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TRANSVERSE KÄHLER GEOMETRY OF SASAKI MANIFOLDS AND TORIC SASAKI-EINSTEIN MANIFOLDS

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Abstract

In this paper we study compact Sasaki manifolds in view of transverse Kähler geometry and extend some results in Kähler geometry to Sasaki manifolds. In particular we define integral invariants which obstruct the existence of transverse Kähler metric with harmonic Chern forms. The integral invariant f_1 for the first Chern class case becomes an obstruction to the existence of transverse Kähler metric of constant scalar curvature. We prove the existence of transverse Kähler-Ricci solitons (or *Sasaki-Ricci soliton*) on compact toric Sasaki manifolds whose basic first Chern form of the normal bundle of the Reeb foliation is positive and the first Chern class of the contact bundle is trivial. We will further show that if S is a compact toric Sasaki manifold with the above assumption then by deforming the Reeb field we get a Sasaki-Einstein structure on S. As an application we obtain Sasaki-Einstein metrics on the $U(1)$-bundles associated with the canonical line bundles of toric Fano manifolds, including as a special case an irregular toric Sasaki-Einstein metrics on the unit circle bundle associated with the canonical bundle of the two-point blow-up of the complex projective plane.

1. Introduction

Sasaki manifolds can be studied from many view points as they have many structures. They are characterized by having a Kähler structure on the cone. They have a one dimensional foliation, called the Reeb foliation or Reeb flow, which has a transverse Kähler structure. They also have a contact structure, which provides us a moment map. These structures are intimately related each other, but when we study the deformations of Sasaki structures it is efficient to fix some of the structures and vary other structures. In this paper we study the deformations of Sasaki structure fixing the Reeb foliation together with its transverse holomorphic structure and the holomorphic structure of the cone, while

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varying the Kähler metric on the transverse holomorphic structure and, as a result, the contact structure.

Sasaki geometry is often described as an odd dimensional analogue of Kähler geometry. The above deformations on a Sasaki manifold correspond to the deformations of Kähler forms in a fixed Kähler class on a Kähler manifold. Therefore we may try to extend results related to Calabi's extremal problem in Kähler geometry to the above setting in Sasaki geometry. The normal bundle of the Reeb foliation has basic Chern forms which are expressed by basic differential forms. The basic forms are those differential forms obtained by pulling back differential forms on the local leaf spaces of the Reeb foliation, see section 4 for the precise definition. The basic first Chern class c_1^B of the normal bundle of the Reeb foliation is said to be positive if it is represented by a positive basic 2-form in the sense of Kähler geometry on the local leaf spaces; We will describe this condition by the notation $c_1^B > 0$. When one considers the problem of finding a Sasaki-Einstein metric, which is one of the main interests of this paper, it is reduced to the problem of finding a transverse Kähler-Einstein metric. A Sasaki-Eisntein metric must satisfy $Ric = 2m\, g$ if $\dim S = 2m+1$, and then the transverse Kähler-Einstein metric satisfies

$$\rho^T = (2m+2)\omega^T$$

where ω^T and ρ^T are respectively the transverse Kähler form and the transverse Ricci form. See Section 3 below. Since $\rho^T/2\pi$ represents the basic first Chern class c_1^B and ω^T is given by $\frac{1}{2}d\eta$, η being the contact 1-form dual to the Reeb vector field, one is naturally lead to another assumption $c_1(D) = 0$ as a de Rham cohomology class where $D = \mathrm{Ker}\,\eta$ is the contact bundle, see Proposition 4.3 for the detail. These two conditions $c_1^B > 0$ and $c_1(D) = 0$ are primary obstructions for the existence of a transverse Kähler-Einstein metric of positive scalar curvature, or equivalently for the existence of a Sasaki-Einstein metric. Proposition 4.3 asserts that the two conditions $c_1^B > 0$ and $c_1(D) = 0$ are equivalent to that c_1^B is represented by $\tau d\eta$ for some positive constant τ. By changing η by homothety we may then assume that $2\pi c_1^B = (2m+2)[\omega^T]$.

In this paper we first extend obstructions to the existence of Kähler metric of harmonic Chern forms [**2**] to the transverse Kähler geometry of compact Sasaki manifolds. The invariant f_1 for the first Chern form is an obstruction to the existence of transverse Kähler metric of constant scalar curvature, which is a secondary obstruction to the existence of transverse Kähler-Einstein metric when $c_1^B > 0$ and $c_1(D) = 0$. This extension has been obtained recently by Boyer, Galicki and Simanca [**12**] independently.

More generally than transverse Kähler-Einstein metrics of positive scalar curvature we consider the transverse Kähler-Ricci solitons, or *Sasaki-Ricci solitons* for short. A Sasaki metric is called a Sasaki-Ricci soliton if there exists a Hamiltonian holomorphic vector field X such that

$$\rho^T - (2m+2)\omega^T = \mathcal{L}_X \omega^T.$$

Hamiltonian holomorphic vector fields are defined in Definition 4.5 and $\mathcal{L}_X$ stands for the Lie derivative by X. In the above equation the imaginary part of X is necessarily a Killing vector field. In general a Sasaki-Ricci soliton is a Sasaki-Einstein metric if and only if f_1 vanishes.

By definition a Sasaki manifold is toric if its Kähler cone is toric (see also Definition 6.5). One of our main results is stated as follows.

Theorem 1.1. *Let S be a compact toric Sasaki manifold with $c_1^B > 0$ and $c_1(D) = 0$. Then there exists a Sasaki metric which is a Sasaki-Ricci soliton. In particular S admits a Sasaki-Einstein metric if and only if f_1 vanishes.*

This is a Sasakian version of a result of Wang and Zhu [**41**] in the Kählerian case. As an application of Theorem 1.1 we obtain the following.

Theorem 1.2. *Let S be a compact toric Sasaki manifold with $c_1^B > 0$ and $c_1(D) = 0$. Then by deforming the Sasaki structure varying the Reeb field we get a Sasaki-Einstein structure.*

Recently, new obstructions to Sasaki-Einstein metrics were studied by Gauntlett, Martelli, Sparks and Yau [**25**]. Theorem 1.2 matches their expectation that those new obstructions are not obstructions in the toric case, see section 6 of [**25**].

The results obtained in this paper has many interesting applications. A straightforward application is the existence of Sasaki-Einstein metrics on the total space of the $U(1)$-bundles associated with the canonical line bundles of toric Fano manifolds. A Sasaki manifold is said to be quasi-regular if all the leaves of the Reeb foliation are compact, and irregular otherwise. A quasi-regular Sasaki manifold is said to be regular if the Reeb foliation is obtained by a free S^1-action.

Corollary 1.3. *There exist Sasaki-Einstein metrics on the total spaces of the $U(1)$-bundles associated with the powers of canonical line bundles of toric Fano manifolds. In particular there exists an irregular toric Sasaki-Einstein metric on the circle bundle associated with the powers of the canonical line bundle of the two-point blow-up of the complex projective plane.*

Examples of irregular Sasaki-Einstein metrics have been only recently known by Gauntlett, Martelli, Sparks and Waldrum [**23**], [**24**], [**31**], [**32**]. They include an irregular toric Sasaki-Einstein metric on the circle bundle associated with the canonical bundle of the one-point blow-up of the complex projective plane. As far as the authors know an irregular example for the two-point blow-up case has not been known. The irregularity of the example mentioned in Corollary 1.3 follows from the computations given by Martelli, Sparks and Yau [**33**]; there is an earlier computation by Bertolini, Bigazzi and Cotrone for the dual theory through AdS/CFT correspondence [**4**]. For a different derivation of irregular Sasaki-Einstein metric on $S^2 \times S^3$ see Hashimoto, Sakaguchi and Yasui [**27**], and for other new Sasaki-Einstein metrics see also [**17**] and [**15**]. It should be mentioned that, before those irregular examples were found, many quasi-regular examples were constructed by Boyer, Galicki, Kollár and their collaborators. These constructions are surveyed in the article [**7**] (see also [**10**]).

As other applications, in [**16**] K. Cho and the first and the second authors applied the existence obtained in this paper to obtain an infinite family of inequivalent toric Sasaki-Einstein metrics on the k-fold connected sum $S^5\#k(S^2 \times S^3)$ of $S^2 \times S^3$ with S^5. In [**16**] it is also shown that the condition for a toric Sasaki manifold to satisfy $c_1^B > 0$ and $c_1(D) = 0$ is equivalent to that the toric diagram has fixed height, and is also equivalent to that $K_{C(S)}^{\otimes \ell}$ is trivial for some ℓ. For further related topics refer to [**9**] and [**37**].

This paper is organized as follows. In section 2 and 3 we review Sasaki geometry, and give a rigorous treatment for the transverse Kähler geometry. In section 4 we introduce Hamiltonian holomorphic vector fields, and define the integral invariants f_k as Lie algebra characters of the Lie algebra of all Hamiltonian holomorphic vector fields. In section 5 we introduce Sasaki-Ricci soliton and a generalized integral invariant which is useful for the study of Sasaki-Ricci soliton. We also set up Monge-Ampère equation on Sasaki manifolds to prove the existence of Sasaki-Ricci soliton. In section 6 we review known facts about toric Sasaki manifolds. In particular we describe the space of all Kähler cone metrics of Sasaki structures whose basic Kähler class is equal to $1/(2m+2)$ times basic first Chern class. This space plays an important role in the study of the volume functional. In section 7 we show that we can give a C^0 estimate and completes the proof of the existence of Sasaki-Ricci solitons on compact toric manifolds of positive basic first Chern class. The point is that we can use the Guillemin metric to get a nice initial metric so that the analysis of Wang and Zhu can be used. In section 8 we will see that on compact toric Sasaki manifolds we can always find a Sasaki structure with $f_1 = 0$, thus proving Theorem 1.2. Then we prove Corollary 1.3.

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2. Sasaki manifolds

In this section we describe a Sasaki manifold in terms of two complex structures. One is given on the cone over the Sasaki manifold, and the other is given on the transverse holomorphic structure for a one dimensional foliation, called the Reeb foliation. Proofs of the results in this section are concisely summarized in the papers of Boyer and Galicki [**5**] and Martelli, Sparks and Yau [**34**], and the recent monograph [**8**] is also good to refer to.

Let (S, g) be a Riemannian manifold, ∇ the Levi-Civita connection of the Riemannian metric g, and let $R(X, Y)$ and Ric respectively denote the Riemann curvature tensor and Ricci tensor of ∇.

Definition 2.1. (S, g) is said to be a Sasaki manifold if the cone manifold $(C(S), \overline{g}) = (\mathbb{R}_+ \times S,\ dr^2 + r^2 g)$ is Kähler.

S is often identified with the submanifold $\{r = 1\} = \{1\} \times S \subset C(S)$. Thus the dimension of S is odd and denoted by $2m + 1$. Hence $\dim_{\mathbb{C}} C(S) = m+1$. Let J denote the complex structure of $C(S)$. Define a vector field ξ on S and a 1-form η on S by

$$\xi = J\frac{\partial}{\partial r}, \quad \eta(Y) = g(\xi, Y) \tag{1}$$

where Y is any smooth vector field on S. One can see that

(a) ξ is a Killing vector field on S;
(b) the integral curve of ξ is a geodesic;
(c) $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any vector field X on S.

The vector field ξ is called the *characteristic vector field* or the *Reeb field*. The 1-dimensional foliation generated by ξ is called the *Reeb foliation* or *Reeb flow*. The 1-form η defines a $2m$-dimensional vector sub-bundle D of the tangent bundle TS, where at each point $p \in S$ the fiber D_p of D is given by

$$D_p = \operatorname{Ker} \eta_p.$$

There is an orthogonal decomposition of the tangent bundle TS

$$TS = D \oplus L_\xi,$$

where L_ξ is the trivial bundle generated by the Reeb field ξ.

We next define a section Φ of the endomorphism bundle $\mathrm{End}(TS)$ of the tangent bundle TS by setting $\Phi|_D = J|_D$ and $\Phi|_{L_\xi} = 0$ where we identified S with the submanifold $\{r = 1\} \subset C(S)$. Φ satisfies

$$\Phi^2 = -I + \eta \otimes \xi$$

and

$$g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

One can see that Φ can also be defined by

$$\Phi(X) = \nabla_X \xi \tag{2}$$

for any smooth vector field X on S. From these descriptions it is clear that $g|_D$ is an Hermitian metric on D. An important property of Sasaki manifolds is that $\frac{1}{2}(d\eta)|_D$ is the associated 2-form of the Hermitian metric $g|_D$:

$$(d\eta)(X, Y) = 2g(\Phi(X), Y) \tag{3}$$

for all smooth vector fields X and Y. Hence $d\eta$ defines a symplectic form on D, and η is a contact form on S in the sense that $\eta \wedge \left(\frac{1}{2}d\eta\right)^m$ is nowhere vanishing. In particular $\eta \wedge \left(\frac{1}{2}d\eta\right)^m$ defines a volume element on S.

We have ξ and η also on the cone $C(S)$ by putting

$$\xi = Jr\frac{\partial}{\partial r}, \qquad \eta(Y) = \frac{1}{r^2}\overline{g}(\xi, Y)$$

where Y is any smooth vector field on $C(S)$. Of course η on $C(S)$ is the pull-back of η on S by the projection $C(S) \to S$. Since $L_\xi J = 0$ the complex vector field $\xi - iJ\xi = \xi + ir\frac{\partial}{\partial r}$ is a holomorphic vector field on $C(S)$, which preserves the cone structure. There is a holomorphic $\widetilde{\mathbb{C}^*}$-action on $C(S)$ generated by $\xi - iJ\xi$ where $\widetilde{\mathbb{C}^*}$ denotes the universal cover of $\mathbb{C}^*$. If the Sasaki manifold is quasi-regular $\xi - iJ\xi$ generates a $\mathbb{C}^*$-action. The local orbits of this action then give the Reeb foliation a transversely holomorphic structure. As we will see in the next section this transversely holomorphic foliation on S is a Kähler foliation in the sense that it has a bundle-like transverse Kähler metric.

It is this transverse Kähler structure that we wish to study in this paper. We will study in later sections the deformations of transverse Kähler structures. They correspond to the deformations of η fixing ξ. A fixed choice of ξ is called the polarization of the Sasaki manifold in [**12**]. As η varies the contact bundle D varies, while we fix the transverse holomorphic structure of the Reeb foliation and the holomorphic structure of the cone $C(S)$, see Proposition 4.2 below.

To conclude this section we summarize the well-known facts about Sasaki geometry. First of all Φ defined above on the Sasaki manifold S satisfies

$$(\nabla_X \Phi)(Y) = g(\xi, Y)X - g(X, Y)\xi \tag{4}$$

for any pair of vector fields X and Y on S.

Conversely, if ξ is a Killing vector field of unit length on a Riemannian manifold S and the $(1,1)$-tensor Φ defined by $\Phi(X) = \nabla_X \xi$ satisfies (4), then the cone $(C(S), dr^2 + r^2 g)$ is a Kähler manifold and thus S becomes a Sasaki manifold. Here the complex structure J on $C(S)$ is defined as follows :

$$Jr\frac{\partial}{\partial r} = \xi, \qquad JY = \Phi(Y) - \eta(Y)r\frac{\partial}{\partial r} \qquad \text{for } Y \in \Gamma(TS).$$

Moreover the following conditions are equivalent and can be used as a definition of Sasaki manifolds.

(2.a) There exists a Killing vector field ξ of unit length on S so that the tensor field Φ of type $(1,1)$, defined by $\Phi(X) = \nabla_X\xi$, satisfies (4).

(2.b) There exists a Killing vector field ξ of unit length on S so that the Riemann curvature satisfies the condition

$$R(X,\xi)Y = g(\xi,Y)X - g(X,Y)\xi,$$

for any pair of vector fields X and Y on S.

(2.c) There exists a Killing vector field ξ of unit length on S so that the sectional curvature of every section containing ξ equals one.

(2.d) The metric cone $(C(S),\bar g) = (\mathbb{R}_+ \times S,\ dr^2 + r^2 g)$ over S is Kähler.

Returning to the geometry of $C(S)$, we recall the following facts. The 1-form η is expressed on $C(S)$ as

$$\eta = 2d^c \log r \tag{5}$$

where $d^c = \frac{i}{2}(\bar\partial - \partial)$. This easily follows from

$$\eta(Y) = \frac{1}{r^2}\bar g(\xi, Y) = \frac{1}{r^2}\bar g\left(Jr\frac{\partial}{\partial r}, Y\right).$$

From (5), the Kähler form of the cone $(C(S), dr^2 + r^2 g)$ is expressed as

$$\frac{1}{2}d\left(r^2\eta\right) = \frac{1}{2}dd^c r^2. \tag{6}$$

3. Transverse holomorphic structures and transverse Kähler structures

In the sequel, we always assume that S is a Sasaki manifold with (ξ,η,g,Φ). To understand the Sasaki structure well, we need to exploit the *transverse Kähler* structure on S. Let $\mathcal{F}_\xi$ be the *Reeb foliation* generated by ξ. As we saw in the previous section $\xi - iJ\xi = \xi + ir\frac{\partial}{\partial r}$ is a holomorphic vector field on $C(S)$, and there is an action on $C(S)$ of the holomorphic flow generated by $\xi - iJ\xi$. The local orbits of this action defines a transversely holomorphic structure on the Reeb foliation $\mathcal{F}_\xi$ in the following sense.

Let $\{U_\alpha\}_{\alpha\in A}$ be an open covering of S and $\pi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{C}^m$ submersions such that when $U_\alpha \cap U_\beta \neq \emptyset$

$$\pi_\alpha \circ \pi_\beta^{-1} : \pi_\beta(U_\alpha \cap U_\beta) \to \pi_\alpha(U_\alpha \cap U_\beta)$$

is biholomorphic. On each V_α we can give a Kähler structure as follows. Let $D = \mathrm{Ker}\ \eta \subset TS$. There is a canonical isomorphism

$$d\pi_\alpha : D_p \to T_{\pi_\alpha(p)}V_\alpha,$$

for any $p \in U_\alpha$. Since ξ generates isometries the restriction of the Sasaki metric g to D gives a well-defined Hermitian metric g^T_α on V_α. This Hermitian structure is in fact Kähler, which can be seen as follows.

