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Traveling Fronts in Monostable Equations with Nonlocal Delayed Effects

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Abstract In this paper, we study the existence, uniqueness and stability of traveling wave fronts in the following nonlocal reaction–diffusion equation with delay

$$\frac{\partial u(x,t)}{\partial t} = d\Delta u(x,t) + f\left(u(x,t), \int_{-\infty}^{\infty} h(x-y)\,u(y,t-\tau)\,dy\right).$$

Under the monostable assumption, we show that there exists a minimal wave speed $c^* > 0$, such that the equation has no traveling wave front for $0 < c < c^*$ and a traveling wave front for each $c \geq c^*$. Furthermore, we show that for $c > c^*$, such a traveling wave front is unique up to translation and is globally asymptotically stable. When applied to some population models, these results cover, complement and/or improve a number of existing ones. In particular, our results show that (i) if $\partial_2 f(0,0) > 0$, then the delay can slow the spreading speed of the wave fronts and the nonlocality can increase the spreading speed; and (ii) if $\partial_2 f(0,0) = 0$, then the delay and nonlocality do not affect the spreading speed.

Keywords Existence · Uniqueness · Asymptotic stability · Traveling wave front · Nonlocal reaction–diffusion equation · Delay · Monostable equation

Mathematics Subject Classification (2000) 35K57 · 35R10 · 35B40 · 34K30 · 58D25

1 Introduction

Traveling wave solutions for reaction–diffusion equations with local and nonlocal delays have been extensively studied in the last two decades ([22]). For reaction–diffusion equations with

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time delay (local delay), Schaaf [39] considered two scalar reaction–diffusion equations with a discrete delay for both Huxley nonlinearity and Fisher nonlinearity. Wu and Zou [47] considered more general reaction–diffusion systems with finite delay using the classical monotone iteration technique coupled with the sub- and supersolutions method. Following [47], Ma [28] employed the Schauder's fixed point theorem to an operator used in [47] in a properly chosen subset in the Banach space $C(\mathbb{R}, \mathbb{R}^n)$ equipped with the so-called exponential decay norm, and showed the existence of traveling wave fronts for a class of delayed systems with quasimonotonicity reaction terms. However, the reaction term in a model system arising from a practical problem may satisfy neither the quasimonotonicity condition nor the nonquasimonotonicity condition considered in [47]. A typical example is the Lotka–Volterra competition system with delays. Recently, Li et al. [23] developed a new cross iteration scheme, which is different from that defined in [28,47]. By using such a scheme to the Lotka–Volterra competition system with delays, we constructed a subset in a suitable Banach space equipped with the exponential decay norm and reduced the existence of traveling wave solutions to the existence of an admissible pair of sub- and supersolutions which are easy to construct in practice.

For the stability and uniqueness of traveling wave solutions in reaction–diffusion equations with a discrete delay, we should mention the work of Smith and Zhao [40]. They first established the existence and comparison theorem of solutions in a quasimonotone reaction–diffusion *bistable* equation with a discrete delay by appealing to the theory of abstract functional differential equations [33], and the global asymptotic stability, Liapunov stability and uniqueness of traveling wave solutions are proved by the elementary sub- and supersolutions comparison and the squeezing technique developed by Chen [9] (see also [5,11,12,15,19] for this technique). In fact, the earlier results concerning with this topic are due to Schaaf [39]. It is worth mentioning that Ma and Zou [32] generalized the method of Chen and Guo [11,12] to a class of discrete reaction–diffusion monostable equation with delay and obtained the existence, uniqueness and stability of traveling wave fronts.

A nonlocal (or spatio-temporal) delay is a term that involves a weighted average over the whole infinite spatial domain and all previous times. Britton [6,7] made the first comprehensive attempt to study the periodic traveling wave solutions in reaction–diffusion equations with such nonlocal delays. Since then, quite a few methods have been developed to prove the existence of traveling wave solutions in these types of equations. The first is to use the perturbation theory of ordinary differential equations coupled with the Fredholm alterative, see Al-Omari and Gourley [3] for an age-structured reaction–diffusion model with nonlocal delay and Gourley [20] for a nonlocal Fisher equation. The second is using the geometric singular perturbation theory of Fenichel [18], see Ai [1], Ashwin et al. [4], Gourley and Ruan [21], Ruan and Xiao [36], etc. The third is recently developed by Wang et al. [43]. The main idea is to use a monotone iteration and the non-standard ordering in a profile set for the corresponding wave system to develop a new monotone iteration scheme, then apply it to establish the existence of solutions for a second order system of functional differential equations if the nonlinear term satisfies certain monotone conditions. Applying this approach, Li et al. [24] and Li and Wang [25] studied the existence of traveling wave fronts for the diffusive Nicholson's Blowflies equation with nonlocal delay and the diffusive and cooperative Lotka–Volterra system with nonlocal delay, respectively. Faria et al. [16] developed a new approach to obtain the existence of traveling wave solutions for delayed monostable reaction–diffusion equations with global response, which is based on a combination of some nonlinear perturbation analysis, the Fredholm theory and the Banach fixed point theorem, so that the existence of traveling wave solutions is dependent of the existence of heteroclinic connecting orbits of a corresponding functional differential equation. Recently, using a similar idea to that of Faria et al. [16], Ou and Wu [34] showed the persistence of traveling wave solutions for reaction–diffusions with nonlocal and delayed nonlinearities when the time lag is relatively small, which can be applied to not only the monostable case but also the bistable case.

In addition to the existence of traveling wave solutions, the spreading speed has been widely studied by many researchers, see [27,42,48,50] and the references therein. Thieme and Zhao [42] generalized the spreading speeds and monotone traveling waves to a large class of scalar nonlinear integral equation so that the results can be applied to some nonlocal reaction–diffusion population models with delay by recasting the reaction–diffusion equation into a integral equation, see also Xu and Zhao [48]. They showed the minimal wave speed coincides with the spreading speed and also discussed the uniqueness of traveling wave solutions. Recently, the theory of spreading speeds and monotone traveling waves for monotone semiflows has been developed by Liang and Zhao [27] in such a way that it can be applied to various evolution equations admitting the comparison principle, see also [50]. For more results and details about traveling wave solutions in reaction–diffusion equations with both local and nonlocal delays, we refer to the survey by Gourley and Wu [22]. We also refer to Ruan [35] for some results in nonlocal epidemiological models.

Due to its significant nature in biology, there are particular interests in studying the following structured single species population model of the form

$$\frac{\partial u(x,t)}{\partial t} = d\Delta u(x,t) - a_0 u(x,t) + \varepsilon \int_{-\infty}^{\infty} h(x-y) b\left(u\left(y, t-\tau\right)\right) dy. \tag{1.1}$$

Under the bistable assumptions, for example, $b(u) = pu^2 e^{-\alpha u}$, Ma and Wu [30] proved the existence, uniqueness and asymptotic stability of traveling wave solutions with a unique velocity of (1.1) by using a slight modification of the method developed by Chen [9] where the so-called squeezing technique and a new method were developed to establish the asymptotic stability and existence of traveling wave solutions for a class of bistable evolution equations satisfying the comparison principle, respectively. Ma and Zou [31] also used this method to prove the stability of traveling wave solutions for a delayed lattice differential equation with global interaction which was proposed by Weng et al. [44]. Under the monostable case, in particular $b(u) = pue^{-\alpha u}$, So et al. [41] applied the method of Wu and Zou [47] to (1.1) with $1 < \varepsilon p/a_0 \le e$ and obtained the existence of monotone traveling wave solutions. Furthermore, under the condition $e < \varepsilon p/a_0 \le e^2$, Faria et al. [16,17] and Ma [29] considered (1.1) by using different techniques and established the existence of traveling wave solutions for large c ([16]) and all $\tau \ge 0$ ([29]) and non-monotone traveling wave solution for large c ([17]), respectively.

Although there are some results on Eq. 1.1, the existence, uniqueness and stability of traveling wave solutions of many biological and epidemiological models with nonlocal delays, such as the age-structured population model proposed by Al-Omari and Gourley [3]

$$\frac{\partial u_m}{\partial t} = d_m \frac{\partial^2 u_m}{\partial x^2} + a_0 e^{-\gamma\tau} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi d_i \tau}} e^{\frac{-(x-y)^2}{4 d_i \tau}} u_m\left(y, t-\tau\right) dy - b_0 u_m^2, \tag{1.2}$$

the vector disease model proposed by Ruan and Xiao [36]

$$\frac{\partial u}{\partial t} = d\Delta u - a_0 u + b_0 \left[1-u\right] \int_{-\infty}^{\infty} h\left(x-y\right) u\left(y, t-\tau\right) dy, \tag{1.3}$$

and the Nicholson's blowflies equation with nonlocal delay ([24])

$$\frac{\partial u}{\partial t} = d\Delta u - \tau u + b_0\tau \int_{-\infty}^{\infty} g(x-y)u(x,t-\tau)dy \exp\left[-\int_{-\infty}^{\infty} g(x-y)u(x,t-\tau)dy\right], \tag{1.4}$$

are not well-understood. In fact, by using the results of Diekmann and Kaper [14] (see also [42,Theorem 3.2 and Theorem 4.3]), Thieme and Zhao [42] showed the uniqueness of monotone traveling wave solutions of (1.1) and (1.2), but it is invalid for non-monotone waves. For the other results on the uniqueness of traveling wave solutions, we refer to [8,10]. Also, there are some results considering the effect of the time delay on the spreading speed (minimal wave speed), see [39,51], but there are few results for the influence of the nonlocality on the spreading speed which only considered by Li et al. [24] for Eq. 1.4.

