TRAPPED MODES FOR AN ELASTIC STRIP WITH PERTURBATION OF THE MATERIAL PROPERTIES

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Summary

The elasticity operator, for zero Poisson coefficient, with stress-free boundary conditions on a two-dimensional strip with local perturbation of Young's modulus, is considered. We prove the existence of embedded eigenvalues and describe their asymptotic behaviour.

1. Introduction

We consider a two-dimensional strip $\Gamma = \{x \in \mathbb{R}^2 \colon |x_2| < 2^{-1}\pi\}$ of a homogeneous and isotropic linear elastic material. Let

$$a_0[u,u] = \int_\Gamma \left(2\left|\frac{\partial u_1}{\partial x_1}\right|^2 + 2\left|\frac{\partial u_2}{\partial x_2}\right|^2 + \left|\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right|^2 \right) dx, \quad u \in H^1(\Gamma, \mathbb{C}^2), \tag{1.1}$$

be the quadratic form of the elasticity operator

$$A_0 = -(\Delta + \text{grad div}) \tag{1.2}$$

for zero Poisson coefficient with stress-free boundary conditions on Γ; see (2.4) below. The spectrum of the operator A_0 is purely absolutely continuous and coincides with $[0, +\infty)$.

Let $f \in L_\infty(\mathbb{R}; (-\infty, 1])$ be a function of compact support, extended to Γ by $f(x_1, x_2) = f(x_1)$ for $x \in \Gamma$. The function f describes a local perturbation of Young's modulus: for $\alpha \in (0, 1)$ we consider the perturbed operator A_α corresponding to the quadratic form

$$a_\alpha[u,u] = \int_\Gamma (1 - \alpha f) \left(2\left|\frac{\partial u_1}{\partial x_1}\right|^2 + 2\left|\frac{\partial u_2}{\partial x_2}\right|^2 + \left|\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right|^2 \right) dx, \quad u \in H^1(\Gamma, \mathbb{C}^2). \tag{1.3}$$

We shall discuss the existence of embedded eigenvalues of A_α for $\alpha \in (0, 1)$, and we describe the asymptotic behaviour of these eigenvalues as $\alpha \to 0$.

The topic of this paper is closely related to a series of works on trapped modes for perturbed quantum and acoustic waveguides; see among others (**1** to **6**) and the references therein. These papers study the operator $-\Delta$ on some infinite domain and discuss the existence and the asymptotics

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of eigenvalues, appearing for certain perturbations of the domain, such as a bending of the domain, a local deformation of the boundary, an inclusion of an obstacle or a local change of the boundary conditions.

In contrast to the Laplacian with Dirichlet boundary conditions (quantum waveguides), the essential spectrum of the Laplacian with Neumann boundary conditions (acoustic waveguides) on a strip-like domain fills the non-negative semi-axes. Therefore, any eigenvalue is embedded into essential spectrum, and it is not possible to apply variational techniques directly. However, if the perturbed domain satisfies a certain spatial symmetry, the Laplacian splits into the orthogonal sum of two operators. Eventually the essential spectrum of the first operator is separated from zero, and the lower discrete portion of its spectrum can be studied in the usual way (**1**). It is not difficult to extend the results of (**2**) to the case of Neumann boundary conditions, if one considers the Laplacian being reduced to antisymmetric functions on a symmetric domain; the results on the Dirichlet Laplacian in (**2**, **6**) do not require such a symmetry.

Passing to elliptic systems of equations one finds new effects. For instance it has been shown in (**7**) that, in contrast to the Neumann Laplacian, the elasticity operator with stress-free boundary conditions on a semi-strip has at least one positive eigenvalue. This effect is related to the so-called edge resonance and it is due to an interaction between the spatial and the internal degrees of freedom of the operator.

While the method in (**7**) is based on a variational argument which only allows to treat the case of zero Poisson's ratio ν, in (**8**) a semi-analytical method was presented which indicates, supported by numerical evidence, that also for a certain $\nu \neq 0$ a trapped mode can be observed. Further new results on trapped modes in elastic media can be found in (**9** to **11**).

To obtain mathematically rigorous results we restrict ourselves to the case of zero Poisson's ratio which corresponds to the investigation of the operator given by (1.2). In this case, beside the spatial symmetries, an additional hidden symmetry can be observed which allows to remove the essential spectrum below a certain cut-off frequency $\Lambda > 0$ and to investigate trapped modes arising below Λ. The importance of this internal symmetry for similar problems has already been pointed out in (**12**).

Besides applying these symmetries, the proof of the existence of the edge resonance in (**7**) exploits another interesting fact. Note that the separation of variables for the Laplacian on Γ leads to parabolic eigenvalue branches, which achieve their minima at zero frequency. In contrast to this, separating variables in the x_1-direction for the reduced operator A_0 on Γ, one finds that the branch of the lowest eigenvalues of the respective reduced fibre operators achieves its minimal value at two different points $\xi = \pm\varkappa$ of the Fourier coordinate ξ, corresponding to two opposite elastic waves with non-zero frequencies; see Lemma 3.2. This fact also implies edge resonances for the elasticity operator on three-dimensional semi-rods with appropriate cross-sections (**13**).

In some sense this paper can be considered as a continuation of (**7**). It is also closely related to (**14**). The proof of the existence of trapped modes applies arguments of (**15**), where the appearance of virtual bound states has been discussed in the general case. After the existence and the number of the trapped modes have been established, we use variational methods to calculate the asymptotic behaviour of these bound states. In the given case this seems to be easier than to deduce the number of trapped modes and their asymptotics at once.

1.1 *Notation*

Statements or formulae containing the index $\pm$ have to be read independently with the index $+$ and the index $-$.

2. Statement of the problem

We put $\Gamma = \mathbb{R} \times J$ with $J = (-\frac{1}{2}\pi, \frac{1}{2}\pi)$ and consider the quadratic form

$$\tilde{a}[u,u] = \int_\Gamma \left(c_l^2 \left| \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right|^2 - 4c_t^2 \Re \frac{\partial u_1}{\partial x_1} \frac{\partial \overline{u}_2}{\partial x_2} + c_t^2 \left| \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \right|^2 \right) dx, \tag{2.1}$$

which is well defined on functions $u = (u_1, u_2)^T \in d[a] = H^1(\Gamma, \mathbb{C}^2)$. The form (2.1) appears, for instance, in elasticity theory for plane stress or plane strain problems. In both models the positive constants c_l and c_t depend upon the density of the material, the Young modulus and the Poisson coefficient; see (**16**, **17**).

In this paper we stress the special case of zero Poisson coefficient. Then both physical models yield $c_l^2 = 2c_t^2$, and choosing a suitable set of units, we shall study the form

$$a_0[u,u] = \int_\Gamma \left(2 \left| \frac{\partial u_1}{\partial x_1} \right|^2 + 2 \left| \frac{\partial u_2}{\partial x_2} \right|^2 + \left| \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \right|^2 \right) dx, \tag{2.2}$$

which is (2.1) for $c_l = \sqrt{2}$, $c_t = 1$. This form is associated with the positive self-adjoint operator

$$A_0 = -(\Delta + \operatorname{grad}\operatorname{div}) \tag{2.3}$$

on the domain

$$\operatorname{Dom} A_0 = \left\{ u \in H^2(\Gamma, \mathbb{C}^2) \colon \left. \frac{\partial u_2}{\partial x_2} \right|_{x_2 = \pm\frac{1}{2}\pi} = \left. \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right|_{x_2 = \pm\frac{1}{2}\pi} = 0 \right\}. \tag{2.4}$$

The inequality

$$a_0[u,u] \leqslant 2\|u\|^2_{H^1(\Gamma,\mathbb{C}^2)}, \quad u \in H^1(\Gamma, \mathbb{C}^2), \tag{2.5}$$

is obvious. On the other hand the class of functions $u \in L_2(\Gamma, \mathbb{C}^2)$, for which the integral (2.2) is well defined and finite, coincides with $H^1(\Gamma, \mathbb{C}^2)$. Moreover, the reverse estimate

$$a_0[u,u] + \|u\|^2_{L_2(\Gamma,\mathbb{C}^2)} \geqslant c(\Gamma)\|u\|^2_{H^1(\Gamma,\mathbb{C}^2)}, \quad u \in H^1(\Gamma, \mathbb{C}^2), \ \ c(\Gamma) > 0, \tag{2.6}$$

holds, which is an extension of the well-known Korn inequality (**18**).

Considering now the form a_α for $\alpha \in (0,1)$, as given in (1.3), we see that this form is also closed on the domain $d[a_\alpha] = d[a_0] = H^1(\Gamma, \mathbb{C}^2)$ in $H = L_2(\Gamma, \mathbb{C}^2)$, where it induces a positive self-adjoint operator A_α in H.

The spectrum of the operator A_0 is purely absolutely continuous and fills the non-negative semi-axis. It is well-known (**19**), that a local change of the boundary conditions or a local change of the quadratic form will not change the essential spectrum. Therefore the essential part of the spectrum of A_α fills the non-negative semi-axes. In this paper we shall discuss the existence of positive eigenvalues of the operator A_α which are embedded into its continuous spectrum.