Let $z^1, z^2, \cdots z^m$ be the local holomorphic coordinates on V_α. We pull back these to U_α and still write them as $z^1, z^2, \cdots z^m$. Let x be the coordinate along the leaves with $\xi = \frac{\partial}{\partial x}$. Then $x, z^1, z^2, \cdots z^m$ form local coordinates on U_α. $(D \otimes \mathbb{C})^{1,0}$ is spanned by the vectors of the form

$$\frac{\partial}{\partial z^i} + a_i\xi, \quad i = 1, 2 \cdots, m.$$

It is clear that

$$a_i = -\eta\left(\frac{\partial}{\partial z^i}\right).$$

Since $i(\xi)d\eta = 0$,

$$d\eta\left(\frac{\partial}{\partial z^i} + a_i\xi, \overline{\frac{\partial}{\partial z^j} + a_j\xi}\right) = d\eta\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\overline{\partial z^j}}\right).$$

Thus the fundamental 2-form ω_α of the Hermitian metric g^T_α on V_α is the same as the restriction of $\frac{1}{2}d\eta$ to the slice $\{x = \text{constant}\}$ in U_α. Since the restriction of a closed 2-form to a submanifold is closed in general, then ω_α is closed. By this construction

$$\pi_\alpha \circ \pi_\beta^{-1} : \pi_\beta(U_\alpha \cap U_\beta) \to \pi_\alpha(U_\alpha \cap U_\beta)$$

gives an isometry of Kähler manifolds. Therefore, the foliation thus defined is a transversely Kähler foliation. The collection of Kähler metrics $\{g^T_\alpha\}$ on $\{V_\alpha\}$ is called a transverse Kähler metric. Since they are isometric over the overlaps we suppress α and denote by g^T. We also write ∇^T, R^T, Ric^T and s^T for its Levi-Civita connection, the curvature, the Ricci tensor and the scalar curvature; of course these are collections of those defined on $\{V_\alpha\}$. It should be emphasized that, though g^T are defined only locally on each V_α, the pull-back to U_α of the Kähler forms ω_α on V_α patch together and coincide with the global form $\frac{1}{2}d\eta$ on S, and $\frac{1}{2}d\eta$ can even be lifted to the cone $C(S)$ by the pull-back. For this reason we often refer to $\frac{1}{2}d\eta$ as the Kähler form of the transverse Kähler metric g^T. Although it is possible to re-define g^T at this stage as a global tensor on S by setting $g^T = \frac{1}{2}d\eta(\cdot, \Phi\cdot)$, we do not take this view point because this may lead to a confusion about the space where the Kählerian geometry is performed. However, notice also that the transverse scalar curvature s^T also lifts to S as a global function which together with the lifted Kähler form $\frac{1}{2}d\eta$ on S is often used later in order to study the global properties of the Sasaki structures, e.g. to define integral invariants.

One can easily check the following. For $\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{W} \in \Gamma(D)$ and $X, Y, Z, W \in \Gamma(TV_\alpha)$ with

$$d\pi_\alpha(\widetilde{X}) = X, \quad d\pi_\alpha(\widetilde{Y}) = Y, \quad d\pi_\alpha(\widetilde{Z}) = Z, \quad d\pi_\alpha(\widetilde{W}) = W$$

we have

$$\begin{aligned}
\nabla^T_X Y &= d\pi_\alpha(\nabla_{\widetilde{X}} \widetilde{Y}), && (7)\\
\nabla_{\widetilde{X}} \widetilde{Y} &= \widetilde{\nabla^T_X Y} - g(JX, Y)\xi, && (8)\\
R(\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{W}) &= R^T(X, Y, Z, W) - g(J\widetilde{Y}, \widetilde{W})g(J\widetilde{X}, \widetilde{Z}) \\
&\quad + g(J\widetilde{X}, \widetilde{W})g(J\widetilde{Y}, \widetilde{Z}), && (9)\\
Ric^T(X, Z) &= Ric(\widetilde{X}, \widetilde{Z}) + 2g(\widetilde{X}, \widetilde{Z}). && (10)
\end{aligned}$$

Definition 3.1. A Sasaki manifold (S, g) is η-Einstein if there are two constants λ and ν such that

$$Ric = \lambda g + \nu \eta \otimes \eta.$$

In this case $\lambda + \nu = 2m$ always holds since $Ric(\xi, \xi) = 2m$ (see (2.b) of the previous section).

Definition 3.2. A Sasaki-Einstein manifold is a Sasaki manifold $(S, g)$with $Ric = 2m\, g$.

Notice that a Sasaki manifold satisfying the Einstein condition is necessarily Ricci positive.

Definition 3.3. A Sasaki manifold S is said to be transversely Kähler-Einstein if

$$Ric^T = \tau g^T,$$

for some real constant τ.

It is well-known that if S is a transversely Kähler-Einstein Sasaki manifold if and only if (S, g) is η-Einstein (cf. [**11**]). In fact, if $Ric^T = \tau g^T$ then

$$Ric = (\tau - 2)g + (2m + 2 - \tau)\eta \otimes \eta. \tag{11}$$

Conversely if $Ric = \lambda g + \nu \eta \otimes \eta$ then

$$Ric^T = (\lambda + 2)g^T. \tag{12}$$

Thus if $\lambda > -2$ then Ric^T is positive definite. In this case by the $\mathcal{D}$-homothetic transformation $g' = \alpha g + \alpha(\alpha - 1)\eta \otimes \eta$ with $\alpha = \frac{\lambda+2}{2m+2}$ we get a Sasaki-Einstein metric g' with $Ric(g') = 2m\, g'$ (Tanno [**38**]).

Thus there exists a Sasaki-Einstein metric if and only if there exists a transverse Kähler-Einstein metric of positive transverse Ricci curvature. In other words if we can find an obstruction to the existence of transverse

Kähler-Einstein metric of positive transverse Ricci tensor then it is an obstruction to the existence of Sasaki-Einstein metric. The invariant f given in Theorem 4.9 is one of such obstructions. The invariant f has been also defined in [**12**].

On the other hand, the Gauss equation relating the curvature of submanifolds to the second fundamental form shows that *a Sasaki metric is Einstein if and only if the cone metric on $C(S)$ is Ricci-flat Kähler.* In particular the Kähler cone of a Sasaki-Einstein manifold has trivial canonical bundle. In section 8 we will reformulate f as an obstruction for a Kähler cone $C(S)$ with flat canonical bundle to admit a Ricci-flat Kähler cone metric. From (11) we see that $\tau = 2m+2$ for a Sasaki-Einstein metric, and thus we have

$$Ric = 2mg$$

and

$$Ric^T = (2m+2)g^T. \tag{13}$$

This also follows from (10).

4. Basic forms and deformations of Sasaki structures

Let S be a compact Sasaki manifold.

Definition 4.1. A p-form α on S is called basic if

$$i(\xi)\alpha = 0, \quad \mathcal{L}_\xi \alpha = 0.$$

Let Λ^p_B be the sheaf of germs of basic p-forms and $\Omega^p_B = \Gamma(S, \Lambda^p_B)$ the set of all global sections of Λ^p_B.

Let $(x, z^1, \cdots, z^m)$ be the coordinates system U_α given above. We call such a coordinate system a *foliation chart*. If $U_\alpha \cap U_\beta \neq \emptyset$ and $(y, w^1, \cdots, w^m)$ is the foliation chart on U_β, then

$$\frac{\partial z^i}{\partial \overline{w}^j} = 0, \quad \frac{\partial z^i}{\partial y} = 0.$$

These mean that the form of type (p,q)

$$\alpha = a_{i_1\cdots i_p \bar{j}_1 \cdots \bar{j}_q} dz^{i_1} \wedge \cdots dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}$$

is also of type (p,q) with respect to $(y, w^1, \cdots, w^m)$. If α is basic, then $a_{i_1\cdots i_p \bar{j}_1 \cdots \bar{j}_q}$ does not depend on x. We thus have the well-defined operators

$$\begin{array}{rcl} \partial_B : \Lambda^{p,q}_B & \to & \Lambda^{p+1,q}_B, \\ \overline{\partial}_B : \Lambda^{p,q}_B & \to & \Lambda^{p,q+1}_B. \end{array}$$

It is easy to see that $d\alpha$ is basic if α is basic. If we set $d_B = d|_{\Omega^p_B}$ then we have $d_B = \partial_B + \bar{\partial}_B$. Let $d^c_B = \frac{i}{2}(\overline{\partial}_B - \partial_B)$. It is clear that

$$d_B d^c_B = i\partial_B \overline{\partial}_B, \qquad d^2_B = (d^c_B)^2 = 0.$$

Let $d_B^*: \Omega_B^{p+1} \to \Omega_B^p$ be the adjoint operator of $d_B: \Omega_B^p \to \Omega_B^{p+1}$. The basic Laplacian Δ_B is defined

$$\Delta_B = d_B^* d_B + d_B d_B^*.$$

Thus we can consider the *basic* de Rham complex (Ω_B^*, d_B) and the *basic* Dolbeault complex $(\Omega^{p,*}, \overline{\partial}_B)$ whose cohomology groups are called the *basic* cohomology groups. Similarly, we can consider the *basic* harmonic forms. Results of El Kacimi-Alaoui [**18**] assert that we have the expected isomorphisms between basic cohomology groups and the space of basic harmonic forms.

Suppose (ξ, η, Φ, g) defines a Sasaki structure on S. We define a new Sasaki structure fixing ξ and varying η as follows. Let $\varphi \in \Omega_B^0$ be a smooth basic function. Put

$$\tilde{\eta} = \eta + 2d_B^c\varphi.$$

It is clear that

$$d\tilde{\eta} = d\eta + 2d_B d_B^c \varphi = d\eta + 2i\partial_B \bar{\partial}_B \varphi.$$

For small φ, $\tilde{\eta}$ is non-degenerate in the sense that $\tilde{\eta} \wedge (d\tilde{\eta})^m$ is nowhere vanishing.

Proposition 4.2. *Given a small smooth basic function φ, there exists a Sasaki structure on S with the same ξ, the same holomorphic structure on the cone $C(S)$ and the same transversely holomorphic structure of the Reeb foliation $\mathcal{F}_\xi$ but with the new contact form $\widetilde{\eta} = \eta + 2d_B^c\varphi$.*

Proof. Put

$$\widetilde{r} = r \exp(\varphi). \tag{14}$$

Then $\frac{1}{2} dd^c \widetilde{r}^2$ gives a new Kähler structure on $C(S)$ and thus a new Sasaki structure on S. Obviously the holomorphic structure on $C(S)$ is unchanged for the new Kähler structure. By (14),

$$\widetilde{\eta} = 2d^c \log \widetilde{r} = \eta + 2d_B^c \varphi.$$

It follows from this that the Reeb field is also unchanged because φ is basic. From the expression (14) one sees

$$\widetilde{r}\frac{\partial}{\partial \widetilde{r}} = r\frac{\partial}{\partial r}.$$

This shows that the transverse holomorphic structures of the Reeb foliation is also unchanged. q.e.d.

Thus the deformation

$$\eta \to \tilde{\eta} = \eta + 2d_B^c \varphi$$

gives a deformation of Sasaki structure with the same transversely holomorphic foliation and the same holomorphic structure on the cone $C(S)$ and deforms the transverse Kähler form in the same basic (1,1) class.

We call this class the *basic Kähler class* of the Sasaki manifold S. Note that, however, the contact bundle D may change under such a deformation.

Now we define a 2-form ρ^T called the *transverse Ricci form* as follows. This is first defined as a collection of $(1,1)$ forms ρ^T_α on V_α given by

$$\rho^T_\alpha = -i\partial\overline{\partial} \log \det \left(g^T_\alpha\right).$$

These are just the Ricci forms of the transverse Kähler metrics g^T_α. One can see that the pull backs $\pi^*_\alpha \rho^T_\alpha$ by $\pi_\alpha : U_\alpha \to V_\alpha$ patch together to give a global basic 2-form on S, which is our ρ^T. As in the Kähler case ρ^T is d_B closed and define a basic cohomology class of type $(1,1)$. The basic cohomology class $[\rho^T]$ is independent of the choice of the transverse Kähler form $\frac{1}{2}d\eta$ in the fixed basic $(1,1)$ class. The basic cohomology class $[\frac{1}{2\pi}\rho^T]$ is called the *basic first Chern class* of S, and is denoted by $c^B_1(S)$. We say that the basic first Chern class of S is positive (resp. negative) if $c^B_1(S)$ (resp. $-c^B_1(S)$) is represented by a positive basic form; This condition is expressed by $c^B_1 > 0$ (resp. $c^B_1 < 0$). When $c^B_1 > 0$ or $c^B_1 < 0$ then we say that c^B_1 is definite. If there exists a transverse Kähler-Einstein metric then the basic first Chern class has to be positive, zero or negative according to the sign of the constant τ with $Ric^T = \tau g^T$. In particular if S has a Sasaki-Einstein metric then the basic first Chern class is positive. There are further necessary condition for the existence of positive or negative transverse Kähler-Einstein metric:

Proposition 4.3. *The basic first Chern class is represented by $\tau d\eta$ for some constant τ if and only if $c_1(D) = 0$ where $D = \mathrm{Ker}\,\eta$ is the contact bundle.*

Proof. This proposition follows from the long exact sequence (c.f. [**40**]):

$$\longrightarrow H^0_B(S) \xrightarrow{\delta} H^2_B(S) \xrightarrow{i_2} H^2(S;\mathbb{R}) \longrightarrow$$

where $\delta a = a[d\eta]$ and $i_2[\rho] = [\rho]$. Note $i_2 c^B_1 = c_1(D)$ in general. q.e.d.

Thus if S admits a transverse Kähler-Einstein metric $\rho^T = \tau\omega^T$ then $c_1(D) = 0$. As was pointed out by Boyer, Galicki and Matzeu [**11**], in the negative and zero basic first class case with $c_1(D) = 0$ the results of El Kacimi-Alaoui [**18**] together with Yau's estimate [**42**] imply that the existence of transverse Kähler-Einstein metric. One of main purposes of the present paper is to consider the remaining positive case, assuming $c_1(D) = 0$.

Proposition 4.4. *The quantity*

$$\int_S \rho^T \wedge (d\eta)^{m-1} \wedge \eta$$

is independent of the choice of Sasaki structures with the same basic Kähler class.

Proof. Let $\varphi \in \Omega^0_B$ and put $\eta_t = \eta + t d^c_B\varphi$ for small t. It is enough to show that

$$\frac{d}{dt}\int_S \rho^T_t \wedge (d\eta_t)^{m-1} \wedge \eta_t = 0.$$

It is not difficult to check that

$$\begin{aligned}\frac{d}{dt}\int_S \rho^T_t \wedge (d\eta_t)^{m-1} \wedge \eta_t \;&=\; \int_S d_B d^c_B(\Delta_B\varphi) \wedge (d\eta_t)^{m-1} \wedge \eta_t \\ &\quad +(m-1)\int_S \rho^T_t \wedge (d\eta_t)^{m-2} \wedge d_B d^c_B\varphi \wedge \eta_t \\ &\quad + \int_S \rho^T_t \wedge (d\eta_t)^{m-1} \wedge d^c_B\varphi.\end{aligned}$$

Since ρ^T_t, $d\eta_t$, $d^c_B\varphi$ are basic, we have $\rho^T_t \wedge (d\eta_t)^{m-1} \wedge d^c_B\varphi = 0$. It is clear that $d_B d^c_B \Delta_B\varphi = d d^c_B \Delta_B\varphi$, from which we have

$$\int_S d_B d^c_B(\Delta_B\varphi) \wedge (d\eta_t)^{m-1} \wedge \eta_t = -\int_S d^c_B \Delta_B\varphi \wedge (d\eta_t)^m.$$

Since $d^c_B\Delta_B\varphi \wedge (d\eta_t)^m$ is a basic $(2m+1)$-form, it is zero. Similarly, we have

$$\int_S \rho^T_t \wedge (d\eta_t)^{m-2} \wedge d_B d^c_B\varphi \wedge \eta_t = \int \rho^T_t \wedge (d\eta_t)^{m-2} \wedge d^c_B\varphi \wedge d\eta_t = 0.$$

This proves the proposition. q.e.d.

This proposition means that the average $\overline{s}$ of the transverse scalar curvature s^T depends only on the basic Kähler class where

$$\overline{s} = \frac{\int_S m\rho^T \wedge \left(\frac{1}{2}d\eta\right)^{m-1} \wedge \eta}{\int_S \left(\frac{1}{2}d\eta\right)^m \wedge \eta} = \frac{\int_S s^T \left(\frac{1}{2}d\eta\right)^m \wedge \eta}{\int_S \left(\frac{1}{2}d\eta\right)^m \wedge \eta}. \tag{15}$$

Recall, for the next definition, that $\frac{1}{2}d\eta$ is the transverse Kähler form.

Definition 4.5. A complex vector field X on a Sasaki manifold is called a Hamiltonian holomorphic vector field if

(1) $d\pi_\alpha(X)$ is a holomorphic vector field on V_α;
(2) the complex valued function $u_X := \sqrt{-1}\eta(X)$ satisfies

$$\bar\partial_B u_X = -\frac{\sqrt{-1}}{2} i(X) d\eta.$$

Such a function u_X is called a Hamiltonian function.

