

# Trapezoidal Approximation of a Fuzzy Number Preserving the Expected Interval and Including the Core

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## ABSTRACT

In this paper, we introduce a method to obtain the nearest trapezoidal approximation of fuzzy numbers so that preserving conditions expect interval and include the core of a fuzzy number.

**Keywords:** Trapezoidal Approximation; Fuzzy Number; Fuzzy Partition; Fuzzy Approximation; Trapezoidal Fuzzy Numbers

## 1. Introduction

Trapezoidal fuzzy intervals are often used in practice. An interesting problem is to approximate general fuzzy intervals by means of trapezoidal ones, so as to simplify calculations. The investigations in this area were started by Ma *et al.* [1] that proposed the symmetric triangular approximation. Actually, symmetric triangular approximation is a particular case of the trapezoidal approximation that was discussed by many authors including Abbasbandy, Amirfakhrian [2], Abbasbandy and Asady [3], Coroianu [4], Ban [5-8], Grzegorzewski [9-11], Grzegorzewski and Mrowka [12,13], Grzegorzewski, K. Pasternak-Winiarska [14,15] and Yeh [16-21] and other methods same as [22-27]. Although, there are many scholars who have investigated interval, triangular and trapezoidal approximation of fuzzy numbers, but the result of approximation is not always a fuzzy number, sometimes it is not a fuzzy set. For example Grzegorzewski and Mrowka proposed in [12] a method to find the nearest (with respect to a well-known metric between fuzzy numbers) trapezoidal approximation operator that preserving the expected interval. Unfortunately, there was a gap in the suggested solution which was later improved by Grzegorzewski and Mrowka [13] and finally solved by Ban [7] and Yeh [18]. In this paper, we combine the ideas proposed in papers [3,12], so that preserving conditions expect interval and include the core of a fuzzy number. In Section 2, we recall some fundamental results on fuzzy numbers. Trapezoidal approximation and examples are in Section 3.

## 2. Preliminaries

Definition 1. (cf. [28]) A fuzzy number is a fuzzy set like

$A: R \rightarrow I = [0,1]$  which satisfies:

- 1)  $A$  is the strictly quasi-convex,
- 2)  $A(x) = 0$  outside some interval  $[c,d]$ ,
- 3) There are real numbers  $a, b$  such that  $c \leq a \leq b \leq d$  and
  - a)  $A(x)$  is monotonic increasing on  $[c,a]$ ,
  - b)  $A(x)$  is monotonic decreasing on  $[b,d]$ ,
  - c)  $A(x) = 1, a \leq x \leq b$ .

The set of all these fuzzy numbers is denoted by  $F(R)$ . The  $\alpha$ -cut,  $\alpha \in ]0,1]$ , of a fuzzy number  $A$  is a crisp set defined as

$$A\alpha = \{x \in R : A(x) \geq \alpha\}.$$

Every  $\alpha$ -cut of a fuzzy number  $A$  is a closed interval  $A\alpha = [AL(\alpha), AU(\alpha)]$ , where

$$AL(\alpha) = \inf \{x \in R : A(x) \geq \alpha\},$$

$$AU(\alpha) = \sup \{x \in R : A(x) \geq \alpha\}.$$

The set of all elements that have a nonzero degree of membership in  $A$  is called the support of  $A$  [27], *i.e.*

$$\text{supp}A = \{x \in R : A(x) > 0\}.$$

Definition 2. (complement) The complement of a fuzzy number  $A$  is defined as

$$(-A)(x) = 1 - A(x).$$

The pair of functions  $(AL, AU)$  gives a parametric representation of fuzzy number  $A$  (see [29]). Another important type of fuzzy numbers was introduced in [23] as follows.