3. Auxiliary material

3.1 *Spatial and internal symmetries*

For $H = L_2(\Gamma, \mathbb{C}^2)$ let H_j be the subspaces of vector functions

$$H_j := \left\{ u \in H \colon u_l(x_1, -x_2) = (-1)^{l+j} u_l(x_1, x_2), \quad l = 1, 2 \right\}, \quad j = 1, 2.$$

Then $H = H_1 \oplus H_2$. Further let H_3 be the set

$$H_3 = \{u \in H : u = (u_1(x_1), 0)\}.$$

This forms a subspace in H_1. The orthogonal complement H_4 to H_3 in H_1 consists of all functions $w = (w_1, w_2) \in H_1$ for which

$$\int_J w_1(x_1, x_2) dx_2 = 0$$

for almost every x_1. Let P_j be the orthogonal projections onto H_j, $j = 1, \dots, 4$. Then $P_j P_1 = P_1 P_j = P_j$ for $j = 3, 4$. A simple calculation shows that

$$d[a_\alpha^{(j)}] := P_j d[a_\alpha] \subset d[a_\alpha], \quad j = 1, \dots, 4,$$

and

$$a_\alpha[u, w] = 0 \quad \text{for all } u \in d[a_\alpha^{(l)}],\ w \in d[a_\alpha^{(j)}] \quad \text{if } l, j = 2, 3, 4 \text{ and } l \neq j.$$

Hence, these subspaces are reducing for the operator A_α and

$$A_\alpha = A_\alpha^{(3)} \oplus A_\alpha^{(4)} \oplus A_\alpha^{(2)} \quad \text{on } H = H_3 \oplus H_4 \oplus H_2, \tag{3.1}$$

where the operators $A_\alpha^{(j)}$ are the restrictions of A_α to Dom $A_\alpha^{(j)} =$ Dom $A_\alpha \cap H_j$ and correspond to the closed forms $a_\alpha^{(j)}$, given by the differential expression (1.3) on $d[a_\alpha^{(j)}]$, $j = 2, 3, 4$. Put

$$A_\alpha^{(1)} = A_\alpha^{(3)} \oplus A_\alpha^{(4)} \quad \text{on } H_1 = H_3 \oplus H_4, \tag{3.2}$$

being the restriction of A_α to Dom $A_\alpha \cap H_1$. Then it holds that

$$A_\alpha = A_\alpha^{(1)} \oplus A_\alpha^{(2)} \quad \text{on } H = H_1 \oplus H_2. \tag{3.3}$$

The decomposition (3.3) reflects the spatial symmetry of the operator A_α, while the decomposition (3.2) exploits the specific internal structure of A_α. We point out that the latter symmetry fails for elasticity operators with non-zero Poisson coefficients.

3.2 *Separation of variables for A_0*

Applying the unitary Fourier transform Φ in the x_1-direction and its inverse Φ^*, one finds that $\Phi A_0 \Phi^*$ permits the orthogonal decomposition

$$\Phi A_0 \Phi^* = \int_{\mathbb{R}}^{\oplus} A(\xi) d\xi \quad \text{on } H = \int_{\mathbb{R}}^{\oplus} h d\xi, \;\; h = L_2(J, \mathbb{C}^2).$$

The self-adjoint operators $A(\xi)$ are given by the differential expressions

$$A(\xi) = \begin{pmatrix} 2\xi^2 - \partial^2/\partial x_2^2 & -i\xi\, \partial/\partial x_2 \\ -i\xi\, \partial/\partial x_2 & \xi^2 - 2\partial^2/\partial x_2^2 \end{pmatrix} \tag{3.4}$$

on the domains

$$\text{Dom } A(\xi) = \left\{ w \in H^2(J, \mathbb{C}^2) \colon \left.\frac{\partial w_2}{\partial x_2}\right|_{x_2=\pm\frac{1}{2}\pi} = \left.\frac{\partial w_1}{\partial x_2} + i\xi w_2\right|_{x_2=\pm\frac{1}{2}\pi} = 0 \right\}. \tag{3.5}$$

The symmetry (3.1) extends to the operators $A(\xi)$. Put

$$h_j := \{h \in w \colon w_l(x_2) = (-1)^{j+l} w_l(-x_2),\ l = 1, 2\}, \quad j = 1, 2.$$

Let h_3 be the one-dimensional subspace spanned by the constant vector function $(1, 0)$, and set $h_4 := h_1 \ominus h_3$ with respect to the scalar product in h. Then we have

$$H_j = \int_{\mathbb{R}}^{\oplus} h_j d\xi \quad \text{and} \quad \Phi A_0^{(j)} \Phi^* = \int_{\mathbb{R}}^{\oplus} A^{(j)}(\xi) d\xi, \quad j = 1, \dots, 4, \tag{3.6}$$

where the operators $A^{(j)}(\xi)$ are the restrictions of $A(\xi)$ to $\text{Dom } A^{(j)}(\xi) = \text{Dom } A(\xi) \cap h_j$. Moreover, it holds that

$$\begin{aligned} A(\xi) &= A^{(1)}(\xi) \oplus A^{(2)}(\xi) \quad \text{on } h = h_1 \oplus h_2, \\ A(\xi) &= A^{(3)}(\xi) \oplus A^{(4)}(\xi) \oplus A^{(2)}(\xi) \quad \text{on } h = h_3 \oplus h_4 \oplus h_2. \end{aligned} \tag{3.7}$$

The operators $A^{(j)}(\xi)$ correspond to the quadratic forms

$$a^{(j)}(\xi)[w, w] = \int_{-\pi/2}^{\pi/2} \left(2\xi^2 |w_1|^2 + 2 \left| \frac{\partial w_2}{\partial x_2} \right|^2 + \left| \frac{\partial w_1}{\partial x_2} + i\xi w_2 \right|^2 \right) dx_2 \tag{3.8}$$

being closed on the domains $d[a^{(j)}(\xi)] = H^1(J, \mathbb{C}^2) \cap h_j$, $j = 1, \dots, 4$.

3.3 *The spectral analysis of the operator* $A_0^{(4)}$

During this paper the spectral decomposition of the operator $A_0^{(4)}$ will be of particular interest. Because of the decomposition (3.6) we have in fact to carry out the spectral analysis of the operators $A^{(4)}(\xi)$. Being the restrictions of the non-negative second-order Sturm–Liouville systems (3.4) to $\text{Dom } A(\xi) \cap h_4$, the operators $A^{(4)}(\xi)$ have a non-negative discrete spectrum, which accumulates to infinity only. Let $\{\lambda_j(\xi)\}_{j=1}^{\infty}$ be the non-decreasing sequence of the eigenvalues of $A^{(4)}(\xi)$. The quantities $\lambda_j(\xi)$ are the solutions of the well-known *Rayleigh–Lamb dispersion equation*

$$\beta_j^{-1} \sin\left(\frac{\pi\beta_j}{2}\right) \gamma_j^2 \cos\left(\frac{\pi\gamma_j}{2}\right) + \xi^2 \cos\left(\frac{\pi\beta_j}{2}\right) \gamma_j^{-1} \sin\left(\frac{\pi\gamma_j}{2}\right) = 0, \tag{3.9}$$

where

$$\beta_j = \beta_j(\xi) := \sqrt{\lambda_j(\xi) - \xi^2}, \quad \gamma_j = \gamma_j(\xi) := \sqrt{\frac{\lambda_j(\xi)}{2} - \xi^2}, \tag{3.10}$$

cf. (**20**) or (**21**). The functions β_j and γ_j take either real or purely imaginary values. It is easy to see that the actual choice of the branch of the square root is of no importance.

An elementary but careful analysis of the boundary problem (3.4) on Dom $A(\xi) \cap h_4$ shows that these eigenvalues are simple for any fixed $\xi \in \mathbb{R}$.[1] The form $a^{(4)}(\xi)$ is a holomorphic family of the Kato type (a), hence the operators $A^{(4)}(\xi)$ form a holomorphic family of the Kato type (B); see (**22**, p. 395). Thus the even functions $\lambda_j(\xi)$ are real analytic in ξ. We shall need the following simple assertion, the proof of which we attach as an Appendix to this paper.

LEMMA 3.1 *For all $w \in P_4 H^1(J, \mathbb{C}^2)$ and $\xi \in \mathbb{R}$ the following estimate holds:*

$$a(\xi)[w, w] \geqslant \max\{(8\sqrt{3} - 12), 2^{-1}\xi^2\}\|w\|^2_{L_2(J,\mathbb{C}^2)}. \tag{3.11}$$

Hence the lowest eigenvalue $\lambda_1(\xi)$ of $A^{(4)}(\xi)$ satisfies the bound

$$\lambda_1(\xi) \geqslant \max\{8\sqrt{3} - 12, 2^{-1}\xi^2\}, \quad \xi \in \mathbb{R}. \tag{3.12}$$

The constants in (3.11), (3.12) are not sharp but suffice for our purposes. In particular we conclude that the spectrum $\sigma(A_0^{(4)})$, which by (3.6) coincides with the union of the images of the spectral branches $\lambda_j(\xi)$ over all $j \in \mathbb{N}$ and $\xi \in \mathbb{R}$, is absolutely continuous and given by

$$\sigma(A^{(4)}) = [\Lambda, \infty), \quad \Lambda = \min_{\xi \in \mathbb{R}} \lambda_1(\xi) \geqslant 8\sqrt{3} - 12 > 1{\cdot}856.$$

The following lemma describes the structure of the global minima of the function $\lambda_1(\xi)$. Its proof uses entirely elementary tools, but since this statement is crucial for what follows, we shall provide a sketch of the proof at the end of the paper.

