\lambda_1 a \in l_-(a, X)$, $\lambda_2 a \in l_+(a, X)$ and $\lambda_2 > \lambda_1$. In other words, $(a, x) > 0$ for all $x \in l_+(a, X) - l_-(a, X)$ (where (a, x) is the scalar product). Note that Minkowski's sum of two parallel lines is a line parallel to them both. We denote

$$m(a, X) = 1/2(l_+(a, X) + l_-(a, X)) \quad \text{and} \quad l(a) = l_+(a, X) - l_+(a, K).$$

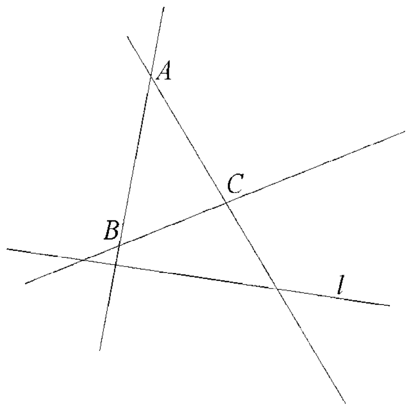

Fig. 1

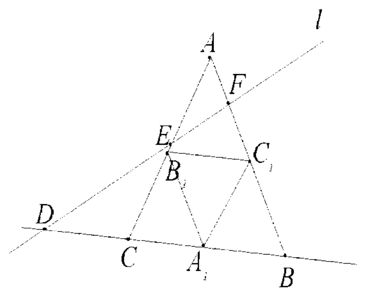

Fig. 2

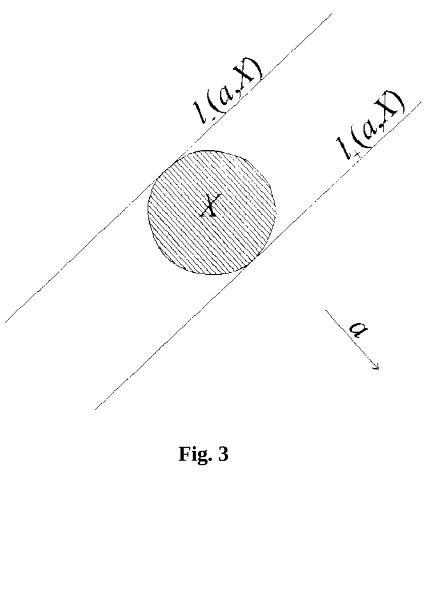

Fig. 3

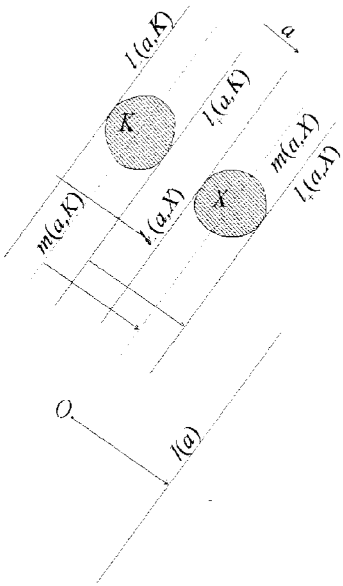

Fig. 4

It is clear that $m(a, X)$ is the equidistant line between $l_+(a, X)$ and $l_-(a, X)$.

Proof of Theorem 3. We first construct explicitly three translates of K and then prove that they cover X. Since $w(X, K, a) = 1$, it easily follows that

$$l(a) = l_-(a, X) - l_-(a, K) = m(a, X - K)$$

and hence $l(a) = l(-a)$ (see Fig. 4).