Let  $a_1, a_2, a_3, a_4 \in R$  such that  $a_1 \leq a_2 \leq a_3 \leq a_4$ . A fuzzy number defined as

$$A(x) = \begin{cases} 0 & \text{if } x \leq a_1 \\ \left(\frac{x-a_1}{a_2-a_1}\right)^l & \text{if } a_1 \leq x \leq a_2 \\ 1 & \text{if } a_2 \leq x \leq a_3 \\ \left(\frac{a_4-x}{a_4-a_3}\right)^r & \text{if } a_3 \leq x \leq a_4 \\ 0 & \text{if } x \geq a_4 \end{cases}$$

where  $l, r > 0$ , is denoted by  $A = (a_1, a_2, a_3, a_4)$   $lr$  and if  $r = 1$  by  $A = (a_1, a_2, a_3, a_4)r$ . Its parametric form is

$$A_\alpha = [A_L(\alpha), A_U(\alpha)] \\ = \left[ a_1 + \alpha^{\frac{1}{l}}(a_2 - a_1), a_4 - \alpha^{\frac{1}{r}}(a_4 - a_3) \right], \alpha \in [0, 1]$$

A popular fuzzy number is the trapezoidal fuzzy number, completely characterized by Equation (1) when  $l = r = 1$ , denoted by  $A = (a_1, a_2, a_3, a_4)$ . We denoted by  $F(R)$  the set of all fuzzy numbers and  $FT(R)$  the set of all trapezoidal fuzzy numbers.

Definition 3. For arbitrary fuzzy numbers  $A$ ,

$$A\alpha = [AL(\alpha), AU(\alpha)] \text{ and } B, \\ B\alpha = [BL(\alpha), BU(\alpha)] \text{ the quantity}$$

$$D(A, B) = \sqrt{\int_0^1 (A_L(\alpha) - B_L(\alpha))^2 d\alpha} \\ + \int_0^1 (A_U(\alpha) - B_U(\alpha))^2 d\alpha} \quad (2)$$

is the distance between  $A$  and  $B$ , [3,24,25]. The function  $D(A, B)$  is a metric in  $F(R)$  and  $(F(R), D)$  is a complete metric space. The expected interval  $EI(A)$  of a fuzzy number  $AA\alpha = [AL(\alpha), AU(\alpha)]$  introduced independently by Dubois and Prade [30] and Heilpern [31]. It is defined by

$$EI(A) = [E^*(A), E^*(A)] \\ = \left[ \int_0^1 A_L(\alpha) d\alpha, \int_0^1 A_U(\alpha) d\alpha \right] \quad (3)$$

Grzegorzewski [9] shows that the interval  $EI(A)$  is the nearest interval to the fuzzy number  $A$ . Hence in [31] the expected value of a fuzzy number  $A$  defined following as

$$EV(A) = \frac{1}{2}(E_*(A) + E^*(A)) \quad (4)$$

and B. Asady and M. Zendehnam in [32] show that  $EV(A)$  is the best approximation of a fuzzy number  $A$ . Finally, let us recall that  $coreA = \{xR : A(x) = 1\}$ .

### 3. Trapezoidal Approximation of Fuzzy Numbers

Suppose we want to approximate a fuzzy number by a

trapezoidal fuzzy number. Thus, we use an operator  $T : F(R) \rightarrow FT(R)$  which transforms fuzzy numbers into family of trapezoidal fuzzy number.

Abbasbandy and Asady [3] considered a trapezoidal approximation that includes the  $\alpha$ -cut superset, i.e.,

$$(T_1) core(A) \subseteq coreT(A) \quad (5)$$

Grzegorzewski and Mrówka [12] said that an operator  $T : F(R) \rightarrow FT(R)$  fulfills the criterion if for any fuzzy number  $A$  its output value  $T(A)$  preserves the expected interval, i.e.,

$$(T_2) EI(A) = EI(T(A)) \quad (6)$$

In this part, we combine ideas proposed in the papers [3,12] to obtain the nearest trapezoidal fuzzy number respect to the original fuzzy number so that it preserves  $T_1$  and  $T_2$  conditions. Since, any  $x$  belongs to the core  $A$  of a fuzzy number  $A$  if and only if it does not belong to the complement of  $A$

$$(x \in coreA \Leftrightarrow x \notin -A)$$

Therefore, for preserving of these points we consider  $coreA \subseteq coreT(A)$ . Also, we are going to preserve the expected interval of the fuzzy number that this additional requirement by the significant role of the expected interval is in many situations and applications (see, e.g., [5-11, 33]. Additionally, the propose approach can provide decision makers with a new alternative to trapezoidal approximation of fuzzy numbers.