LEMMA 3.2 *The eigenfunction $\lambda_1(\xi)$ achieves its minimal value Λ at exactly two points $\xi = \pm\varkappa$, $\varkappa > 0$, and there exists a value $q > 0$ such that*

$$\lambda_1(\epsilon \pm \varkappa) = \Lambda + q^2\epsilon^2 + O(\epsilon^3) \quad \text{as } \epsilon \to 0. \tag{3.13}$$

Being solutions of transcendent equations, $\varkappa$, Λ and q do not have explicit analytic expressions. A numerical evaluation for these values gives

$$\begin{aligned} \varkappa &= 0{\cdot}632138 \pm 10^{-6}, \\ \Lambda &= 1{\cdot}887837 \pm 10^{-6}, \\ q &= 0{\cdot}849748 \pm 10^{-6}. \end{aligned} \tag{3.14}$$

The eigenfunction corresponding to λ_j can be given by $\psi_j = \tilde{\psi}_j / \|\tilde{\psi}_j\|_{L_2(J,\mathbb{C}^2)}$, where

$$\tilde{\psi}_j = \tilde{\psi}_j(\xi, x_2) = \begin{pmatrix} i\xi \cos\left(\frac{\beta_j \pi}{2}\right) \cos(\gamma_j x_2) + \frac{i\xi\gamma_j^2}{\xi^2} \cos\left(\frac{\gamma_j \pi}{2}\right) \cos(\beta_j x_2) \\ -\gamma_j \cos\left(\frac{\beta_j \pi}{2}\right) \sin(\gamma_j x_2) + \frac{\gamma_j^2}{\beta_j} \cos\left(\frac{\gamma_j \pi}{2}\right) \sin(\beta_j x_2) \end{pmatrix} \tag{3.15}$$

[1] In particular, the trivial eigenfunction $u = (1, 0)$ with the eigenvalue $2\xi^2$ of (3.4), (3.5) does not belong to h_4 and has to be excluded.

if $\gamma_j \neq 0$, or

$$\tilde{\psi}_j = \tilde{\psi}(\xi, x_2) = \begin{pmatrix} i\left(\cos((2l-1)x_2) + \dfrac{2(-1)^l}{\pi(2l-1)}\right) \\ \sin((2l-1)x_2) \end{pmatrix}, \qquad j = \left|l - \frac{1}{2}\right| + \frac{1}{2}, \tag{3.16}$$

in the case $\gamma_j = 0$, which occurs for $\xi = (2l-1)$ and $\lambda_j(\xi) = 2\xi^2 = 2(2l-1)^2$, $l \in \mathbb{Z}$.

4. Statement of the main result

Let $\phi = (\phi^{(1)}, \phi^{(2)}) = \psi_1(\varkappa, \cdot)$ be the normalized eigenfunction (3.15) of $A^{(4)}(\varkappa)$ corresponding to the eigenvalue Λ. Put

$$\theta = \int_{-\pi/2}^{\pi/2} \left(2\varkappa^2 \left|\phi^{(1)}\right|^2 + 2\left|\frac{\partial \phi^{(2)}}{\partial x_2}\right|^2 - \left|\frac{\partial \phi^{(1)}}{\partial x_2} + i\varkappa\phi^{(2)}\right|^2 \right) dx_2.$$

A numerical evaluation with the values for $\varkappa$ and Λ as in (3.14) gives

$$\theta = 1{\cdot}816478 \pm 10^{-6}. \tag{4.1}$$

Moreover, for a given function $f \in L_\infty(\mathbb{R}; (-\infty, 1])$ of bounded support put

$$\mu_j = \Lambda \int_{\mathbb{R}} f(x_1)dx_1 + (-1)^j\theta \left| \int_{\mathbb{R}} e^{2i\varkappa x_1} f(x_1)dx_1 \right|, \quad j = 1, 2. \tag{4.2}$$

Let q be the respective parameter in (3.13).

THEOREM 4.1 *If*

$$\mu_1 > 0 \quad \textit{and} \quad \mu_2 > 0, \tag{4.3}$$

then for all sufficiently small positive α the spectrum of $A_\alpha^{(4)}$ below Λ consists of two eigenvalues

$$\nu_j(\alpha) = \Lambda - \frac{\alpha^2\pi^2}{q^2}\mu_j + o(\alpha^2), \tag{4.4}$$

where $j = 1, 2$. If

$$\mu_1 > 0 \quad \textit{and} \quad \mu_2 < 0, \tag{4.5}$$

then for all sufficiently small positive α the spectrum of $A_\alpha^{(4)}$ below Λ consists of one eigenvalue $\nu_1(\alpha)$, satisfying (4.4) *for $j = 1$. If*

$$\mu_1 < 0 \quad \textit{and} \quad \mu_2 < 0, \tag{4.6}$$

then $A_\alpha^{(4)}$ does not have spectrum below Λ for all sufficiently small positive α.

Obviously the eigenvalues $\nu_j(\alpha)$ of $A_\alpha^{(4)}$ are embedded eigenvalues for the complete elasticity operator A_α.

5. On the existence of discrete spectrum

The effect that arbitrarily small perturbations will draw bound states from the threshold of the continuous spectrum is sometimes depicted as the existence of 'virtual bound states' for the unperturbed operator at the edge of its spectrum. One of the main difficulties in this context is that Kato's perturbation theory is not applicable in a straightforward way. There is extensive literature on this subject; see, for example, (**14**) and the more operator theoretical treatment in (**15**).

In particular, the later approach allows one to deal with the appearance of several virtual bound states, as is the case in our setting. Indeed, in contrast to many analogous problems for the Laplace operator, where only one trapped mode appears for small couplings, the matrix structure of the elasticity operator gives rise to two small coupling modes. This is due to the structure of the band functions near the spectral minimum as described in Lemma 3.2.

Applying (**15**) we reduce the initial problem to the study of an isolated eigenvalue of multiplicity two (corresponding to the two 'virtual eigenvalues' of $A_0^{(4)}$ at Λ) of an auxiliary Birman–Schwinger type operator $\mathfrak{B}$. Using standard analytic perturbation theory we discuss the splitting of this double eigenvalue and discover which branches actually give rise to trapped modes.

5.1 *Preliminary estimates* I

We recall that Φ is the Fourier transform in x_1-direction and Φ^* is its inverse. Let χ_+ be the characteristic function of the interval $(0, 2\varkappa)$ and let χ_- be the characteristic function of the interval $(-2\varkappa, 0)$. For $u \in L_2(\Gamma, \mathbb{C}^2)$ and $j \in \mathbb{N}$ we define

$$\hat{u}^{(j)}(\xi) = \big\langle (\Phi u)(\xi, \cdot), \psi_j(\xi, \cdot)\big\rangle_{L_2(J,\mathbb{C}^2)} \quad \text{and} \quad \hat{u}^{\pm}(\xi) = \chi_{\pm}(\xi)\hat{u}^{(1)}(\xi).$$

Moreover, put

$$u^{(j)} = (\Pi_j u) = \Phi^*(\hat{u}^{(j)}\psi_j) \quad \text{and} \quad u^{\pm} = (\Pi_{\pm} u) = \Phi^*(\hat{u}^{\pm}\psi_1).$$

The operators Π_j and $\Pi_{\pm}$ are orthogonal projections onto invariant subspaces for $A_0^{(4)}$ in H_4,

$$\Pi_+\Pi_- = 0 \quad \text{and} \quad \Pi_j\Pi_k = 0 \ \text{ for } j \neq k.$$

Moreover, $P_4 = \sum_{j=1}^{\infty} \Pi_j$. Since $\Pi_- + \Pi_+ \leqslant \Pi_1$, the operator

$$\Pi = P_4 - \Pi_+ - \Pi_-$$

is also an orthogonal projection onto an invariant subspace of $A_0^{(4)}$ in H_4, and we set $\tilde{u} = \Pi u$. Hence for $u \in P_4 H^1(\Gamma, \mathbb{C}^2)$ we have $\tilde{u}, u^{\pm} \in P_4 H^1(\Gamma, \mathbb{C}^2) \subset H^1(\Gamma, \mathbb{C}^2)$, and the form a_0 can be written as

$$\begin{aligned} a_0[u,u] &= a_0[\tilde{u},\tilde{u}] + a_0[u^-,u^-] + a_0[u^+,u^+] \\ &= \sum_{j\geqslant 2}\int_{\mathbb{R}} \lambda_j(\xi)|\hat{u}^{(j)}(\xi)|^2 d\xi + \int_{|\xi|\geqslant 2\varkappa} \lambda_1(\xi)|\hat{u}^{(1)}(\xi)|^2 d\xi \\ &\quad + \int_{-2\varkappa}^{0} \lambda_1(\xi)|\hat{u}^-(\xi)|^2 d\xi + \int_0^{2\varkappa} \lambda_1(\xi)|\hat{u}^+(\xi)|^2 d\xi. \end{aligned} \tag{5.1}$$

Since $\lambda_j(\xi)$ is separated from Λ for all ξ if $j \geqslant 2$ or for $|\xi| \geqslant 2\varkappa$ if $j = 1$, we have a two-sided estimate

$$a_0[\tilde{u}, \tilde{u}] - \Lambda \int_\Gamma |\tilde{u}|^2 dx \asymp \sum_{j \geqslant 2} \int_{\mathbb{R}} (1 + \lambda_j(\xi)) |\hat{u}^{(j)}|^2 d\xi + \int_{|\xi| \geqslant 2\varkappa} (1 + \lambda_1(\xi)) |\hat{u}^{(1)}|^2 d\xi$$
$$\asymp a_0[\tilde{u}, \tilde{u}] + \|\tilde{u}\|^2_{L_2(\Gamma, \mathbb{C}^2)} \asymp \|\tilde{u}\|^2_{H^1(\Gamma, \mathbb{C}^2)}. \tag{5.2}$$

On the last line we make use of Korn's inequality. Moreover, since $\lambda_1(\xi) - \Lambda \asymp (\xi \mp \varkappa)^2$ with the minus sign if $\xi \in (0, 2\varkappa)$ and the plus sign if $\xi \in (-2\varkappa, 0)$, we have

$$a_0[u^\pm, u^\pm] - \Lambda \int_\Gamma |u^\pm|^2 dx \asymp \int_{\mathbb{R}} (\xi \mp \varkappa)^2 |\hat{u}^\pm|^2 d\xi \asymp \int_\Gamma \left| \frac{\partial e^{\mp i \varkappa x_1} u^\pm}{\partial x_1} \right|^2 dx. \tag{5.3}$$

Combining (5.1) and (5.3) we obtain

$$a_0[u, u] - \Lambda \int_\Gamma |u|^2 dx \asymp \|\tilde{u}\|^2_{H^1(\Gamma, \mathbb{C}^2)} + \int_\Gamma \left\{ \left| \frac{\partial e^{-i\varkappa x_1} u^+}{\partial x_1} \right|^2 + \left| \frac{\partial e^{i\varkappa x_1} u^-}{\partial x_1} \right|^2 \right\} dx \tag{5.4}$$

for all $u \in P_4 H^1(\Gamma, \mathbb{C}^2)$.

