

Transverse vibrations of double-tapered cantilever beams with end support and with end mass

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The free vibrations of a double-tapered cantilever beam with (1) end support and (2) end mass have been investigated using the Bernoulli-Euler equation. The beam was tapered linearly in the horizontal and in the vertical planes simultaneously with the taper ratio in the horizontal plane equal to that in the vertical plane. A table is presented for the first case from which the fundamental frequency, second, third, fourth, and fifth harmonic can easily be obtained for various taper ratios. A chart, plotted from this table, shows the effect of taper ratio on the various harmonics. For the second case, a table and resulting charts show the effect of taper ratio and ratio of end mass to beam mass on the fundamental frequency and higher harmonics. Although previously presented, the case of the beam with free end is also included for purposes of comparison.

Subject Classification: 40.22.

INTRODUCTION

This analysis is a continuation of the work¹⁻³ started by the authors several years ago on the vibration of tapered cantilever beams. This type of beam tapered linearly in either the horizontal or the vertical plane is widely used for electrical contacts and for springs in electro-mechanical devices. Occasionally, however, the desired spring rate cannot be achieved by tapering the beam in only one plane so that it is necessary to resort to a taper in the horizontal and in the vertical plane simultaneously. Both the single-tapered and the double-tapered beams will require an end mass or an end support depending upon whether the beam is used for an electrical contact (normally open) or for a spring. Single-tapered cantilever beams with end mass and with end support have been treated by the authors.^{1,2}

This paper deals with the vibration of double-tapered cantilever beams with end support and with end mass for the case where the taper ratio in the horizontal plane equals that in the vertical plane. Tables have been developed from which the fundamental frequency, second, third, fourth, and fifth harmonic can be obtained for various taper ratios and ratios of end mass to beam mass. Although the case of the double-tapered cantilever beam with free end has been presented in Ref. 3, it is included in this work for comparative purposes.

I. BEAM OF LINEARLY VARIABLE THICKNESS AND OF LINEARLY VARIABLE WIDTH

In Ref. 3 the differential equation of motion for a vibrating beam tapered in two planes as shown in Fig. 1 was developed from the Bernoulli-Euler equation

$$\frac{\partial^2}{\partial x^2} \left(\frac{EI \partial^2 y}{\partial x^2} \right) = - \left(\frac{\rho A}{g} \right) \frac{\partial^2 y}{\partial t^2}, \quad (1)$$

where $\rho A/g$ is the mass per unit length (ρ weight density, A cross-sectional area, g gravitational constant), E the modulus of elasticity, and I the moment of inertia. A sustained free vibration at a frequency ω of $y(x, t)$

$= z(x) \sin \omega t$ was assumed which gave the following:

$$\begin{aligned} \frac{d^4 z}{du^4} + \frac{2d^3 z}{du^3} \left[\frac{3(\alpha-1)}{1+(\alpha-1)u} + \frac{\beta-1}{1+(\beta-1)u} \right] + \frac{6d^2 z}{du^2} \\ \times \left\{ \frac{(\beta-1)(\alpha-1)}{[1+(\beta-1)u][1+(\alpha-1)u]} + \frac{(\alpha-1)^2}{[1+(\alpha-1)u]^2} \right\} \\ = \frac{(lk)^4 z}{[1+(\alpha-1)u]^2}, \end{aligned} \quad (2)$$