(The terminology "Hamiltonian" may be misleading because X does not preserve the symplectic forms on local orbit spaces unless u_X is a real valued function.) If S is a compact Sasaki manifold then the Lie algebra of all Hamiltonian holomorphic vector fields is isomorphic to the Lie algebra of the automorphism group of the transverse holomorphic structure, see Proposition 2.2 in [**16**]. If $(x, z^1, \cdots, z^m)$ is a foliation chart on U_α, then X is written as

$$X = \eta(X)\frac{\partial}{\partial x} + \sum_{i=1}^{m} X^i \frac{\partial}{\partial z^i} - \eta\left(\sum_{i=1}^{m} X^i \frac{\partial}{\partial z^i}\right)\frac{\partial}{\partial x},$$

where X^i are local holomorphic basic functions.

Note that $X + i\eta(X)r\frac{\partial}{\partial r}$ is a holomorphic vector field on $C(S)$. A Hamiltonian holomorphic vector field X is the orthogonal projection of a Hamiltonian holomorphic vector field $\widetilde{X}$ on $C(S)$ to $S = \{r = 1\}$, whose Hamiltonian function $\tilde{u}$ satisfies $\xi(\tilde{u}) = \frac{\partial}{\partial r}\tilde{u} = 0$, i.e., $\tilde{u}$ is basic and homogeneous of degree zero with respect to r.

Remark 4.6. If $u_X = c$ ($=$ constant) then by (2) of Definition 4.5, $X = c\xi$ and $d\pi_\alpha(X) = 0$. In the case of regular Sasakian manifolds this corresponds to the fact that the Hamiltonian function for a Hamiltonian vector field on a symplectic manifold is unique up to constant.

Let $\mathfrak{h}$ denote the set of all Hamiltonian holomorphic vector fields. One can easily check that $\mathfrak{h}$ is a Lie algebra. Nishikawa and Tondeur [**35**] proved that if the scalar curvature s^T of the transverse Kähler metric is constant then $\mathfrak{h}$ is reductive, extending Lichnerowicz-Matsushima theorem in the Kähler case.

Let ω_α be the Kähler metric on V_α, *i.e.*

$$\pi_\alpha^*\omega_\alpha = \frac{1}{2}d\eta.$$

Then as is well-known, there is a smooth function f_α on V_α such that

$$\omega_\alpha = i\partial\bar{\partial}f_\alpha.$$

Hence in terms of the foliation chart $(x, z^1, \cdots, z^m)$ on U_α,

$$\eta = dx + 2d_B^c f_\alpha$$

and

$$d\eta = 2d_B d_B^c f_\alpha = 2i\partial_B\bar{\partial}_B f_\alpha. \tag{16}$$

Proposition 4.7. *By the deformation $\eta \to \tilde{\eta} = \eta + 2d_B^c\phi$, u_X is deformed to $u_X + X\phi$.*

Proof. Note that $u_X = i\eta(X)$ and $\tilde{u}_X = i\tilde{\eta}(X)$. q.e.d.

Proposition 4.8. *A complex valued basic function u is a Hamiltonian function for some Hamiltonian holomorphic vector field X if and only if*

$$\Delta_B^2 u + \left(i\partial_B\bar{\partial}_B u, \rho^T\right) + \left(\bar{\partial}_B u, \bar{\partial}_B s^T\right) = 0 \tag{17}$$

where ρ^T and s^T are the transverse Ricci form and transverse scalar curvature.

Proof. From $\bar{\partial}_B u = -\frac{i}{2} i(X)d\eta$, we have $\frac{\partial u}{\partial \bar{z}^j} = g^T_{i\bar{j}} X^i$, ie. $X^i = (g^T)^{i\bar{j}} \frac{\partial u}{\partial \bar{z}^j}$. For simplicity of the notation, we omit T and B in the proof. $X^i = (g^T)^{i\bar{j}} \frac{\partial u}{\partial \bar{z}^j}$ is holomorphic if and only if

$$\begin{aligned}
&\Leftrightarrow \quad \nabla_{\bar{i}} \nabla_{\bar{j}} u = 0, \quad \forall i,\, j \\
&\Leftrightarrow \quad \nabla^{\bar{i}} \nabla^{\bar{j}} \nabla_{\bar{i}} \nabla_{\bar{j}} u = 0, \\
&\Leftrightarrow \quad \Delta^2 u + R^{i\bar{j}} u_{i\bar{j}} + u^i s_i = 0 \\
&\Leftrightarrow \quad \Delta^2 u + (i\partial\bar{\partial} u, \rho) + (\bar{\partial} u, \bar{\partial} s) = 0
\end{aligned}$$

q.e.d.

The following result was obtained also by Boyer, Galicki and Simanca [**12**].

Theorem 4.9. *Let $\eta_t = \eta + t d^c_B \phi$ be the deformation of Sasaki structures. Let X be a Hamiltonian holomorphic vector field on S and u_t the Hamiltonian functions with respects to X and η_t. Then*

$$f(X) = -\int_S u_t \left(m\rho^T_t \wedge \left(\frac{1}{2} d\eta_t\right)^{m-1} \wedge \eta_t - \bar{s} \left(\frac{1}{2} d\eta_t\right)^m \wedge \eta_t \right)$$

is independent of t, where $\overline{s}$ was defined in (15).

Proof. A direct computation gives

$$\begin{aligned}
& -\frac{d}{dt} f(X,t) \\
(18) = \quad & \int_S (\bar{\partial}_B u, \bar{\partial}_B \varphi) \left(m\rho^T_t \wedge \left(\frac{1}{2} d\eta_t\right)^{m-1} \wedge \eta_t - \bar{s} \left(\frac{1}{2} d\eta_t\right)^m \wedge \eta_t \right) \\
(19) \quad & + \int_S u m d_B d^c_B (\Delta_B \varphi) \wedge \left(\frac{1}{2} d\eta_t\right)^{m-1} \wedge \eta_t \\
(20) \quad & + \int_S u m \rho^T_t \wedge (m-1) \left(\frac{1}{2} d\eta_t\right)^{m-2} \wedge d_B d^c_B \varphi \wedge \eta_t \\
(21) \quad & + \int_S u \left(m\rho^T_t \wedge \left(\frac{1}{2} d\eta_t\right)^{m-1} - \bar{s} \left(\frac{1}{2} d\eta_t\right)^m \right) \wedge d^c_B \varphi \\
(22) \quad & + \int u \bar{s} \Delta_B \varphi \left(\frac{1}{2} d\eta_t\right)^m \wedge \eta_t
\end{aligned}$$

Here, and later too, we suppressed the suffix t in the notations of inner product and the basic Laplacian. It is easy to check that (22) + the second term of (18) vanishes. (21) vanishes, for its integrand is basic of degree $(2m+1)$. (20) equals to

$$\int_S \varphi d_B d^c_B u \wedge m\rho^T_t \wedge (m-1) \left(\frac{1}{2} d\eta_t\right)^{m-2} \wedge \eta_t.$$

Thus,

$$
\begin{aligned}
(23)\qquad & -\frac{d}{dt}f(X,t) \\
= & \int_S \left(\bar\partial_B u, \bar\partial_B \varphi\right)\, s^T \left(\frac{1}{2}d\eta_t\right)^m \wedge \eta_t - \int_S u\,\Delta_B^2\varphi \left(\frac{1}{2}d\eta_t\right)^m \wedge \eta \\
& + \int \varphi\, d_B d_B^c u \wedge m\rho^T \wedge (m-1)\left(\frac{1}{2}d\eta_t\right)^{m-2} \wedge \eta_t.
\end{aligned}
$$

Take any point $p \in U_\alpha \subset S$ and a foliation chart $(x, z^1, \cdots, z^m)$ on U_α such that, on V_α, $\partial/\partial z^1, \ \cdots, \partial/\partial z^m$ are orthonormal and that either $(\partial_i \partial_{\bar j} u)$ or $(R^T_{i\bar j})$ is diagonal. Then the second term on the right hand side of (23) is equal to

$$
\begin{aligned}
& \int_S \varphi \sum_{I\neq j} u_{i\bar i} R^T_{j\bar j} \left(\frac{1}{2}d\eta\right)^m \wedge \eta \\
= & \int_S \varphi\, \left(-\Delta_B u\, s^T - \left(d_B d_B^c u, \rho^T\right)\right)\left(\frac{1}{2}d\eta\right)^m \wedge \eta.
\end{aligned}
$$

From this and Proposition 4.8 we finally get

$$
\begin{aligned}
& -\frac{d}{dt}f(X,t) = \\
& -\int_S \varphi\left(\Delta_B^2 u + \left(\bar\partial_B u, \bar\partial_B s^T\right) + \left(d_B d_B^c u, \rho^T\right)\right)\left(\frac{1}{2}d\eta\right)^m \wedge \eta = 0.
\end{aligned}
$$

q.e.d.

The linear function f on the Lie algebra $\mathfrak{h}$ of all Hamiltonian holomorphic vector fields is obviously an obstruction to the existence of transverse Kähler metric of constant scalar curvature in the fixed basic Kähler class. In particular it obstructs the existence of transverse Kähler-Einstein metric, extending earlier result of the first author for Fano manifolds, see [**20**].

The invariant f has different expressions. In the positive case, we assume that $c_1^B = (2m+2)[\frac{1}{2}d\eta]_B$ which is assured by the assumption $c_1(D) = 0$, see Proposition 4.3. By a result of El Kacimi-Alaoui [**18**], there is a basic function h such that

$$
(24)\qquad \rho^T - (2m+2)\frac{1}{2}d\eta = i\partial_B\bar\partial_B h.
$$

In this case, the average of the scalar curvature is m. Thus we have, by the definition of f, that

(25)
$$\begin{aligned} f(X) &= -\int_S u_X m \left(\rho \wedge \left(\frac{1}{2}d\eta\right)^{m-1} \wedge \eta - (2m+2)\left(\frac{1}{2}d\eta\right)^m \wedge \eta\right) \\ &= -m\int_S u_X i\partial_B\bar\partial_B h \wedge \left(\frac{1}{2}d\eta\right)^{m-1} \wedge \eta \\ &= -\int_S u_X \Delta_B h \left(\frac{1}{2}d\eta\right)^m \wedge \eta = \int_S Xh \ \left(\frac{1}{2}d\eta\right)^m \wedge \eta. \end{aligned}$$

From (25) it is clear that when the Sasaki manifold S has a transverse Kähler-Einstein metric in its basic Kähler class, then f vanishes.

We also have the following generalization. Let η_0 be a contact form of a compact Sasaki manifold, and let Ω be the set of all contact forms giving Sasaki structures with the same Reeb field as the Sasaki structure given by η_0, thus with the fixed transverse holomorphic structure, namely

$$\Omega = \{\eta = \eta_0 + 2d_B^c\varphi \,|\, \varphi \text{ is basic and } (d\eta)^m \wedge \eta \text{ is nowhere vanishing }\}.$$

Choose any $\eta \in \Omega$. Let $\omega_B = \frac{1}{2}d\eta$ be the transverse Kähler form and Θ be the curvature matrix of the transverse Levi-Civita connection of ω_B. Consider the basic $2m$-form

$$c_m\left(\omega_B \otimes I + \frac{\sqrt{-1}}{2\pi}\lambda\Theta\right),$$

where I denotes the identity matrix, c_m the invariant polynomial corresponding to the determinant. We can expand it as follows

$$c_m\left(\omega_B \otimes I + \frac{\sqrt{-1}}{2\pi}\lambda\Theta\right) = \omega_B^m + \lambda c_1(w_B) \wedge \omega_B^{m-1} + \cdots + \lambda^m c_m(\omega_B),$$

where $c_i(\omega_B)$ is the i-th Chern form with respect to ω_B. It is clear that a tangential vector field to Ω at η can be expressed by a basic function ψ with a normalization

$$\int_S \psi \left(\frac{1}{2}d\eta\right)^m \wedge \eta = 0.$$

Hence the space of all basic functions is the tangent space to Ω. We define a one-form α on Ω by

(26)
$$\psi \to \int_S \phi c_m \left(\omega_B \otimes I + \frac{\sqrt{-1}}{2\pi}\lambda\Theta\right) \wedge \eta.$$

Define $f_k : \mathfrak{h} \to \mathbb{C}$ by

$$f_k(X) = \int_S u_X c_k(\omega_B) \wedge \left(\frac{1}{2}d\eta\right)^{m-k} \wedge \eta,$$

where $c_k(\omega_B)$ is the kth Chern form defined above and u_X is the Hamiltonian function for X with the normalization

$$\int_S u_X \left(\frac{1}{2}d\eta\right)^m \wedge \eta = 0.$$

Theorem 4.10. *The functional f_k is independent of the choice of η. Moreover f_k is a Lie algebra homomorphism and obstructs the existence of a Sasaki metric of harmonic basic kth Chern form. Moreover $f_1 = 1/2\pi\, f$.*

Proof. The proof is similar to the proof of Theorem 4.9. See Appendix. q.e.d.

Lemma 4.11. *The one-form α defined by* (26) *is a closed form on Ω.*

Proof. To show the Lemma, it is convenient to consider an arbitrary map $\gamma : [0,1] \times [0,1] \to \Omega$. Let (s,t) be the coordinates on $[0,1] \times [0,1]$. By definition, we know that

$$(\gamma^*\alpha)\left(\frac{\partial}{\partial s}\right) = \alpha\left(\gamma_*\frac{\partial}{\partial s}\right) = \alpha\left(\frac{\partial \gamma}{\partial s}\right).$$

Thus,

$$\begin{aligned}\gamma^*\alpha \quad = \quad & \left(\int_S \frac{\partial \gamma}{\partial s} c_m\left(\omega_B \otimes I + \frac{\sqrt{-1}}{2\pi}\lambda\Theta\right) \wedge \eta\right) ds \\ & + \left(\int_S \frac{\partial \gamma}{\partial t} c_m\left(\omega_B \otimes I + \frac{\sqrt{-1}}{2\pi}\lambda\Theta\right) \wedge \eta\right) dt,\end{aligned}$$

and α is closed if only if $\gamma^*\alpha$ is closed for any γ. In view of Lemma 9.1 and Lemma 9.2, the latter can be proved as in the Kähler case (see [**20**]). q.e.d.

Theorem 4.12. *Let η and η' be two Sasaki structures in Ω and $\eta_t = \eta + 2d^c_B\varphi_t$ ($t \in [a,b]$) be a path in Ω connecting η and η'. Then*

$$\mathcal{M}_k(\eta,\eta') = \int_a^b \int_S \dot{\varphi}_t(c_k(\omega_B) - H(c_k(\omega_B))) \wedge \left(\frac{1}{2}d\eta\right)^{m-k} \wedge \eta$$

is independent of the path η_t, where $H(c_k(\omega_B))$ is the basic harmonic part of $c_k(\omega_B)$ with respect to the transverse Kähler form $\omega_B = \frac{1}{2}d\eta$.

Proposition 4.13. *Fix $\eta_0 \in \Omega$ and consider the functional $m_k : \Omega \to \mathbb{R}$ defined by*

$$m_k(\eta) = -\mathcal{M}_k(\eta_0, \eta).$$

then the critical points of m_k are the Sasaki metrics of harmonic basic k-th Chern form. When $k = 1$ and $[c_1\left(\frac{1}{2}d\eta\right)]_B = const.[\frac{1}{2}d\eta]_B$, then the critical points are the metrics of transverse Kähler-Einstein.

The proofs of Theorem 4.12 and Proposition 4.13 can be given by the principle stated in the Appendix. Theorem 4.10 and Theorem 4.12 respectively extend results of Bando [**2**] and Bando and Mabuchi [**3**].

5. Sasaki-Ricci solitons

To study the existence of Sasaki-Einstein metrics, or equivalently transversely Kähler-Einstein metrics of positive scalar curvature (or η-Einstein metric), a natural analogue of Kähler-Ricci flow was introduced in [**36**]. Assume that $c_1^B = \kappa[\frac{1}{2}d\eta]_B$, where κ is normalized so that $\kappa = -1, 0$, or 1 for simplicity. We consider the following flow $(\xi, \eta(t), \Phi(t), g(t))$ with initial data $(\xi, \eta(0), \Phi(0), g(0)) = (\xi, \eta, \Phi, g)$

$$\frac{d}{dt}g^T(t) = -\left(Ric^T_{g(t)} - \kappa g^T(t)\right), \tag{27}$$

or equivalently

$$\frac{d}{dt}d\eta(t) = -\left(2\rho^T_{g(t)} - \kappa d\eta(t)\right). \tag{28}$$

This flow is called *Sasaki-Ricci flow.* Locally, if we write

$$\frac{1}{2}d\eta = \sqrt{-1}g^T_{i\bar{j}}dz^i \wedge d\bar{z}^j,$$

we can check that $\rho^T = -\sqrt{-1}\partial\bar{\partial}\log\det(g^T_{k\bar{l}})$. Let $\eta(t) = \eta + 2d^c_B\varphi(t)$ for a family of basic functions $\varphi(t)$. Then the flow can be written as

$$\frac{d}{dt}\varphi = \log\det\left(g^T_{i\bar{j}} + \varphi_{i\bar{j}}\right) - \log\left(\det g^T_{i\bar{j}}\right) + \kappa\varphi - h, \tag{29}$$

where h is a basic function defined by

$$\rho^T_g - \frac{\kappa}{2}d\eta = d_B d^c_B h. \tag{30}$$

The solvability of (30) was proved in [**18**]. The well-posedness of the flow was proved in [**36**]. Like the Kähler-Ricci flow [**13**], the long-time existence can be also proved. When the flow converges, then the limit is a transverse Kähler-Einstein metric. In fact, one can show that when $\kappa = -1$, or 0, then the flow globally converges to an η-Einstein metric. See also [**18**] and [**11**]. Hence, the remaining interesting case is when $\kappa = 1$, namely the basic first Chern form of the Sasaki manifold is of positive definite. But from now on, we assume that $\kappa = 2m + 2$ because this normalization fits to the study of Sasaki-Einstein metric, see (13). In this case, in general the convergence of the flow could not be obtained. What one can hope is the limit converges in some sense to a soliton solution, as in the Kähler case. A Sasaki structure (S, ξ, η, Φ, g) with a Hamiltonian holomorphic vector field X is called a *transverse Kähler-Ricci soliton* or *Sasaki-Ricci soliton* if

$$Ric^T - (2m+2)g^T = \mathcal{L}_X g^T, \tag{31}$$

or equivalently,

$$2\rho^T - (2m+2)d\eta = \mathcal{L}_X(d\eta). \tag{32}$$

In the next section, we will prove that on any toric Sasaki manifold there always exists a Sasaki-Ricci soliton. To end this section, we give a generalization of the invariant f, which is an obstruction of the existence of the Sasaki-Ricci solitons.