Obviously, $l(a)$ is a continuous function of $a \in S^1$. For any three mutually noncollinear vectors $a_1, a_2, a_3 \in S^1$, let $S(a_1, a_2, a_3)$ be the area of the triangle $T(a_1, a_2, a_3)$ formed by $l(a_1)$, $l(a_2)$, $l(a_3)$. If a_1, a_2 are noncollinear, then $x = m(a_1, X - K) \cap m(a_2, X - K)$ is the center of the parallelogram formed by the supporting lines $l_+(a_1, X - K)$, and $l_-(a_1, X - K)$, $l_+(a_2, X - K)$, and $l_-(a_2, X - K)$ of $X - K$. Therefore $x \in X - K$. Hence,

$$S(a_1, a_2, a_3) \le 1/2(\mathrm{diam}(X - K))^2 \sin\varphi,$$

where φ is the angle between $l(a_1)$ and $l(a_2)$. Therefore $S(a_1, a_2, a_3)$ tends to zero when the directions of any two $l(a_i)$ tend to each other. This implies that $S(a_1, a_2, a_3)$ may be regarded as a continuous function of arbitrary three unit vectors a_1, a_2 and a_3. Thus, S is a continuous function on the compact set $S^1 \times S^1 \times S^1$, and hence attains its maximum value M at a certain $(a_1, a_2, a_3) \in S^1 \times S^1 \times S^1$.

Now we consider two cases.

Case 1: $M = 0$. Then any three, and therefore all of $l(a)$, have the common point t. We

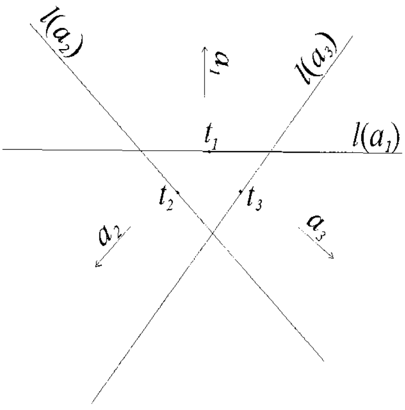

Fig. 5

claim that $X = K + t$. Indeed, we have

$$t \in l_+(a, X) - l_+(a, K) \quad \Rightarrow \quad 0 \in l_+(a, X) - l_+(a, K) - t,$$

i.e.,

$$0 \in l_+(a, X) - l_+(a, K + t) \quad \text{and} \quad l_+(a, K + t) = l_+(a, X).$$

Therefore, X and $K + t$ have the same supporting lines in all directions and we obtain $X = K + t$.

Case 2: *Assume that* $M = S(a_1, a_2, a_3) > 0$. Let t_1, t_2, and t_3 be the midpoints of the sides of the triangle $T(a_1, a_2, a_3)$. Consider the translates $K_i = K + t_i$. Since $l(a) = l(-a)$, we may assume that $(a_i, t_i - t_j) > 0$ $(i \neq j)$, changing signs of a_i where needed (see Fig. 5). For any $i = 1, 2, 3$ we take some points $y_i \in l_-(a_i, K)$ on the boundary of K. Now we prove that $y_i + t_i - t_j \in K$ or, equivalently, $y_i + t_i \in K_j$, $i, j = 1, 2, 3$ (see Fig. 6). It is sufficient to prove this assertion for $y_1 + t_1 - t_2$, $y_1 + t_1 - t_3$, since the argument does not depend on i.

Since $t_k - t_j \parallel l_-(a_i, K)$ $(i \neq j \neq k)$, it follows that

$$l_-(a_i, K) + t_i - t_j = l_-(a_i, K) + t_i - t_k \qquad \text{(see Fig. 6).}$$

Also since $t_i \in l(a_i) = l_-(a_i, X) - l_-(a_i, K)$, we have $l_-(a_i, K) + t_i = l_-(a_i, X)$.