where

$$u = x/l,$$

$$\alpha = h_0/h_1,$$

$$\beta = b_0/b_1,$$

$$k^4 = 12 \rho \omega^2 / Eg h_1^2.$$

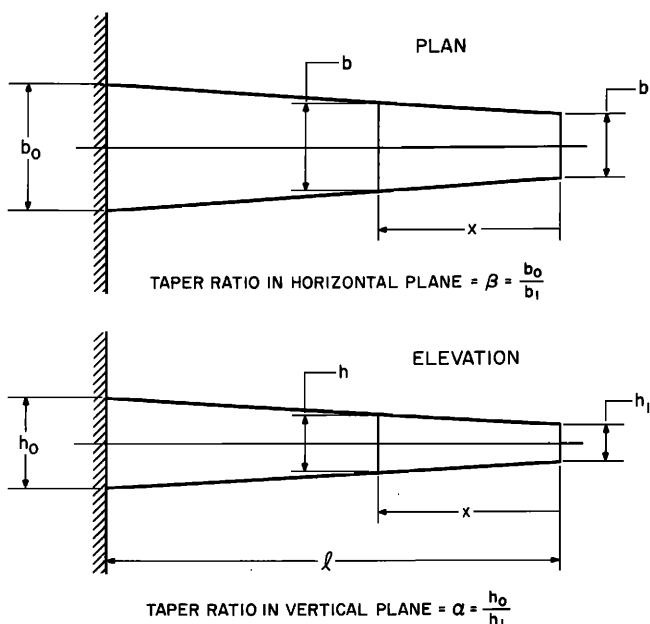


FIG. 1. Cantilever beam tapered linearly in horizontal and in vertical planes simultaneously.

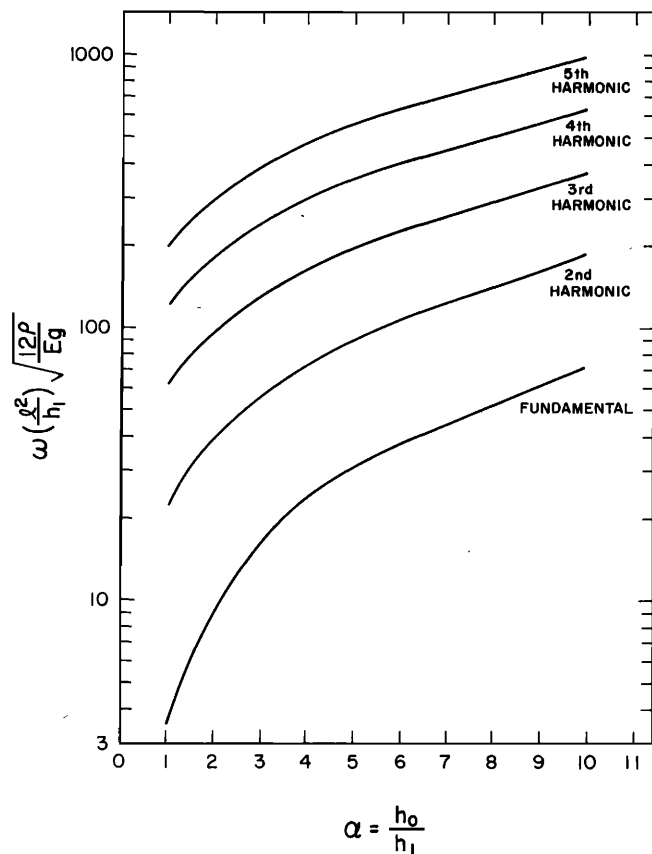


FIG. 2. Frequencies for double-tapered cantilever beam with free end with $\beta = \alpha$.

A formal solution for this equation could not be obtained, and it was solved by numerical integration to give values of (lk) for various taper ratios of α and β .

A solution to Eq. 2 can be obtained by considering the special case where the taper ratios α and β are equal. The resulting differential equation can then be solved in terms of Bessel functions. If $\beta = \alpha$ is substituted in Eq. 2, the following equation results:

$$\frac{d^4 z}{du^4} + \frac{8d^3 z}{du^3} \left[\frac{(\alpha - 1)}{1 + (\alpha - 1)u} \right] + \frac{12d^2 z}{du^2} \left[\frac{(\alpha - 1)}{1 + (\alpha - 1)u} \right]^2 = \frac{(lk)^4 z}{[1 + (\alpha - 1)u]^2} \tag{3}$$

Equation 3 may be placed in a more recognizable form if $\phi = 1 + (\alpha - 1)u$; this substitution yields

$$\phi^4 \frac{d^4 z}{d\phi^4} + 8\phi^3 \frac{d^3 z}{d\phi^3} + 12\phi^2 \frac{d^2 z}{d\phi^2} = \left(\frac{lk}{\alpha - 1} \right)^4 \phi^2 z \tag{4}$$

Siddall and Isackson⁴ list the steps to put Eq. 4 into operator notation for which Watson⁵ gives the solution as

$$z = \frac{1}{\phi} \left[AJ_2 \left(\frac{2lk}{\alpha - 1} \sqrt{\phi} \right) + BY_2 \left(\frac{2lk}{\alpha - 1} \sqrt{\phi} \right) + CI_2 \left(\frac{2lk}{\alpha - 1} \sqrt{\phi} \right) + DK_2 \left(\frac{2lk}{\alpha - 1} \sqrt{\phi} \right) \right] \tag{5}$$

where J_2 and Y_2 are Bessel functions of the first and second kind and I_2 and K_2 are modified Bessel functions of the first and second kind.