In this paper we are concerned with the following nonlocal reaction–diffusion equation with delay

$$\frac{\partial u(x,t)}{\partial t} = d\Delta u(x,t) + f\left(u(x,t), \int_{-\infty}^{\infty} h(x-y)\,u(y,t-\tau)\,dy\right), \quad x\in\mathbb{R}, t>0, \tag{1.5}$$

where $d > 0$ and $\tau \geq 0$ are constants. The functions $f(u,v)$ and $h(x)$ satisfy the following assumptions:

- **(A1)** $f \in C^2\left([0,K]^2, \mathbb{R}\right)$, $f(0,0) = f(K,K) = 0$, $f(u,u) > 0$ for $u \in (0,K)$, and $\partial_2 f(u,v) \geq 0$ for $(u,v) \in [0,K]^2$, where K is a positive constant.
- **(A2)** $\partial_1 f(0,0)\,u + \partial_2 f(0,0)\,v \geq f(u,v)$ for any $(u,v) \in [0,K]^2$ and $\partial_1 f(K,K) + \partial_2 f(K,K) < 0$.
- **(A3)** For every $\delta \in (0,1)$, there exist $a = a(\delta) > 0$, $\alpha = \alpha(\delta) \geq 0$ and $\beta = \beta(\delta) \geq 0$ with $\alpha + \beta > 0$ such that for any $\theta \in (0,\delta]$ and $(u,v) \in [0,K]^2$,

$$(1-\theta)\,f(u,v) - f((1-\theta)u, (1-\theta)v) \leq -a\theta u^{\alpha} v^{\beta}.$$

- **(G1)** $h(x)$ is nonnegative and integrable, and satisfies

$$\int_{-\infty}^{\infty} h(x)\,dx = 1 \text{ and } h(x) = h(-x), x \in \mathbb{R}.$$

- **(G2)** One of the following is satisfied:
  - (i) For any $\lambda > 0$, $\int_0^{\infty} h(x)\,e^{\lambda x}dx < \infty$.
  - (ii) There exists $\lambda_0 > 0$ such that $\int_0^{\infty} h(x)\,e^{\lambda x}dx < \infty$ for any $\lambda < \lambda_0$ and $\lim_{\lambda\to\lambda_0-0}\int_0^{\infty} h(x)\,e^{\lambda x}dx = +\infty$.

We would like to point out that the assumptions (A1) and (A2) are standard and the assumption (A3) is not a more restrictive condition. Indeed, the assumption (A3) is a convex condition and in general, monostable nonlinearities satisfy it. For example, the nonlinearities $f(u) = u(1-u)$ in [9] and $f(u,v) = -du + b(v)$ in [32] with many well-used birth functions $b(\cdot)$ (see [26] and the references therein) satisfy (A3), the nonlinearities of Eqs. (1.2–1.4) also satisfy (A3), see also Sect. 5 for applications. However, it could be invalid for bistable nonlinearities, for example, $f(u,v) = -du + pv^2e^{-\alpha v}$ with suitable $p > 0$ and $\alpha > 0$, see [31]. From (A1) we can see that (1.5) has two equilibria 0 and K. Furthermore, condition (A2) together with (A1) implies that $\partial_1 f(0,0) + \partial_2 f(0,0) \geq \frac{2}{K} f\left(\frac{K}{2}, \frac{K}{2}\right) > 0$, hence 0 is unstable and K is stable. Throughout this paper, *a traveling wave solution* of (1.5) always refers to as a pair (U, c), where $U = U(\xi)$ is a function on $\mathbb{R}$ and $c > 0$ is a constant, such that $u(x,t) := U(x+ct) = U(\xi)$ is a solution of (1.5) and

$$\lim_{\xi\to\infty} U(\xi) = K, \quad \lim_{\xi\to-\infty} U(\xi) = 0. \tag{1.6}$$

We call c the *traveling wave speed* and U the *profile of the wave front*.

The purpose of the current paper is to establish the existence, nonexistence, uniqueness and stability of traveling wave solutions of (1.5), see Theorem 2.6, Theorem 4.1, Theorem 4.8 and Corollary 4.9. We then apply the results to some specific biological and epidemiological models with nonlocal delay. In particular, our results for the uniqueness of traveling wave solutions is valid for nonmonotone waves, see Corollary 4.9.

The rest of the paper is organized as follows. In Sect. 2, we use the sub- and supersolution method of Wang et al. [43] to obtain the existence of traveling wave fronts. In order to consider the uniqueness and stability of traveling wave solutions, in Sect. 3 we establish the existence and comparison principle of solutions and construct some sub- and supersolutions for the initial value problem of (1.5). In Sect. 4, by using the comparison principle and squeezing technique of Chen and Guo [11], we consider the asymptotic stability and uniqueness of traveling wave fronts, respectively. In Sect. 5 we apply our results to the age-structured model (1.2), the vector disease model (1.3) and the Nicholson's blowflies model (1.4), some new results are obtained, which conclude, improve and/or complement a number of existing ones in [3,24,36,39,42]. Finally, in Sect. 6, we consider the effect of the delay and nonlocality in (1.5) on the spreading speed (the minimal wave speed), respectively.

2 Existence of Traveling Wave Fronts

In this section we consider the existence of traveling wave fronts of (1.5). In fact, there are some known results on the existence of traveling wave fronts of (1.5), see [27,42,50]. In order to prove the stability of traveling wave fronts, we give the proof of the existence of traveling wave fronts of (1.5) by using the theory developed by Wang et al. [43].

Let $u(x,t) = U(\xi)$, $\xi = x + ct$. Then Eq. 1.5 reduces to the following equation

$$dU''(\xi) - cU'(\xi) + f(U(\xi), (h*U)(\xi)) = 0, \xi \in \mathbb{R}, \tag{2.1}$$

where $(h*U)(\xi) = \int_{-\infty}^{\infty} h(y)\, U(\xi - y - c\tau)\, dy$.

Now we consider the existence of monotone solutions of Eq. 2.1 satisfying (1.6). Let $g(x,t) = \delta(t-\tau)\, h(x)$. Then it is easy to see that $g(x,t)$ satisfies (H_0) in [43]. Furthermore, the condition $\partial_2 f(u,v) \geq 0$ for $(u,v) \in [0,K]^2$ implies that the function $f(U(\xi), (h*U)(\xi))$ satisfies the following monotonicity condition:

(H_1) There exists a positive constant γ such that

$$f(\phi_2(\xi), (h*\phi_2)(\xi)) + \gamma\phi_2(\xi) \geq f(\phi_1(\xi), (h*\phi_1)(\xi)) + \gamma\phi_1(\xi),$$

where $\phi_1, \phi_2 \in C(\mathbb{R}, \mathbb{R})$ satisfy $0 \leq \phi_1(\xi) \leq \phi_2(\xi) \leq K$ in $\xi \in \mathbb{R}$.

Let $\bar{\lambda} = +\infty$ if (G2)(i) holds and $\bar{\lambda} = \lambda_0$ if (G2)(ii) holds. For $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda < \bar{\lambda}$, define a function

$$G(\lambda) = \int_{-\infty}^{\infty} h(y) e^{-\lambda y} dy = \int_{0}^{\infty} h(y) \left(e^{\lambda y} + e^{-\lambda y}\right) dy.$$

Since $e^{-i\mathrm{Im}\lambda y}$ is bounded, $G(\lambda)$ is well defined for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda < \bar{\lambda}$. Obviously, $G(0) = 1$. Here we give a property of $G(\lambda)$ which can be proved by using Lebesgue's dominated convergence theorem and we omit its proof.

Lemma 2.1 *$G(\lambda)$ is twice differentiable in $[0, \bar{\lambda})$, $G'(\lambda) = \int_0^\infty yh(y)\left(e^{\lambda y} - e^{-\lambda y}\right) dy > 0$ and $G''(\lambda) = \int_0^\infty y^2 h(y)\left(e^{\lambda y} + e^{-\lambda y}\right) dy > 0$.*

Let $\lambda^+ = \bar{\lambda}$ if $\partial_2 f(0,0) > 0$ and $\lambda^+ = +\infty$ if $\partial_2 f(0,0) = 0$. For $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda < \lambda^+$ and $c \in \mathbb{R}$ with $c \geq 0$, we define a function

$$\Delta(\lambda, c) = d\lambda^2 - c\lambda + \partial_1 f(0,0) + \partial_2 f(0,0) e^{-\lambda c\tau} G(\lambda).$$

Note that for $\lambda \in \mathbb{R}$ with $0 < \lambda < \lambda^+$,

$$\frac{\partial^2}{\partial \lambda^2}\Delta(\lambda, c) = 2d + \partial_2 f(0,0) e^{-\lambda c\tau} \times \int_0^\infty h(y)\left(e^{\lambda y}(y - c\tau)^2 + e^{-\lambda y}(y + c\tau)^2\right) dy > 0,$$

$$\frac{\partial}{\partial c}\Delta(\lambda, c) = -\lambda - \lambda\tau\partial_2 f(0,0) e^{-\lambda c\tau} G(\lambda) < 0,$$

$$\Delta(0, c) = \partial_1 f(0,0) + \partial_2 f(0,0) > 0,$$

$$\Delta(\lambda, 0) = d\lambda^2 + \partial_1 f(0,0) + \partial_2 f(0,0) G(\lambda) > 0,$$

and $\lim_{\lambda \to \lambda^+ - 0} \Delta(\lambda, c) = +\infty$, then it is easy to see that the following result holds.

Lemma 2.2 *There exist $c^* > 0$ and $\lambda^* \in \left(0, \lambda^+\right)$ such that $\frac{\partial}{\partial \lambda}\Delta(\lambda, c^*)\big|_{\lambda=\lambda^*} = 0$ and $\Delta(\lambda^*, c^*) = 0$. Furthermore,*

- **(i)** *if $0 < c < c^*$, then $\Delta(\lambda, c) > 0$ for any $\lambda \in \left(0, \lambda^+\right)$;*
- **(ii)** *if $c > c^*$, then the equation $\Delta(\lambda, c) = 0$ has two positive real roots $\lambda_1(c)$ and $\lambda_2(c)$ with $0 < \lambda_1(c) < \lambda^* < \lambda_2(c) < \lambda^+$ such that $\lambda_1'(c) < 0$, $\lambda_2'(c) > 0$ and*

$$\Delta(\lambda, c) \begin{cases} >0 & \text{for } \lambda \in (0, \lambda_1(c)), \\ <0 & \text{for } \lambda \in (\lambda_1(c), \lambda_2(c)), \\ >0 & \text{for } \lambda \in \left(\lambda_2(c), \lambda^+\right). \end{cases}$$

Now we give definitions of the sub- and supersolutions of (2.1).

Definition 2.3 A continuous function $\varphi : \mathbb{R} \to \mathbb{R}^n$ is called *a supersolution* of (2.1) if φ' and φ'' exist almost everywhere and are essentially bounded on $\mathbb{R}$, and φ satisfies

$$-d\varphi''(\xi) + c\varphi'(\xi) - f(\varphi(\xi), (h * \varphi)(\xi)) \geq 0 \text{ a.e. on } \mathbb{R}. \tag{2.2}$$

A subsolution of (2.1) is defined in a similar way by reversing the inequality in (2.2).