5.2 *Preliminary estimates* II

Put

$$b[u, u] := \int_\Gamma \left(\left| \frac{\partial u}{\partial x_1} \right|^2 + \left| \frac{\partial u}{\partial x_2} \right|^2 + |u|^2 \right) \frac{dx}{1 + x_1^2}, \quad u \in P_4 H^1(\Gamma, \mathbb{C}^2). \tag{5.5}$$

In view of (5.4) we have obviously

$$b[\tilde{u}, \tilde{u}] \leqslant c \left(a_0 \left[\tilde{u}, \tilde{u} \right] - \Lambda \|\tilde{u}\|^2_{L_2(\Gamma, \mathbb{C}^2)} \right), \quad u \in P_4 H^1(\Gamma, \mathbb{C}^2). \tag{5.6}$$

The analogous bound fails for the components $u^\pm$, but it can be replaced by the following statement.

LEMMA 5.1 *Assume* $u \in P_4 H^1(\Gamma, \mathbb{C}^2)$ *and*

$$\int_{\mathbb{R}} \hat{u}^\pm(\xi) d\xi = 0. \tag{5.7}$$

Then

$$b[u^\pm, u^\pm] \leqslant c \left(a_0[u^\pm, u^\pm] - \Lambda \|u^\pm\|^2_{L_2(\Gamma, \mathbb{C}^2)} \right). \tag{5.8}$$

Proof. First note that $u^\pm \in P_4 H^1(\Gamma, \mathbb{C}^2) \subseteq H^1(\Gamma, \mathbb{C}^2)$ implies that

$$\int_\Gamma \left(|u^\pm|^2 + \left| \frac{\partial u^\pm}{\partial x_1} \right|^2 \right) dx = \int_{\mathbb{R}} d\xi (1 + \xi^2) |\hat{u}^\pm(\xi)|^2 \int_J dx_2 |\psi_1(\xi, x_2)|^2$$
$$= \int_{\mathbb{R}} (1 + \xi^2) |\hat{u}^\pm(\xi)|^2 d\xi < \infty,$$

and by Hölder's inequality $\hat{u}^{\pm} \in L_1(\mathbb{R}, \mathbb{C})$. Thus condition (5.7) is justified. Put $\zeta(x) = (1 + x_1^2)^{-1/2}$. Since

$$\left\| \zeta \frac{\partial u^{\pm}}{\partial x_1} \right\|_{L_2(\Gamma, \mathbb{C}^2)} \leqslant |\varkappa| \left\| \zeta u^{\pm} \right\|_{L_2(\Gamma, \mathbb{C}^2)} + \left\| \frac{\partial e^{\mp i \varkappa x_1} u^{\pm}}{\partial x_1} \right\|_{L_2(\Gamma, \mathbb{C}^2)},$$

in view of (5.3) it is sufficient to prove that

$$\left\| \zeta u^{\pm} \right\|^2_{L_2(\Gamma, \mathbb{C}^2)} + \left\| \zeta \frac{\partial u^{\pm}}{\partial x_2} \right\|^2_{L_2(\Gamma, \mathbb{C}^2)} \leqslant c \left\| \frac{\partial e^{\mp i \varkappa x_1} u^{\pm}}{\partial x_1} \right\|^2_{L_2(\Gamma, \mathbb{C}^2)}. \tag{5.9}$$

Let $Q_{\pm} \colon L_2(\mathbb{R}, \mathbb{C}) \to L_2(\Gamma, \mathbb{C}^2)$ be the integral operators

$$(Q_{\pm} h)(x_1, x_2) := \frac{e^{\pm i \varkappa x_1}}{\sqrt{2\pi}} \int_{-\varkappa}^{\varkappa} e^{i t x_1} \psi_1(t \pm \varkappa, x_2) |t|^{-1} h(t) dt,$$

being defined on all appropriate functions h. Set $\hat{w}^{\pm}(t) = |t| \hat{u}^{\pm}(t \pm \varkappa)$. Then we have

$$u^{\pm} = Q_{\pm} \hat{w}^{\pm} \quad \text{and} \quad \| \partial (e^{\mp i \varkappa x_1} u^{\pm}) / \partial x_1 \|_{L_2(\Gamma, \mathbb{C}^2)} = \| \hat{w}^{\pm} \|_{L_2(\mathbb{R}, \mathbb{C})}. \tag{5.10}$$

Developing the eigenfunction $\psi_1(\xi, x_2)$, given in (3.15), (3.16) in a Taylor series near $\pm\varkappa$, we find

$$\psi_1(t \pm \varkappa, x_2) = \psi_1(\pm \varkappa, x_2) + t \tau^{\pm}(t, x_2) \quad \text{for } t \in [-\varkappa, \varkappa],$$

where

$$\psi_1, \ \frac{\partial}{\partial x_2} \psi_1, \ \tau^{\pm}, \ \frac{\partial}{\partial x_2} \tau^{\pm} \in L_{\infty}([-\varkappa, \varkappa] \times J, \mathbb{C}^2). \tag{5.11}$$

Moreover it holds that

$$\begin{aligned} \zeta Q_{\pm} &= \frac{e^{\pm i \varkappa x_1}}{\sqrt{2\pi}} \psi_1(\pm \varkappa, x_2) Q_0 + \frac{e^{\pm i \varkappa x_1}}{\sqrt{2\pi}} Q_1 \tau^{\pm}, \\ \zeta \frac{\partial}{\partial x_2} Q_{\pm} &= \frac{e^{\pm i \varkappa x_1}}{\sqrt{2\pi}} \frac{\partial \psi_1(\pm \varkappa, x_2)}{\partial x_2} Q_0 + \frac{e^{\pm i \varkappa x_1}}{\sqrt{2\pi}} Q_1 \frac{\partial \tau^{\pm}}{\partial x_2}, \end{aligned} \tag{5.12}$$

where Q_0 and Q_1 are the integral operators

$$(Q_0 h_0)(x_1) := \zeta \int_{-\varkappa}^{\varkappa} e^{i t x_1} h_0(t) \frac{dt}{|t|} \quad \text{and} \quad (Q_1 h_1)(x) := \zeta \int_{-\varkappa}^{\varkappa} e^{i t x_1} h_1(t, x_2) \frac{t dt}{|t|}.$$

The operator Q_1 is obviously bounded in $L_2(\Gamma, \mathbb{C}^2)$. Next note that for functions $h_2 \in H^1(\mathbb{R}, \mathbb{C})$ with $h_2(0) = 0$ Hardy's inequality

$$\| \zeta h_2 \|_{L_2(\mathbb{R}, \mathbb{C})} \leqslant 2 \| \partial h_2 / \partial x_1 \|_{L_2(\mathbb{R}, \mathbb{C})}$$

holds. Because of (5.7) we can apply this to $h_2 = e^{\mp i \varkappa x_1} \Phi^* \hat{u}^{\pm}$, which leads to

$$\| Q_0 \hat{w}^{\pm} \|_{L_2(\mathbb{R}, \mathbb{C})} \leqslant 2 \| \hat{w}^{\pm} \|_{L_2(\mathbb{R}, \mathbb{C})}.$$

Combining this with (5.11) and (5.12), we conclude that

$$\max\left\{\|\zeta Q_\pm \hat{w}^\pm\|_{L_2(\Gamma,\mathbb{C}^2)}, \left\|\zeta \frac{\partial}{\partial x_2} Q_\pm \hat{w}^\pm\right\|_{L_2(\Gamma,\mathbb{C}^2)}\right\} \leqslant c\|\hat{w}^\pm\|_{L_2(\mathbb{R},\mathbb{C})}.$$

Then (5.10) implies (5.9).

5.3 *The Birman–Schwinger principle* I

On the domain $d[m] = P_4 H^1(\Gamma, \mathbb{C}^2)$ we define the quadratic form

$$m[u,u] := a_0[u,u] - \Lambda\|u\|^2_{L_2(\Gamma,\mathbb{C}^2)} + b[u,u]. \tag{5.13}$$

Then $P_4 H^1(\Gamma, \mathbb{C}^2)$ is a pre-Hilbert space with respect to the scalar product m. Let the Hilbert space $\mathfrak{H}$ be the completion of $P_4 H^1(\Gamma, \mathbb{C}^2)$ with respect to m. Since $a_0[u,u] - \Lambda\|u\|^2_{L_2(\Gamma,\mathbb{C}^2)} \geqslant 0$ for $u \in P_4 H^1(\Gamma, \mathbb{C}^2)$, the form b extends to a bounded form on $\mathfrak{H}$, where it induces a non-negative operator $\mathfrak{B}$. The operator norm of $\mathfrak{B}$ does not exceed one. In fact the following holds.