Now, given a fuzzy number  $B$  with  $\alpha$ -cut set

$[BL(\alpha), BU(\alpha)]$ , the problem is to find a trapezoidal fuzzy number  $T(B) = (t_1, t_2, t_3, t_4)$  which is the nearest to  $B$  with respect to metric  $D$  and preserves the conditions  $T_1$  and  $T_2$  i.e. we have

$$\min \{D(B, T(B))\} = \left\{ \int_0^1 (B_L(\alpha) - T(B)_L(\alpha))^2 d\alpha \right. \\ \left. + \int_0^1 (B_U(\alpha) - T(B)_U(\alpha))^2 d\alpha \right\}^{\frac{1}{2}} \quad (7)$$

Subject to

$$[t_2, t_3] \subseteq [B_L(1), B_U(1)], \quad (8)$$

$$\left[ \frac{t_1+t_2}{2}, \frac{t_3+t_4}{2} \right] = \left[ \int_0^1 B_L(\alpha) d\alpha, \int_0^1 B_U(\alpha) d\alpha \right], \quad (9)$$

As, in order to minimize  $D(B, T(B))$  it suffices to minimize function

$$\begin{aligned}
 F(h_1, h_2) &= D^2(B, T(B)) \\
 &= \int_0^1 (B_L(\alpha) - (t_1 + (t_2 - t_1)\alpha))^2 d\alpha \quad (10) \\
 &\quad + \int_0^1 (B_U(\alpha) - (t_4 + (t_3 - t_4)\alpha))^2 d\alpha,
 \end{aligned}$$

with respect to  $h_1$  and  $h_2$ . Subject to

$$t_2 = B_L(1) - h_1^2 \quad (11)$$

and

$$t_3 = B_U(1) - h_2^2 \quad (12)$$

and

$$t_1 = 2 \int_0^1 B_L(\alpha) d\alpha - t_2, \quad (13)$$

and

$$t_4 = 2 \int_0^1 B_U(\alpha) d\alpha - t_3, \quad (14)$$

Clearly, with note to Equations (8), (9), we can say the conditions (11)-(14) are suitable for finding the nearest trapezoidal fuzzy number  $T(B)$  to a fuzzy number  $B$  with the conditions as the expected interval and  $\text{core}B\alpha\text{core}(T(B))$  are preserved. Theorem 1: Let  $B$  with  $\alpha$ -cut set  $[BL(\alpha), BU(\alpha)]$ , be a fuzzy number and

$$\begin{aligned}
 d_1 &= B_L(1) - \int_0^1 B_L(\alpha) d\alpha \\
 &\quad - 3 \int_0^1 B_L(\alpha)(2\alpha - 1) d\alpha, \quad (15)
 \end{aligned}$$

$$\begin{aligned}
 d_2 &= -B_U(1) + \int_0^1 B_U(\alpha) d\alpha \\
 &\quad + 3 \int_0^1 B_U(\alpha)(2\alpha - 1) d\alpha, \quad (16)
 \end{aligned}$$

Case 1: If  $d_1 \leq 0$  and  $d_2 \leq 0$  then, optimal solution of problem (10) is

$$h_1 = h_2 = 0$$

and consequently, we have

$$\begin{aligned}
 t_1 &= -B_L(1) + 2 \int_0^1 B_L(\alpha) d\alpha, \\
 t_2 &= B_L(1), \\
 t_3 &= B_U(1), \\
 t_4 &= -B_U(1) + 2 \int_0^1 B_U(\alpha) d\alpha
 \end{aligned} \quad (17)$$