Lemma 2.4 *Assume that (A1), (A2), (G1) and (G2) hold. Let c^*, $\lambda_1(c)$ and $\lambda_2(c)$ be defined as in Lemma 2.2. Let $c > c^*$ be any number. Then for every $\eta \in \left(1, \min\left\{2, \frac{\lambda_2(c)}{\lambda_1(c)}\right\}\right)$, there exists $Q(c,\eta) \geq 1$ such that for any $q \geq Q(c,\eta)$ and any $\xi^{\pm} \in \mathbb{R}$, the functions ϕ and ψ defined by*

$$\phi(\xi) = \min\left\{K, e^{\lambda_1(c)(\xi+\xi^+)} + qe^{\eta\lambda_1(c)(\xi+\xi^+)}\right\}, \xi \in \mathbb{R} \tag{2.3}$$

and

$$\psi(\xi) = \max\left\{0, e^{\lambda_1(c)(\xi+\xi^-)} - qe^{\eta\lambda_1(c)(\xi+\xi^-)}\right\}, \xi \in \mathbb{R} \tag{2.4}$$

are a supersolution and a subsolution to (2.1), respectively. Furthermore,

$$\begin{aligned}\phi(x+ct) \geq{}& \frac{1}{\sqrt{4\pi d(t-s)}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4d(t-s)}} \phi(y+cs)\,dy \\ &+ \int_s^t \frac{1}{\sqrt{4\pi d(t-z)}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4d(t-z)}} F(\phi)(y,z)\,dydz,\ t > s \geq 0,\end{aligned} \tag{2.5}$$

$$\begin{aligned}\psi(x+ct) \leq{}& \frac{1}{\sqrt{4\pi d(t-s)}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4d(t-s)}} \psi(y+cs)\,dy \\ &+ \int_s^t \frac{1}{\sqrt{4\pi d(t-z)}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4d(t-z)}} F(\psi)(y,z)\,dydz,\ t > s \geq 0,\end{aligned} \tag{2.6}$$

where

$$F(\phi)(y,z) = f\left(\phi(y+cz), \int_{-\infty}^{\infty} h(y-\theta)\,\phi(\theta+cz-c\tau)\,d\theta\right).$$

Proof We begin by proving that $\phi(\xi)$ and $\psi(\xi)$ are a pair of super- and subsolutions of (2.1). We first consider the case $\partial_2 f(0,0) > 0$. In this case, $\eta\lambda_1(c) < \bar{\lambda}$. From (2.3), it is easy to see that there exists a $\xi^* < -\xi^+ - \frac{1}{\eta\lambda_1(c)} \ln \frac{q}{K}$ satisfying $e^{\lambda_1(c)(\xi^*+\xi^+)} + qe^{\eta\lambda_1(c)(\xi^*+\xi^+)} = K$, $\phi(\xi) = K$ for $\xi > \xi^*$ and $\phi(\xi) = e^{\lambda_1(c)(\xi+\xi^+)} + qe^{\eta\lambda_1(c)(\xi+\xi^+)}$ for $\xi \leq \xi^*$.

For $\xi > \xi^*$, $\phi''(\xi) = 0$, $\phi'(\xi) = 0$, then, by (A1), we have

$$d\phi''(\xi) - c\phi'(\xi) + f(\phi(\xi), (h*\phi)(\xi)) = f(K, (h*\phi)(\xi)) \leq f(K,K) = 0.$$

For $\xi \leq \xi^*$, note that $\Delta(\eta\lambda_1(c), c) < 0$, $\phi''(\xi) = \lambda_1^2(c)e^{\lambda_1(c)(\xi+\xi^+)} + q\eta^2\lambda_1^2(c)$ $e^{\eta\lambda_1(c)(\xi+\xi^+)}$, $\phi'(\xi) = \lambda_1(c)e^{\lambda_1(c)(\xi+\xi^+)} + q\eta\lambda_1(c)e^{\eta\lambda_1(c)(\xi+\xi^+)}$ and

$$(h*\phi)(\xi) \leq e^{\lambda_1(c)(\xi+\xi^+-c\tau)} G(\lambda_1(c)) + qe^{\eta\lambda_1(c)(\xi+\xi^+-c\tau)} G(\eta\lambda_1(c)), \tag{2.7}$$

then

$$\begin{aligned}
&d\phi''(\xi) - c\phi'(\xi) + f(\phi(\xi), (h*\phi)(\xi)) \\
&= \left(d\lambda_1^2(c) - c\lambda_1(c)\right) e^{\lambda_1(c)(\xi+\xi^+)} + q\left(d\eta^2\lambda_1^2(c) - c\eta\lambda_1(c)\right) e^{\eta\lambda_1(c)(\xi+\xi^+)} \\
&\quad + f(\phi(\xi), (h*\phi)(\xi)) \\
&= \Delta(\lambda_1(c), c)\, e^{\lambda_1(c)(\xi+\xi^+)} + q\Delta(\eta\lambda_1(c), c)\, e^{\eta\lambda_1(c)(\xi+\xi^+)} + f(\phi(\xi), (h*\phi)(\xi)) \\
&\quad - \partial_1 f(0,0)\phi(\xi) - \partial_2 f(0,0)\left(e^{\lambda_1(c)(\xi+\xi^+-c\tau)} G(\lambda_1(c)) + q e^{\eta\lambda_1(c)(\xi+\xi^+-c\tau)} G(\eta\lambda_1(c))\right) \\
&\le q\Delta(\eta\lambda_1(c), c)\, e^{\eta\lambda_1(c)(\xi+\xi^+)} + f(\phi(\xi), (h*\phi)(\xi)) \\
&\quad - \partial_1 f(0,0)\,\phi(\xi) - \partial_2 f(0,0)\,(h*\phi)(\xi) \\
&\le q\Delta(\eta\lambda_1(c), c)\, e^{\eta\lambda_1(c)(\xi+\xi^+)} \\
&\le 0,
\end{aligned}$$

where we have used the condition (A2) in the last inequality. Therefore, ϕ is a supersolution of (2.1).

Let

$$M = \max_{(u,v)\in[0,K]^2} \left\{\max\left\{|\partial_{11} f(u,v)|, |\partial_{12} f(u,v)|, |\partial_{21} f(u,v)|, |\partial_{22} f(u,v)|\right\}\right\}$$

and

$$\xi_* = -\xi^- - \frac{1}{(\eta-1)\lambda_1(c)} \ln q, \quad Q(c,\eta) = \max\left\{1, -\frac{10MG^2(\eta\lambda_1(c))}{\Delta(\eta\lambda_1(c))}\right\}.$$

If $q \ge Q(c,\eta)$, then $\xi_* \le -\xi^-$. Obviously, $\psi(\xi) = 0$ for $\xi > \xi_*$ and $\psi(\xi) = e^{\lambda_1(c)(\xi+\xi^-)} - q e^{\eta\lambda_1(c)(\xi+\xi^-)}$ for $\xi \le \xi_*$. Then for $\xi > \xi_*$, we have

$$d\psi''(\xi) - c\psi'(\xi) + f(\psi(\xi), (h*\psi)(\xi)) = f(0, (h*\psi)(\xi)) \ge f(0,0) = 0.$$

For $\xi \le \xi_*$, by the definition of ξ_*, one can see that $\xi + \xi^- \le -\frac{1}{(\eta-1)\lambda_1(c)} \ln q$. Note that $\eta \in \left(1, \min\left\{2, \frac{\lambda_2(c)}{\lambda_1(c)}\right\}\right)$, we have $e^{(2-\eta)\lambda_1(c)(\xi+\xi_-)} \le e^{\frac{\eta-2}{\eta-1}\ln q} = q^{\frac{\eta-2}{\eta-1}} \le 1$, $q e^{\lambda_1(c)(\xi+\xi_-)} \le q e^{-\frac{1}{\eta-1}\ln q} = q^{\frac{\eta-2}{\eta-1}} \le 1$, $q^2 e^{\eta\lambda_1(c)(\xi+\xi_-)} \le q^2 e^{-\frac{\eta}{\eta-1}\ln q} = q^{\frac{\eta-2}{\eta-1}} \le 1$ and

$$\begin{aligned}
&e^{\lambda_1(c)(\xi+\xi^--c\tau)} G(\lambda_1(c)) - q e^{\eta\lambda_1(c)(\xi+\xi^--c\tau)} G(\eta\lambda_1(c)) \\
&\quad \le (h*\psi)(\xi) \le e^{\lambda_1(c)(\xi+\xi^--c\tau)} G(\lambda_1(c)) + q e^{\eta\lambda_1(c)(\xi+\xi^--c\tau)} G(\eta\lambda_1(c)). \quad (2.8)
\end{aligned}$$

Then

$$
\begin{aligned}
& d\psi''(\xi) - c\psi'(\xi) + f(\psi(\xi), (h*\psi)(\xi)) \\
&\quad = \left(d\lambda_1^2(c) - c\lambda_1(c)\right) e^{\lambda_1(c)(\xi+\xi^-)} - q\left(d\eta^2\lambda_1^2(c) - c\eta\lambda_1(c)\right) e^{\eta\lambda_1(c)(\xi+\xi^-)} \\
&\qquad + f(\psi(\xi), (h*\psi)(\xi)) \\
&\quad = -q\Delta(\eta\lambda_1(c), c) e^{\eta\lambda_1(c)(\xi+\xi^-)} - \partial_1 f(0,0)\,\psi(\xi) \\
&\qquad - \partial_2 f(0,0)\left[e^{\lambda_1(c)(\xi+\xi^- - c\tau)} G(\lambda_1(c)) - q e^{\eta\lambda_1(c)(\xi+\xi^- - c\tau)} G(\eta\lambda_1(c))\right] \\
&\qquad + f(\psi(\xi), (h*\psi)(\xi)) \\
&\quad \geq -q\Delta(\eta\lambda_1(c), c) e^{\eta\lambda_1(c)(\xi+\xi^-)} - M\psi^2(\xi) \\
&\qquad - 2M\psi(\xi)\left[e^{\lambda_1(c)(\xi+\xi^- - c\tau)} G(\lambda_1(c)) + q e^{\eta\lambda_1(c)(\xi+\xi^- - c\tau)} G(\eta\lambda_1(c))\right] \\
&\qquad - M\left[e^{\lambda_1(c)(\xi+\xi^- - c\tau)} G(\lambda_1(c)) + q e^{\eta\lambda_1(c)(\xi+\xi^- - c\tau)} G(\eta\lambda_1(c))\right]^2 \\
&\quad \geq -q\Delta(\eta\lambda_1(c), c) e^{\eta\lambda_1(c)(\xi+\xi^-)} - M\left[1 + 2G(\lambda_1(c)) + G^2(\lambda_1(c))\right] e^{2\lambda_1(c)(\xi+\xi^-)} \\
&\qquad - 2MG(\lambda_1(c))\,G(\eta\lambda_1(c))\, q e^{(\eta+1)\lambda_1(c)(\xi+\xi^-)} \\
&\qquad - M\left[1 + 2G(\eta\lambda_1(c)) + G^2(\eta\lambda_1(c))\right] q^2 e^{2\eta\lambda_1(c)(\xi+\xi^-)} \\
&\quad \geq \left[-q\Delta(\eta\lambda_1(c), c) - 2M\left(1 + G(\lambda_1(c)) + G(\eta\lambda_1(c)) + 2G^2(\eta\lambda_1(c))\right)\right] e^{\eta\lambda_1(c)(\xi+\xi^-)} \\
&\quad \geq \left[-q\Delta(\eta\lambda_1(c), c) - 10MG^2(\eta\lambda_1(c))\right] e^{\eta\lambda_1(c)(\xi+\xi^-)} \\
&\quad \geq 0.
\end{aligned}
$$

Thus, ψ is a subsolution of (2.1).