Recall that there is a basic function satisfying (24). As in [**20**], we define the following operator

$$\Delta_B^h u = \Delta_B u - \nabla^i u \nabla_i h,$$

where $\nabla = \nabla^T$ is the Levi-Civita connection of the transverse Kähler metric. One can show that the operator Δ_B^h is self-adjoint in the following sense

$$\begin{aligned} \int_S \Delta_B^h u\ \overline{v}\ e^h \left(\frac{1}{2}d\eta\right)^m \wedge \eta &= \int_S u\ \overline{\Delta_B^h v}\ e^h \left(\frac{1}{2}d\eta\right)^m \wedge \eta \\ &= \int_S (\bar\partial_B u, \bar\partial_B v)\ e^h \left(\frac{1}{2}d\eta\right)^m \wedge \eta. \end{aligned} \tag{33}$$

We need to consider "*normalized* Hamiltonian holomorphic vector fields", whose corresponding Hamiltonian functions u_X satisfying

$$\int_S u_X e^h \left(\frac{1}{2}d\eta\right)^m \wedge \eta = 0. \tag{34}$$

For any Hamiltonian holomorphic vector field X, there is a unique constant $c \in \mathbb{R}$ such that $X + c\xi$ is a normalized Hamiltonian holomorphic vector field. For simplicity of notation, from now on any holomorphic vector field X we consider is normalized and its Hamiltonian function is denoted by θ_X. Hence θ_X satisfies (34).

The operator Δ_B^h has the following properties, whose proof can be given as in [**20**].

Theorem 5.1. *We have*

(1) *The first eigenvalue λ_1 of Δ_B^h is greater than or equal to $2m+2$.*
(2) *The equality $\lambda_1 = 2m+2$ holds if and only if there exists a non-zero Hamiltonian holomorphic vector field X.*
(3) *$\{u \in \Omega_B(S)^{\mathbb{C}} \mid \Delta_B^h u = (2m+2)u\}$ is isomorphic to $\{X \mid$ normalized Hamiltonian holomorphic vector fields$\}$. The correspondence is given, in a local foliation chart, by*

$$u \to u\xi + \nabla^i u \frac{\partial}{\partial z^i} - \eta\left(\nabla^i u \frac{\partial}{\partial z^i}\right)\xi.$$

Remark 5.2. (1) $\nabla^i \theta = g_T^{i\bar j} \frac{\partial u}{\partial \bar z^j}$ is holomorphic.

(2) Let $u \in \{u \in \Omega_B(S)^{\mathbb{C}} \mid \Delta_B^h u = (2m+2)u\}$. By putting $v = 1$ in (33), we get

$$\int_S ue^h \left(\frac{1}{2}d\eta\right)^m \wedge \eta = 0.$$

Now, as in Tian and Zhu [**39**] we define a generalized invariant f_X for a given Hamiltonian holomorphic vector field X by

$$f_X(v) = -\int_S \theta_v e^{\theta_X} \left(\frac{1}{2}d\eta\right)^m \wedge \eta.$$

We will leave the proof of the invariance of f_X to the reader. It is also easy to check that f_X is an obstruction of the existence of Sasaki-Ricci solitons as follows.

Let (S, g, X) be a Sasaki-Ricci soliton, i.e., we have (32). From the above discussion we know that

$$dd_B^c h = L_X\left(\frac{1}{2}d\eta\right) = d\left(i_X\left(\frac{1}{2}d\eta\right)\right) = id\bar{\partial}_B\theta_X.$$

It follows that $\theta_X = h + \text{constant}$. Hence $f_X(v) = 0$ for all $v \in \mathfrak{h}$. The next proposition shows that we can always find such an $X \in \mathfrak{h}(S)$.

Proposition 5.3. *There exists an $X \in \mathfrak{h}(S)$ such that*

$$f_X(v) = 0, \quad \forall v \in \mathfrak{h}_r(S)$$

where $\mathfrak{h}_r(S)$ denotes the reductive part of $\mathfrak{h}(S)$. If S is toric v may be in $\mathfrak{h}(S)$.

Proof. The proof can be given by arguments similar to [**39**], and we will not reproduce them here. The last statement follows from similar arguments as in Lemma 2.1 of [**41**] q.e.d.

Now we wish to set up the Monge-Ampère equation to prove the existence of a transverse Kähler-Ricci soliton for the choice of X in Proposition 5.3. Choose an initial Sasaki metric g such that the transverse Kähler form $\omega^T = \frac{1}{2}d\eta$ represents the basic first Chern class of the normal bundle of the Reeb foliation. There exists a smooth basic function h such that

$$\rho^T - (2m+2)\omega^T = i\partial_B\overline{\partial}_B h. \tag{35}$$

Suppose we can get a new Sasaki metric $\widetilde{g}$ satisfying the Sasaki-Ricci soliton equation by a transverse Kähler deformation. Let

$$\widetilde{\omega}^T = i\,\widetilde{g}^T_{i\bar{j}}\,dz^i \wedge d\overline{z}^j = i\left(g^T_{i\bar{j}} + \varphi_{i\bar{j}}\right) dz^i \wedge d\overline{z}^j,$$

$\widetilde{\rho}^T$ and $\widetilde{\theta}_X$ respectively denote the transverse Kähler form, transverse Ricci form and the Hamiltonian function for the normalized Hamiltonian function X with respect to $\widetilde{g}$ where φ is a smooth basic function S. Then by the Sasaki-Ricci soliton equation we have

$$\widetilde{\rho}^T - (2m+2)\widetilde{\omega}^T = \mathcal{L}_X \widetilde{\omega}^T = i\partial_B \overline{\partial}_B \widetilde{\theta}_X. \tag{36}$$

As one can see easily (c.f. Appendix 2, [**22**])

$$\widetilde{\theta}_X = \theta_X + X\varphi. \tag{37}$$

From (35), (36) and (37) we get

$$\frac{\det(g^T_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g^T_{i\bar{j}})} = \exp(-(2m+2)\varphi - \theta_X - X\varphi + h) \tag{38}$$

with $(g^T_{i\bar{j}} + \varphi_{i\bar{j}})$ positive definite (recall that φ is a basic function). In order to prove the existence of a solution to (38) we consider a family of equations parametrized by $t \in [0,1]$:

$$\frac{\det(g^T_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g^T_{i\bar{j}})} = \exp(-t(2m+2)\varphi - \theta_X - X\varphi + h) \tag{39}$$

with $(g^T_{i\bar{j}} + \varphi_{i\bar{j}})$ positive definite. It is sufficient to show that the subset of $[0,1]$ consisting of all t for which (39) has a solution is non-empty, open and closed. Combining the arguments of [**18**], [**42**] and [**43**] one can show that (39) has a solution at $t = 0$ and that the openness is also satisfied by the implicit function theorem. By El Kacimi-Alaoui's generalization of Yau's estimates [**42**] for the transverse Monge-Ampr̀e equations it suffices to show the C^0 estimate for φ to prove the closedness. To get the C^0 estimate for φ it is sufficient to get the C^0 estimate of on an open dense subset of S. We prove in section 7 that this C^0 estimate can be obtained for toric Sasaki manifolds.

6. Toric Sasaki manifolds

In this section, we recall known facts about toric Sasaki manifolds, following [**28**], [**6**], and a slight modification of some arguments of [**33**].

Definition 6.1. A co-oriented contact structure D on a manifold M is a codimension 1 distribution such that

(1) $D^0 = \{\alpha \in T^*M \,|\, \alpha(X) = 0, \forall X \in D\}$ is an oriented real line bundle, and a component D^0_+ of $D^0/\{0\}$ is chosen.
(2) $D^0/\{0\}$ is a symplectic submanifold of T^*M.

Note that T^*M has a natural symplectic structure $\sum_{i=1}^n dq^i \wedge dp_i$ where $q^1, \cdots, q^n$ is local coordinates on M and $p = \sum_{i=1}^n p_i dq^i$ is a cotangent vector.

Definition 6.2. Suppose a Lie group G acts on M preserving D, and consequently D^0. The moment map for the action of G is a map $\mu : D^0_+ \to \mathfrak{g}^*$ defined by

$$\langle \mu(q,p), X\rangle = \langle p, X_M(q)\rangle,$$

where $X_M(q) := \frac{d}{dt}\big|_{t=0} exp(tX) \cdot q$ is a vector field on M induced by $X \in \mathfrak{g}$. Here $\mathfrak{g}$ is Lie algebra of G and $\mathfrak{g}^*$ is its dual.

Definition 6.3. A contact toric G-manifold is a co-oriented contact manifold (M, D) with an action of a torus G preserving D and with $2 \dim G = dimM + 1$.

From [**28**], we have

Lemma 6.4. *Let (M, D) be a toric contact manifold with an action of a torus G. Then zero is not in the image of the contact moment map $\mu : D^0_+ \to \mathfrak{g}^*$. Moreover the moment cone $C(\mu)$ defined by*

$$C(\mu) := \mu(D^0_+) \cup \{0\}$$

is a convex rational polyhedral cone.

Recall that a subset $\mathcal{C} \subset \mathfrak{g}^*$ is a convex rational polyhedral set, if there exists a finite set of vectors $\{\lambda_j\}$ in the integral lattice $\mathbb{Z}_G :=$ Ker$\{\exp : \mathfrak{g} \to G\}$ and $\mu_i \in \mathbb{R}$ such that

$$\mathcal{C} = \bigcap_j \{X \in \mathfrak{g}^* \,|\, \langle \lambda_j, X\rangle + \mu_i \geq 0\}.$$

Clearly it is a cone if all μ_i are 0.

Let η be a contact form, i.e., $\operatorname{Ker} \eta = D$ and $d\eta|_D$ non-degenerate. Let G be a torus action on M preserving η. The moment map $\mu_\eta : M \to \mathfrak{g}^*$ is defined by

$$\langle \mu_\eta(x), X\rangle = \eta(X_M(x)), \quad \forall x \in M.$$

The contact form η is a section $\eta : M \to D^0_+$ and we obviously have

$$\eta^*\mu = \mu_\eta,$$

since $\langle \eta^*\mu, X\rangle = \eta(X_M)$. For a G-invariant contact form η we define the moment cone $C(\mu_\eta)$ by

$$C(\mu_\eta) = \{re \,|\, e \in \mu_\eta(M), r \in [0, \infty)\}.$$

It is clear that

$$C(\mu_\eta) = C(\mu).$$

Now we introduce the notion of toric Sasaki manifolds.

Definition 6.5. A toric Sasaki manifold S is a Sasaki manifold of dimension $2m + 1$ with Sasaki structure (ξ, η, Φ, g) such that there is an effective action of $(m+1)$-dimensional torus G preserving the Sasaki structure and that ξ is an element of the Lie algebra $\mathfrak{g}$ of G. Equivalently, a toric Sasaki manifold is a Sasaki manifold whose Kähler cone is a toric Kähler manifold.

Proposition 6.6. *Let S be a toric Sasaki manifold with Sasaki structure (ξ, η, Φ, g) and $\eta_t = \eta + 2td^c_B\varphi$ a G-invariant basic deformation of Sasaki structure where φ is a G-invariant basic smooth function. Then the moment cones $C(\mu_{\eta_t})$ are the same for all t, i.e.,*

$$C(\mu_{\eta_t}) = C(\mu_\eta), \quad \forall t.$$

Proof. This follows since every moment cone is rational polyhedral cone. q.e.d.

Let $G^c \cong (\mathbb{C}^*)^{m+1}$ denote the complexification of G. Then G^c acts on the cone $C(S)$ as biholomorphic automorphisms. The moment map on $C(S)$ with respect to the Kähler form $\omega = d(\frac{1}{2}r^2\eta)$ is given by

$$\begin{aligned} \mu : C(S) &\rightarrow \mathfrak{g}^* \\ \langle \mu(x), X\rangle &= r^2\eta(X_S(x)), \end{aligned}$$

where we have used the natural diffeomorphism $C(S) \cong \mathbb{R}^+ \times S$ and view vector fields on S as vector fields on $C(S)$. Notice that we deleted $1/2$ so that there is a consistency with the moment maps for contact manifolds. It is clear that the image of μ is the same with the moment cone defined above, which is denoted by $C(\mu)$. Let $\mathrm{Int}C(\mu)$ denote the interior of $C(\mu)$. Then the action of G on $\mu^{-1}(\mathrm{Int}C(\mu))$ is free and the orbit space is $\mathrm{Int}C(\mu)$. This means that $\mu^{-1}(\mathrm{Int}C(\mu))$ is a torus bundle over $\mathrm{Int}C(\mu)$. On the other hand the image $\mathrm{Im}(\mu_\eta)$ of the moment map $\mu_\eta : S \to \mathfrak{g}^*$ is given by

$$\mathrm{Im}(\mu_\eta) = \{\alpha \in C(\mu) \,|\, \alpha(\xi) = 1\}.$$

The hyperplane $\{\alpha \in \mathfrak{g}^* \,|\, \alpha(\xi) = 1\}$ is called *characteristic plane* in [**6**]. Notice that the constants differs by $1/2$ from [**33**] because we use the moment map as a contact manifold.

In the rest of this section we study the Guillemin metric obtained by the Kähler reduction through the Delzant construction (c.f. [**1**], [**26**], [**28**]).

Assume that the moment cone $C(\mu)$ of our Sasaki manifold S is described by

$$C(\mu) = \{y \in \mathfrak{g}^* \mid l_i(y) = \lambda_i \cdot y \geq 0,\ i = 1, \cdots, d\}, \tag{40}$$

and let $C(\mu)^*$ be its dual cone

$$C(\mu)^* = \{\widetilde{x} \in \mathfrak{g} \mid \widetilde{x} \cdot y \geq 0 \text{ for all } y \in C(\mu)\}. \tag{41}$$

Then the Reeb field ξ is considered as an element of the interior of $C(\mu)^*$ since $\frac{1}{2}r^2\eta(\xi) = \frac{1}{2}r^2 > 0$. We identify $\mathfrak{g}^* \cong \mathbb{R}^{m+1} \cong \mathfrak{g}$, and regard

$$\lambda_i = \left(\lambda_i^1, \cdots, \lambda_i^{m+1}\right), \quad \xi = \left(\xi^1, \cdots, \xi^{m+1}\right).$$

As was shown in [**33**] the symplectic potential G^{can} of the canonical metric in the above sense is expressed by

$$G^{can} = \frac{1}{2}\sum_{i=1}^{d} l_i(y)\log l_i(y).$$

If we put

$$G_\xi = \frac{1}{2}l_\xi(y)\log l_\xi(y) - \frac{1}{2}l_\infty(y)\log l_\infty(y)$$

where

$$l_\xi(y) = \xi\cdot y, \quad l_\infty(y) = \sum_{i=1}^{d}\lambda_i\cdot y,$$

then

$$\begin{aligned} G_\xi^{can} &= G^{can} + G_\xi \\ &= \frac{1}{2}\sum_{i=1}^{d} l_i(y)\log l_i(y) + \frac{1}{2}l_\xi(y)\log l_\xi(y) - \frac{1}{2}l_\infty(y)\log l_\infty(y) \end{aligned} \tag{42}$$

gives a symplectic potential of a Kähler metric on $C(S)$ such that the induced Sasaki structure on S has ξ as the Reeb field. To see this, compute

$$(G_\xi^{can})_{ij} = \frac{1}{2}\sum_{k=1}^{d}\frac{\lambda_k^i\lambda_k^j}{l_k(y)} + \frac{1}{2}\frac{\xi^i\xi^j}{l_\xi(y)} - \frac{1}{2}\frac{\sum_{k=1}^d\lambda_k^i\sum_{\ell=1}^d\lambda_\ell^j}{l_\infty(y)} \tag{43}$$

and observe that $\xi^i = 2\sum_j(G_\xi^{can})_{ij}y_j$ and that

$$\sum_{i,j}(G_\xi^{can})_{ij}y_iy_j = \frac{1}{2}l_\xi(y) > 0.$$

Hence $(G_\xi^{can})_{ij}$ is positive definite.

Since any two complex structures associated to a polytope are equivariantly biholomorphic (c.f. Proposition A.1 in [**1**]) we may assume that the complex structure obtained by the Delzant construction is the same with the complex structure of the Kähler cone $C(S)$ of the Sasaki manifold S under consideration. Thus the Sasaki structure induced by the above Delzant construction has the same complex structure and Reeb field. If we denote by

$$\overline{\widetilde{g}} = d\widetilde{r}^2 + \widetilde{r}^2\widetilde{g}$$

the Kähler cone metric of this Sasaki structure then we have

$$\widetilde{r}\frac{\partial}{\partial\widetilde{r}} = J\xi = r\frac{\partial}{\partial r}.$$

This implies that $\widetilde{r} = r\exp(\varphi)$ for some basic smooth function φ. Taking $d^c\log$ we get

$$\widetilde{\eta} = \eta + 2d^c\varphi.$$

This is a transverse Kähler deformation described in Proposition 4.2.

Thus we have proved the following.

Proposition 6.7. *Let S be a compact toric Sasaki manifold and $C(S)$ its Kähler cone. Let ξ be the Reeb field. Then we may assume that there is a transverse Kähler deformation of the Sasaki structure of S whose symplectic potential is of the form* (42).