TABLE I. Factor $(lk)^2 \beta = \alpha$, free end.

α	Funda- mental frequency	Second harmonic	Third harmonic	Fourth harmonic	Fifth harmonic
1.0	3.51602	22.03449	61.69721	120.9019	199.8595
1.2	4.54997	25.46665	68.98686	133.9884	220.6668
1.4	5.64828	28.87848	76.14960	146.7374	240.8424
1.6	6.80221	32.28233	83.21828	159.2265	260.5263
1.8	8.00478	35.68615	90.21556	171.5096	279.8154
2.0	9.25030	39.09523	97.15780	183.6255	298.7798
2.5	12.5226	47.6622	114.3459	213.3596	345.0885
3.0	15.9785	56.3146	131.3862	242.5213	390.2255
3.5	19.5781	65.0655	148.3511	271.2901	434.5191
4.0	23.2923	73.9188	165.2854	299.7799	478.1801
5.0	30.9820	91.9273	199.1682	356.2088	564.1394
10.0	72.0487	186.802	371.238	635.049	981.657

A. Beam with free end

For a beam with a free end at $x = 0$, the boundary conditions are

$$\begin{aligned} \text{at } x = 0 \text{ or } u = 0, \quad d^2 z / du^2 = 0 \text{ and } d^3 z / du^3 = 0, \\ \text{at } x = l \text{ or } u = 1, \quad dz / du = 0 \text{ and } z = 0. \end{aligned}$$

With these boundary conditions, the solution becomes that of a double-tapered cantilever beam which is truncated and tapers from the fixed end only. Imposing the above boundary conditions on the general solution in Eq. 5 gives the following determinantal equation for obtaining the natural frequencies of the beam:

$$\begin{vmatrix} J_2(\Theta\sqrt{\alpha}) & Y_2(\Theta\sqrt{\alpha}) & I_2(\Theta\sqrt{\alpha}) & K_2(\Theta\sqrt{\alpha}) \\ J_3(\Theta\sqrt{\alpha}) & Y_3(\Theta\sqrt{\alpha}) - I_3(\Theta\sqrt{\alpha}) & K_3(\Theta\sqrt{\alpha}) \\ J_4(\Theta) & Y_4(\Theta) & I_4(\Theta) & K_4(\Theta) \\ J_5(\Theta) & Y_5(\Theta) & -I_5(\Theta) & K_4(\Theta) \end{vmatrix} = 0 \tag{6}$$

The various values of $\Theta = [2lk/(\alpha - 1)]$ were found. To compare with values tabulated in Ref. 3, Table I was developed giving values of $(lk)^2$ corresponding to fundamental, second harmonic, third harmonic, fourth harmonic, and fifth harmonic frequencies for the free-end case with $\beta = \alpha$. Rearranging the equation $k^4 = 12\rho\omega^2/Eg h_1^2$ to obtain the term $(lk)^2$, it follows that

$$\omega(l^2/h_1)(12\rho/Eg)^{1/2} = (lk)^2 \tag{7}$$

TABLE II. Factor $(lk)^2 \beta = \alpha$, end support.

α	Funda- mental frequency	Second harmonic	Third harmonic	Fourth harmonic	Fifth harmonic
1.0	15.4182	49.9649	104.248	178.270	272.031
1.2	17.5615	55.4779	115.077	196.337	299.264
1.4	19.6505	60.8528	125.577	213.801	325.540
1.6	21.6980	66.1184	135.817	230.790	351.061
1.8	23.7128	71.2951	145.847	247.391	375.967
2.0	25.7010	76.3976	155.702	263.671	400.361
2.5	30.5821	88.8974	179.735	303.257	459.572
3.0	35.3696	101.117	203.108	341.627	516.841
3.5	40.0904	113.129	225.989	379.089	572.656
4.0	44.7611	124.980	248.488	415.841	627.331
5.0	53.9936	148.315	292.615	487.727	734.077
6.0	63.1233	171.289	335.881	558.001	838.218
10.0	99.0859	261.086	503.841	829.474	1239.11