For the case $\partial_2 f(0,0) = 0$, the proof is similar. In this case, we do not need the estimates (2.7) and (2.8).

We now prove that (2.5) and (2.6) hold. For the supersolution ϕ, let

$$
H(x+ct) = d\phi''(x+ct) - c\phi'(x+ct) + F(\phi)(x,t),
$$

where a prime indicates differentiation with respect to $\xi = x + ct$. Then $H(x+ct) \leq 0$. Again let

$$
\begin{aligned}
g(x,t,z) &= \frac{1}{\sqrt{4\pi d(t-z)}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4d(t-z)}} \phi(y+cz)\,dy \\
&= \frac{1}{\sqrt{4\pi d(t-z)}} \int_{-\infty}^{\xi^*-cz} e^{\frac{-(x-y)^2}{4d(t-z)}} \phi(y+cz)\,dy \\
&\quad + \frac{K}{\sqrt{4\pi d(t-z)}} \int_{\xi^*-cz}^{\infty} e^{\frac{-(x-y)^2}{4d(t-z)}}\,dy,
\end{aligned}
$$

$t > z \geq 0$. Then

$$\frac{\partial}{\partial z} g(x,t,z) = \frac{1}{2(t-z)\sqrt{4\pi d(t-z)}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4d(t-z)}} \phi(y+cz)\,dy$$
$$+\frac{1}{\sqrt{4\pi d(t-z)}} \int_{-\infty}^{\infty} \frac{-(x-y)^2}{4d(t-z)^2} e^{\frac{-(x-y)^2}{4d(t-z)}} \phi(y+cz)\,dy$$
$$+\frac{1}{\sqrt{4\pi d(t-z)}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4d(t-z)}} d\phi''(y+cz)\,dy$$
$$+\frac{1}{\sqrt{4\pi d(t-z)}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4d(t-z)}} [F(\phi)(y,z) - H(y+cz)]\,dy.$$

Let $\phi'_-(\xi^*) = \lim_{\xi\to\xi^*-0}\phi'(\xi)$ and $\psi'_-(\xi_*) = \lim_{\xi\to\xi_*-0}\psi'(\xi)$. From (2.3) and (2.4), we then know that $\phi'_-(\xi^*)$ and $\psi'_-(\xi_*)$ exist, $\phi'_-(\xi^*) \geq 0$, $\psi'_-(\xi_*) \leq 0$. Since

$$\frac{d}{\sqrt{4\pi d(t-z)}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4d(t-z)}} \phi''(y+cz)\,dy$$
$$= \frac{d}{\sqrt{4\pi d(t-z)}} \int_{-\infty}^{\xi^*-cz} e^{\frac{-(x-y)^2}{4d(t-z)}} \phi''(y+cz)\,dy$$
$$= \frac{d}{\sqrt{4\pi d(t-z)}} \phi'_-\left(\xi^*\right) e^{\frac{-(x-\xi^*+cz)^2}{4d(t-z)}} - \frac{d}{\sqrt{4\pi d(t-z)}} \frac{2(x-y)}{4d(t-z)} e^{\frac{-(x-y)^2}{4d(t-z)}} \phi(y+cz)\Big|_{-\infty}^{\xi^*-cz}$$
$$+\frac{d}{\sqrt{4\pi d(t-z)}} \int_{-\infty}^{\xi^*-cz} \frac{-2}{4d(t-z)} e^{\frac{-(x-y)^2}{4d(t-z)}} \phi(y+cz)\,dy$$
$$+\frac{d}{\sqrt{4\pi d(t-z)}} \int_{-\infty}^{\xi^*-cz} \frac{(x-y)^2}{4d^2(t-z)^2} e^{\frac{-(x-y)^2}{4d(t-z)}} \phi(y+cz)\,dy,$$

and

$$\frac{K}{\sqrt{4\pi d(t-z)}} \int_{\xi^*-cz}^{\infty} \frac{-(x-y)^2}{4d(t-z)^2} e^{\frac{-(x-y)^2}{4d(t-z)}}\,dy$$
$$= \frac{K(x-\xi^*+cz)}{2(t-z)\sqrt{4\pi d(t-z)}} e^{\frac{-(x-\xi^*+cz)^2}{4d(t-z)}} - \frac{K}{2(t-z)\sqrt{4\pi d(t-z)}} \int_{\xi^*-cz}^{\infty} e^{\frac{-(x-y)^2}{4d(t-z)}}\,dy,$$

it follows that

$$
\begin{aligned}
\frac{\partial}{\partial z} g(x,t,z) &= \frac{d}{\sqrt{4\pi d(t-z)}}\phi'_-\left(\xi^*\right)e^{\frac{-(x-\xi^*+cz)^2}{4d(t-z)}} - \frac{K(x-\xi^*+cz)}{2(t-z)\sqrt{4\pi d(t-z)}}e^{\frac{-(x-\xi^*+cr)^2}{4d(t-z)}} \\
&\quad + \frac{K}{2(t-z)\sqrt{4\pi d(t-z)}}\int_{\xi^*-cz}^{\infty} e^{\frac{-(x-y)^2}{4d(t-z)}}dy \\
&\quad + \frac{K}{\sqrt{4\pi d(t-z)}}\int_{\xi^*-cz}^{\infty} \frac{-(x-y)^2}{4d(t-z)^2}e^{\frac{-(x-y)^2}{4d(t-z)}}dy \\
&\quad + \frac{1}{\sqrt{4\pi d(t-z)}}\int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4d(t-z)}}\left[F(\phi)(y,z)-H(y+cz)\right]dy \\
&= \frac{d}{\sqrt{4\pi d(t-z)}}\phi'_-\left(\xi^*\right)e^{\frac{-(x-\xi^*+cz)^2}{4d(t-z)}} \\
&\quad + \frac{1}{\sqrt{4\pi d(t-z)}}\int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4d(t-z)}}\left[F(\phi)(y,z)-H(y+cz)\right]dy.
\end{aligned}
$$

Since $\frac{\partial}{\partial z}g(x,t,z)$ is continuous in $z \in [0,t)$, $\frac{d}{\sqrt{4\pi d(t-z)}}\phi'_-(\xi^*)\exp\left\{\frac{-(x-\xi^*+cz)^2}{4d(t-z)}\right\}$ is integrable on $z \in [0,t)$, and $\lim_{z\to t-0}\frac{1}{\sqrt{4\pi d(t-z)}}\int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4d(t-z)}}\phi(y+cz)\,dy = \phi(x+ct)$, we have for $0 \le s < t$ that

$$
\begin{aligned}
\phi(x+ct) &= \lim_{\eta\to 0+0} g(x,t,t-\eta) \\
&= \lim_{\eta\to 0+0}\int_s^{t-\eta}\frac{\partial}{\partial z}g(x,t,z)\,dz + g(x,t,s) \\
&= \frac{1}{\sqrt{4\pi d(t-s)}}\int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4d(t-s)}}\phi(y+cs)\,dy \\
&\quad + \int_s^t \frac{d}{\sqrt{4\pi d(t-z)}}\phi'_-\left(\xi^*\right)e^{\frac{-(x-\xi^*+cr)^2}{4d(t-z)}}dz \\
&\quad + \int_s^t \frac{1}{\sqrt{4\pi d(t-z)}}\int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4d(t-z)}}\left[F(\phi)(y,z)-H(y+cz)\right]dydz.
\end{aligned}
$$

In view of $\phi'_-(\xi^*) \ge 0$, we can see that (2.5) holds.

In a similar manner, we can show that (2.6) holds together with $\psi'_-(\xi_*) \le 0$. This completes the proof. □

Remark 2.5 The supersolution and subsolution ϕ and ψ in Lemma 2.4 were firstly used by Chen and Guo [11] and Ma and Zou [32]. The inequalities (2.5) and (2.6) imply that $\phi(x+ct)$ and $\psi(x+ct)$ are a supersolution and a subsolution of (1.5), respectively, see Definition 3.1.

Theorem 2.6 *Assume that* (*A1*), (*A2*), (*G1*) *and* (*G2*) *hold. Let* c^* *and* $\lambda_1(c)$ *be defined as in Lemma 2.2. Then for each* $c \geq c^*$, *equation* (1.5) *has a traveling wave front* $u(x,t) = U(x+ct)$ *being increasing and satisfying* (1.6). *Moreover, if* $c > c^*$, *then*

$$\lim_{\xi\to-\infty} U(\xi)e^{-\lambda_1(c)\xi} = 1, \quad \lim_{\xi\to-\infty} U'(\xi)e^{-\lambda_1(c)\xi} = \lambda_1(c). \tag{2.9}$$

Proof We first consider the case $c > c^*$. By Lemma 2.4, the functions $\phi(\xi)$ and $\psi(\xi)$ defined by (2.3) and (2.4) are a pair of super- and subsolutions of (2.1) and satisfy $\psi(\xi) \leq \phi(\xi)$ on $\mathbb{R}$. Then Theorem 4.8(i) of [43] implies that (1.5) has a traveling wave front $U(\xi)$ being increasing and satisfying (1.6) and $\psi(\xi) \leq U(\xi) \leq \phi(\xi)$. Furthermore, it is easy to prove that $\lim_{\xi\to-\infty} U(\xi)e^{-\lambda_1(c)\xi} = 1$.