Now we assume hereafter that the initial Sasaki structure is so chosen that the symplectic potential G is written as (42). Let

$$\widetilde{x}^j = \frac{\partial G}{\partial y_j} = \frac{\partial G^{can}_\xi}{\partial y_j}$$

be the inverse Legendre transform of y_j. Then

$$\left(z^1, \cdots, z^{m+1}\right) = \left(\widetilde{x}^1 + i\widetilde{\phi}^1, \cdots, \widetilde{x}^{m+1} + i\widetilde{\phi}^{m+1}\right)$$

is the affine logarithmic coordinate system on for $\mu^{-1}(\mathrm{Int}C(\mu)) \cong (\mathbb{C}^*)^{m+1}$, i.e. the standard holomorphic coordinates are given by

$$\left(e^{\widetilde{x}^1 + i\widetilde{\phi}^1}, \cdots, e^{\widetilde{x}^{m+1} + i\widetilde{\phi}^{m+1}}\right).$$

(We reserve the notation $x^j + i\phi^j$ for a subtorus H^c of $G^c \cong (\mathbb{C}^*)^{m+1}$ which will appear later.)

Suppose that the basic first Chern class c_1^B of the normal bundle of the Reeb foliation is positive and that $c_1(D) = 0$. Then the basic Kähler class $[\omega^T] = \frac{1}{2}[d\eta]$ can be so chosen that $c_1^B = (2m+2)[\omega^T]$. If we denote by F the Kähler potential on $C(S)$, F and $G = G^{can}_\xi$ are related by

$$F = \widetilde{x} \cdot y - G.$$

Then the Ricci form ρ on $C(S)$ must be written as

$$\rho = -i\partial\overline{\partial} \log \det(F_{ij}) = i\partial\overline{\partial} h \tag{44}$$

with

$$r\frac{\partial}{\partial r} h = 0, \qquad \xi h = 0,$$

i.e. h is a pull-back of a basic smooth function S. This of course is equivalent to

$$\rho^T = (2m+2)\frac{1}{2} d\eta + i\partial_B \overline{\partial}_B h.$$

Since T^{m+1}-invariant pluriharmonic harmonic function is an affine function we see from (44) that there exist $\gamma_1, \cdots, \gamma_{m+1} \in \mathbb{R}$ such that, replacing h by $h + \mathrm{const}$ if necessary,

$$\log \det(F_{ij}) = -2\gamma_i \widetilde{x}^i - h. \tag{45}$$

In terms of G, (45) can be written as

$$\det(G_{ij}) = \exp\left(2\sum_{i=1}^{m+1} \gamma_i G_i + h\right). \tag{46}$$

Recall from [**33**] that

$$r\frac{\partial}{\partial r} = 2\sum_{j=1}^{m+1} y_j \frac{\partial}{\partial y_j}, \tag{47}$$

$$\xi_i = 2\sum_j G_{ij} y_j. \tag{48}$$

The left hand side of (46) is homogeneous of degree $-(m+1)$ by (43) while applying $\sum_{j=1}^{m+1} y_j \frac{\partial}{\partial y_j}$ to the right hand side of (46) gives

$$\sum_{j=1}^{m+1} y_j \frac{\partial}{\partial y_j} \exp\left(2\sum_{i=1}^{m+1} \gamma_i G_i + h\right) = (\xi, \gamma) \exp\left(2\sum_{i=1}^{m+1} \gamma_i G_i + h\right)$$

where we used (48). Hence we obtain

$$(\xi, \gamma) = -(m+1). \tag{49}$$

One can compute the right hand side of (46) using (42) to get

$$\det(G_{ij}) = \Pi_j \left(\frac{l_j(y)}{l_\infty(y)}\right)^{(\lambda_j, \gamma)} (l_\xi(y))^{-(m+1)} \exp(h). \tag{50}$$

On the other hand using (43) we can compute the left hand side of (46) to get

$$\det(G_{ij}) = f(y) \Pi_j (l_j(y))^{-1}, \tag{51}$$

where f is a smooth positive function on $C(\mu)$. It follows from (50) and (51) that

$$(\lambda_j, \gamma) = -1, \qquad j = 1, \cdots, d. \tag{52}$$

Since the cone $C(\mu)^*$ is a cone over a finite polytope there are $(m+1)$ linearly independent λ_i's. Thus γ is uniquely determined from the moment cone $C(\mu)$, and is rational. The equalities (52) show that the structure of the cone is very special. If γ is a primitive lattice vector then the apex is a Gorenstein singularity as Martelli, Sparks and Yau pointed out in section 2.2 of [**34**] (Professor James Sparks kindly pointed out to us that their paper assumed the existence of Killing spinor so that the singularity must be Gorenstein. The present paper does not assume the existence of Killing spinor and the singularity is $\mathbb{Q}$-Gorenstein in general as is claimed in [**16**].).

The condition (44) says that the Hermitian metric $e^h \det(F_{ij})$ gives a flat metric on the canonical bundle $K_{C(S)}$. Consider a holomorphic $(m+1)$-form Ω of the form

$$\Omega = e^{i\alpha} e^{\frac{h}{2}} \det(F_{ij})^{\frac{1}{2}} dz^1 \wedge \cdots \wedge dz^{m+1}.$$

From (45) Ω is written as

$$\Omega = e^{i\alpha} \exp\left(-\sum_{i=1}^{m+1} \gamma_i \widetilde{x}^i\right) dz^1 \wedge \cdots \wedge dz^{m+1}.$$

Hence taking $-\alpha = \sum_{i=1}^{m+1} \gamma_i \widetilde{\phi}^i$ we have

$$\Omega = e^{-\sum_{i=1}^{m+1} \gamma_i z^i} dz^1 \wedge \cdots \wedge dz^{m+1}. \tag{53}$$

Note that when γ is not integral but only rational then Ω is multi-valued. Let ℓ be a positive integer such that $\ell\gamma$ is a primitive element of the integer lattice. Then $\Omega^{\otimes \ell}$ defines a section of $K_{C(S)}^{\otimes \ell}|_U$ where U is the open dense subset of $C(S)$ corresponding to the interior of the moment cone. But since $||\Omega^{\otimes \ell}|| = 1$, $\Omega^{\otimes \ell}$ extends to a smooth section of $K_{C(S)}^{\otimes \ell}$.

Since ξ is expressed as $\xi = \sum_i \xi^i \frac{\partial}{\partial \widetilde{\phi}^i}$ we have

$$\mathcal{L}_\xi \Omega = -i(\xi, \gamma)\Omega = i(m+1)\Omega. \tag{54}$$

From (53) and (45) we see

$$\left(\frac{i}{2}\right)^{m+1} (-1)^{m(m+1)/2} \Omega \wedge \overline{\Omega} = \exp(h) \frac{1}{(m+1)!} \omega^{m+1} \tag{55}$$

where ω denotes the Kähler form of $C(S)$.

To sum up we have obtained the following.

Proposition 6.8. *There exist a unique rational vector $\gamma \in \mathfrak{g}^*$ such that (52) holds and a multi-valued holomorphic $(m+1, 0)$-form Ω with the following properties. For some positive integer ℓ, $\Omega^{\otimes \ell}$ defines a holomorphic section of $K_{C(S)}^{\otimes \ell}$. Further for any Kähler cone metric on $C(S)$ such that the Ricci form ρ is written as*

$$\rho = i\partial\overline{\partial}h \tag{56}$$

where h is the pull-back of a smooth basic function on S,

(1) *the Reeb field ξ satisfies (49) and (54);*
(2) *if we denote by ω the Kähler form of the Kähler cone metric then the equation (55) is satisfied.*

Conversely if the Reeb field ξ satisfies either (49) or (54) then the Kähler cone metric satisfies (56) for some basic function h on S.

7. Sasaki-Ricci soliton on toric Sasaki manifolds

In this section we want to show the existence of Sasaki-Ricci solitons on any toric Sasaki manifolds with $c_1^B > 0$ as stated in Theorem 1.1 in the introduction. As was explained in section 5 we have only to give a C^0 estimate for the family of Monge-Ampère equations (39):

Lemma 7.1. *Let S be a compact toric Sasaki manifold with $c_1^B > 0$ and $c_1(D) = 0$. Then there exists a constant $C > 0$ independent of $t \in [0,1]$ such that $\sup_S |\varphi| \le C$ for any solution for* (39).

The rest of this section is devoted to the proof of Lemma 7.1, which will complete the proof of Theorem 1.1.

First let us take any subtorus $H \subset G$ of codimension 1 such that its Lie algebra $\mathfrak{h}$ does not contain ξ. Let $H^c \cong (\mathbb{C}^*)^m$ denote the complexification of H. Take any point $p \in \mu^{-1}(\mathrm{Int}C(\mu))$ and consider the orbit $Orb_{C(S)}(H^c, p)$ of the H^c-action on $C(S)$ through p. Since H^c-action preserves $-J\xi = r\partial/\partial r$, it descends to an action on the set $\{r = 1\} \subset C(S)$, which we identify with the Sasaki manifold S. More precisely this action is described as follows. Let $\gamma : H^c \times C(S) \to C(S)$ denote the H^c-action on $C(S)$. Let $\overline{p}$ and $\overline{\gamma(g,p)}$ respectively be the points on $\{r = 1\}$ at which the flow lines through p and $\gamma(g,p)$ generated by $r\partial/\partial r$ respectively meet $\{r = 1\}$. Then the H^c-action on $S \cong \{r = 1\}$ is given by $\bar{\gamma} : H^c \times \{r = 1\} \to \{r = 1\}$ where

$$\bar{\gamma}(g, \overline{p}) = \overline{\gamma(g,p)}.$$

Let $Orb_S(H^c, \overline{p})$ be the orbit of the induced action of H^c on $\{r = 1\} \cong S$.

Proposition 7.2. *The transverse Kähler structure of the Sasaki manifold S over the open dense subset $\mu_\eta^{-1}(\mathrm{IntIm}(\mu_\eta))$, which is the inverse image by μ_η of the interior of the intersection of the characteristic hyperplane and $C(\mu)$, is completely determined by the restriction of $\frac{1}{2}d\eta$ to $Orb_{C(S)}(H^c, p)$.*

Proof. Let $q \in Orb_{C(S)}(H^c, p) \subset C(S)$ be any point. Since $r\partial/\partial r - i\xi = r\partial/\partial r - J(r\partial/\partial r)$ is preserved by H^c, a neighborhood V_q of q in $Orb_{C(S)}(H^c, p)$ is mapped biholomorphically to some V_α where $\overline{q} \in U_\alpha \subset S$ and $\pi_\alpha : U_\alpha \to V_\alpha$ is given by the transversely holomorphic structure of the Reeb foliation. Thus the transverse Kähler structure on V_α is determined by $d\eta|_{V_q}$ because $(V_q, \frac{1}{2}d\eta|_{V_q})$ is isometric to $(V_\alpha, \frac{1}{2}d\eta|_{V_\alpha})$ as Kähler manifolds; this is because $\eta = 2d^c \log r$ is homogeneous of degree 0. But for any $q' \in \mu_\eta^{-1}(\mathrm{IntIm}(\mu_\eta))$ the trajectory through q' generated by ξ meets $Orb_S(H^c, \overline{p})$ and ξ generates a one parameter subgroup of isometries. So, the transverse Kähler geometry at any q' is determined by the transverse Kähler geometry along the points on $Orb_S(H^c, \overline{p})$. This trajectory may meet $Orb_S(H^c, \overline{p})$ infinitely many times when the Sasaki structure is irregular. But the transverse structures at all of them define the same Kähler structure because ξ generates a subtorus in T^{m+1} and we assumed that T^{m+1} preserves the Sasaki structure. Thus $\frac{1}{2}d\eta|_{Orb_{C(S)}(H^c,p)}$ completely determines the transverse Kähler structure of S on $\mu_\eta^{-1}(\mathrm{IntIm}(\mu_\eta))$. q.e.d.

$(Orb_{C(S)}(H^c,p), \frac{1}{2}d\eta|_{Orb_{C(S)}(H^c,p)})$ and $(Orb_S(H^c,\overline{p}), \frac{1}{2}d\eta|_{Orb_S(H^c,\overline{p})})$ are essentially the same in that if we give them the holomorphic structures induced from the holomorphic structure of H^c then they are isometric Kähler manifolds. The difference between them is that $Orb_{C(S)}(H^c,p)$ is a complex submanifold of the complex manifold $C(S)$ while $Orb_S(H^c,\overline{p})$ is a complex submanifold in the real Sasaki manifold S. In what follows we are interested in the Kähler potential of $\frac{1}{2}d\eta$. For this purpose $Orb_{C(S)}(H^c,p)$ is better to treat.

On $Orb_{C(S)}(H^c,p) \cong (\mathbb{C}^*)^m$ we use the affine logarithm coordinates

$$\left(w^1, w^2, \cdots, w^m\right) = \left(x^1+\sqrt{-1}\theta^1, x^2+\sqrt{-1}\theta^2, \cdots, x^m+\sqrt{-1}\theta^m\right)$$

for a point

$$\left(e^{x^1+\sqrt{-1}\theta^1}, e^{x^2+\sqrt{-1}\theta^2}, \cdots, e^{x^m+\sqrt{-1}\theta^m}\right) \in (\mathbb{C}^*)^m \cong H^c.$$

Since η is H-invariant, so is $d\eta$. Therefore $\frac{1}{2}d\eta|_{Orb_{C(S)}(H^c,p)}$ is determined by a convex function u^0 on $\mathbb{R}^m$, namely

$$\begin{aligned}\frac{1}{2}d\eta|_{Orb_{C(S)}(H^c,p)} &= \sqrt{-1}\partial\bar{\partial}u^0 \\ &= \frac{\sqrt{-1}}{4}\frac{\partial^2 u^0}{\partial x^i \partial x^j} dw^i \wedge d\overline{w^j}.\end{aligned}$$

Hence we have

$$i\left(\frac{\partial}{\partial\theta^i}\right)\frac{1}{2}d\eta|_{Orb_{C(S)}(H^c,p)} = -\frac{1}{2}\frac{\partial^2 u^0}{\partial x^i\partial x^j}dx^j = -\frac{1}{2}\,d\left(\frac{\partial u^0}{\partial x^i}\right). \tag{57}$$

On the other hand since $\mathcal{L}_{\frac{\partial}{\partial\theta^i}}\eta = 0$, we have

$$i\left(\frac{\partial}{\partial\theta^i}\right)\frac{1}{2}d\eta = -\frac{1}{2}d\left(\eta\left(\frac{\partial}{\partial\theta^i}\right)\right)$$

and from this and (57) it follows that

$$\eta\left(\frac{\partial}{\partial\theta^i}\right) = \frac{\partial u^0}{\partial x^i} + c_i, \tag{58}$$

where $c_i \in \mathbb{R}$ is a constant.

Now we wish to know more about the Kähler potential u^0. One way of expressing u^0 is

$$u^0 = \log r|_{Orb_{C(S)}(H^c,p)} + \text{const}. \tag{59}$$

For, since $Orb_{C(S)}(H^c,p)$ is a complex submanifold of $C(S)$ and $\eta = 2d^c\log r$,

$$\left(\frac{1}{2}d\eta\right)|_{Orb_{C(S)}(H^c,p)} = (dd^c\log r)|_{Orb_{C(S)}(H^c,p)} = dd^c(\log r|_{Orb_{C(S)}(H^c,p)}).$$

If we take the Kähler metric on $C(S)$ as the canonical metric obtained by the Kähler reduction through the Delzant construction, we get a

Sasaki structure for which the transverse Kähler potential u^0 has a more explicit description as explained below.

By (42) the real part $\widetilde{x}^i$ of the affine logarithm coordinates in $C(S)$ is given by

$$\begin{aligned}(60)\qquad \widetilde{x}^j &= \frac{\partial G_\xi^{can}}{\partial y_j}\\ &= \frac{1}{2}\sum_{i=1}^{d}\lambda_i^j(1+\log l_i(y))\\ &\quad +\frac{1}{2}\xi^j(1+\log l_\xi(y)) - \frac{1}{2}\sum_{i=1}^{d}\lambda_i^j(1+\log l_\infty(y))\\ &= \frac{1}{2}\sum_{i=1}^{d}\lambda_i^j\log l_i(y)\\ &\quad +\frac{1}{2}\xi^j(1+\log l_\xi(y)) - \frac{1}{2}\sum_{i=1}^{d}\lambda_i^j\log l_\infty(y).\end{aligned}$$

The Kähler potential F_ξ^{can} on $C(S)$ is then obtained by the Legendre transform:

$$(61)\qquad F_\xi^{can}(\widetilde{x}) = \widetilde{x}\cdot y - G_\xi^{can}(y) = \frac{1}{2}l_\xi(y).$$

Now we know that $\frac{r^2}{2}$ is also a Kähler potential on $C(S)$, and hence $\frac{r^2}{2} - \frac{1}{2}l_\xi(y)$ is a harmonic function on $\mathbb{R}^{m+1}$. But $\frac{1}{2}l_\xi(y)$ is bounded from below as (65), which appears in the Proposition 7.3, is satisfied with $\mathbf{u} = \xi$, and $\frac{r^2}{2}$ is also clearly bounded from below. Thus $\frac{r^2}{2} - \frac{1}{2}l_\xi(y)$ must be a constant. But $\frac{r^2}{2} - \frac{1}{2}l_\xi(y)$ tends to 0 as r tends to 0. Therefore

$$F_\xi^{can} = \frac{1}{2}l_\xi(y) = \frac{r^2}{2}.$$

It follows that

$$\begin{aligned}(62)\qquad u^0 &= \log r|_{Orb_{C(S)}(H^c,p)} + \mathrm{const}\\ &= \frac{1}{2}\log(2F_\xi^{can})|_{Orb_{C(S)}(H^c,p)} + \mathrm{const}\\ &= \frac{1}{2}\log l_\xi|_{Orb_{C(S)}(H^c,p)} + \mathrm{const}.\end{aligned}$$

Now we consider the moment map on the Kähler manifold

$$Orb_{C(S)}(H^c,p) \cong Orb_S(H^c,\overline{p})$$

for the action of $H \cong T^m$. This is defined as

$$j^*\circ\mu_\eta : Orb_{C(S)}(H^c,p) \to \mathfrak{h}^*$$

where $j : \mathfrak{h} \to \mathfrak{g}$ is the inclusion and

$$\langle \mu_\eta(y), X\rangle = \eta(X)(y), \qquad X \in \mathfrak{h},\ y \in Orb_{C(S)}(H^c, p),$$

X being identified with a vector field on $Orb_{C(S)}(H^c, p)$. This is essentially the same as the restriction of the moment map $\mu_\eta : S \to \mathfrak{h}^*$ to $Orb_S(H^c, \overline{p})$. Hence the image of $j^* \circ \mu_\eta$ is equal to

$$j^*(\mathrm{Im}(\mu_\eta)) = \{j^*\alpha \,|\, \alpha \in C(\mu), \alpha(\xi) = 1\}.$$

This is a (possibly irrational) compact convex polyhedron. Identifying $\mathfrak{h}$ with $\mathbb{R}^m$ in the canonical way, the interior $\mathrm{Int} j^*\mathrm{Im}(\mu_\eta)$ of $j^*\mathrm{Im}(\mu_\eta)$ coincides up to translation with

$$\Sigma := \left\{ Du^0(x) = \left(\frac{\partial u^0}{\partial x^1}(x), \cdots, \frac{\partial u^0}{\partial x^m}(x) \right) \,|\, x \in \mathbb{R}^m \right\} \tag{63}$$

because of (58). Let $p_1, \cdots, p_\ell$ be the vertices of the closure $\overline{\Sigma}$ of Σ.