Case 2: If  $d_1 > 0$  and  $d_2 \leq 0$  then

$$\begin{aligned}
 h_1 &= \sqrt{d_1}, h_2 = 0, \\
 t_1 &= 4 \int_0^1 B_L(\alpha) d\alpha - 6 \int_0^1 \alpha B_L(\alpha) d\alpha, \\
 t_2 &= -2 \int_0^1 B_L(\alpha) d\alpha + 6 \int_0^1 \alpha B_L(\alpha) d\alpha, \quad (18) \\
 t_3 &= B_U(1), \\
 t_4 &= -B_U(1) + 2 \int_0^1 B_U(\alpha) d\alpha,
 \end{aligned}$$

Case 3: If  $d_1 \leq 0$  and  $d_2 > 0$  then

$$\begin{aligned}
 h_1 &= 0, h_2 = \sqrt{d_2}, \\
 t_1 &= -B_L(1) + 2 \int_0^1 B_L(\alpha) d\alpha, \\
 t_2 &= B_L(1), \\
 t_3 &= -2 \int_0^1 B_U(\alpha) d\alpha + 6 \int_0^1 \alpha B_U(\alpha) d\alpha, \\
 t_4 &= 4 \int_0^1 B_U(\alpha) d\alpha - 6 \int_0^1 \alpha B_U(\alpha) d\alpha, \quad (19)
 \end{aligned}$$

Case 4: If  $d_1 > 0$  and  $d_2 > 0$  then

$$\begin{aligned}
 h_1 &= \sqrt{d_1}, h_2 = \sqrt{d_2} \\
 t_1 &= 4 \int_0^1 B_L(\alpha) d\alpha - 6 \int_0^1 \alpha B_L(\alpha) d\alpha, \\
 t_2 &= -2 \int_0^1 B_L(\alpha) d\alpha + 6 \int_0^1 \alpha B_L(\alpha) d\alpha, \quad (20) \\
 t_3 &= -2 \int_0^1 B_U(\alpha) d\alpha + 6 \int_0^1 \alpha B_U(\alpha) d\alpha, \\
 t_4 &= 4 \int_0^1 B_U(\alpha) d\alpha - 6 \int_0^1 \alpha B_U(\alpha) d\alpha,
 \end{aligned}$$

Proof: In order to minimize  $F(h_1, h_2)$  it suffices to minimize function  $F2(h_1, h_2) = f(h_1, h_2)$  with the partial derivatives

$$\begin{cases} \frac{\partial f(h_1, h_2)}{\partial h_1} \\ = \frac{4}{3} h_1 \left( h_1^2 - B_L(1) + \int_0^1 B_L(\alpha) d\alpha + 3 \int_0^1 B_L(\alpha)(2\alpha - 1) d\alpha \right) \\ \frac{\partial f(h_1, h_2)}{\partial h_2} \\ = \frac{4}{3} h_2 \left( h_2^2 + B_U(1) - \int_0^1 B_U(\alpha) d\alpha - 3 \int_0^1 B_U(\alpha)(2\alpha - 1) d\alpha \right) \end{cases} \quad (21)$$

$$\begin{cases} \frac{\partial^2 f(h_1, h_2)}{\partial h_1^2} = 4h_1^2 - \frac{4}{3}d_1 \\ \frac{\partial^2 f(h_1, h_2)}{\partial h_2^2} = 4h_2^2 - \frac{4}{3}d_2 \\ \frac{\partial^2 f(h_1, h_2)}{\partial h_1 \partial h_2} = 0, \end{cases} \quad (22)$$

system (10) is  $h_1 = h_2 = 0$  and the point  $(0,0)$  minimizes the function  $F$  in the Equation (10). Because  $F(h_1, h_2)$  is lower bounded function and has only a critical point. Also, with note to Equations (11)-(14) and optimal solution  $h_1 = h_2 = 0$  formula (17) is correct.