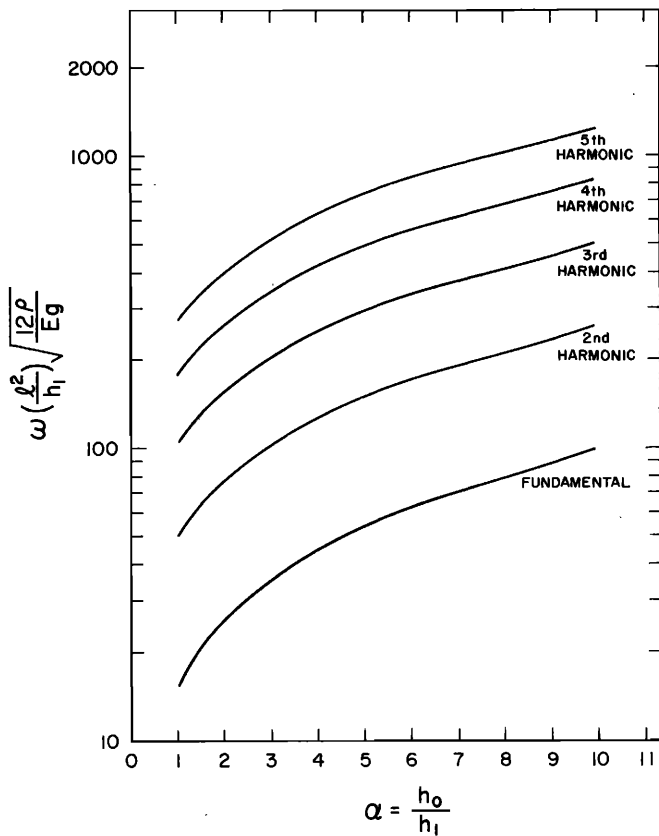


FIG. 3. Frequencies for double-tapered cantilever beam with end support with $\beta = \alpha$.

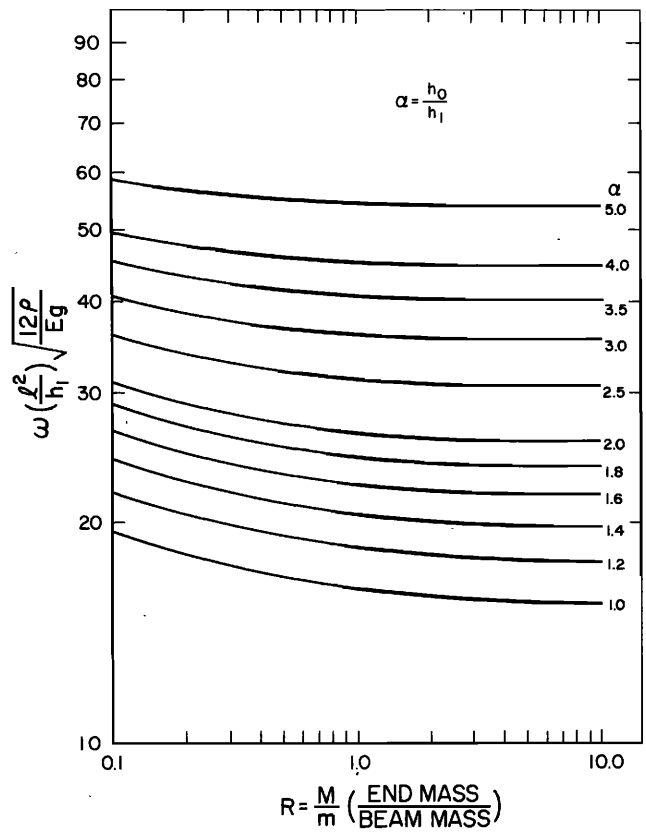


FIG. 5. Second-harmonic frequency for double-tapered cantilever beam with end mass with $\beta = \alpha$.