Proposition 7.3. *Consider the Sasaki structure defined by the Kähler cone metric on $C(S)$ with the symplectic potential (42). Let u^0 be the Kähler potential of $(Orb_{C(S)}(H^c, p), \frac{1}{2}d\eta)$. Define $\overline{v} : \mathbb{R}^m \to \mathbb{R}$ by*

$$\overline{v}(x) = \max_{1 \le i \le \ell} \langle p_i, x\rangle.$$

Then there exists a constant C such that $|u^0 - \overline{v}| \le C$.

Proof. From (62) it is sufficient to show

$$|2\overline{v}(x) - \log l_\xi(y)| \le C$$

where x and y are related as follows: first write $\widetilde{x}$ for $jx \in \mathbb{R}^{m+1}$ take the Legendre transform

$$y_j = \frac{\partial F_\xi^{can}}{\partial \widetilde{x}^j}$$

and then restrict y_j to $Orb_{C(S)}(H^c, p)$. For any vertex q_i of

$$\{\alpha \in C(\mu) \,|\, \alpha(\xi) = 2\}$$

such that

$$j^* q_i = 2p_i, \qquad i = 1, \cdots, \ell$$

we have

$$\langle 2p_i, x\rangle = \langle j^* q_i, x\rangle = \langle q_i, jx\rangle = q_i \cdot \widetilde{x}$$

where we again wrote $\widetilde{x}$ for jx in the last term (we do so throughout the rest of the proof of Proposition 7.3). Then from (60) we have

$$q_j \cdot \widetilde{x} - \log l_\xi(y) = \frac{1}{2}\sum_{i=1}^{d} l_i(q_j) \log l_i(y) - \frac{1}{2} l_\infty(q_j) \log l_\infty(y) + 1. \tag{64}$$

In this proof we use the following simple fact repeatedly: *Let $\mathbf{u}$ be a non-zero vector in $\mathbb{R}^{m+1}$ and V be a closed strictly convex polyhedral* *cone in the open half space* $\{y \in \mathbb{R}^{m+1} \,|\, \mathbf{u} \cdot y > 0\}$. *Then there are positive constants* c *and* C *such that for any* $y \in V$ *we have*

$$c|y| \leq \mathbf{u} \cdot y \leq C|y|. \tag{65}$$

These constants c and C will appear many times and take different values, but we will use the same notation by taking smaller value of c and lager value of C. This will not cause any problem as they appear only finitely many times. Recall that $\sum_{i=1}^{d} \lambda_i$ is the Reeb field for the canonical metric [**33**]. Since $\sum_{i=1}^{d} \lambda_i$ is in the interior of $C(\mu)^*$ we have from the above fact that for any $y \in C(\mu)$

$$c|y| \leq l_\infty(y) \leq C|y|. \tag{66}$$

On the other hand by the Schwarz inequality we have for each i

$$l_i(y) \leq C|y|.$$

Let $q_1, \cdots, q_\ell$ be the vertices of

$$\{y \in C(\mu) \,|\, \ell_\xi(y) = 2\} = \{\alpha \in C(\mu) \,|\, \alpha(\xi) = 2\}.$$

Suppose

$$q_j \in \bigcap_{k=1}^{m} L_{i_k}$$

where L_i denotes the hyperplane $\{l_i(y) = 0\}$. It follows from (64) that

$$\begin{aligned} q_j \cdot \widetilde{x} - \log l_\xi(y) &= \frac{1}{2} \sum_{i \notin \{i_1, \cdots, i_m\}} l_i(q_j) \log l_i(y) \\ &\quad - \frac{1}{2} \sum_{i \notin \{i_1, \cdots, i_m\}} l_i(q_j) \log l_\infty(y) + 1 \\ &\leq \frac{1}{2} \sum_{i \notin \{i_1, \cdots, i_m\}} l_i(q_j)(\log |y| + \log C) \\ &\quad - \frac{1}{2} \sum_{i \notin \{i_1, \cdots, i_m\}} l_i(q_j)(\log |y| + \log c) + 1 \\ &\leq C. \end{aligned} \tag{67}$$

This proves

$$2\overline{v}(x) - u^0(x) \leq C. \tag{68}$$

On the other hand

$$\begin{aligned} q_j \cdot \widetilde{x} - \log l_\xi(y) &\geq \frac{1}{2} \sum_{i \notin \{i_1, \cdots, i_m\}} l_i(q_j) \log l_i(y) \\ &\quad - \frac{1}{2} \sum_{i \notin \{i_1, \cdots, i_m\}} l_i(q_j)(\log |y| + \log C) + 1. \end{aligned} \tag{69}$$

For each hyperplane $L_i = \{l_i(y) = 0\}$ we take a hyperplane $L'_i = \{l'_i(y) = 0\}$ which is close to L'_i such that $(C(\mu) - \text{apex}) \cap L'_i$ is non-empty and included in $\{l_i(y) > 0\}$ and that $(C(\mu) - \text{apex}) \cap L_i$ is included in $\{l'_i(y) < 0\}$. Put

$$D_i := \{y \in C(\mu) \,|\, l'_i(y) \le 0\}.$$

Define the following sets successively:

$$\begin{aligned}
&C_0 = C(\mu) - \textstyle\bigcup_{j=1}^d D_j,\\
&C_i = D_i - D_i \cap (\textstyle\bigcup_{j \ne i} D_j),\\
&C_{i_1 i_2} = D_{i_1} \cap D_{i_2} - D_{i_1} \cap D_{i_2} \cap (\textstyle\bigcup_{j \ne i_1, i_2} D_j),\\
&\cdots\\
&C_{i_1 i_2 \cdots i_k} = \textstyle\bigcap_{n=1}^k D_{i_n} - \bigcap_{n=1}^k D_{i_n} \cap (\bigcup_{j \ne i_1, \cdots, i_k} D_j),\\
&\cdots\\
&C_{i_1 \cdots i_m} = \textstyle\bigcap_{n=1}^m D_{i_n}.
\end{aligned}$$

The union of all these sets is $C(\mu)$. We exclude from above the empty sets. First of all, we have on C_0

$$c|y| \le l_j(y) \le C|y| \tag{70}$$

for any j. Hence for any $q_s \in L_{j_1} \cap \cdots \cap L_{j_m}$ we have

$$\begin{aligned}
&q_s \cdot \widetilde{x} - \log l_\xi(y)\\
&\ge \frac{1}{2} \sum_{j \ne j_1, \cdots, j_m} l_j(q_s) \log l_j(y) - \frac{1}{2} \sum_{j \ne j_1, \cdots, j_m} l_j(q_s)(\log |y| + C) + 1\\
&\ge \frac{1}{2} \sum_{j \ne j_1, \cdots, j_m} l_j(q_s)(\log |y| + c) - \frac{1}{2} \sum_{j \ne j_1, \cdots, j_m} l_j(q_s)(\log |y| + C) + 1\\
&\ge c.
\end{aligned} \tag{71}$$

On C_i (70) holds for any $j \ne i$. Take a vertex $q_s \in L_i \cap L_{j_1} \cap \cdots \cap L_{j_{m-1}}$. Then

$$\begin{aligned}
q_s \cdot \widetilde{x} &- \log l_\xi(y)\\
&\ge \frac{1}{2} \sum_{j \ne i, j_1, \cdots, j_{m-1}} l_j(q_s) \log l_j(y)\\
&\quad - \frac{1}{2} \sum_{j \ne i, j_1, \cdots, j_{m-1}} l_j(q_s)(\log |y| + C) + 1\\
&\ge \frac{1}{2} \sum_{j \ne i, j_1, \cdots, j_{m-1}} l_j(q_s)(\log |y| + c)\\
&\quad - \frac{1}{2} \sum_{j \ne i, j_1, \cdots, j_{m-1}} l_j(q_s)(\log |y| + C) + 1\\
&\ge c.
\end{aligned} \tag{72}$$

Continuing this way, on $C_{i_1 i_2 \cdots i_k}$ (70) holds for any $j \neq i_1, \cdots, i_k$. Take a vertex $q_s \in L_{i_1} \cap \cdots \cap L_{i_k} \cap L_{j_1} \cap \cdots \cap L_{j_{m-k}}$. Then

$$\begin{aligned}
(73) \qquad & q_s \cdot \widetilde{x} - \log l_\xi(y) \\
&\geq \frac{1}{2} \sum_{j \neq i_1, \cdots, i_k j_1, \cdots, j_{m-k}} l_j(q_s) \log l_j(y) \\
&\quad - \frac{1}{2} \sum_{j \neq i_1, \cdots, i_k, j_1, \cdots, j_{m-k}} l_j(q_s)(\log |y| + C) + 1 \\
&\geq \frac{1}{2} \sum_{j \neq i_1, \cdots, i_k, j_1, \cdots, j_{m-k}} l_j(q_s)(\log |y| + c) \\
&\quad - \frac{1}{2} \sum_{j \neq i_1, \cdots, i_k, j_1, \cdots, j_{m-k}} l_j(q_s)(\log |y| + C) + 1 \\
&\geq c.
\end{aligned}$$

It follows from (71), (72) and (73) that

$$\begin{aligned}
(74) \qquad 2\overline{v}(x) - 2u^0(x) &= 2 \max_{1 \leq i \leq \ell} p_i \cdot x - \log l_\xi(y) \\
&\geq q_s \cdot \widetilde{x} - \log l_\xi(y) \geq c.
\end{aligned}$$

Then (68) and (74) give the desired estimate. This completes the proof. q.e.d.

Lemma 7.4. *Let $X_i = -\frac{i}{2}\partial/\partial w^i$. Then we have*

$$\theta_{X_i} = \frac{\partial u^0}{\partial x^i}.$$

Proof. It is easy to show

$$i\left(-\frac{\sqrt{-1}}{2}\partial/\partial w^i\right)\sqrt{-1}\frac{\partial^2 u^0}{\partial x^i \partial x^j} dw^i \wedge d\overline{w}^j = \overline{\partial}\left(\frac{\partial u^0}{\partial x^i}\right).$$

Thus it is sufficient to show

$$\int_S \frac{\partial u^0}{\partial x^i} e^h \left(\frac{1}{2}d\eta\right)^m \wedge \eta = 0.$$

But $\rho^T - \frac{1}{2}d\eta = i\partial_B \overline{\partial}_B h$ implies

$$e^h \left(\frac{1}{2}d\eta\right)^m = e^{-u^0} dx^1 \wedge \cdots \wedge dx^m \wedge d\theta^1 \wedge \cdots \wedge d\theta^m.$$

Since ξ generates isometries and $\mathcal{L}_\xi \eta = 0$, we have

$$\int_S \frac{\partial u^0}{\partial x^i} e^h \left(\frac{1}{2}d\eta\right)^m \wedge \eta = \text{const} \int_{\mathbb{R}^m} \frac{\partial u^0}{\partial x^i} e^{-u^0} dx = 0$$

because u^0 is strictly convex and $u^0 \to \infty$ as $x \to \infty$ by Proposition 7.3. q.e.d.

Lemma 7.5. *If the vector field obtained in Proposition* 5.3 *is denoted by* $X = \sum_{i=1}^m c_i(-\frac{i}{2}\frac{\partial}{\partial w^i})$*, Then we have*

$$\int_\Sigma y_j e^{\sum_{i=1}^m c_i y_i} dy = 0, \quad \forall j = 1, 2, \cdots, m.$$

Proof. We have

$$\begin{aligned} f_X(X_j) &= -\int_S \theta_{X_j} e^{\theta_X} \left(\frac{1}{2} d\eta\right)^m \wedge \eta \\ &= -\int_S \frac{\partial u^0}{\partial x^j} e^{\sum_i c_i \frac{\partial u^0}{\partial x^i}} \det\left(u^0_{ij}\right) dx \wedge d\theta \wedge \eta \\ &= -\text{const.} \int_\Sigma y_j e^{\sum_{i=1}^m c_i y_i} dy. \end{aligned}$$

Hence the Lemma follows from Proposition 5.3. q.e.d.

Proposition 7.6. *Let* $\gamma \in \mathfrak{g}^*$ *be as in Proposition* 6.8*, and* H *be the subtorus of* $G = T^{m+1}$ *whose Lie algebra is* $\mathfrak{h} := \{x \,|\, (\gamma, x) = 0\}$*. Then there is a constant* C *such that*

$$\left|\log\det(u^0_{ij}) + (2m+2)u^0\right| \le C. \tag{75}$$

Proof. We keep the same notations as before. Let x^i be the real part of the affine logarithmic coordinates on $Orb_{C(S)}(H^c, p) \cong (\mathbb{C}^*)^m$. The natural inclusion $Orb_{C(S)}(H^c, p) \cong (\mathbb{C}^*)^m \to \mu^{-1}(\mathrm{Int}(C(\mu))$ is induced by $j : \mathfrak{h} \to \mathfrak{g}$. So, we denote by $\widetilde{x} = jx$ the real part of the affine logarithmic coordinates on $\mu^{-1}(\mathrm{Int}(C(\mu)) \cong (\mathbb{C}^*)^{m+1}$ corresponding to x. Let y be the Legendre transform on $C(S)$ of $\widetilde{x}$. Then the Legendre transform v of x on the Kähler manifold $Orb_{C(S)}(H^c, p)$ is given by

$$v = \frac{j^* y}{l_\xi(y)} \tag{76}$$

where $j^* : \mathfrak{g}^* \to \mathfrak{h}^*$. Define $\lambda'_i \in \mathfrak{h} \cong \mathbb{R}^m$ by the decomposition

$$\lambda_i = j\lambda'_i + \mu_i \xi. \tag{77}$$

By our choice of γ and $\mathfrak{h}$ we have

$$\mu_i = \frac{l_i(\gamma)}{l_\xi(\gamma)} = \frac{1}{m+1}. \tag{78}$$

Moreover

$$\begin{aligned} \frac{l_i(y)}{l_\xi(y)} &= \frac{(j\lambda'_i + \mu_i\xi, y)}{l_\xi(y)} \\ &= (\lambda'_i, v) + \mu_i = l'_i(v) \end{aligned} \tag{79}$$

where we have set $l'_i(v) = (\lambda'_i, v) + \mu_i$. Similarly if we set $\lambda'_\infty = \sum_{i=1}^d \lambda'_i$, $\mu_\infty = \sum_{i=1}^d \mu_i (= \frac{d}{m+1})$, and $l'_\infty(v) = (\lambda'_\infty, v) + \mu_\infty$ then

$$\frac{l_\infty(y)}{l_\xi(y)} = l'_\infty(v). \tag{80}$$

We see from (60), (79) and (80) that

$$\begin{aligned} u^0(x) &= \frac{1}{2}\log l_\xi(y) \\ &= -\frac{1}{2}\left(\sum_{i=1}^d \mu_i \log \frac{l_i(y)}{l_\xi(y)} - \mu_\infty \log \frac{l_\infty(y)}{l_\xi(y)} + 1\right) \\ &= -\frac{1}{2}\left(\sum_{i=1}^d \mu_i \log l'_i(v) - \mu_\infty \log l'_\infty(v) + 1\right) \end{aligned} \tag{81}$$

The symplectic potential G_0 on $Orb_{C(S)}(H^c, p)$ is also computed using (60), (79) and (80) as

$$\begin{aligned} G_0(v) &= (v, x) - u^0(x) \\ &= \frac{(j^*y, x)}{l_\xi(y)} - \frac{1}{2}\log l_\xi(y) \\ &= \frac{1}{l_\xi(y)}\Bigg(y, \frac{1}{2}\sum_{i=1}^d \lambda_i \log l_i(y) + \frac{1}{2}\xi(1 + \log l_\xi(y)) \\ &\qquad -\frac{1}{2}\lambda_\infty \log l_\infty(y)\Bigg) - \frac{1}{2}\log l_\xi(y) \\ &= \frac{1}{2}\sum_{i=1}^d \frac{l_i(y)}{l_\xi(y)} \log \frac{l_i(y)}{l_\xi(y)} + \frac{1}{2}(1 + \log l_\xi(y)) \\ &\qquad -\frac{1}{2}\frac{l_\infty(y)}{l_\xi(y)} \log \frac{l_\infty(y)}{l_\xi(y)} - \frac{1}{2}\log l_\xi(y) \\ &= \frac{1}{2}\left(\sum_{i=1}^d l'_i(v) \log l'_i(v) - l'_\infty(v) \log l'_\infty(v) + 1\right) \end{aligned} \tag{82}$$

Since

$$\overline{\Sigma} = \bigcap_{i=1}^d \{l'_i(v) \geq 0\}$$

we have

$$\begin{aligned} \det \mathrm{Hess}\, u^0(x) &= (\det \mathrm{Hess}\, G_0(v))^{-1} \\ &= \delta(v)\Pi_{i=1}^d l'_i(v) \end{aligned} \tag{83}$$

where δ is a strictly positive function on $\overline{\Sigma}$. Since $\mu_i = \frac{1}{m+1}$ by (78) we have

$$\begin{aligned} \left|\log \det \operatorname{Hess} u^0 + (2m+2)u^0\right| &\leq \left|\log \delta + \frac{1}{2}\mu_\infty \log l'_\infty(v) - \frac{1}{2}\right| \\ &\leq C. \end{aligned}$$

q.e.d.