2) If  $d_1 > 0$  and  $d_2 \leq 0$  are satisfied then the solutions of the system (10) are

Case 1:  $h_1 = 0, h_2 = 0,$

Case 2:  $h_1 = \sqrt{d_1}, h_2 = 0$

Moreover, since

$$\begin{cases} \frac{\partial^2 f(0,0)}{\partial h_1^2} < 0, \\ \frac{\partial^2 f(0,0)}{\partial h_2^2} \geq 0, \\ \frac{\partial^2 f(\sqrt{d_1},0)}{\partial h_1^2} = \frac{8}{3}d_1 > 0, \\ \frac{\partial^2 f(\sqrt{d_1},0)}{\partial h_2^2} = -\frac{4}{3}d_2 \geq 0, \end{cases}$$

therefore case 2 such that  $h_1 = \sqrt{d_1}, h_2 = 0$  minimizes the function  $F$ . Because the Hessian matrix for it is positive definite and for case 1 is not positive definite. Also, with note to Equations (11)-(14) and  $h_1 = \sqrt{d_1}, h_2 = 0$  Formulae (18) are satisfied.

3) Proof is similar with 2)

4) If  $d_1 > 0$  and  $d_2 > 0$  are satisfied then the solutions of the system (10) are given by

Case 1:  $h_1 = 0, h_2 = 0;$

Case 2:  $h_1 = 0, h_2 = \sqrt{d_2};$

Case 3:  $h_1 = \sqrt{d_1}, h_2 = 0;$

Case 4:  $h_1 = \sqrt{d_1}, h_2 = \sqrt{d_2}.$

Because the Hessian matrix for of case 4 is positive definite and for others cases are not positive definite, then the solution of case 4 is minimizer and formulae (20) is correct. 2 Now we compare our method with the other works [3,7,20] in following examples.

Example 1. [7] Let us consider the fuzzy number  $A = (1, 2, 3, 30) \times 2$ , in the parametric forms,

$$A_L(\alpha) = 1 + \sqrt{\alpha}, A_U(\alpha) = 30 - 27\sqrt{\alpha}, \alpha \in [0, 1].$$

By substitute above equations on the (2) and (3) equations, we obtain the expect interval and core of the fuzzy number A same as follows

$$EI(A) = \left[ \frac{5}{3}, 12 \right], \text{core}(A) = [2, 3]$$

Also, trapezoidal approximation of it by the other methods and proposed method is shown in **Table 1**.

Clearly, in **Table 1**, by Ban and Yeh methods, we have

$$EI(A) = EI(T(A))$$

and

$$\text{core}A \subsetneq \text{core}T(A).$$

and, by Abbasbandy and Asady method, get

$$EI(A) \neq EI(T(A))$$

and

$$\text{core}A \subseteq \text{core}T(A).$$

Consequently, with note above results proposed method preserved T1 and T2 conditions (see **Figures 1-4**).

Example 2. Consider the fuzzy number  $B = (-10, 10, 10, 14)_{41/3}$  that the parametric form of it is as follows

**Table 1. Comparative results of example 1.**

Authors	The trapezoidal approximation of fuzzy numbers A	$EI(T(A))$	$\text{core}T(A)$
Ban	(5/3, 5/3, 5/3, 67/3)	$\left[ \frac{5}{3}, 12 \right]$	$\frac{5}{3}$
Yeh	(47/35, 71/35, 78/35, 726/35)	$\left[ \frac{5}{3}, 12 \right]$	$\left[ \frac{71}{35}, \frac{78}{35} \right]$
Abbasbandy Asady	(13/10, 2, 3, 219/10)	$\left[ \frac{33}{20}, \frac{240}{20} \right]$	[2, 3]
Proposed method	(3/4, 2, 3, 21)	$\left[ \frac{5}{3}, 12 \right]$	[2, 3]