We now prove that $\lim_{\xi\to-\infty} e^{-\lambda_1(c)\xi}U'(\xi) = \lambda_1(c)$. From Lebesgue's dominated convergence theorem, we know that

$$\begin{aligned}\lim_{\xi\to-\infty} e^{-\lambda_1(c)\xi}(h*U)(\xi) &= \int_{-\infty}^{\infty} h(y)\, e^{-\lambda_1(c)(y+c\tau)} \\ &\qquad \left[\lim_{\xi\to-\infty} e^{-\lambda_1(c)(\xi-y-c\tau)}U(\xi-y-c\tau)\right]dy \\ &= \int_{-\infty}^{\infty} h(y)\, e^{-\lambda_1(c)(y+c\tau)}dy = e^{-\lambda_1(c)c\tau}G(\lambda_1(c)).\end{aligned} \tag{2.10}$$

Let $V(\xi) = (h*U)(\xi)$. Then

$$\begin{aligned}&\lim_{\xi\to-\infty} e^{-\lambda_1(c)\xi} f(U(\xi), V(\xi)) \\ &= \lim_{\xi\to-\infty} e^{-\lambda_1(c)\xi}\partial_1 f(0,0)\, U(\xi) + \lim_{\xi\to-\infty} e^{-\lambda_1(c)\xi}\partial_2 f(0,0)\, V(\xi) \\ &\quad + \lim_{\xi\to-\infty} e^{-\lambda_1(c)\xi} o\left(\sqrt{U^2(\xi)+V^2(\xi)}\right) \\ &= \partial_1 f(0,0) + \partial_2 f(0,0)\, e^{-\lambda_1(c)c\tau}G(\lambda_1(c)).\end{aligned}$$

Using $\lim_{\xi\to-\infty} U'(\xi) = 0$ and integrating both sides of (2.1) from $-\infty$ to ξ, we have $dU'(\xi) = cU(\xi) - \int_{-\infty}^{\xi} f(U(t), (h*U)(\xi))\,dt$. Thus,

$$\begin{aligned}\lim_{\xi\to-\infty} e^{-\lambda_1(c)\xi}U'(\xi) &= \frac{c}{d} - \frac{1}{d}\lim_{\xi\to-\infty} e^{-\lambda_1(c)\xi}\int_{-\infty}^{\xi} f(U(t), V(t))\,dt \\ &= \frac{c}{d} - \lim_{\xi\to-\infty}\frac{e^{-\lambda_1(c)\xi} f(U(\xi), V(\xi))}{d\lambda_1(c)} \\ &= \frac{c\lambda_1(c) - \partial_1 f(0,0) - \partial_2 f(0,0)\, e^{-\lambda_1(c)c\tau}G(\lambda_1(c))}{d\lambda_1(c)} \\ &= \lambda_1(c).\end{aligned}$$

The remainder is to consider the existence of traveling wave fronts when $c = c^*$. In fact, it could be obtained by a limiting argument similar to that of ([49], Theorem 3.1). We omit the details. The proof is complete. □

3 The Initial Value Problem

In order to study the uniqueness and asymptotic stability of the traveling waves, we need to consider the following initial value problem

$$\begin{cases} \frac{\partial u}{\partial t} = d\Delta u + f\left(u(x,t), \int\limits_{-\infty}^{\infty} h(x-y)\, u(y,t-\tau)\, dy\right) \\ u(x,s) = \varphi(x,s), \end{cases} \tag{3.1}$$

where $d > 0, \tau \geq 0, \ x \in \mathbb{R}, t > 0, s \in [-\tau, 0]$.

Let $X = BUC(\mathbb{R}, \mathbb{R})$ be the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}$ into $\mathbb{R}$ with the supremum norm $|\cdot|_X$. Let $X^+ = \{\varphi \in X; \varphi(x) \geq 0, x \in \mathbb{R}\}$. It is easy to see that X^+ is a closed cone of X and X is a Banach lattice under the partial ordering induced by X^+. By ([13], Theorem 1.5), it follows that the X-realization $d\Delta_X$ of $d\Delta$ generates a strongly continuous analytic semigroup $T(t)$ on X and $T(t)X^+ \subset X^+, t \geq 0$. Moreover, we have for $x \in \mathbb{R}, t > 0, \varphi(\cdot) \in X$ that

$$T(t)\varphi(x) = \frac{1}{\sqrt{4\pi dt}} \int\limits_{-\infty}^{\infty} \exp\left(-\frac{(x-y)^2}{4dt}\right)\varphi(y)\, dy. \tag{3.2}$$

Let $C = C([-\tau, 0], X)$ be the Banach space of continuous functions from $[-\tau, 0]$ into X with the supremum norm $\|\cdot\|$ and $C^+ = \left\{\varphi \in C; \varphi(s) \in X^+, s \in [-\tau, 0]\right\}$. Then C^+ is a closed (positive) cone of C. As usual, we identify an element $\varphi \in C$ as a function from $\mathbb{R} \times [-\tau, 0]$ into $\mathbb{R}$ defined by $\varphi(x,s) = \varphi(s)(x)$. For any continuous function $w : [-\tau, b) \to X, b > 0$, we define $w_t \in C, t \in [0, b)$, by $w_t(s) = w(t+s), s \in [-\tau, 0]$. Then $t \mapsto w_t$ is a continuous function from $[0, b)$ to C. For any $\varphi \in [0, K]_C = \{\varphi \in C; \varphi(x,s) \in [0, K], x \in \mathbb{R}, s \in [-\tau, 0]\}$, define

$$F(\varphi)(x) = f\left(\varphi(x,0), \int\limits_{-\infty}^{\infty} h(x-y)\,\varphi(y,-\tau)\, dy\right).$$

Then $F(\varphi) \in X$ and $F : [0, K]_C \to X$ is globally Lipschitz continuous.

Definition 3.1 A continuous function $v : [-\tau, b) \to X, b > 0$, is called a *supersolution* (*subsolution*) of (3.1) on $[0, b)$ if

$$v(t) \geq (\leq)\, T(t-s)\, v(s) + \int\limits_{s}^{t} T(t-\theta)\, F(v_\theta)\, d\theta \tag{3.3}$$

for all $0 \leq s < t < b$. If v is both a supersolution and a subsolution on $[0, b)$, then it is said to be a *mild solution* of (3.1).

Remark 3.2 Assume that there is a $v \in BUC(\mathbb{R} \times [-\tau, b), \mathbb{R}), b > 0$, such that v is C^2 in $x \in \mathbb{R}$, C^1 in $t \in (0, b)$ and for $x \in \mathbb{R}, t \in (0, b)$,

$$\frac{\partial v}{\partial t} \geq (\leq)\, d\Delta v + f\left(v(x,t), \int\limits_{-\infty}^{\infty} h(x-y)\, v(y, t-\tau)\, dy\right).$$

Then, the positivity of the linear semigroup $T(t) : X^+ \to X^+$ implies that (3.3) holds and hence v is a supersolution (subsolution) of (3.1) on $[0, b)$.

We now have the following existence and comparison result. Since its proof is similar to that of ([40], Theorem 2.2) (see also [33,46]), we omit it.

Theorem 3.3 *Assume that (A1), (A2), (G1) and (G2) hold. Then for any* $\varphi \in [0, K]_C$*, equation (3.1) has a unique mild solution* $u(x, t, \varphi)$ *on* $[0, \infty)$ *and* $u(x, t, \varphi)$ *is a classical solution to (3.1) for* $(x, t) \in \mathbb{R} \times (\tau, \infty)$*. Furthermore, for any pair of supersolution* $\overline{w}(x, t)$ *and subsolution* $\underline{w}(x, t)$ *of (3.1) on* $[0, \infty)$ *with* $0 \le \underline{w}(x, t), \overline{w}(x, t) \le K$ *for* $x \in \mathbb{R}$, $t \in [-\tau, \infty)$*, and* $\overline{w}(x, s) \ge \underline{w}(x, s)$ *for* $x \in \mathbb{R}$, $s \in [-\tau, 0]$*, there holds* $\overline{w}(x, t) \ge \underline{w}(x, t)$ *for* $x \in \mathbb{R}$, $t \ge 0$*, and*

$$\overline{w}(x, t) - \underline{w}(x, t) \ge \Theta(J, t - t_0) \int_z^{z+1} \left(\overline{w}(y, t_0) - \underline{w}(y, t_0)\right) dy \tag{3.4}$$

for any $J \ge 0$*,* x *and* $z \in \mathbb{R}$ *with* $|x - z| \le J$*, and* $t > t_0 \ge 0$*, where*

$$\Theta(J, t) = \frac{1}{\sqrt{4\pi dt}} \exp\left(-L_1 t - \frac{(J+1)^2}{4dt}\right), \quad J \ge 0, \ t > 0.$$

and $L_1 = \max_{(u,v)\in[0,K]^2} |\partial_1 f(u, v)|$*. In particular, if there exists* $x_0 \in \mathbb{R}$ *such that* $\overline{w}(x_0, 0) > \underline{w}(x_0, 0)$*, then* $\overline{w}(x, t) > \underline{w}(x, t)$ *for any* $x \in \mathbb{R}$ *and* $t > 0$*.*

Remark 3.4 From Theorem 3.3, if $\widetilde{U}(\xi)$ is a traveling wave solution of (1.5) satisfying $0 \le \widetilde{U}(\xi) \le K$ and (1.6), then $0 < \widetilde{U}(\xi) < K$ for any $\xi \in \mathbb{R}$.

Remark 3.5 Assume that $U(\xi)$ is a traveling wave front of (1.5) given in Theorem 2.6. Then $U'(\xi) > 0$ for $\xi \in \mathbb{R}$.