LEMMA 5.2 *The point* 1 *is an isolated eigenvalue of multiplicity two of the operator* $\mathfrak{B}$. *The respective eigenspace can be represented by the two-dimensional linear set of fundamental sequences* $\tilde{u}^\varsigma = \{u_k^\varsigma\}_{k=1}^\infty$,

$$u_k^\varsigma = \vartheta(k^{-1}x_1)(\varsigma_+ e^{i\varkappa x_1}\psi_1(\varkappa, x_2) + \varsigma_- e^{-i\varkappa x_1}\psi_1(-\varkappa, x_2)), \tag{5.14}$$

where $\varsigma = (\varsigma_+, \varsigma_-) \in \mathbb{C}^2$, $\vartheta \in C_0^\infty(\mathbb{R}, \mathbb{C})$ *and* $\vartheta(x_1) = 1$ *in some neighbourhood of* $x_1 = 0$.

Proof. The spectrum of $\mathfrak{B}$ is a subset of the interval $[0, 1]$. By (5.6) and (5.8) there exists $\delta > 0$ such that

$$b[u,u] \leqslant (1-\delta)m[u,u] \tag{5.15}$$

for all functions $u \in P_4 H^1(\Gamma, \mathbb{C}^2)$ satisfying (5.7). Since this set of functions is of codimension two in $P_4 H^1(\Gamma, \mathbb{C}^2)$, and the latter set is dense in $\mathfrak{H}$, the total multiplicity of the spectrum of $\mathfrak{B}$ above $1-\delta$ does not exceed two.

Obviously $u_k^\varsigma \in P_4 H^1(\Gamma, \mathbb{C}^2)$. Using the two-sided bound (5.4) it is easy to verify that $\tilde{u}^\varsigma$ is fundamental with respect to m, and

$$a^\Lambda[u_k^\varsigma, u_k^\varsigma] := a_0[u_k^\varsigma, u_k^\varsigma] - \Lambda\|u_k^\varsigma\|^2_{L_2(\Gamma,\mathbb{C}^2)} \to 0 \quad \text{as } k \to \infty. \tag{5.16}$$

By continuity the form a^Λ extends to a bounded non-negative form on $\mathfrak{H}$. The union of the representative sequences (5.14) over $\varsigma \in \mathbb{C}^2$ forms a two-dimensional subspace $\mathfrak{H}_1$ in $\mathfrak{H}$, on which a^Λ vanishes. But then it holds that

$$m[\tilde{u}^\varsigma, \tilde{w}] - b[\tilde{u}^\varsigma, \tilde{w}] = a^\Lambda[\tilde{u}^\varsigma, \tilde{w}] = 0$$

for all $\tilde{u}^\varsigma \in \mathfrak{H}_1$ and $w \in \mathfrak{H}$, or equivalently $\mathfrak{B}\tilde{u}^\varsigma = \tilde{u}^\varsigma$. Hence the point 1 is an isolated eigenvalue of multiplicity two for $\mathfrak{B}$.

5.4 *The Birman–Schwinger principle* II

Below $\chi_{[0,\Lambda)}$ and $\chi_{(1,\infty)}$ are the characteristic functions for the respective intervals and

$$\upsilon[u,u] := \int_\Gamma f \left(2 \left| \frac{\partial u_1}{\partial x_1} \right|^2 + 2 \left| \frac{\partial u_2}{\partial x_2} \right|^2 + \left| \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \right|^2 \right) dx, \quad u \in H^1(\Gamma, \mathbb{C}^2). \tag{5.17}$$

Glazmann's lemma and (1.3) imply that

$$\operatorname{rank} \chi_{[0,\Lambda)}(A_\alpha^{(4)}) = \max \dim L,$$

where the supremum is taken over all linear sets $L \subset P_4 H^1(\Gamma, \mathbb{C}^2)$ such that

$$a_0[u,u] - \alpha \upsilon[u,u] < \Lambda \|u\|^2_{L_2(\Gamma,\mathbb{C}^2)} \quad \text{for all } u \in L,\ u \not\equiv 0. \tag{5.18}$$

Because of the boundedness of f the form υ can be extended to a bounded hermitian form on $\mathfrak{H}$, where it induces the bounded self-adjoint operator $\mathfrak{V}$. Put $\mathfrak{B}(\alpha) := \mathfrak{B} + \alpha\mathfrak{V}$. Applying Glazmann's lemma to this operator, one finds

$$\operatorname{rank} \chi_{(1,\infty)}(\mathfrak{B}(\alpha)) = \max \dim \ L,$$

where the supremum is taken over all linear sets L from the subset $P_4 H^1(\Gamma, \mathbb{C}^2)$ being dense in $\mathfrak{H}$, such that

$$m[u,u] < b[u,u] + \alpha \upsilon[u,u] \quad \text{for all } u \in L,\ u \not\equiv 0. \tag{5.19}$$

Comparing (5.18) and (5.19), one obtains the following variation of the Birman–Schwinger principle:

$$\operatorname{rank} \chi_{[0,\Lambda)}(A_\alpha^{(4)}) = \operatorname{rank} \chi_{(1,+\infty)}(\mathfrak{B}(\alpha)), \quad 0 < \alpha < 1. \tag{5.20}$$

5.5 *Proof of Theorem* 4.1—*Existence of eigenvalues*

According to Lemma 5.2 the point 1 is an isolated eigenvalue of multiplicity two of $\mathfrak{B} = \mathfrak{B}(0)$ and $\mathfrak{B}$ has no spectrum above 1. The perturbation family $\mathfrak{B}(\alpha)$ is analytic of Kato type (A) in α (**22**). Thus for small $\alpha > 0$ the spectrum of $\mathfrak{B}(\alpha)$ near or above 1 will consist of two eigenvalues, which form two analytic branches

$$\kappa_j(\alpha) = 1 + \alpha\kappa_j^{(1)} + O(\alpha^2), \quad j = 1, 2.$$

Hence by (5.20) the value $\lim_{\alpha\to+0} \operatorname{rank} \chi_{[0,\Lambda)}(A_\alpha^{(4)})$ coincides with the quantity of the branches $\kappa_j(\alpha)$ satisfying $\kappa_j(\alpha) > 1$ for all sufficiently small $\alpha > 0$.

Obviously $\kappa_j^{(1)} > 0$ implies $\kappa_j(\alpha) > 1$ and $\kappa_j^{(1)} < 0$ implies $\kappa_j(\alpha) < 1$ for small α. From standard analytic perturbation theory we know (**22**) that the values $\kappa_j^{(1)}$ are the eigenvalues of the form υ, being reduced to the two-dimensional eigenspace $\mathfrak{H}_1$ of $\mathfrak{B}$ at 1. Since we are interested in the signs of these values only, according to (5.14) we have to calculate the signs of the eigenvalues of the matrix

$$M = \lim_{k\to\infty} \begin{pmatrix} \upsilon[u_k^{(1,0)}, u_k^{(1,0)}] & \upsilon[u_k^{(1,0)}, u_k^{(0,1)}] \\ \upsilon[u_k^{(0,1)}, u_k^{(1,0)}] & \upsilon[u_k^{(0,1)}, u_k^{(0,1)}] \end{pmatrix}$$
$$= \begin{pmatrix} \Lambda \int f(x_1)dx_1 & \theta \int e^{2i\varkappa x_1} f(x_1)dx_1 \\ \theta \int e^{-2i\varkappa x_1} f(x_1)dx_1 & \Lambda \int f(x_1)dx_1. \end{pmatrix}. \tag{5.21}$$

The eigenvalues of M are μ_1 and μ_2 from (4.2). Then the conditions (4.3), (4.5), or (4.6) correspond to $\kappa_1^{(1)} > 0$ and $\kappa_2^{(1)} > 0$, $\kappa_1^{(1)} > 0$ and $\kappa_2^{(1)} < 0$, or $\kappa_1^{(1)} < 0$ and $\kappa_2^{(1)} < 0$, respectively. This concludes the proof.

6. The asymptotic behaviour of trapped modes

We have shown that in the setting of Theorem 4.1 the spectrum of the operator $A_\alpha^{(4)}$ below Λ consists of exactly two eigenvalues $\nu_1(\alpha) \leqslant \nu_2(\alpha)$ in the case (4.3), or exactly one eigenvalue $\nu_1(\alpha)$ in the case (4.5), if the positive parameter α is sufficiently small. In this section we shall calculate the asymptotic behaviour of these eigenvalues in the cases (4.3) and (4.5) as $\alpha \to 0$.

6.1 *Preliminary estimates* III

We take a finite interval I such that supp $f \subset I$, and let χ_I be the characteristic function for I. For $\nu < \Lambda$ we consider on H_4 the two rank one operators

$$(T_\nu^{\pm} w)(x) = \psi_1(\pm\varkappa, x_2)e^{\pm i\varkappa x_1}\chi_I(x_1) \int_\Gamma \frac{\overline{\psi_1(\pm\varkappa, x_2')}\hat{w}(\xi, x_2')d\xi dx_2'}{\sqrt{q^2(\xi \mp \varkappa)^2 + \Lambda - \nu}}.$$

Put $T_\nu = T_\nu^+ + T_\nu^-$. Then the form

$$y_\nu[w, w] = \upsilon[T_\nu w, T_\nu w]$$

is well defined and bounded on $L_2(\Gamma, \mathbb{C}^2)$. Let Y_ν be the associated self-adjoint operator of rank two.