First we prove that $l_+(a_2, K)$ and $l_+(a_3, K)$ cannot intersect K strictly between the lines $l_-(a_1, K)$ and $l_-(a_1, K) + t_1 - t_2$. Show, e.g., that $l_+(a_3, K) \cap K$ does not lie between $l_-(a_1, K)$ and $l_-(a_1, K) + t_1 - t_2$. We make a translation of all objects by t_2. Then K becomes K_2. Now we have to prove that $l_+(a_3, K_2) \cap K_2$ does not lie between $l_-(a_1, K_2)$ and $l_-(a_1, K_2) + t_1 - t_2 = l_-(a_1, K) + t_1 = l_-(a_1, X)$.

Assume the contrary and take $a \in S^1$ which becomes a_3 after a sufficiently small rotation toward a_1. If the rotation was small enough, then the lines $l_+(a_3, K_2)$ and $l_+(a, K_2)$ will intersect between $l_-(a_1, K_2)$ and $l_-(a_1, X)$ (see Fig. 7).

Now consider the lines $l_+(a_3, X)$ and $l_+(a, X)$. The line $l_+(a_3, X)$ is obtained from $l_+(a_3, K_2)$ by a translation by $t_3 - t_2$. Since $t_3 \in l_+(a_3, X) - l_+(a_3, K)$, it follows that $t_3 - t_2 \in l_+(a_3, X) - l_+(a_3, K_2)$.

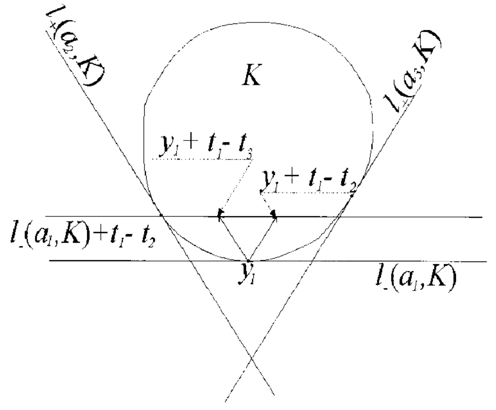

Fig. 6

Lemma 4 asserts that $l(a)$ intersects $\triangle t_1t_2t_3$ and, in this case, intersects the side t_2t_3. Since

$$l(a) = l_+(a, X) - l_+(a, K) = l_+(a, X) - l_+(a, K_2) + t_2,$$

$l_+(a, X)$ is obtained from $l_+(a, K_2)$ by translation by a vector collinear to $t_3 - t_2$ of smaller length. This means that the point $l_+(a_3, X) \cap l_+(a, X)$, as well as $l_+(a_3, K_2) \cap l_+(a, K_2)$, lies on the other side of $l_-(a_1, X)$ relative to X (see Fig. 7, the translations of $l_+(a_3, K_2)$ and $l_+(a, K_2)$ move its intersection point within the same open half-plane relative to $l_-(a_1, X)$).

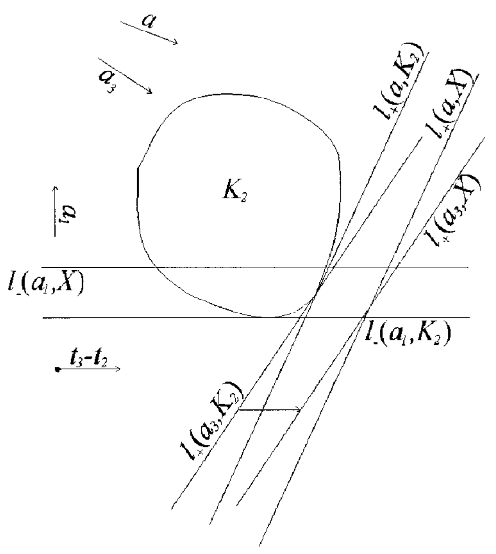

Fig. 7

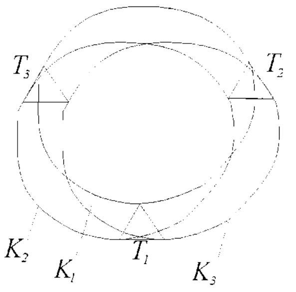

Fig. 8

Figure 7 also shows that in such an arrangement all three lines $l_+(a, X)$, $l_+(a_3, X)$, and $l_-(a_1, X)$ cannot be supporting lines for X. A contradiction.