Lemma 3.6 *Assume that* $u(x, t)$ *and* $v(x, t)$ *are two mild solutions to (3.1) with initial values* $\phi(x, s)$ *and* $\psi(x, s)$*, respectively, where* $\phi, \psi \in [0, K]_C$*. Then*

$$|u(\cdot, t) - v(\cdot, t)|_X \le \sup_{s\in[-\tau,0]} |\phi(\cdot, s) - \psi(\cdot, s)|_X e^{\mu t} \tag{3.5}$$

for any $t \ge 0$*, where* $\mu = L_1 + L_2$ *and* $L_i = \max_{(u,v)\in[0,K]^2} |\partial_i f(u, v)|$*.*

Proof By Definition 3.1, we have

$$\begin{aligned}
&|u(\cdot, t) - v(\cdot, t)|_X \\
&\quad \le |T(t)\phi(0) - T(t)\psi(0)|_X + \int_0^t |T(t-\theta) F(u_\theta) - T(t-\theta) F(v_\theta)|_X d\theta \\
&\quad \le |\phi(\cdot, 0) - \psi(\cdot, 0)|_X + \int_0^t |F(u_\theta) - F(v_\theta)|_X d\theta \\
&\quad \le |\phi(\cdot, 0) - \psi(\cdot, 0)|_X \\
&\qquad + \int_0^t (L_1 |u(\cdot, \theta) - v(\cdot, \theta)|_X + L_2 |u(\cdot, \theta - \tau) - v(\cdot, \theta - \tau)|_X) d\theta \\
&\quad \le |\phi(\cdot, 0) - \psi(\cdot, 0)|_X + \mu \int_0^t \sup_{s\in[-\tau,\theta]} |u(\cdot, s) - v(\cdot, s)|_X d\theta.
\end{aligned}$$

Consequently,

$$\sup_{s\in[-\tau,t]} |u(\cdot,s)-v(\cdot,s)|_X \le \sup_{s\in[-\tau,0]} |\phi(\cdot,s)-\psi(\cdot,s)|_X + \mu\int_0^t \sup_{s\in[-\tau,\theta]} |u(\cdot,s)-v(\cdot,s)|_X \, d\theta.$$

Then it follows from the Gronwall's inequality that

$$\sup_{s\in[-\tau,t]} |u(\cdot,s)-v(\cdot,s)|_X \le \sup_{s\in[-\tau,0]} |\phi(\cdot,s)-\psi(\cdot,s)|_X \, e^{\mu t},$$

which implies that (3.5) holds. The proof is complete. □

Lemma 3.7 *Assume that (A1), (A2), (A3), (G1) and (G2) hold, and that U is the traveling wave front with wave speed $c > c^*$ given in Theorem 2.6. Then for each $\delta \in (0,1)$, there exist $\rho > 0$ and $\sigma > 0$ such that for each $\epsilon \in (0,\delta]$ and for any $\xi^\pm \in \mathbb{R}$, the following functions*

$$w_+(x,t) = \min\left\{\left(1+\epsilon e^{-\rho t}\right) U\left(x+ct+\xi^+ - \sigma\epsilon e^{-\rho t}\right), K\right\} \tag{3.6}$$

and

$$w_-(x,t) = \left(1-\epsilon e^{-\rho t}\right) U\left(x+ct+\xi^- + \sigma\epsilon e^{-\rho t}\right) \tag{3.7}$$

are a pair of super- and subsolutions to (3.1), respectively.

Proof First, we define a new function $\widehat{f} : [0,K]\times[0,2K] \to \mathbb{R}$ by

$$\widehat{f}(u,v) = \begin{cases} f(u,v) & \text{for } (u,v)\in[0,K]^2, \\ f(u,K) + (v-K)\,\partial_2 f(u,K) & \text{for } (u,v)\in[0,K]\times[K,2K]. \end{cases}$$

Obviously, $\partial_1\widehat{f}(u,v)$ exists and is continuous on $[0,K]^2$, $\partial_2\widehat{f}(u,v)$ exists and is continuous on $[0,K]\times[0,2K]$. In fact,

$$\partial_1\widehat{f}(u,v) = \partial_1 f(u,v) \text{ on } (u,v)\in[0,K]^2$$

and

$$\partial_2\widehat{f}(u,v) = \begin{cases} \partial_2 f(u,v) \ge 0 & \text{for } (u,v)\in[0,K]^2, \\ \partial_2 f(u,K) \ge 0 & \text{for } (u,v)\in[0,K]\times[K,2K]. \end{cases}$$

Note that $\vartheta := \partial_1 f(K,K) + \partial_2 f(K,K) < 0$. There exists θ with $0<\theta<K/2$ such that for any $(u,v)\in[K-\theta,K]^2$, $|\partial_1 f(u,v) - \partial_1 f(K,K)| < -\vartheta/8$, $|\partial_2 f(u,v) - \partial_2 f(K,K)| < -\vartheta/8$, and for any $(u,v)\in[K-\theta,K]\times[K-\theta,2K]$, $|\partial_2\widehat{f}(u,v) - \partial_2 f(K,K)| < -\vartheta/8$.

Let $\xi_2 > 0$ be such that for any $\xi > \xi_2$, $U(\xi)\in[K-\theta,K]$, $V(\xi)\in[K-\theta,K]$ and

$$\frac{9\partial_1 f(K,K)+\partial_2 f(K,K)}{8}(K-U(\xi)) + \frac{\partial_1 f(K,K)+9\partial_2 f(K,K)}{8}(K-V(\xi)) + \frac{\partial_1 f(K,K)-7\partial_2 f(K,K)}{8}V(\xi) - \frac{3\partial_1 f(K,K)-5\partial_2 f(K,K)}{8}U(\xi_2) \ge 0 \tag{3.8}$$

From (2.9) and (2.10), there exists ξ_1 such that for any $\xi \le \xi_1$,

$$U(\xi) < \frac{K}{2},\ V(\xi) < \frac{K}{2},\ \frac{1}{2} \le U(\xi)e^{-\lambda_1(c)\xi} \le \frac{3}{2},\ \frac{1}{2}\lambda_1(c) \le U'(\xi)e^{-\lambda_1(c)\xi} \le \frac{3}{2}\lambda_1(c),$$

and

$$\frac{1}{2}e^{-\lambda_1(c)c\tau}G(\lambda_1(c)) \le e^{-\lambda_1(c)\xi}V(\xi) \le \frac{3}{2}e^{-\lambda_1(c)c\tau}G(\lambda_1(c)).$$

Fix $0 < \rho < 1$ such that $e^{\rho\tau} < \frac{1+\delta}{2\delta}$ and

$$-\rho K - \left(e^{\rho\tau}-1\right)KL_2 - \frac{1}{2}\left(\partial_1 f(K,K) + \partial_2 f(K,K)\right)U(\xi_2) \ge 0, \tag{3.9}$$

$$\rho K + L_2 K\left(e^{\rho\tau}-1\right) - aU^{\alpha}(\xi_1)V^{\beta}(\xi_1) \le 0, \tag{3.10}$$

where $a = a\left(\frac{1+\delta}{2}\right)$, $\alpha = \alpha\left(\frac{1+\delta}{2}\right)$ and $\beta = \beta\left(\frac{1+\delta}{2}\right)$ are determined in (A3).

Let $\varrho = \min\left\{U'(\xi) : \xi_1 \le \xi \le \xi_2\right\} > 0$. Now let $\sigma > 0$ be sufficiently large and satisfy

$$-\frac{3}{2}\rho + \frac{1}{2}\sigma\rho\lambda_1(c) - 3L_1 - \frac{3}{2}L_2\left(e^{\rho\tau}+1\right)e^{-\lambda_1(c)c\tau}G(\lambda_1(c)) \ge 0, \tag{3.11}$$

$$-\rho K + \sigma\rho\varrho - 2KL_1 - \left(e^{\rho\tau}+1\right)KL_2 \ge 0, \tag{3.12}$$

$$\frac{3}{2}\rho - \frac{1}{4}(1-\delta)\sigma\rho\lambda_1(c) + \frac{3}{2}L_2\left(e^{\rho\tau}-1\right)e^{-\lambda_1(c)c\tau}G(\lambda_1(c)) \le 0. \tag{3.13}$$

Define

$$\begin{aligned} B^+ &= \left\{(x,t) : \left(1+\epsilon e^{-\rho t}\right)U\left(x+ct+\xi^+-\sigma\epsilon e^{-\rho t}\right) > K\right\}, \\ B^- &= \left\{(x,t) : \left(1+\epsilon e^{-\rho t}\right)U\left(x+ct+\xi^+-\sigma\epsilon e^{-\rho t}\right) < K\right\}. \end{aligned}$$

Let $\xi = x+ct+\xi^+-\sigma\epsilon e^{-\rho t}$. Since $\left(1+\epsilon e^{-\rho t}\right)U\left(x+ct+\xi^+-\sigma\epsilon e^{-\rho t}\right)$ is continuous, $B^{\pm}$ are open sets. If $(x,t) \in B^+$, then $w_+(x,t) = K$ and so

$$\begin{aligned} \frac{\partial w_+}{\partial t} - d\Delta w_+ - f\left(w_+(x,t), (h*w_+)(x,t)\right) &= -f\left(K, (h*w_+)(x,t)\right) \\ &\ge -f(K,K) = 0. \end{aligned}$$

If $(x,t) \in B^-$, then

$$\begin{aligned} (h*w_+)(x,t) &\le \int_{-\infty}^{\infty} h(x-y)\left(1+\epsilon e^{-\rho(t-\tau)}\right)U\left(y+c(t-\tau)+\xi^+-\sigma\epsilon e^{-\rho(t-\tau)}\right)dy \\ &= \left(1+\epsilon e^{-\rho(t-\tau)}\right)\int_{-\infty}^{\infty} h(y)\,U\left(\xi-y-c\tau-\sigma\epsilon e^{-\rho t}\left(e^{\rho\tau}-1\right)\right)dy \\ &= \left(1+\epsilon e^{-\rho(t-\tau)}\right)V\left(\xi-\sigma\epsilon e^{-\rho t}\left(e^{\rho\tau}-1\right)\right) \\ &\le \left(1+\epsilon e^{-\rho(t-\tau)}\right)V(\xi). \end{aligned}$$