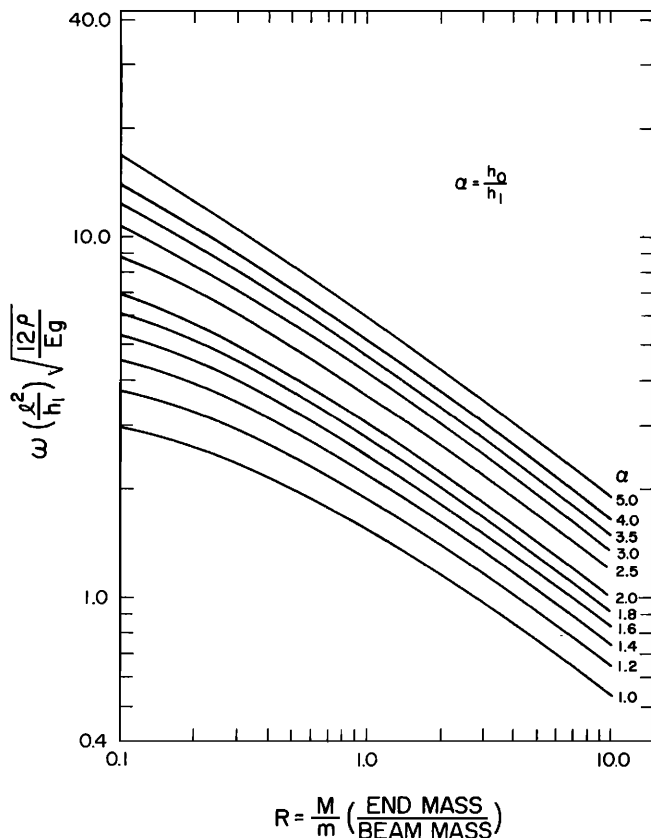


FIG. 4. Fundamental frequency for double-tapered cantilever beam with end mass with $\beta = \alpha$.

Using this equation and the values of $(lk)^2$ from Table I, curves were plotted of $\omega(l^2/h_1)(12\rho/Eg)^{1/2}$ vs α for the five harmonics as shown in Fig. 2.

B. Beam with end support

For a beam with end support at $x=0$, the boundary conditions are:

$$\begin{aligned} \text{at } x=0 \text{ or } u=0, \quad d^2z/du^2=0 \text{ and } z=0; \\ \text{at } x=l \text{ or } u=1, \quad dz/du=0 \text{ and } z=0. \end{aligned}$$

Imposing these boundary conditions gives the following determinantal equation for obtaining the natural frequencies of the beam with end support:

$$\begin{vmatrix} J_2(\Theta \sqrt{\alpha}) & Y_2(\Theta \sqrt{\alpha}) & I_2(\Theta \sqrt{\alpha}) & K_2(\Theta \sqrt{\alpha}) \\ J_3(\Theta \sqrt{\alpha}) & Y_3(\Theta \sqrt{\alpha}) & -I_3(\Theta \sqrt{\alpha}) & K_3(\Theta \sqrt{\alpha}) \\ J_4(\Theta) & Y_4(\Theta) & I_4(\Theta) & K_4(\Theta) \\ J_2(\Theta) & Y_2(\Theta) & I_2(\Theta) & K_2(\Theta) \end{vmatrix} = 0. \quad (8)$$

Table II was developed to give values of $(lk)^2$ corresponding to the fundamental, second, third, fourth, and fifth harmonic frequencies for the end support case with $\beta = \alpha$. From Table II, curves were plotted of $\omega(l^2/h_1) \cdot (12\rho/Eg)^{1/2}$ vs α for the five harmonics as shown in Fig. 3.

C. Beam with end mass

For the case of a concentrated mass M located at $x=0$, the boundary conditions are:

Therefore,

$$\frac{d^3 z}{dx^3} = \left(\frac{12\rho\omega^2}{EgI_1^2} \right) \frac{M \left[\frac{1}{3} l (\alpha^2 + \alpha + 1) \right]}{(\rho/g) b_1 h_1 \left[\frac{1}{3} l (\alpha^2 + \alpha + 1) \right]} z_0$$

$$= k^4 (M/m) (l/3) (\alpha^2 + \alpha + 1) z_0.$$

Substituting $\phi = 1 + [(\alpha - 1)/l] x$,

$$\left(\frac{d^3 z}{d\phi^3} \right)_{\phi=1} = k^4 \left(\frac{M}{m} \right) \left(\frac{l^4}{3} \right) \left[\frac{\alpha^2 + \alpha + 1}{(\alpha - 1)^3} \right] (z)_{\phi=1}.$$