LEMMA 6.1 *Let q be the respective parameter in* (3.13) *and let μ_j be the eigenvalues of M in* (5.21). *The eigenvalues $\mu_j(\nu)$, corresponding to the non-trivial part of Y_ν, satisfy the asymptotic equation*

$$\mu_j(\nu) = \frac{\pi}{q\sqrt{\Lambda - \nu}}\mu_j + o\left(\frac{1}{\sqrt{\Lambda - \nu}}\right) \quad \text{as } \nu \to \Lambda - 0, \quad j = 1, 2.$$

Proof. Let W_δ be the unitary scaling operator

$$(W_\delta w)(x) = \sqrt{\delta}w(\delta x_1, x_2), \quad \delta > 0.$$

Put

$$\eta_\delta^{\pm}(\xi, x_2) = \sqrt{\frac{q}{\pi}} \frac{\psi_1(\pm\varkappa, x_2)}{\sqrt{q^2(\xi \mp \delta^{-1}\varkappa)^2 + 1}}.$$

These functions are normed in $L_2(\Gamma, \mathbb{C}^2)$. Let $\tilde{T}_\nu^\pm$ be the rank one operators

$$(\tilde{T}_\nu^\pm w)(x) = \psi_1(\pm\varkappa, x_2)e^{\pm i\varkappa x_1}\chi_I(x_1)\left\langle w, \eta_\delta^\pm\right\rangle_{L_2(\Gamma,\mathbb{C}^2)}, \quad \delta = \sqrt{\Lambda - \nu}.$$

Then it holds that

$$\sqrt{\pi^{-1}q\delta}T_\nu^\pm = \tilde{T}_\nu^\pm W_\delta \Phi, \quad \delta = \sqrt{\Lambda - \nu}. \tag{6.1}$$

Let $\tilde{Y}_\nu$ be the rank two self-adjoint operator, corresponding to the quadratic form

$$\tilde{y}_\nu[w, w] = v[\tilde{T}_\nu w, \tilde{T}_\nu w], \quad \tilde{T}_\nu = \tilde{T}_\nu^+ + \tilde{T}_\nu^-.$$

Further, set

$$\tilde{\eta}_\delta = \frac{\eta_\delta^- - \eta_\delta^+\left\langle \eta_\delta^-, \eta_\delta^+\right\rangle_{L_2(\Gamma,\mathbb{C}^2)}}{\|\eta_\delta^- - \eta_\delta^+\left\langle \eta_\delta^-, \eta_\delta^+\right\rangle_{L_2(\Gamma,\mathbb{C}^2)}\|_{L_2(\Gamma,\mathbb{C}^2)}}.$$

Let $S_\nu, \tilde{S}_\nu : H^4 \mapsto \mathbb{C}^2$ be the operators

$$S_\nu = \begin{pmatrix} \left\langle \cdot, \eta_\delta^+\right\rangle_{L_2(\Gamma,\mathbb{C}^2)} \\ \left\langle \cdot, \eta_\delta^-\right\rangle_{L_2(\Gamma,\mathbb{C}^2)} \end{pmatrix} \quad \text{and} \quad \tilde{S}_\nu = \begin{pmatrix} \left\langle \cdot, \eta_\delta^+\right\rangle_{L_2(\Gamma,\mathbb{C}^2)} \\ \langle \cdot, \tilde{\eta}_\delta\rangle_{L_2(\Gamma,\mathbb{C}^2)} \end{pmatrix}, \quad \delta = \sqrt{\Lambda - \nu}.$$

The operator $\tilde{S}_\nu$ is a partial isometric mapping from the linear span of $\eta_\delta^\pm$ onto $\mathbb{C}^2$. The identity $\tilde{y}_\nu[w, w] = \langle MS_\nu w, S_\nu w\rangle_{\mathbb{C}^2}$ implies $\tilde{Y}_\nu = S_\nu^* M S_\nu$. The eigenvalues of the non-trivial part of $\tilde{S}_\nu^* M \tilde{S}_\nu$ are μ_j. Since $\left\langle \eta_\delta^+, \eta_\delta^-\right\rangle_{L_2(\Gamma,\mathbb{C}^2)} \to 0$ as $\delta \to 0$, we have

$$\tilde{S}_\nu^* M \tilde{S}_\nu - \tilde{Y}_\nu = \tilde{S}_\nu^* M \tilde{S}_\nu - S_\nu^* M S_\nu \to 0 \quad \text{as } \nu \to \Lambda - 0.$$

By (6.1) the eigenvalues $\mu_j(\nu)$ of Y_ν coincide with the eigenvalues of the non-trivial part of the operator $\pi q^{-1}\delta^{-1}\tilde{Y}_\nu$, $\delta = \sqrt{\Lambda - \nu}$. But then

$$q\pi^{-1}\mu_j(\nu)\sqrt{\Lambda - \nu} \to \mu_j \quad \text{as } \nu \to \Lambda - 0, \quad j = 1, 2.$$

6.2 *Preliminary estimates* IV

Let $R_\nu = (A_0^{(4)} - \nu)^{-1}$ be the resolvent of $A_0^{(4)}$ at the spectral point ν. For $\nu < \Lambda$ the operator $R_\nu^{1/2}$ is a bounded mapping from H_4 to $d[a^{(4)}] = P_4 H^1(\Gamma, \mathbb{C}^2) \subseteq H^1(\Gamma, \mathbb{C}^2)$. Hence the form

$$x_\nu[w, w] = v[R_\nu^{1/2} w, R_\nu^{1/2} w]$$

is well defined and bounded on H_4. Let X_ν be the associated bounded self-adjoint operator on H_4.

LEMMA 6.2 *There exists a positive constant C such that the estimate*

$$\|X_\nu - Y_\nu\| \leqslant C(1 + 1/\sqrt[4]{\Lambda - \nu}) \tag{6.2}$$

holds for all $\nu < \Lambda$.

Proof. Put $\delta = \sqrt{\Lambda - \nu}$. By Korn's inequality the operator $\nabla R_\nu^{1/2}$ is bounded on H_4 for fixed $\nu < \Lambda$. Since $R_\nu^{1/2}\Pi$ is uniformly bounded for all $\nu \leqslant \Lambda$, it is then easy to see that the operator

$\nabla R_\nu^{1/2}\Pi$ is uniformly bounded for all $\nu \leqslant \Lambda$. Moreover, for $\nu < \Lambda$ the operators $\chi_I \nabla R_\nu^{1/2}\Pi_\pm$ are Hilbert–Schmidt, and

$$\begin{aligned}\|\chi_I \nabla R_\nu^{1/2}\Pi_\pm u\|^2_{L_2(\Gamma,\mathbb{C}^2)} &\leqslant c_1 \|u\|^2_{L_2(\Gamma,\mathbb{C}^2)} \int_{0<\pm\xi<\varkappa} \frac{\xi^2 d\xi}{\lambda_1(\xi)-\nu} \\ &\leqslant c_2 \|u\|^2_{L_2(\Gamma,\mathbb{C}^2)} \int \frac{d\xi}{q^2(\xi\mp\varkappa)^2+\delta^2} \leqslant c_3\delta^{-1}\|u\|^2_{L_2(\Gamma,\mathbb{C}^2)} \qquad (6.3)\end{aligned}$$

for all $u \in H_4$. The same type of estimate shows that

$$\|\chi_I \nabla T_\nu^\pm u\|^2 \leqslant c_4\delta^{-1}\|u\|^2_{L_2(\Gamma,\mathbb{C}^2)}, \qquad u \in H_4. \qquad (6.4)$$

Computing the corresponding Taylor series with remainder estimates we see that

$$\frac{e^{i\xi x_1}\psi_1(\xi,x_2)\overline{\psi_1(\xi,x_2')}}{\sqrt{\lambda_1(\xi)-\nu}} = \frac{e^{\pm i\varkappa x_1}\psi_1(\pm\varkappa,x_2)\overline{\psi_1(\pm\varkappa,x_2')}}{\sqrt{q^2(\xi\mp\varkappa)^2+\delta^2}}(1+(\xi\mp\varkappa)R^\pm(\xi,x,x')),$$

where the functions $R^\pm$ are uniformly bounded on $(-2\varkappa,2\varkappa)\times(I\times J)^2$. But then

$$\begin{aligned}\|\chi_I\nabla(R_\nu^{1/2}(\Pi_+ + \Pi_-) - T_\nu)u\|^2_{L_2(\Gamma,\mathbb{C}^2)} &\leqslant c_5\|u\|^2_{L_2(\Gamma,\mathbb{C}^2)} \int_0^\varkappa \frac{(\xi-\varkappa)^2 d\xi}{q^2(\xi-\varkappa)^2+\delta^2} \\ &\leqslant c_6\|u\|^2_{L_2(\Gamma,\mathbb{C}^2)}. \qquad (6.5)\end{aligned}$$

Recall that

$$\mathfrak{v}[u,u] \leqslant c\|\chi_I\nabla u\|^2_{L_2(\Gamma,\mathbb{C}^2)} \qquad (6.6)$$

for $u \in P_4H^1(\Gamma,\mathbb{C}^2)$. We decompose the form x_ν as follows:

$$x_\nu[u,u] = \mathfrak{v}[R_\nu^{1/2}(\Pi_+ + \Pi_-)u, R_\nu^{1/2}(\Pi_+ + \Pi_-)u] + r[u,u],$$

where by (6.3), (6.6) the form

$$r[u,u] = \mathfrak{v}[R_\nu^{1/2}\Pi u, R_\nu^{1/2}\Pi u] + 2\Re\mathfrak{v}[R_\nu^{1/2}\Pi u, R_\nu^{1/2}(\Pi_+ + \Pi_-)u]$$

satisfies the estimate

$$|r[u,w]| \leqslant C(1+\delta^{-1/2})\|u\|_{L_2(\Gamma,\mathbb{C}^2)}\|w\|_{L_2(\Gamma,\mathbb{C}^2)}.$$