Thus, by (3.8) and (3.9), we have for $\xi > \xi_2$ that

$$
\begin{aligned}
&\frac{\partial w_+}{\partial t} - d\Delta w_+ - f\left(w_+(x,t), (h*w_+)(x,t)\right) \\
&= -\rho\epsilon e^{-\rho t}U(\xi) + \left(1+\epsilon e^{-\rho t}\right)\left(c+\sigma\rho\epsilon e^{-\rho t}\right)U'(\xi) - d\left(1+\epsilon e^{-\rho t}\right)U''(\xi) \\
&\quad - f\left(\left(1+\epsilon e^{-\rho t}\right)U(\xi), (h*w_+)(x,t)\right) \\
&\geq -\rho\epsilon e^{-\rho t}U(\xi) + \sigma\rho\epsilon e^{-\rho t}\left(1+\epsilon e^{-\rho t}\right)U'(\xi) + \left(1+\epsilon e^{-\rho t}\right)f\left(U(\xi), V(\xi)\right) \\
&\quad - \widehat{f}\left(\left(1+\epsilon e^{-\rho t}\right)U(\xi), \left(1+\epsilon e^{-\rho(t-\tau)}\right)V(\xi)\right) \\
&\geq -\rho\epsilon e^{-\rho t}U(\xi) + \sigma\rho\epsilon e^{-\rho t}U'(\xi) + \epsilon e^{-\rho t}f\left(U(\xi), V(\xi)\right) + \widehat{f}\left(U(\xi), V(\xi)\right) \\
&\quad - \widehat{f}\left(\left(1+\epsilon e^{-\rho t}\right)U(\xi), \left(1+\epsilon e^{-\rho t}\right)V(\xi)\right) + \widehat{f}\left(\left(1+\epsilon e^{-\rho t}\right)U(\xi), \left(1+\epsilon e^{-\rho t}\right)V(\xi)\right) \\
&\quad - \widehat{f}\left(\left(1+\epsilon e^{-\rho t}\right)U(\xi), \left(1+\epsilon e^{-\rho(t-\tau)}\right)V(\xi)\right) \\
&\geq -\rho\epsilon e^{-\rho t}U(\xi) + \sigma\rho\epsilon e^{-\rho t}U'(\xi) \\
&\quad + \epsilon e^{-\rho t}\partial_1 f\left(U(\xi) + \theta_1\left(K - U(\xi)\right), V(\xi) + \theta_1\left(K - V(\xi)\right)\right)\left(K - U(\xi)\right) \\
&\quad + \epsilon e^{-\rho t}\partial_2 f\left(U(\xi) + \theta_1\left(K - U(\xi)\right), V(\xi) + \theta_1\left(K - V(\xi)\right)\right)\left(K - V(\xi)\right) \\
&\quad - \epsilon e^{-\rho t}\partial_1 \widehat{f}\left(\left(1+\theta_2\epsilon e^{-\rho t}\right)U(\xi), V(\xi)\right)U(\xi) \\
&\quad - \epsilon e^{-\rho t}\partial_2 \widehat{f}\left(\left(1+\epsilon e^{-\rho t}\right)U(\xi), \left(1+\theta_3\epsilon e^{-\rho t}\right)V(\xi)\right)V(\xi) \\
&\quad - \epsilon e^{-\rho t}\left(e^{\rho\tau}-1\right)V(\xi)\partial_2 \widehat{f}\left(\left(1+\epsilon e^{-\rho t}\right)U(\xi), \left(1+\epsilon e^{-\rho t} + \theta_4\epsilon e^{-\rho t}\left(e^{\rho\tau}-1\right)\right)V(\xi)\right) \\
&\geq -\rho\epsilon e^{-\rho t}U(\xi) + \epsilon e^{-\rho t}\frac{9\partial_1 f(K,K) + \partial_2 f(K,K)}{8}\left(K - U(\xi)\right) \\
&\quad + \epsilon e^{-\rho t}\frac{\partial_1 f(K,K) + 9\partial_2 f(K,K)}{8}\left(K - V(\xi)\right) \\
&\quad + \epsilon e^{-\rho t}\frac{-7\partial_1 f(K,K) + \partial_2 f(K,K)}{8}U(\xi) \\
&\quad + \epsilon e^{-\rho t}\left[\frac{\partial_1 f(K,K) - 7\partial_2 f(K,K)}{8} - L_2\left(e^{\rho\tau}-1\right)\right]V(\xi) \geq 0.
\end{aligned}
$$

For $\xi < \xi_1$, by (3.11), we have

$$
\begin{aligned}
&\frac{\partial w_+}{\partial t} - d\Delta w_+ - f\left(w_+(x,t), (h*w_+)(x,t)\right) \\
&= -\rho\epsilon e^{-\rho t}U(\xi) + \left(1+\epsilon e^{-\rho t}\right)\left(c+\sigma\rho\epsilon e^{-\rho t}\right)U'(\xi) - d\left(1+\epsilon e^{-\rho t}\right)U''(\xi) \\
&\quad - f\left(\left(1+\epsilon e^{-\rho t}\right)U(\xi), (h*w_+)(x,t)\right) \\
&\geq -\rho\epsilon e^{-\rho t}U(\xi) + \sigma\rho\epsilon e^{-\rho t}\left(1+\epsilon e^{-\rho t}\right)U'(\xi) + \left(1+\epsilon e^{-\rho t}\right)f\left(U(\xi), V(\xi)\right) \\
&\quad - f\left(\left(1+\epsilon e^{-\rho t}\right)U(\xi), \left(1+\epsilon e^{-\rho(t-\tau)}\right)V(\xi)\right) \\
&\geq -\rho\epsilon e^{-\rho t}U(\xi) + \sigma\rho\epsilon e^{-\rho t}U'(\xi) - 2L_1\epsilon e^{-\rho t}U(\xi) - L_2\epsilon e^{-\rho t}\left(e^{\rho\tau}+1\right)V(\xi) \\
&\geq \epsilon e^{-\rho t}e^{\lambda_1(c)\xi}\left[-\frac{3}{2}\rho + \frac{1}{2}\sigma\rho\lambda_1(c) - 3L_1 - \frac{3}{2}L_2\left(e^{\rho\tau}+1\right)e^{-\lambda_1(c)c\tau}G\left(\lambda_1(c)\right)\right] \geq 0,
\end{aligned}
$$

and for $\xi_1 \leq \xi \leq \xi_2$, by (3.12) and using the estimate for the case $\xi > \xi_2$, we have

$$
\begin{aligned}
&\frac{\partial w_+}{\partial t} - d\Delta w_+ - f\left(w_+(x,t), (h*w_+)(x,t)\right) \\
&\geq -\rho\epsilon e^{-\rho t}U(\xi) + \sigma\rho\epsilon e^{-\rho t}U'(\xi) - 2KL_1\epsilon e^{-\rho t} - 2KL_2\epsilon e^{-\rho t} - \epsilon e^{-\rho t}\left(e^{\rho\tau}-1\right)KL_2 \\
&\geq \epsilon e^{-\rho t}\left[-\rho K + \sigma\rho\varrho - 2KL_1 - \left(e^{\rho\tau}+1\right)KL_2\right] \geq 0.
\end{aligned}
$$

Note that for every $t \geq 0$, there exists a unique $x^*(t)$ such that $w_+(x^*(t), t) = K$, $(x, t) \in B^+$ for $x > x^*(t)$ and $(x, t) \in B^-$ for $x < x^*(t)$. In addition,

$$\frac{\partial}{\partial x} w_+(x, t)\bigg|_{x=x^*(t)-0} = \left(1 + \epsilon e^{-\rho t}\right) U'\left(x^*(t) + ct + \xi^+ - \sigma\epsilon e^{-\rho t}\right) > 0.$$

Thus, we can use a similar argument, which has been used to prove (2.5) in Lemma 2.4, to show that (3.3) holds for $w_+(x, t)$. Therefore, w_+ is a supersolution of (3.1).

Let $\xi = x + ct + \xi^- + \sigma\epsilon e^{-\rho t}$. Then (A3) implies that

$$\begin{aligned}
&\frac{\partial w_-}{\partial t} - d\Delta w_- - f\left(w_-(x, t), (h * w_-)(x, t)\right) \\
&\quad = \rho\epsilon e^{-\rho t} U(\xi) + \left(1 - \epsilon e^{-\rho t}\right)\left(c - \sigma\rho\epsilon e^{-\rho t}\right) U'(\xi) - d\left(1 - \epsilon e^{-\rho t}\right) U''(\xi) \\
&\qquad - f\left(\left(1 - \epsilon e^{-\rho t}\right) U(\xi), \left(1 - \epsilon e^{-\rho(t-\tau)}\right) V\left(\xi + \sigma\epsilon e^{-\rho t}\left(e^{\rho\tau} - 1\right)\right)\right) \\
&\quad \leq \rho\epsilon e^{-\rho t} U(\xi) - \sigma\rho\epsilon e^{-\rho t}\left(1 - \epsilon e^{-\rho t}\right) U'(\xi) + \left(1 - \epsilon e^{-\rho t}\right) f\left(U(\xi), V(\xi)\right) \\
&\qquad - f\left(\left(1 - \epsilon e^{-\rho t}\right) U(\xi), \left(1 - \epsilon e^{-\rho(t-\tau)}\right) V(\xi)\right) \\
&\quad \leq \rho\epsilon e^{-\rho t} U(\xi) - \sigma\rho\epsilon e^{-\rho t}\left(1 - \epsilon e^{-\rho t}\right) U'(\xi) + \left(1 - \epsilon e^{-\rho t}\right) f\left(U(\xi), V(\xi)\right) \\
&\qquad - f\left(\left(1-\epsilon e^{-\rho t}\right) U(\xi), \left(1-\epsilon e^{-\rho t}\right) V(\xi)\right) + f\left(\left(1-\epsilon e^{-\rho t}\right) U(\xi), \left(1-\epsilon e^{-\rho t}\right) V(\xi)\right) \\
&\qquad - f\left(\left(1 - \epsilon e^{-\rho t}\right) U(\xi), \left(1 - \epsilon e^{-\rho(t-\tau)}\right) V(\xi)\right) \\
&\quad \leq \rho\epsilon e^{-\rho t} U(\xi) - \frac{1-\delta}{2}\sigma\rho\epsilon e^{-\rho t} U'(\xi) - a\epsilon e^{-\rho t} U^\alpha(\xi) V^\beta(\xi) + L_2\epsilon e^{-\rho t}\left(e^{\rho\tau} - 1\right) V(\xi) \\
&\quad \leq \epsilon e^{-\rho t}\left[\rho U(\xi) - \frac{1-\delta}{2}\sigma\rho U'(\xi) - aU^\alpha(\xi) V^\beta(\xi) + L_2\left(e^{\rho\tau} - 1\right) V(\xi)\right].
\end{aligned}$$

For $\xi > \xi_1$, by (3.10) we have

$$\begin{aligned}
&\rho U(\xi) - \frac{1-\delta}{2}\sigma\rho U'(\xi) - aU^\alpha(\xi) V^\beta(\xi) + L_2\left(e^{\rho\tau} - 1\right) V(\xi) \\
&\quad \leq \rho K - aU^\alpha(\xi) V^\beta(\xi) + L_2 K\left(e^{\rho\tau} - 1\right) \leq 0,
\end{aligned}$$

and for $\xi \leq \xi_1$, (3.13) implies that

$$\begin{aligned}
&e^{-\lambda_1(c)\xi}\left[\rho U(\xi) - \frac{1-\delta}{2}\sigma\rho U'(\xi) - aU^\alpha(\xi) V^\beta(\xi) + L_2\left(e^{\rho\tau} - 1\right) V(\xi)\right] \\
&\quad \leq e^{-\lambda_1(c)\xi}\left[\rho U(\xi) - \frac{1-\delta}{2}\sigma\rho U'(\xi) + L_2\left(e^{\rho\tau} - 1\right) V(\xi)\right] \\
&\quad \leq \frac{3}{2}\rho - \frac{1}{4}(1-\delta)\sigma\rho\lambda_1(c) + \frac{3}{2}L_2\left(e^{\rho\tau} - 1\right) e^{-\lambda_1(c)c\tau} G\left(\lambda_1(c)\right) \leq 0.
\end{aligned}$$

Thus Remark 3.2 implies that w_- is a subsolution of (3.1). The proof is complete. □

Remark 3.8 We note that the supersolution w_+ in Lemma 3.7 is only a slight modification of the supersolution of ([32], Lemma 4.6) and the subsolution w_- is same as that of ([32], Lemma 4.6). In fact, these types of supersolutions and subsolutions were firstly used by Chen and Guo ([12], Lemma 3.7).