The boundary conditions at $x=0$ or $\phi=1$ are therefore

$$\frac{d^2 z}{d\phi^2} = 0 \tag{9a}$$

and

$$\frac{d^3 z}{d\phi^3} = \left(\frac{M}{m} \right) \left[\frac{(lk)^4}{3} \right] \left[\frac{\alpha^2 + \alpha + 1}{(\alpha - 1)^3} \right] z.$$

The boundary conditions at $x=l$ or $\phi=\alpha$ become

$$z = 0 \tag{9b}$$

and

$$\frac{dz}{d\phi} = 0.$$

Imposing the boundary conditions given by Eqs. 9 on Eq. 5 gives the following determinantal equation for obtaining the natural frequencies of the beam with an end mass M :

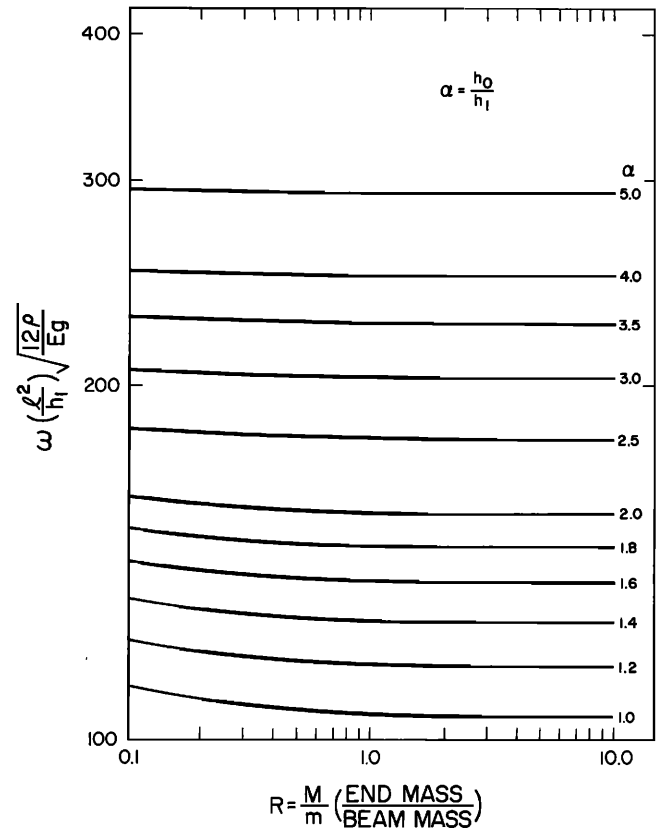


FIG. 7. Fourth-harmonic frequency for double-tapered cantilever beam with end mass with $\beta = \alpha$.

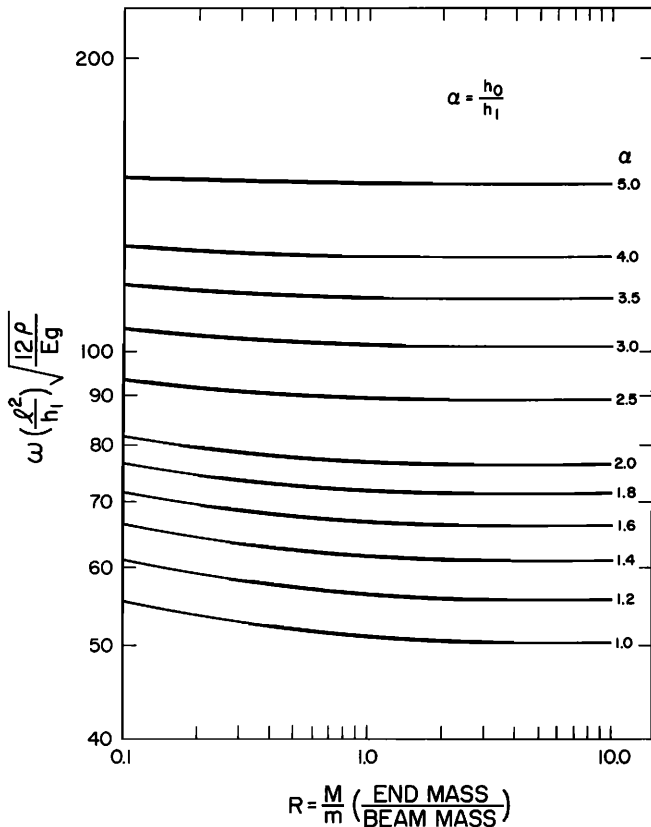


FIG. 6. Third-harmonic frequency for double-tapered cantilever beam with end mass with $\beta = \alpha$.