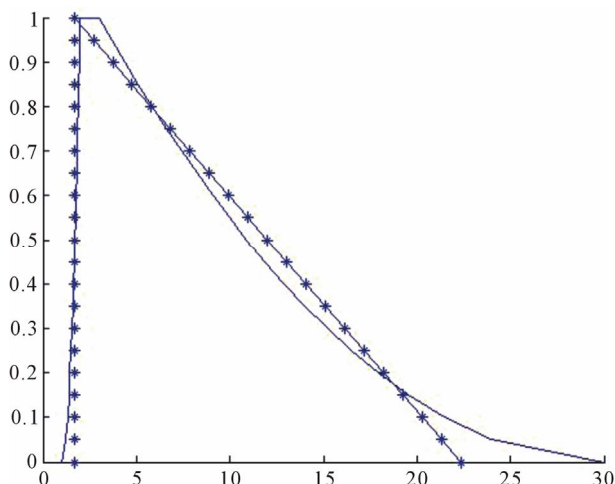


Figure 1. The trapezoidal approximation of A by Ban method.

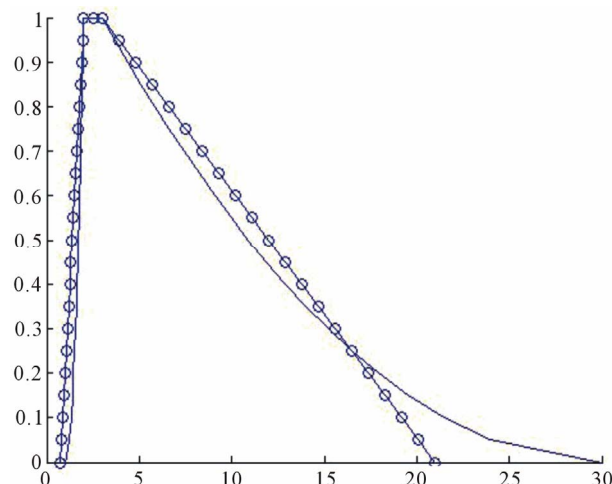


Figure 4. The trapezoidal approximation of A by proposed method.

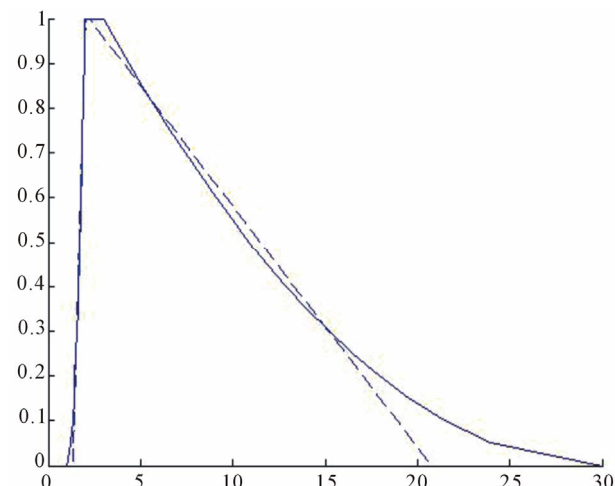


Figure 2. The trapezoidal approximation of A by Yeh method.

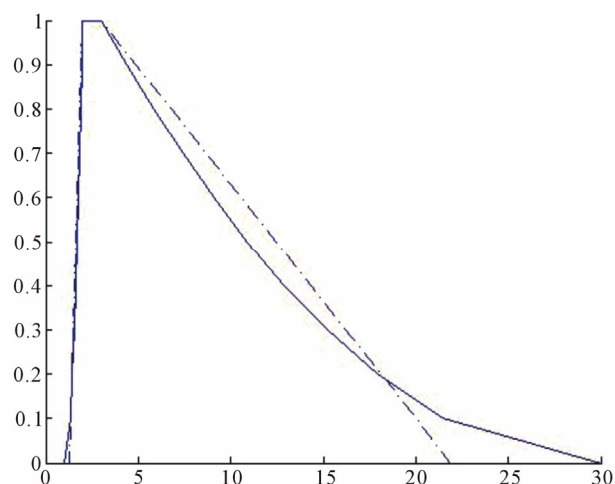


Figure 3. The trapezoidal approximation of A by Abbasbandy and Asady method.