Remark 7.7. One can also show Proposition 7.3 along the line of the proof of Proposition 7.6.

From now on we choose the subtorus H as in Proposition 7.6. Then we have

$$\det\left(u^0_{ij}\right) = \exp\left(-h - u^0\right). \tag{84}$$

Now we can have our equation on $\mathbb{R}^m$. Let $\tilde{\eta} = \eta + 2d^c_B\varphi$ be the solution to (39) where the initial metric is chosen so that the symplectic potential on $C(S)$ is G^{can}_ξ. Then using (39) and (84) one can show that u satisfies

$$\det(u_{ij}) = \exp\left(-tu - (1-t)u^0 - \sum_i c_i \frac{\partial u}{\partial x^i}\right) \quad \text{on } \mathbb{R}^m. \tag{85}$$

Then by the same arguments as Lemma 3.1 - 3.4 in Wang-Zhu [**41**] using our Proposition 7.3 we get an estimate

$$\sup_{Orb_{C(S)}(H^c,p)} \varphi \leq C$$

for some constant $C > 0$ independent of $t \in [0,1]$, or equivalently

$$\sup_{Orb_S(H^c,\overline{p})} \varphi \leq C.$$

But the Reeb flow on S generates isometries and we have C^0 estimate on an open dense subset of S, which gives a C^0 estimate on S naturally. By the same arguments as in either proof (i) or proof (ii) in [**41**] give an estimate

$$\inf_S \varphi \geq -C$$

for some constant $C > 0$ independent of t. In fact we can give all necessary modifications to the arguments of proof (ii) in [**41**], including the arguments of Cao-Tian-Zhu [**14**] and Mabuchi [**29**], [**30**], which we do not re-produce here. This completes the proof of Lemma 7.1, and consequently the proof of Theorem 1.1.

8. The invariant for Kähler cone manifolds

In this section, we will reformulate the invariant $f = 2\pi f_1$ of Sasaki manifolds of positive basic first Chern class as an invariant for Kähler cone manifolds, and then relate the volume function of Sasaki manifolds with the invariant f. This relation was pointed out in section 4.2 of [**34**] in the case of quasi-regular Sasaki manifolds. We wish to relate the invariant further to the existence problem of a Sasaki-Einstein metric.

In the previous sections we used the same notation ξ for the Reeb field $J\frac{\partial}{\partial r}$ and the vector field $Jr\frac{\partial}{\partial r}$ on $C(S)$, but we distinguish them by denoting the vector field on $C(S)$ as

$$\widetilde{\xi} = Jr\frac{\partial}{\partial r}.$$

Definition 8.1. *Let S be a compact $(2m+1)$-dimensional manifold and a complex structure J on $C(S) := \mathbb{R}_+ \times S$. We call a Kähler metric $\bar{g}$ on $(C(S), J)$ a Kähler cone metric if there exist a smooth function $r : C(S) \to \mathbb{R}_+$, a smooth map $p : C(S) \to S$ and a Riemannian metric g on S such that $(r, p) : C(S) \to \mathbb{R}_+ \times S$ is a diffeomorphism and that $(r, p)^*(ds^2 + s^2 g) = \bar{g}$. (Then, of course, g is a Sasaki metric on S.)*

When a Kähler cone metric $\bar{g}$ on $(C(S), J)$ is given, we define the vector field $\widetilde{\xi}$ and the 1-form η on $C(S)$ by

$$\widetilde{\xi} = rJ\frac{\partial}{\partial r}, \ \eta = \frac{1}{r^2}\bar{g}\left(\widetilde{\xi}, \cdot\right) = 2d^c \log r$$

where we use the notation $d^c = \frac{i}{2}(\overline{\partial} - \partial)$. Then we see that $\widetilde{\xi}$ is a holomorphic and Killing vector field. Moreover $\widetilde{\xi}$ lies in the center of the Lie algebra of the group of isometries $\mathrm{Isometry}(C(S), \bar{g})$. The restrictions of $\widetilde{\xi}$ and η to $\{r = 1\} \subset C(S)$, where r is the smooth function on $C(S)$ associated with the Kähler cone metric $\bar{g}$, are the Reeb vector field and the contact 1-form on the Sasaki manifold $\{r = 1\} \simeq (S, g)$. Moreover we see that the Kähler form ω of $\bar{g}$ has the Kähler potential $\frac{1}{2}r^2$;

$$\omega = \frac{1}{2}d\left(r^2\eta\right) = \frac{i}{2}\partial\bar{\partial}r^2. \tag{86}$$

Let S be a compact $(2m+1)$-dimensional manifold and J be a given complex structure on $C(S)$. Suppose that the canonical bundle $K_{C(S)}$ of $C(S)$ is trivial. Then we want to know whether a Ricci-flat Kähler cone metric exists on $(C(S), J)$. In what follows we will reformulate the invariant f obtained in Theorem 4.9 as an obstruction to the existence of a Ricci-flat Kähler cone metric on $C(S)$.

We fix a maximal torus $T^n \subset \mathrm{Aut}(C(S), J)$. (Later we will consider toric Sasaki manifolds, and then $n = m + 1$. But for the moment we do not assume S is toric.) Let $\mathrm{KCM}(C(S), J)$ denote the space of Kähler cone metrics on $(C(S), J)$ such that the maximal torus T^n is contained in the group of isometries and $\widetilde{\xi} \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of T^n. For each metric in $\mathrm{KCM}(C(S), J)$, there is an associated moment map

$$\mu : C(S) \to \mathfrak{g}^*, \quad \langle \mu, X \rangle = r^2 \eta(X)$$

where we identify $X \in \mathfrak{g}$ with the corresponding vector field on $C(S)$ (recall our convention of the moment map in section 6 where we deleted $\frac{1}{2}$). The image of μ is a convex rational polyhedral cone $C(\mu) \subset \mathfrak{g}^*$. Moreover these cones are all isomorphic for all Kähler cone metrics in $\mathrm{KCM}(C(S), J)$. Note that, since the Kähler form of any metric in $\mathrm{KCM}(C(S), J)$ is exact, this is a deformation of Kähler metrics with the same Kähler class.

When we investigate the existence of Ricci-flat Kähler cone metrics, we may restrict the deformation space of Kähler cone metrics to

$$\mathrm{KCM}_c(C(S), J) := \left\{ \bar{g} \in \mathrm{KCM}(C(S), J) \,|\, \rho(\bar{g}) = i\partial\bar{\partial}\tilde{f},\ r\frac{\partial}{\partial r}\tilde{f} = \widetilde{\xi}\tilde{f} = 0 \right\}.$$

Of course a Ricci-flat Kähler cone metric belongs to $\mathrm{KCM}_c(C(S), J)$. In the Sasakian point of view, $\bar{g} \in \mathrm{KCM}_c(C(S), J)$ means that $[\rho^T]_{B,\xi} = 2(m+1)[\omega^T]_{B,\xi}$, where $[\rho^T]_{B,\xi}$ and $[\omega^T]_{B,\xi}$ are respectively the basic cohomology classes of the transverse Ricci form and the transverse Kähler form of the Sasaki metric g on S induced from $\bar{g}$, the Reeb field ξ being with respect to g. This is because $\rho(\bar{g}) = \rho^T - 2(m+1)\omega^T$ and $r\frac{\partial}{\partial r}\tilde{f} = \widetilde{\xi}\tilde{f} = 0$. Proposition 6.8 can be restated as follows.

Proposition 8.2. $\mathrm{KCM}_c(C(S), J)$ *is exactly the set of all Kähler cone metrics such that the associated Reeb fields satisfy* (49)*, or equivalently* (54)*.*

We also consider the subset

$$\mathrm{KCM}_c(C(S), J, \widetilde{\xi}_0) := \left\{ \bar{g} \in \mathrm{KCM}_c(C(S), J) \,|\, \widetilde{\xi} = \widetilde{\xi}_0 \right\}$$

for each fixed $\widetilde{\xi}_0$. This subset corresponds to the set of all the transverse Kähler deformations of Sasaki metrics with the fixed Reeb vector field $\widetilde{\xi}_0$ and varying isometric inclusion $S \subset C(S)$. This fact can be checked as follows. Let $\bar{g}$ and $\bar{g}'$ be Kähler cone metrics on $(C(S), J)$ with $\widetilde{\xi} = \widetilde{\xi}' = \widetilde{\xi}_0$. Then by rotating by J we get $r\frac{\partial}{\partial r} = r'\frac{\partial}{\partial r'}$. Hence there exists a smooth function φ on $C(S)$ such that $r' = r\exp(\varphi)$ and $\frac{\partial}{\partial r}\varphi = \widetilde{\xi}\varphi = 0$. On the other hand, (S, g') is isometrically identified with $\{r' = 1\} \subset C(S)$ by the embedding $i : S \hookrightarrow C(S)$, $i(x) = (\exp(-\varphi)(x), x)$. Thus, on $\{r' = 1\}$, $i_*(\xi) = 0 \oplus \xi = \widetilde{\xi}_{\{r'=1\}} = \widetilde{\xi}'_{\{r'=1\}}$. Since i_* is injective, this suggests that ξ is the Reeb vector field of the Sasaki manifold (S, g'). The one form η on $C(S)$ is deformed to

$$\eta' = 2d^c(\log r + \varphi) = \eta + 2d^c\varphi$$

when we change the metric from $\bar{g}$ to $\bar{g'}$. Hence

$$\omega^T(g') = \omega^T(g) + d_B d_B^c \varphi,$$

where $\omega^T(g), \omega^T(g')$ are the transverse Kähler forms with respect to Sasaki metrics g, g' on S respectively.

Consider the volume functional

$$\tilde{V} : \mathrm{KCM}(C(S), J) \to \mathbb{R}, \quad \tilde{V}(\bar{g}) := \mathrm{Vol}(S, g),$$

where g is the Sasaki metric on S induced from $\bar{g}$. Proposition 8.3 and 8.4 below, which are the first and second variation formulae for the volume functional, were proved in Appendix C of [**34**], but we give slightly more comprehensive proofs in this paper for the reader's convenience.

Proposition 8.3 ([**34**], Appendix C)**.** *Let S be a compact Sasaki manifold. Let $\{\bar{g}(t)\}_{-\varepsilon<t<\varepsilon}$ be a 1-parameter family in* $\mathrm{KCM}(C(S), J)$ *with $\bar{g}(0) = \bar{g}$ and $Y = d\widetilde{\xi}/dt_{|t=0}$. Then*

$$\frac{d}{dt}_{|t=0} \tilde{V}(\bar{g}(t)) = -4(m+1) \int_S \eta(Y) dvol_g. \tag{87}$$

Proof. Let $S(t) \subset C(S)$ denote the subset $\{r(t) = 1\}$ for each t, where $r(t) : C(S) \to \mathbb{R}_+$ is the smooth function associated with the cone metric $\bar{g}(t)$, see Definition 8.1. Then $(S(t), \bar{g}(t)_{|S(t)})$ is isometric to $(S, g(t))$. Since $(d\eta(t))^m \wedge \eta(t)$ is closed for each t, we have

$$\begin{aligned}\tilde{V}(\bar{g}(t)) &= \frac{1}{m!}\int_{S(t)} \left(\frac{1}{2}d\eta(t)\right)^m \wedge \eta(t) = \frac{1}{m!}\int_{S(0)} \left(\frac{1}{2}d\eta(t)\right)^m \wedge \eta(t)\\ &= \frac{1}{m!}\int_{S(0)} \left(\frac{1}{2}d\eta + t dd^c\phi + O(t^2)\right)^m \wedge \left(\eta + 2td^c\phi + O(t^2)\right)\end{aligned}$$

Therefore, by Lemma 8.6, the first variation is

$$\begin{aligned}&\frac{d}{dt}_{|t=0}\tilde{V}(\bar{g}(t))\\ &= \frac{1}{m!}\int_{S(0)} \left(m\left(\frac{1}{2}d\eta\right)^{m-1} \wedge dd^c\phi \wedge \eta + \left(\frac{1}{2}d\eta\right)^m \wedge 2d^c\phi\right)\\ &= \frac{2m+2}{m!}\int_{S(0)} d^c\phi \wedge \left(\frac{1}{2}d\eta\right)^m = -4(m+1)\int_S \eta(Y) dvol_g.\end{aligned}$$

Here the second equality holds since

$$d\left((d\eta)^{m-1} \wedge d^c\phi \wedge \eta\right) = (d\eta)^{m-1} \wedge dd^c\phi \wedge \eta - (d\eta)^m \wedge d^c\phi.$$

q.e.d.

Proposition 8.4 ([**34**], Appendix C)**.** *Let $\{\bar{g}(t)\}_{-\varepsilon<t<\varepsilon}$ be a one parameter family in* $\mathrm{KCM}_c(C(S), J)$ *with $\bar{g}(0) = \bar{g}$ and $Y = d\widetilde{\xi}/dt_{|t=0}$.*

Then

$$\frac{d}{dt}_{|t=0}\left(\int_S \eta(t)(X)dvol_{g(t)}\right) = -(2m+4)\int_S \eta(X)\eta(Y)dvol_g \tag{88}$$

for each $X \in \mathfrak{g}$, where $g(t)$ and $\eta(t)$ are the Sasaki metric and the contact 1-form on S induced from $\bar{g}(t)$.

Proof. Let the notations be as in the proof of the previous proposition. For each sufficiently small t,

$$\widetilde{\xi}(t) = \widetilde{\xi} + tY + O(t^2) \tag{89}$$

and

$$r^2(t) = r^2(1 + 2t\varphi + O(t^2)). \tag{90}$$

Then we have

$$\eta(t) = \eta + 2td^c\varphi + O(t^2). \tag{91}$$

Multiplying (89) by J, we have

$$r(t)\frac{\partial}{\partial r(t)} = r\frac{\partial}{\partial r} - tJY + O(t^2). \tag{92}$$

Expanding $\mathcal{L}_{r(t)\frac{\partial}{\partial r(t)}} r^2(t) = 2r^2(t)$ to first order in t gives

$$2\mathcal{L}_{r\frac{\partial}{\partial r}}\varphi = -2\eta(Y). \tag{93}$$

Since $\mathcal{L}_{r(t)\frac{\partial}{\partial r(t)}} X = 0$ for each $X \in \mathfrak{g}$ and $\mathcal{L}_{r(t)\frac{\partial}{\partial r(t)}}\eta(t) = 0$,

$$\begin{aligned} d\left\{\eta(t)(X)\,(d\eta(t))^m \wedge \eta(t)\right\} &= d\left(\eta(t)(X)\right) \wedge (d\eta(t))^m \wedge \eta(t) \\ &= \frac{\partial(\eta(t)(X))}{\partial r(t)} dr(t) \wedge (d\eta(t))^m \wedge \eta(t) = 0 \end{aligned}$$

on $C(S)$. Hence

$$\begin{aligned} &\int_S \eta(t)(X)dvol_{g(t)} \\ &= \frac{1}{m!}\int_{S(t)} \eta(t)(X)\left(\frac{1}{2}d\eta(t)\right)^m \wedge \eta(t) \\ &= \frac{1}{m!}\int_{S(0)} \eta(t)(X)\left(\frac{1}{2}d\eta(t)\right)^m \wedge \eta(t) \\ &= \frac{1}{m!}\int_{S(0)} \Big\{ \left(\eta + 2td^c\varphi + O(t^2)\right)(X)\left(\frac{1}{2}d\eta + tdd^c\varphi + O(t^2)\right)^m \\ &\qquad \wedge \left(\eta + 2td^c\varphi + O(t^2)\right)\Big\}. \end{aligned}$$

Thus

$$\begin{aligned}\frac{d}{dt}_{|t=0}\left(\int_S \eta(t)(X)dvol_{g(t)}\right) &= \frac{1}{m!}\int_{S(0)}\left\{2d^c\varphi(X)\left(\frac{1}{2}d\eta\right)^m\wedge\eta\right.\\ &\quad + m\eta(X)\left(\frac{1}{2}d\eta\right)^{m-1}\wedge dd^c\varphi\wedge\eta\\ &\quad \left.+\eta(X)\left(\frac{1}{2}d\eta\right)^m\wedge d^c\varphi\right\}\\ &= \frac{2m+4}{m!}\int_{S(0)}\eta(X)\left(\frac{1}{2}d\eta\right)^m\wedge 2d^c\varphi\\ &= -(2m+4)\int_S \eta(X)\eta(Y)dvol_g.\end{aligned}$$

Here the second and third equalities are given by the following lemmas.

q.e.d.