The identity

$$\begin{aligned}x_\nu[u,u] - y_\nu[u,u] = {} & 2\Re\mathfrak{v}[(R_\nu^{1/2}(\Pi_+ + \Pi_-) - T_\nu)u, R_\nu^{1/2}(\Pi_+ + \Pi_-)u] \\ & - \mathfrak{v}[(R_\nu^{1/2}(\Pi_+ + \Pi_-) - T_\nu)u, (R_\nu^{1/2}(\Pi_+ + \Pi_-) - T_\nu)u] + r[u,u]\end{aligned}$$

implies together with (6.4), (6.5) and (6.6) that

$$|\langle (X_\nu - Y_\nu)u, u\rangle_{L_2(\Gamma,\mathbb{C}^2)}| \leqslant C(1+\delta^{-1/2})\|u\|^2_{L_2(\Gamma,\mathbb{C}^2)}$$

as $u \in H_4$. This completes the proof.

6.3 *The proof of Theorem* 4.1—*Formula* (4.4)

For $t \in \mathbb{R}$ let $\chi_{\{t\}}$ be the characteristic function for the point t. The operator αX_ν is the Birman–Schwinger operator for the perturbed operator family $A_\alpha^{(4)}$,

$$\operatorname{rank} \chi_{\{1\}}\alpha X_\nu = \operatorname{rank} \chi_{\{\nu\}}(A_\alpha) \tag{6.7}$$

and

$$\operatorname{rank} \chi_{[1,\infty)}\alpha X_\nu = \operatorname{rank} \chi_{[0,\nu]}(A_\alpha) \tag{6.8}$$

for all $\nu < \Lambda$ and $0 < \alpha < 1$; see (**23**). By (6.3) and (6.6) we see that

$$|\langle X_\nu u, u\rangle_{L_2(\Gamma,\mathbb{C}^2)}| \leqslant c\delta^{-1}\|u\|^2_{L_2(\Gamma,\mathbb{C}^2)}. \tag{6.9}$$

Put $\delta_j(\alpha) = \sqrt{\Lambda - \nu_j(\alpha)}$. Then (6.7) and (6.9) imply

$$\delta_j(\alpha) = O(\alpha) \quad \text{as } \alpha \to +0.$$

The estimate (6.2) transforms into

$$\|\delta_j(\alpha)X_{\nu_j(\alpha)} - \delta_j(\alpha)Y_{\nu_j(\alpha)}\| \leqslant C(\delta_j(\alpha) + \sqrt{\delta_j(\alpha)}) = O(\sqrt{\alpha})$$

as $\alpha \to 0$. The operators $\delta_j(\alpha)Y_{\nu_j(\alpha)}$ are of rank two, and by Lemma 6.1 their non-trivial eigenvalues $\delta_j(\alpha)\mu_j(\nu_j(\alpha))$ satisfy $\delta_j(\alpha)\mu_j(\nu_j(\alpha)) \to q^{-1}\pi\mu_j$, $j = 1, 2$. By standard perturbation theory we conclude that, if $\mu_j \neq 0$, $j = 1, 2$, the operators $\delta_j(\alpha)X_{\nu_j(\alpha)}$ have all spectra in an $O(\sqrt{\alpha})$-neighbourhood of zero, except two eigenvalues $\varrho_j(\alpha) \to q^{-1}\pi\mu_j$ for $j = 1, 2$, respectively. In the cases (4.5), (4.3) $\mu_j > 0$ implies now that the point $\varrho_j(\alpha)$ becomes the jth largest eigenvalue of $\delta_j(\alpha)X_{\nu_j(\alpha)}$ for sufficiently small $\alpha > 0$. That means $\alpha\varrho_j(\alpha)\delta_j^{-1}(\alpha)$ becomes the jth largest eigenvalue of $\alpha X_{\nu_j(\alpha)}$, which on its turn by (6.7), (6.8) equals 1. Hence

$$\alpha^{-1}\delta_j(\alpha) = \varrho_j(\alpha) \to q^{-1}\pi\mu_j$$

as $\alpha \to 0$. This concludes the proof.

7. Conclusions

We have proven the existence of trapped modes in elastic strips perturbed by local changes of Young's modulus. We furthermore have obtained an asymptotic formula which describes the behaviour of the trapped modes in the limit of small differences of Young's modulus.

In a forthcoming paper the results of this article will be extended to three-dimensional elastic plates. Further extensions which are in the focus of our present work concern the treatment of other limit cases like the limit of small or large areas of changed Young's modulus. Also extensions to other types of perturbations like cracks or holes in the three-dimensional medium can be treated by variational techniques.

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APPENDIX

A.1 *Sketch of the Proof of Lemma* 3.1

For brevity we shall write w_j' instead of $\partial w_j/\partial x_2$. The functions w_j are continuous. Since w_1 is symmetric and orthogonal to the constant function, it is easy to see that

$$4\|w_1\|^2_{L_2(J,\mathbb{C})} \leqslant \|w_1'\|^2_{L_2(J,\mathbb{C})} \quad \text{and} \quad \|w_1\|^2_{C(J,\mathbb{C})} \leqslant \|w_1'\|_{L_2(J,\mathbb{C})}\|w_1\|_{L_2(J,\mathbb{C})}. \tag{A.1}$$

On the other hand, for w_2 being antisymmic it holds that

$$\|w_2\|^2_{L_2(J,\mathbb{C})} \leqslant \|w_2'\|^2_{L_2(J,\mathbb{C})} \quad \text{and} \quad \|w_2\|^2_{C(J,\mathbb{C})} \leqslant \|w_2'\|_{L_2(J,\mathbb{C})}\|w_2\|_{L_2(J,\mathbb{C})}. \tag{A.2}$$

Minimizing the expression for $a(\xi)[w,w]$ in ξ and using the first bound in (A.1), (A.2), respectively, one obtains

$$\begin{aligned} a(\xi)[w,w] &\geqslant \|w_1'\|^2_{L_2(J,\mathbb{C})} + 2\|w_2'\|^2_{L_2(J,\mathbb{C})} - \frac{\|w_1'\|^2_{L_2(J,\mathbb{C})}\|w_2\|^2_{L_2(J,\mathbb{C})}}{2\|w_1\|^2_{L_2(J,\mathbb{C})} + \|w_2\|^2_{L_2(J,\mathbb{C})}} \\ &\geqslant 2\frac{4\|w_1\|^4_{L_2(J,\mathbb{C})} + \|w_2\|^4_{L_2(J,\mathbb{C})} + 2\|w_1\|^2_{L_2(J,\mathbb{C})}\|w_2\|^2_{L_2(J,\mathbb{C})}}{2\|w_1\|^2_{L_2(J,\mathbb{C})} + \|w_2\|^2_{L_2(J,\mathbb{C})}}. \end{aligned}$$

Minimizing the right-hand side under the restriction $\|w\|^2_{L_2(J,\mathbb{C}^2)} = \|w_1\|^2_{L_2(J,\mathbb{C})} + \|w_2\|^2_{L_2(J,\mathbb{C})}$ we arrive at

$$a(\xi)[w,w] \geqslant (8\sqrt{3} - 12)\|w\|^2_{L_2(J,\mathbb{C})}. \tag{A.3}$$

For the second estimate we shall use the fact that

$$|\langle w_1', w_2\rangle_{L_2(J,\mathbb{C})}| \leqslant 2\|w_1\|_{C(J,\mathbb{C})}\|w_2\|_{C(J,\mathbb{C})} + \|w_1\|_{L_2(J,\mathbb{C})}\|w_2'\|_{L_2(J,\mathbb{C})}.$$

Then in view of the second of the bounds in (A.1), (A.2), respectively, we have

$$\begin{aligned} a(\xi)[w,w] \geqslant{}& 2\xi^2\|w_1\|^2_{L_2(J,\mathbb{C})} + \xi^2\|w_2\|^2_{L_2(J,\mathbb{C})} + \|w_1'\|^2_{L_2(J,\mathbb{C})} + 2\|w_2'\|^2_{L_2(J,\mathbb{C})} \\ &- \xi\|w_1'\|_{L_2(J,\mathbb{C})}\|w_2\|_{L_2(J,\mathbb{C})} - \xi\|w_1\|_{L_2(J,\mathbb{C})}\|w_2'\|_{L_2(J,\mathbb{C})} \\ &- 2\xi\sqrt{\|w_1\|_{L_2(J,\mathbb{C})}\|w_1'\|_{L_2(J,\mathbb{C})}\|w_2\|_{L_2(J,\mathbb{C})}\|w_2'\|_{L_2(J,\mathbb{C})}}. \end{aligned}$$