$$BL(\alpha) = -10 + 20\alpha^{1/4}, BU(\alpha) = 14 - 4\alpha^3.$$

Clearly, by Equations (2) and (3) the expect interval and core of it are same as follows

$$EI(B) = [6, 13] \text{ and } \text{core}B = \left[10, \frac{56}{5}\right].$$

Also, trapezoidal approximation of the fuzzy number B by the other methods and proposed method is shown in Table 2.

Clearly, in Table 2 by Ban and Yeh methods,  $\text{core}B \subsetneq \text{core}T(B)$  and even  $\text{core}B \cap \text{core}T(B) = \emptyset$  (see Figures 5 and 6). But, for Abbasbandy and Asady method, we get  $EI(B) \neq EI(T(B))$ . Finally, the results of proposed method are

$$EI(A) = EI(T(A))$$

and  $\text{core}A \subseteq \text{core}T(A)$ . Therefore, we can say that proposed method preserves T1 and T2 conditions in example 2. (please see Figures 7 and 8).

Theorem 3. Trapezoidal approximation satisfies the following properties.

1) Translation invariance:  $\forall z \in R, \forall A \in F(R)$ : then we have  $T(A+z) = T(A) + z$ .

2) Scale invariance:  $\forall z \in R, \forall A \in F(R)$ : then we have  $T(zA) = zT(A)$ .

Proof. Proof is similar to proof of Theorem 12 in [7].

Theorem 4. Trapezoidal approximation satisfies the identity property.

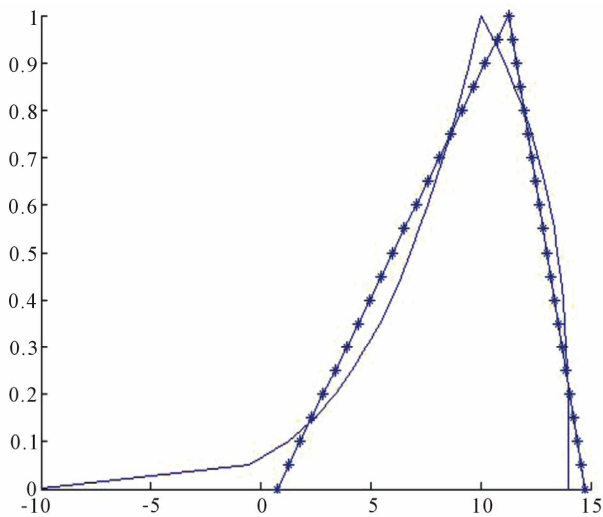
Proof. Let  $B = (b_1, b_2, b_3, b_4)$  be a trapezoidal fuzzy number. By Equations (2) and (3), we get  $d_1 = \frac{t_2 - t_1}{2}$ ,

$$d_2 = \frac{t_4 - t_3}{2}.$$

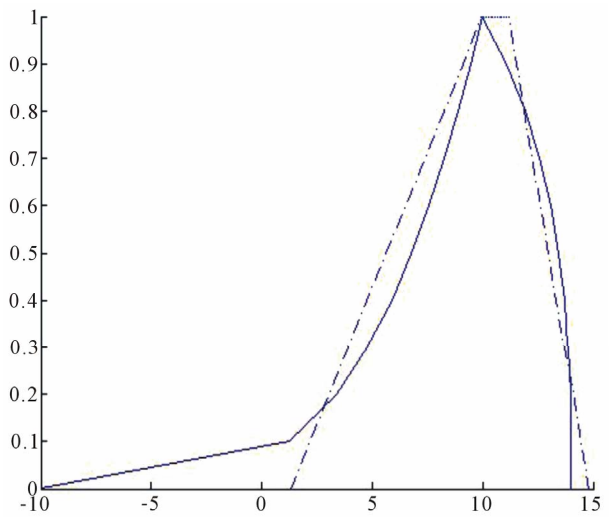
Therefore,  $d_1$  and  $d_2$  are positive and conditions in the