Lemma 8.5.

$$\begin{aligned}&\int_{S(0)}\left\{d^c\varphi(X)\left(\frac{1}{2}d\eta\right)^m\wedge\eta+\frac{m}{2}\eta(X)\left(\frac{1}{2}d\eta\right)^{m-1}\wedge dd^c\varphi\wedge\eta\right\}\\ &= (m+1)\int_{S(0)}\eta(X)\left(\frac{1}{2}d\eta\right)^m\wedge d^c\varphi.\end{aligned}$$

Proof. Since X is tangent to $S(0)$ and $\mathcal{L}_X\eta = 0$,

$$\begin{aligned}0 &= \int_{S(0)}\iota(X)\left(\left(\frac{1}{2}d\eta\right)^m\wedge\eta\wedge d^c\varphi\right)\\ &= \int_{S(0)}\left(m\left(\frac{1}{2}\iota(X)d\eta\right)\wedge\left(\frac{1}{2}d\eta\right)^{m-1}\wedge\eta\wedge d^c\varphi\right.\\ &\qquad \left.+\eta(X)\left(\frac{1}{2}d\eta\right)^m\wedge d^c\varphi - d^c\varphi(X)\left(\frac{1}{2}d\eta\right)^m\wedge\eta\right)\\ &= \int_{S(0)}\left(-\frac{m}{2}d(\eta(X))\wedge\left(\frac{1}{2}d\eta\right)^{m-1}\wedge\eta\wedge d^c\varphi\right.\\ &\qquad \left.+\eta(X)\left(\frac{1}{2}d\eta\right)^m\wedge d^c\varphi - d^c\varphi(X)\left(\frac{1}{2}d\eta\right)^m\wedge\eta\right).\end{aligned}$$

On the other hand

$$\begin{aligned}&\frac{1}{2}d\left(\eta(X)\left(\frac{1}{2}d\eta\right)^{m-1}\wedge\eta\wedge d^c\varphi\right)\\ &\quad = \frac{1}{2}d(\eta(X))\wedge\left(\frac{1}{2}d\eta\right)^{m-1}\wedge\eta\wedge d^c\varphi+\eta(X)\left(\frac{1}{2}d\eta\right)^m\wedge d^c\varphi\\ &\qquad -\frac{1}{2}\eta(X)\left(\frac{1}{2}d\eta\right)^{m-1}\wedge dd^c\varphi\wedge\eta.\end{aligned}$$

Combining these equations, we get the lemma. q.e.d.

Lemma 8.6. *On $S(0)$,*

$$d^c\varphi \wedge (d\eta)^m = -2\eta(Y)\eta \wedge (d\eta)^m. \tag{94}$$

Proof. On $S(0)$,

$$d^c\varphi \wedge (d\eta)^m = (\iota(\widetilde{\xi})d^c\varphi)\eta \wedge (d\eta)^m.$$

On the other hand, by (93),

$$\iota(\widetilde{\xi})d^c\varphi = -2\iota(J\widetilde{\xi})d\varphi = 2\iota\left(r\frac{\partial}{\partial r}\right)d\varphi = -2\eta(Y).$$

Combining these equations, we get (94). q.e.d.

Let $\bar{g} \in \mathrm{KCM}_c(C(S), J)$, and denote by $\mathfrak{h}(C(S), J, \bar{g})$ the space of *normalized* holomorphic Hamiltonian vector fields on $(C(S), J, \bar{g})$ in the following sense. We call a complex vector field $\widetilde{X}$ on $C(S)$ Hamiltonian holomorphic if $\widetilde{X}$ is a holomorphic vector field and $\widetilde{X}_{\mathbb{R}} = (\widetilde{X} + \overline{\widetilde{X}})/2$ is Killing. If $\widetilde{X}$ is a holomorphic Hamiltonian vector field then

$$X = \widetilde{X} - i\eta(\widetilde{X})r\frac{\partial}{\partial r} = \widetilde{X} - i\eta(X)r\frac{\partial}{\partial r}$$

defines a Hamiltonian holomorphic vector field on S in the sense of Definition 4.5. With this remark, we say that a Hamiltonian holomorphic vector field $\widetilde{X}$ on $C(S)$ is normalized if X is normalized in the sense that $u_X := \frac{i}{2}\eta(X)$ satisfies (34).

Then, using the above relation between $\widetilde{X}$ and X, we define a linear function on $\mathfrak{h}(C(S), J, \bar{g})$ by

$$\widetilde{X} \mapsto F(\widetilde{X}) := 2\pi i \int_S Xh\, dvol_g,$$

where g is the Sasaki metric on S induced from the cone metric $\bar{g}$ and h is the basic function on S such that $\widetilde{f} = p^*h$.

Proposition 8.7. *The linear map F defined above coincides with the restriction to $\mathfrak{h}(C(S), J, \bar{g})$ of $2\pi i m! f = m! i f_1$ on (S, g).*

Proof. This follows from (25). q.e.d.

Proposition 8.8. *F is independent of $\bar{g} \in \mathrm{KCM}_c(C(S), J, \widetilde{\xi}_0)$ for each fixed $\widetilde{\xi}_0$.*

Proof. This follows from Proposition 8.7 and Theorem 4.9. But we will give an alternate proof below. q.e.d.

Notice that $\mathfrak{h}(C(S), J, \bar{g})$ may vary as ξ varies as the elements of $\mathfrak{h}(C(S), J, \bar{g})$ have to commute with ξ. Since this linear function F is a character of the Lie algebra $\mathfrak{h}(C(S), J, \bar{g})$, F is only nontrivial on the center, and ξ and the center are included in the maximal torus $\mathfrak{g}$. So we restrict F to $\mathfrak{g}$, but consider it for all $\bar{g} \in \mathrm{KCM}_c(C(S), J)$:

$$F : \mathrm{KCM}_c(C(S), J) \times \mathfrak{g} \to \mathbb{R}.$$

Alternative Proof of Proposition 8.8. Since X is a normalized Hamiltonian holomorphic vector field, $\eta(X) = -2iu_X$ satisfies

$$\begin{aligned}
&2(m+1)\pi \int_S \eta(X)\, dvol_g \\
&= \pi \int_S (-\Delta_B \eta(X) + 2(m+1)\eta(X))\, dvol_g \\
&= -2\pi i \int_S \nabla^i u_X \nabla_i h\, dvol_g \\
&= 2\pi i \int_S Xh\ dvol_g \\
&= F(\widetilde{X}).
\end{aligned} \tag{95}$$

Therefore, the invariance of F on $\mathrm{KCM}_c(C(S), J, \xi_0)$ follows from the following Proposition 8.4 by putting $Y = 0$. q.e.d.

Of course if $\bar{g} \in \mathrm{KCM}_c(C(S), J)$ is Ricci-flat, then F vanishes on $\mathrm{KCM}_c(C(S), J, \widetilde{\xi})$ for the corresponding $\widetilde{\xi}$. Therefore the nonvanishing of F on $\mathrm{KCM}_c(C(S), J)$ obstructs the existence of a Ricci-flat Kähler cone metric.

Now let (S, g_0) be a $(2m+1)$-dimensional compact toric Sasaki manifold (see Definition 6.5). Then the metric cone $(C(S), J, \bar{g}_0)$ is a toric Kähler cone. Here a Kähler metric being toric means that the real torus T^{m+1} acts holomorphically and effectively on $C(S)$ preserving the Kähler form. Note here that if we fix a maximal torus T^{m+1} of $Aut(C(S), J)$, then the metrics in $\mathrm{KCM}(C(S), J)$ are all toric Kähler since we defined $\mathrm{KCM}(C(S), J)$ to be the set of all Kähler cone metric invariant under the maximal torus of $Aut(C(S), J)$. We will see that there is a unique vector field ξ_c on S such that there are Sasaki metrics on S such that the Reeb vector field is ξ_c and that the invariant f vanishes identically. To find such ξ_c, we need to use the relationship between the invariant f and the volume functional of Sasaki manifolds given by the first variation formula, Proposition 8.3. In [**33**] and [**34**], Martelli, Sparks and Yau came up with this idea.

When we fix an angular coordinates $\phi_i \sim \phi_i + 2\pi$ on T^{m+1}, we can identify $\mathfrak{g}$ with $\mathbb{R}^{m+1}$ by identifying $\sum_i X^i \partial/\partial\phi_i$ with $(X^1, \cdots, X^{m+1})$, and $\mathfrak{g}^*$ is also identified with $\mathbb{R}^{m+1}$. Then the cone $C(\mu)$ can be represented as

$$C(\mu) = \{y \in \mathbb{R}^{m+1} | \langle \lambda_j, y\rangle \geq 0, j = 1, \cdots, d\},$$

where λ_j are the inward pointing normal vectors of the d facets of the cone $C(\mu)$. Since $C(\mu)$ is a rational cone, we can normalize the vectors λ_j to be primitive elements of $\mathbb{Z}^{m+1}$.

As is shown in section 2 of [**33**], the image $\mathrm{Im}(Reeb)$ of the map

$$Reeb : \mathrm{KCM}(C(S), J) \to \mathfrak{g} \simeq \mathbb{R}^{m+1}, \quad \bar{g} \mapsto \widetilde{\xi} = rJ\frac{\partial}{\partial r}$$

is $\mathrm{Int}C(\mu)^*$, the interior of the dual cone $C(\mu)^*$ of $C(\mu)$, see (41).

Proposition 8.9. *Let S be a compact toric Sasaki manifold. The volume functional*

$$\widetilde{V} : \mathrm{KCM}(C(S), J) \to \mathbb{R}, \quad \widetilde{V}(\bar{g}) := \mathrm{Vol}(S, g),$$

where g is the Sasaki metric on S induced from $\bar{g}$, descends to a well-defined function V on $\mathrm{Int}C(\mu)^*$ and thus depends only on Reeb fields.

This proposition is a corollary to Proposition 8.3: One just puts $Y = 0$.

Combining Proposition 8.9 and the second variation formula, Proposition 8.4, the volume function $V : \mathrm{Int}C(\mu)^* \to \mathbb{R}$ is a non-negative continuous strictly convex function. In fact, $V(x)$ can be described by the Euclidean volume of the polytope Δ_x, which depends only on $x \in \mathrm{Int}C(\mu)^*$, in $\mathfrak{g}^* \simeq \mathbb{R}^{m+1}$;

$$\Delta_x = \{y \in C(\mu) \mid 2\langle x, y\rangle \leq 1\},$$
$$V(x) = 2(m+1)(2\pi)^{m+1}\,\mathrm{Vol}(\Delta_x),$$

where $\mathrm{Vol}(\Delta_x)$ is the Euclidean volume of the polytope Δ_x, see section 2 of [**33**]. Hence we see that V diverges to infinity when $x \to \partial C(\mu)^*$.

By Proposition 6.8

$$\begin{aligned}&\left\{\widetilde{\xi} \in \mathrm{Int}C(\mu)^* \,|\, \widetilde{\xi} \text{ is the Reeb field for some } \overline{g} \in \mathrm{KCM}_c(C(S), J)\right\}\\ &\qquad = \left\{\widetilde{\xi} \in \mathrm{Int}C(\mu)^* \,|\, (\gamma, \widetilde{\xi}) = -(m+1)\right\}\end{aligned}$$

which is a relatively compact set in $C(S)^*$. Therefore the restriction $V : \mathrm{Int}C(\mu)^* \cap \{x \in \mathbb{R}^{m+1} | (\gamma, \widetilde{\xi}) = -(m+1)\} \to \mathbb{R}$ has unique minimum point x_c.

Proposition 8.10. *Let (S, g_0) be a $(2m+1)$-dimensional compact Sasaki manifold such that the metric cone $(C(S), J, \bar{g}_0)$ is a toric Kähler cone $\bar{g}_0 \in \mathrm{KCM}_c(C(S), J)$. Suppose that $\widetilde{\xi} = x_c \in \mathrm{Int}C(\mu)^*$. Then the invariant f for (S, g_0) vanishes.*

Proof. Let $\bar{g} \in \mathrm{KCM}_c(C(S), J)$ with $\widetilde{\xi} \in \mathrm{Int}C(\mu)^* \cap \{x \in \mathfrak{g}^* \cong \mathbb{R}^{m+1} | (\gamma, x) = -(m+1)\}$. By (95) and the first variation formula (Proposition 8.3) it suffices to prove that

$$\{X \in \mathfrak{g} \,|\, (\gamma, X) = 0\} = \left\{X \in \mathfrak{g} \,|\, \Delta_B^h u_X = 2(m+1)u_X\right\},$$

where $u_X = \frac{i}{2}\eta(X_c)$, X_c is the Hamiltonian holomorphic vector field

$$X_c = X - i\Phi X$$

on S and h is a basic function which satisfies $\rho(\bar{g}) = i\partial\bar{\partial}(p^* h)$. (Note that we use the notation X_c for X in Definition 4.4.)

Then by Proposition 6.8

$$\{X \in \mathfrak{g} \mid (\gamma, X) = 0\} = \left\{X \in \mathfrak{g} \,|\, \mathcal{L}_{\widetilde{X}_c}\Omega = 0\right\},$$

where $\widetilde{X}_c = X_c + i\eta(X_c) r\frac{\partial}{\partial r}$ and Ω is the non-vanishing multi-valued holomorphic $(m+1,0)$-form satisfying (55). Let $X \in \{X \in \mathfrak{g} \mid \mathcal{L}_{\widetilde{X}_c}\Omega = 0\}$ be given. Then the Lie derivative of (55) by $\widetilde{X}_c$ gives along $\{r = 1\} \cong S$

$$\begin{aligned} 0 &= 2\widetilde{X}_c h - \Delta_{C(S)}\left(\frac{i}{2}r^2\eta(\widetilde{X}_c)\right) \\ &= 2X_c h + \left(\frac{\partial^2}{\partial r^2} - \frac{1}{r^2}\Delta_S + \frac{2m+1}{r}\frac{\partial}{\partial r}\right)\left(\frac{i}{2}r^2\eta(X_c)\right) \\ &= 2X_c h + 4(m+1)u_X - \Delta_S u_X \\ &= -2\Delta_B^h u_X + 4(m+1)u_X, \end{aligned} \tag{96}$$

where $\Delta_{C(S)}$ and Δ_S are the positive real Laplacians of $(C(S), \bar{g})$ and (S, g) respectively and we have put $u_X = \frac{i}{2}\eta(X_c)$. Hence

$$\{X \in \mathfrak{g} \mid (\gamma, X) = 0\} \subset \left\{X \in \mathfrak{g} \mid \Delta_B^h u_X = 2(m+1)u_X\right\}.$$

Therefore

$$\{X \in \mathfrak{g} \mid (\gamma, X) = 0\} = \left\{X \in \mathfrak{g} \mid \Delta_B^h u_X = 2(m+1)u_X\right\},$$

since these are hyperplanes in $\mathfrak{g}$. Therefore Y in Proposition 8.3 can be taken as normalized Hamiltonian holomorphic vector fields. q.e.d.

Remark 8.11. In the quasi-regular case, the relationship between the first variation of the volume function V and the invariant in [**19**] of the orbit space of the flow of the Reeb vector field, which is in general Kähler orbifold, was proved in section 4.2 of [**34**], using "Killing spinor" on S. We have proved this relationship without using Killing spinors.

Remark 8.12. *Note that there always exist toric Sasaki metrics such that* $\widetilde{\xi} = x_c \in \mathrm{Int}C(\mu)^*$. *In fact, we can construct symplectic potential of such metrics concretely, see section 2 of* [**33**].

Proof of Theorem 1.2. By Proposition 8.10 there exists $\widetilde{\xi}$ such that the corresponding Kähler cone metric with vanishing invariant f. By Theorem 1.1 there exists a transverse Kähler-Einstein metric satisfying $\rho^T = (2m+2)\omega^T$. This metric is a Sasaki-Einstein metric. q.e.d.

Let $(M, \underline{g}, J)$ be a real $2m$-dimensional compact Kähler manifold such that $[\rho] = 2(m+1)[\omega] \in H^2(M;\mathbb{R})$, where ω and ρ are the Kähler form and the Ricci form respectively. Let N be the maximal integer such that $c_1(M)/N$ is an integral cohomology class and $\pi : S \to M$ the principal S^1-bundle with the first Chern class $c_1(S) = c_1(M)/N$. Then it is well-known that S is simply connected and there is a regular Sasaki metric g on S such that the projection π is a Riemannian submersion. Moreover, this regular Sasaki metric g is Einstein if $\underline{g}$ is Einstein. However, in contrast, g is not Einstein if $\underline{g}$ is not Einstein.

Proof of Corollary 1.3. Let M be the blow up of $\mathbb{C}P^2$ at 2 generic points. Then M is a toric manifold which does not admit Kähler-Einsten metric by Matsushima's theorem. Hence any regular Sasaki metric on S associated with Kähler metric on M is not Einstein.

However, there is a toric Sasaki-Einstein metric on S by Theorem 1.2; in this case, the inward pointing normal vectors of the facets of the moment cone $C(\mu) \subset \mathbb{R}^3$ of $C(S)$ are

$$v_1 = (1,0,0),\ v_2 = (1,0,1),\ v_3 = (1,1,2),\ v_4 = (1,2,1),\ v_5 = (1,1,0).$$

By the calculation in section 3 of [**33**], the Reeb vector of the toric Sasaki Einstein metric is given by

$$x_c = \left(3, \frac{9}{16}\left(-1+\sqrt{33}\right), \frac{9}{16}\left(-1+\sqrt{33}\right)\right).$$

Clearly, since x_c is not a rational point, this is the Reeb vector of an irregular Sasaki metric. q.e.d.

9. Appendix

The proof of Theorem 4.9 is based on the following two lemmas.

Lemma 9.1. *If α is a basic $(2m-1)$-form, then*

$$\int_S d_B\alpha \wedge \eta = 0.$$

This lemma can be seen as a special case of the following

Lemma 9.2. *If α and β are basic forms with $\deg\alpha + \deg\beta = 2m-1$, then*

$$\int_S d_B\alpha \wedge \beta \wedge \eta = (-1)^{\deg\alpha} \int_S \alpha \wedge d_B\beta \wedge \eta.$$

The proof of these lemmas are easy exercises. These lemmas show that if we have a result for a compact Kähler manifold which can be proved only using the Stokes theorem, including the integration by parts, then the result holds true for compact Sasaki manifolds. The proof can be given only by adding "$\wedge\eta$" at each line of the proof.

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