This chain of inequalities can be continued as follows:

$$\begin{aligned} a(\xi)[w,w] \geqslant{}& 2\xi^2\|w_1\|^2_{L_2(J,\mathbb{C})} + \xi^2\|w_2\|^2_{L_2(J,\mathbb{C})} + \|w_1'\|^2_{L_2(J,\mathbb{C})} + 2\|w_2'\|^2_{L_2(J,\mathbb{C})} \\ &- (1+\delta)\xi\|w_1'\|_{L_2(J,\mathbb{C})}\|w_2\|_{L_2(J,\mathbb{C})} - (1+\delta^{-1})\xi\|w_1\|_{L_2(J,\mathbb{C})}\|w_2'\|_{L_2(J,\mathbb{C})} \\ \geqslant{}& \xi^2\left(2 - \frac{(1+\delta^{-1})^2}{8}\right)\|w_1\|^2_{L_2(J,\mathbb{C})} + \xi^2\left(1 - \frac{(1+\delta)^2}{4}\right)\|w_2\|^2_{L_2(J,\mathbb{C})} \end{aligned}$$

for all $\delta > 0$. In particular, for $\delta = \sqrt{2} - 1$ we conclude that

$$a(\xi)[w, w] \geqslant \frac{23 - 16\sqrt{2}}{4(\sqrt{2} - 1)^2}\xi^2 \|w_1\|^2_{L_2(J,\mathbb{C})} + \frac{1}{2}\xi^2\|w_2\|^2_{L_2(J,\mathbb{C})} \geqslant \frac{1}{2}\xi^2\|w\|^2_{L_2(J,\mathbb{C}^2)} . \tag{A.4}$$

It remains to combine (A.3), (A.4) and to apply this to

$$\lambda_1(\xi) = \min_{w \in P_4 H^1(J,\mathbb{C}^2)} \|w\|^{-2}_{L_2(J,\mathbb{C}^2)} a(\xi)[w, w].$$

A.2 *Proof of Lemma* 3.2

First note that by (A.1) and (A.2) we have

$$a(\xi)[w, w] \geqslant 2\min\{\xi^2, 1\}\|w\|^2_{L_2(J,\mathbb{C}^2)} \quad \text{and} \quad a(0)[w, w] \geqslant 2\|w\|^2_{L_2(J,\mathbb{C}^2)}$$

for all $w \in P_4 H^1(J, \mathbb{C}^2)$, $w \not\equiv 0$. Moreover, if

$$w(x_2) = \begin{pmatrix} -\frac{\sqrt{7}}{8}\cos\left(\frac{3x}{4}\right) + \frac{9\sqrt{7}}{56}\cos\left(\frac{5x}{4}\right) + \frac{4\sqrt{7}}{105\pi}\sqrt{2 + \sqrt{2}} \\ \frac{3}{8}\sin\left(\frac{3x}{4}\right) + \frac{9}{40}\sin\left(\frac{5x}{4}\right) \end{pmatrix},$$

then $w \in P_4 H^1(J, \mathbb{C}^2)$ and

$$\frac{a(4^{-1}\sqrt{7})[w, w]}{\|w\|^2_{L_2(J,\mathbb{C}^2)}} = \frac{21468\sqrt{2}\pi - 30330\pi^2 + 1120 + 560\sqrt{2}}{9384\sqrt{2}\pi - 15165\pi^2 + 1280 + 640\sqrt{2}} < 1{\cdot}91. \tag{A.5}$$

Hence

$$\lambda_u = 8\sqrt{3} - 12 \leqslant \Lambda < 1{\cdot}91 = \lambda_o, \tag{A.6}$$

and the non-constant analytic function $\lambda_1(\xi)$ achieves its global minima Λ at a finite number of points ξ_n such that $0 < \xi_n^2 < \lambda_o/2$. In a neighbourhood ε_n of these points ξ_n we have $\lambda_1(\xi) < 2$ and hence $0 \leqslant \gamma_1 < 1$, $\beta_1 > 0$. Now it is easy to see that the equation (3.9) has no solution with $\gamma_1 = 0$ or $\beta_1 \leqslant 1$ as $\xi \in \varepsilon_n$. Hence $1 - \lambda_1(\xi)/2 < \gamma_1^2(\xi) < \lambda_1(\xi)/2$ and

$$\gamma_1^2(\xi)\Upsilon(\beta_1(\xi)) + \xi^2\Upsilon(\gamma_1(\xi)) = 0 \quad \text{for } \xi \in \varepsilon_n, \tag{A.7}$$

where $\Upsilon(x) = x^{-1}\tan(\pi x/2)$. Differentiating (A.7) with respect to $z = \xi^2$ and applying (A.7), (3.10), we claim that

$$\tilde{\Upsilon}(\gamma) := ((2\gamma^2 - \Lambda)\Upsilon(\gamma) + 8\pi^{-1})((2\gamma^2 + \Lambda)\Upsilon(\gamma) - 4\pi^{-1}) - 2\Lambda + 32\pi^{-2} = 0 \tag{A.8}$$

at the points $\gamma = \sqrt{\Lambda/2 - \xi_n^2}$. Note that $2\Lambda - 32\pi^{-2} > 0$. Consider (A.8) as an equation in $\gamma \in (\sqrt{1 - \lambda_o/2}, \sqrt{\lambda_o/2})$. The second factor on the left-hand side is positive and increasing in γ. Using (A.6) it is not difficult to see that the first factor is increasing in γ as well, hence the product is increasing where it is non-negative, and the equation (A.8) has not more than one solution $\gamma \in (\sqrt{1 - \lambda_o/2}, \sqrt{\lambda_o/2})$. We conclude that $\lambda_1(\xi)$ achieves its minimal value at exactly two points $\xi = \pm\xi_0 \neq 0$.

Next we sharpen the estimate on $\gamma = \sqrt{\Lambda/2 - \xi_0^2}$. By (A.6) we see that

$$\tilde{\Upsilon}(\tilde{\gamma}) \leqslant ((2\tilde{\gamma}^2 - \lambda_u)\Upsilon(\tilde{\gamma}) + 8\pi^{-1})((2\tilde{\gamma}^2 + \lambda_o)\Upsilon(\tilde{\gamma}) - 4\pi^{-1}) - 2\lambda_u + 32\pi^{-2} \tag{A.9}$$

if $(2\tilde{\gamma}^2 - \lambda_u)\Upsilon(\tilde{\gamma}) + 8\pi^{-1} \geqslant 0$ and

$$\tilde{\Upsilon}(\tilde{\gamma}) \geqslant ((2\tilde{\gamma}^2 - \lambda_o)\Upsilon(\tilde{\gamma}) + 8\pi^{-1})((2\tilde{\gamma}^2 + \lambda_u)\Upsilon(\tilde{\gamma}) - 4\pi^{-1}) - 2\lambda_o + 32\pi^{-2} \qquad \text{(A.10)}$$

if $(2\tilde{\gamma}^2 - \lambda_o)\Upsilon(\tilde{\gamma}) + 8\pi^{-1} \geqslant 0$. By the same monotonicity argument as above the functions on the right-hand side of (A.9), (A.10) have only one root $\tilde{\gamma}_u$, $\tilde{\gamma}_o$, respectively, within $(\sqrt{1-\lambda_o/2}, \sqrt{\lambda_o/2})$. But then $\tilde{\gamma_u} \leqslant \gamma \leqslant \tilde{\gamma_o}$. Evaluating (A.9), (A.10) at the points $\tilde{\gamma} = \gamma_u = 11/16$ and $\tilde{\gamma} = \gamma_o = 25/32$, where $\Upsilon(\tilde{\gamma})$ can be calculated explicitly, one claims $\gamma_u < \tilde{\gamma_u} \leqslant \gamma \leqslant \tilde{\gamma_o} < \gamma_o$.

Differentiating (A.7) twice with respect to $z = \xi^2$, we see that $d^2\lambda_1(\xi)/d\xi^2|_{\xi=\pm\xi_0} = 0$ would imply

$$\begin{aligned}
0 = &\left(6\lambda^2\gamma^2 - \frac{3}{4}\pi^2\lambda^4 + 16\gamma^6 + 44\lambda\gamma^4 - 28\gamma^8\pi^2 - 6\gamma^6\pi^2\lambda + 10\pi^2\lambda^2\gamma^4 + \frac{3}{2}\pi^2\lambda^3\gamma^2\right)\sin\left(\frac{\pi\gamma}{2}\right) \\
&+ \left(16\gamma^6 + 6\lambda^2\gamma^2 - 2\pi^2\lambda^2\gamma^4 + \frac{1}{4}\pi^2\lambda^4 + 2\gamma^6\pi^2\lambda - \frac{1}{2}\pi^2\lambda^3\gamma^2 + 44\lambda\gamma^4 + 4\gamma^8\pi^2\right)\sin\left(\frac{3\pi\gamma}{2}\right) \\
&+ (10\pi\gamma^5\lambda + 4\pi\gamma^7 + 3\pi\lambda^2\gamma^3 - 2\pi\lambda^3\gamma)\cos\left(\frac{3\pi\gamma}{2}\right) \\
&+ (-15\pi\lambda^2\gamma^3 - 66\pi\gamma^5\lambda - 20\pi\gamma^7 + 2\pi\lambda^3\gamma)\cos\left(\frac{\pi\gamma}{2}\right)
\end{aligned}$$

for $\lambda = \Lambda$ and $\gamma = \sqrt{\Lambda/2 - \xi_0^2}$. However, the function on the right-hand side is negative for all pairs $(\gamma, \lambda) \in (\gamma_u, \gamma_o) \times (\lambda_u, \lambda_o)$ and thus $d^2\lambda_1(\xi)/d\xi^2|_{\xi=\pm\xi_0} \neq 0$. A respective numerical calculation can be made rigorous by estimating the sine and cosine by appropriate finite Taylor series, inserting these estimates into the right-hand side of the above equation, estimating the derivatives of the resulting polynomial and evaluating the polynomial on a sufficiently dense finite set of test points.