**Table 2. Comparative results of Example 2.**

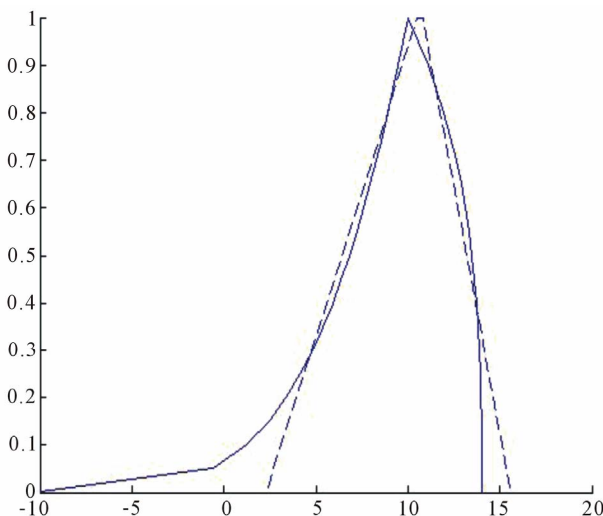
Authors	The trapezoidal approximation of fuzzy numbers $B$	$EI(T(B))$	$coreT(B)$
Ban	$TB(B) = (11/15, 169/15, 169/15, 221/15)$	$[6, 13]$	$\frac{169}{15}$
Yeh	$T_y(B) = (90/39, 410/39, 54/5, 78/5)$	$[6, 13]$	$\left[\frac{410}{39}, \frac{54}{5}\right]$
Abbasbandy, Asady	$T_A(B) = (4/3, 10, 56/5, 74/5)$	$\left[\frac{17}{3}, 13\right]$	$\left[10, \frac{56}{5}\right]$
Proposed method	$T_r(B) = (2, 10, 56/5, 74/5)$	$[6, 13]$	$\left[10, \frac{56}{5}\right]$



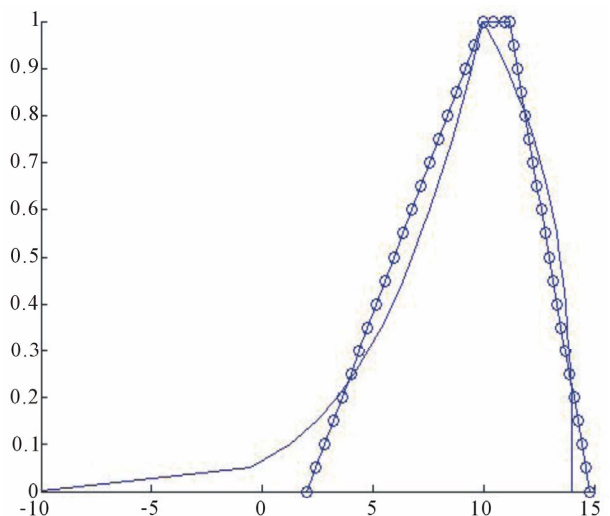
**Figure 5. The trapezoidal approximation of  $B$  by Ban method.**



**Figure 7. The trapezoidal approximation of  $B$  by Abbasbandy and Asady method.**



**Figure 6. The trapezoidal approximation of  $B$  by Yeh method.**



**Figure 8. The trapezoidal approximation of  $B$  by proposed method.**

Case 4) are satisfied, Theorem 1 is applicable.

We obtain  $t_k = b_k$ ,  $k \in \{1, 2, 3, 4\}$ .

#### 4. Conclusion

In this paper, we have been suggested an interesting approach to trapezoidal approximation of general fuzzy numbers. The proposed method leads to the trapezoidal fuzzy number which is the best one with respect to a certain measure of distance between fuzzy numbers,  $D(u, v)$ .

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