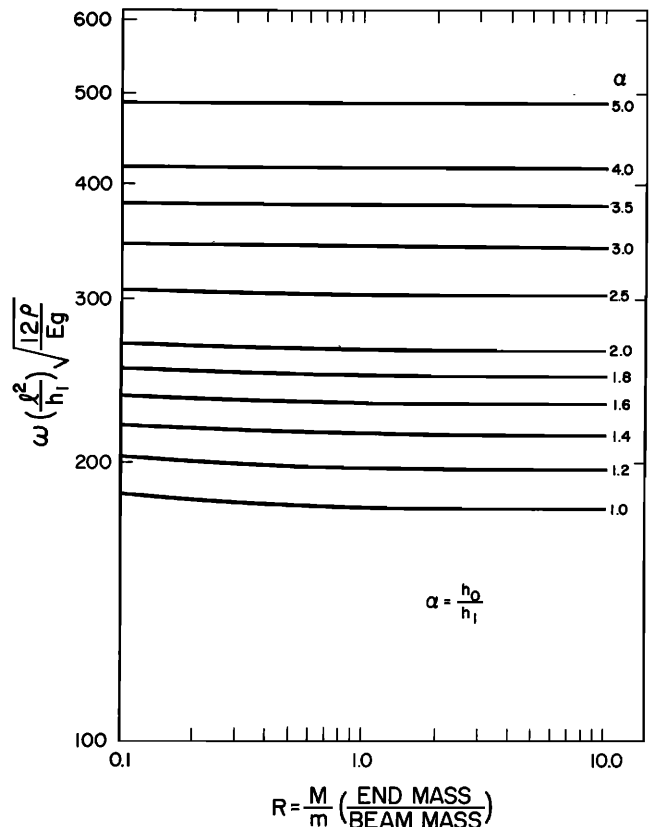


FIG. 8. Fifth-harmonic frequency for double-tapered cantilever beam with end mass with $\beta = \alpha$.