We proved that $l_+(a_2, K)$ and $l_+(a_3, K)$ cannot intersect K strictly between the lines $l_-(a_1, K)$ and $l_-(a_1, K) + t_1 - t_2 = l_-(a_1, K) + t_1 - t_3$ (the latter contains $y_1 + t_1 - t_2$ and $y_1 + t_1 - t_3$). It follows that the convex hull of the intersections of $l_+(a_2, K)$ and $l_+(a_3, K)$ with K and y_1 contains y_1, $y_1 + t_1 - t_2$, and $y_1 + t_1 - t_3$ and lies in K (see Fig. 6).

Thus, $y_i + t_i \in \bigcap_j K_j$.

Denote by T_i the triangle with vertices at $y_i + t_j$ ($j = 1, 2, 3$, see Fig. 8). Let $C = \operatorname{conv} \bigcup_i K_i$. We show that

$$C = \bigcup_i K_i \cup T_i \qquad \text{(see Fig. 8)}.$$

Indeed, the line $l_i = l_-(a_i, K) + t_j = l_-(a_i, K) + t_k$ ($i \neq j \neq k$) is the supporting line of $K + t_j$, $K + t_k$ at $y_i + t_j$, $y_i + t_k$, respectively. The sets $K + t_i$, $K + t_j$, $K + t_k$ lie in the same half-plane relative to l_i. Therefore l_i is the supporting line of C, and we obtain

$$C = \bigcup_i K_i \cup T_i.$$

We prove that $X \subseteq C$. Assume the contrary and take a line separating some part of X from C (it exists, since the latter is convex). Let a be its normal vector. Then the origin does not lie between any two lines from $l_+(a, X) - l_+(a, K_i)$ which means that the t_i are on the same side from $l(a)$. A contradiction to Lemma 4. Since $l(a_i, K) + t_i = l_(a_i, X)$, we see that the triangle T_i is separated from X by $l_-(a_i, X)$ and therefore

$$X \subseteq C \setminus \left(\bigcup_i T_i \cup \{y_1 + t_1, y_2 + t_2, y_3 + t_3, \} \right) \subseteq \bigcup_i K_i.$$

Theorem 3 is proved. □

References

1. Chakerian, G.D., and Groemer, H., Convex bodies of constant width, in *Convexity and Its Applications*, P.M. Gruber and J.M. Wills, eds., Birkhäuser, Basel, 1983.

2. Chakerian, G.D., and Sallee, G.T., An intersection theorem for sets of constant width, *Duke Math. J.* **36** (1969), 165–170.
3. Chakerian, G.D., and Stein, S.K., Some intersection properties of convex bodies, *Proc. Amer. Math. Soc.* **18** (1967), 109–112.
4. Danzer, L., Grünbaum, B., and Klee, V., Helly's theorem and its relatives, in *Convexity*, pp. 101–181, Proc. of Symposia in Pure Mathematics, Vol. 7, American Mathematical Society, Providence, RI, 1963.
5. Eckhoff, J., Helly, Radon, and Carathéodory type theorems, in *Handbook of Convex Geometry*, P.M. Gruber and J.M. Wills, eds., North-Holland, Amsterdam, 1993.
6. Grünbaum, B., On intersection of similar sets, *Portugal. Math.* **18** (1959), 155–164.
7. Grünbaum, B., Borsuk's problem and and related questions, in *Convexity*, pp. 271–284, Proc. of Symposia in Pure Mathematics, Vol. 7, American Mathematical Society, Providence, RI, 1963.
8. Hadwiger, H., Debrunner, H., and Klee, V., *Combinatorial Geometry in the Plane*, Holt, Rinehart, and Winston, New York, 1964.

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