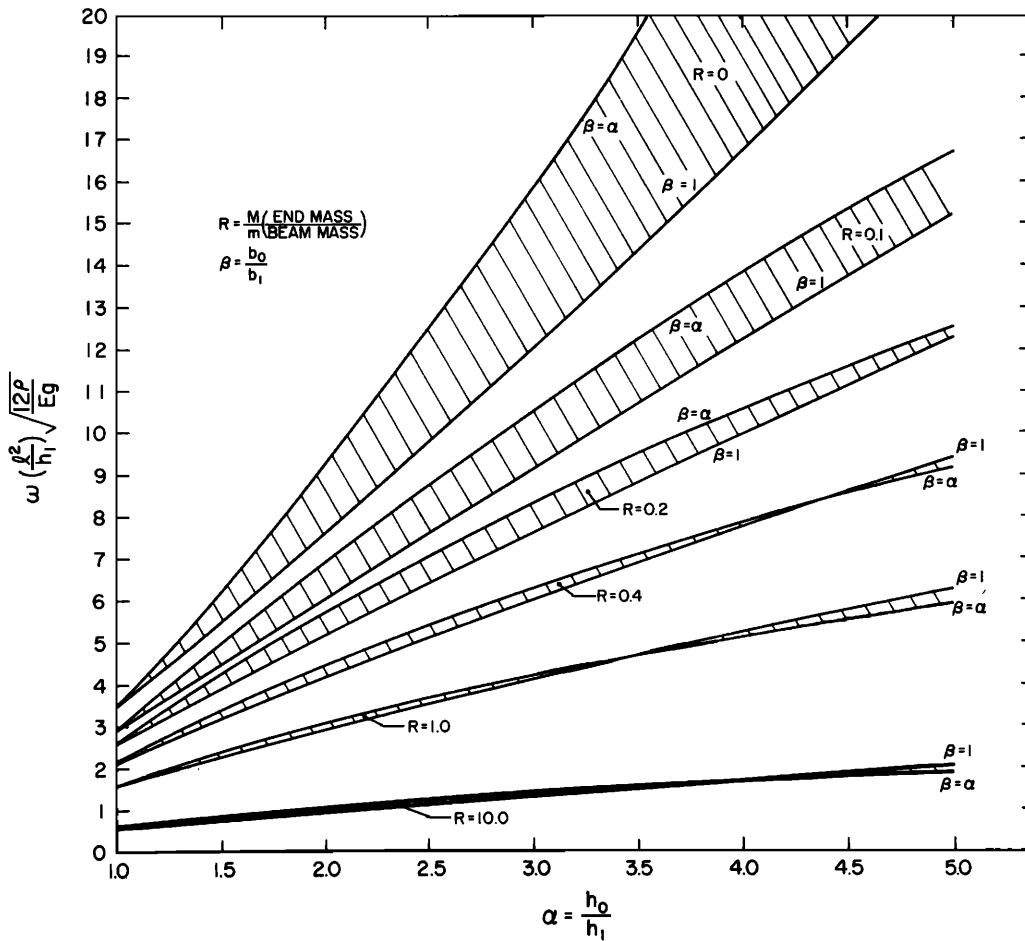


FIG. 9. Fundamental frequency for double-tapered cantilever beam with end mass with $\beta = \alpha$ and $\beta = 1$ for several values of R .

$$\begin{vmatrix} J_2(\Theta\sqrt{\alpha}) & Y_2(\Theta\sqrt{\alpha}) & I_2(\Theta\sqrt{\alpha}) & K_2(\Theta\sqrt{\alpha}) \\ J_3(\Theta\sqrt{\alpha}) & Y_3(\Theta\sqrt{\alpha}) & -I_3(\Theta\sqrt{\alpha}) & K_3(\Theta\sqrt{\alpha}) \\ J_4(\Theta) & Y_4(\Theta) & I_4(\Theta) & K_4(\Theta) \\ A & B & C & D \end{vmatrix} = 0, \quad (10)$$

where

$$\begin{aligned} A &= \left[\frac{M}{m} \right] \left[\frac{lk}{3} \right] (\alpha^2 + \alpha + 1) J_2(\Theta) + J_5(\Theta), \\ B &= \left[\frac{M}{m} \right] \left[\frac{lk}{3} \right] (\alpha^2 + \alpha + 1) Y_2(\Theta) + Y_5(\Theta), \\ C &= \left[\frac{M}{m} \right] \left[\frac{lk}{3} \right] (\alpha^2 + \alpha + 1) I_2(\Theta) - I_5(\Theta), \\ D &= \left[\frac{M}{m} \right] \left[\frac{lk}{3} \right] (\alpha^2 + \alpha + 1) K_2(\Theta) + K_5(\Theta). \end{aligned}$$

Table III was developed to give values of $(lk)^2$ corresponding to the fundamental, second, third, fourth, and fifth harmonic frequencies for a beam with end mass with $\beta = \alpha$. In this table R is the ratio M/m of the mass of the concentrated load to that of the beam.

Figures 4-8 show curves of $\omega(l^2/h_1)(12\rho/Eg)^{1/2}$ vs R for the five harmonics plotted from the data in Table III. It is interesting to note that the curves of $\omega(l^2/h_1)(12\rho/Eg)^{1/2}$ vs R , after the fundamental frequency (Fig. 4), are almost independent of R . This is especially true

for the higher harmonics and higher values of α .

Figure 9 shows a plot of $\omega(l^2/h_1)(12\rho/Eg)^{1/2}$ vs α for the fundamental frequency for $\beta = \alpha$ and $\beta = 1$ for several values of R . The values for $\beta = 1$ were taken from the previously solved case given in Ref. 1. It can be seen that as the value of R increases, the spread of the two curves decreases for a particular R . This is particularly significant for the values of $R > 0.4$. In these cases the value of the fundamental frequency is approximately the same whether the beam has a double taper or whether it tapers only in the vertical plane ($\beta = 1$).

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