4 Stability and Uniqueness of Traveling Fronts

In the following, we first establish the asymptotic stability of traveling wave fronts of (1.5) obtained in Theorem 2.6 by using the method of Chen and Guo [11] and Ma and Zou [32]. We also refer to Alexander et al. [2], Samaey and Sandstede [37] and Sattinger [38] for other techniques in studying stability of traveling waves.

We now state our stability result in this section.

Theorem 4.1 *Assume that* (*A1*), (*A2*), (*A3*), (*G1*) *and* (*G2*) *hold. Let* $c > c^*$ *and* U *be the traveling wave front given in Theorem* 2.6. *Assume that there exists a* $\rho_0 \in (0, +\infty)$ *such that the initial value* $\varphi \in [0, K]_C$ *satisfies* $\liminf_{x\to+\infty} \varphi(x, 0) > 0$ *and*

$$\lim_{x\to-\infty} \max_{s\in[-\tau,0]} \left|\varphi(x, s)\, e^{-\lambda_1(c)x} - \rho_0 e^{\lambda_1(c)cs}\right| = 0. \tag{4.1}$$

Then

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} \left|\frac{u(x, t)}{U(x + ct + \xi_0)} - 1\right| = 0, \tag{4.2}$$

where $\xi_0 = \frac{1}{\lambda_1(c)} \ln \rho_0$.

Lemma 4.2 *Under the assumptions of Theorem* 4.1, *for any* $\epsilon > 0$, *there exists a* $\xi_1(\epsilon) < 0$ *such that*

$$\sup_{t\geq-\tau} u(\xi - 2\epsilon - ct, t) < U(\xi + \xi_0) < \inf_{t\geq-\tau} u(\xi + 2\epsilon - ct, t) \tag{4.3}$$

for all $\xi \leq \xi_1(\epsilon)$.

Proof Let $\varepsilon_1 = \rho_0\left(e^{\lambda_1(c)\epsilon} - 1\right) e^{-\lambda_1(c)c\tau}$. Then by (4.1), there exists an $x^- < 0$ such that for any $x < x^-$ and $s \in [-\tau, 0]$,

$$\begin{aligned}\varphi(x - \epsilon, s)\, e^{-\lambda_1(c)(x-\epsilon)} &< \rho_0 e^{\lambda_1(c)cs} + \varepsilon_1 = \rho_0 e^{\lambda_1(c)cs} + \rho_0\left(e^{\lambda_1(c)\epsilon} - 1\right) e^{-\lambda_1(c)c\tau}\\ &\leq \rho_0 e^{\lambda_1(c)cs} + \rho_0\left(e^{\lambda_1(c)\epsilon} - 1\right) e^{\lambda_1(c)cs} = \rho_0 e^{\lambda_1(c)\epsilon} e^{\lambda_1(c)cs},\end{aligned}$$

that is $\varphi(x - \epsilon, s) < e^{\lambda_1(c)(x+\xi_0+cs)}$. Similarly, let $\varepsilon_2 = \rho_0\left(1 - e^{-\lambda_1(c)\epsilon}\right) e^{-\lambda_1(c)c\tau}$, then there exists an $x^+ < 0$ such that $e^{\lambda_1(c)(x+\xi_0+cs)} < \varphi(x + \epsilon, s)$ for any $x < x^+$ and $s \in [-\tau, 0]$. Take $x_1(\epsilon) = \min\left\{x^- - 1, x^+ - 1\right\}$, then $\varphi(x - \epsilon, s) < e^{\lambda_1(c)(x+\xi_0+cs)} < \varphi(x + \epsilon, s)$ for any $x \leq x_1(\epsilon)$ and $s \in [-\tau, 0]$.

Let $\psi(\xi) = \max\left\{0, e^{\lambda_1(c)(\xi+\xi_0)} - qe^{\eta\lambda_1(c)(\xi+\xi_0)}\right\}$, where $\eta = \frac{1}{2}\left(1 + \min\left\{2, \frac{\lambda_2(c)}{\lambda_1(c)}\right\}\right)$ and $q \geq \max\left\{Q(c, \eta), e^{-(\eta-1)\lambda_1(c)(x_1(\epsilon)+\xi_0-c\tau)}\right\}$. Then by Lemma 2.4, $\psi(x + ct)$ is a subsolution of (3.1). Since $e^{\lambda_1(c)(x+\xi_0+cs)} - qe^{\eta\lambda_1(c)(x+\xi_0+cs)} < 0$ for any $x > x_1(\epsilon)$ and $s \in [-\tau, 0]$, we have $\varphi(x + \epsilon, s) \geq \max\left\{0, e^{\lambda_1(c)(x+\xi_0+cs)} - qe^{\eta\lambda_1(c)(x+\xi_0+cs)}\right\}$ for any $x \in \mathbb{R}$ and $s \in [-\tau, 0]$. By Theorem 3.3, for any $x \in \mathbb{R}$ and $t \geq -\tau$, we have $u(x + \epsilon, t) \geq e^{\lambda_1(c)(x+\xi_0+ct)} - qe^{\eta\lambda_1(c)(x+\xi_0+ct)}$. Since $\lim_{\xi\to-\infty} U(\xi)\, e^{-\lambda_1(c)\xi} = 1$, there exists an $x_2(\epsilon) < 0$ such that $e^{\lambda_1(c)(\xi+\xi_0+\epsilon)} - qe^{\eta\lambda_1(c)(\xi+\xi_0+\epsilon)} > U(\xi + \xi_0)$ foe any $\xi \leq x_2(\epsilon)$. Thus, for any $\xi \leq x_2(\epsilon)$, we have

$$\inf_{t\geq-\tau} u(\xi + 2\epsilon - ct, t) \geq e^{\lambda_1(c)(\xi+\xi_0+\epsilon)} - qe^{\eta\lambda_1(c)(\xi+\xi_0+\epsilon)} > U(\xi + \xi_0).$$

Let $\phi(\xi) = \min\{K, e^{\lambda_1(c)(\xi+\xi_0)} + qe^{\eta\lambda_1(c)(\xi+\xi_0)}\}$. Then by Lemma 2.4, $\phi(x + ct)$ is a supersolution of (3.1). Since $e^{\lambda_1(c)\xi} + qe^{\eta\lambda_1(c)\xi} > K$ for $\xi > -\frac{1}{\eta\lambda_1(c)} \ln \frac{q}{K}$, we can take

q sufficiently large so that $e^{\lambda_1(c)(x+\xi_0+cs)} + qe^{\eta\lambda_1(c)(x+\xi_0+cs)} > K$ for any $x > x_1(\epsilon)$ and $s \in [-\tau, 0]$. Since $\varphi(x-\epsilon, s) < e^{\lambda_1(c)(x+\xi_0+cs)} < e^{\lambda_1(c)(x+\xi_0+cs)} + qe^{\eta\lambda_1(c)(x+\xi_0+cs)}$ for any $x \le x_1(\epsilon)$ and $s \in [-\tau, 0]$, we have $\varphi(x-\epsilon, s) \le \min\{K, e^{\lambda_1(c)(x+\xi_0+cs)} + qe^{\eta\lambda_1(c)(x+\xi_0+cs)}\}$ for any $x \in \mathbb{R}$ and $s \in [-\tau, 0]$. By Theorem 3.3 and Lemma 2.4, we have

$$u(x-\epsilon, t) \le \min\left\{K, e^{\lambda_1(c)(x+\xi_0+ct)} + qe^{\eta\lambda_1(c)(x+\xi_0+ct)}\right\}$$

for any $x \in \mathbb{R}$ and $t \ge -\tau$. Since $\lim_{\xi\to-\infty} \frac{e^{\lambda_1(c)(\xi-\epsilon)} + qe^{\eta\lambda_1(c)(\xi-\epsilon)}}{U(\xi)} = e^{-\lambda_1(c)\epsilon} < 1$, there exists $x_3(\epsilon) < 0$ such that $e^{\lambda_1(c)(\xi+\xi_0-\epsilon)} + qe^{\eta\lambda_1(c)(\xi+\xi_0-\epsilon)} < U(\xi+\xi_0)$ for any $\xi \le x_3(\epsilon)$. Therefore, for any $\xi \le x_3(\epsilon)$, we have

$$\sup_{t\ge-\tau} u(\xi - 2\epsilon - ct, t) \le e^{\lambda_1(c)(\xi+\xi_0-\epsilon)} + qe^{\eta\lambda_1(c)(\xi+\xi_0-\epsilon)} < U(\xi+\xi_0).$$

Setting $\xi_1(\epsilon) = \min\{x_1(\epsilon), x_2(\epsilon), x_3(\epsilon)\}$, then (4.3) holds. The proof is complete. □

Lemma 4.3 *There exist $\delta \in (0, 1)$, $\rho > 0$, $\sigma > 0$ and $z_0 > 0$ such that for all $x \in \mathbb{R}$ and $t \ge 1$,*

$$\begin{aligned}
&\left(1 - \delta e^{-\rho(t-1-\tau)}\right) U\left(x + \xi_0 - z_0 + \delta\sigma e^{-\rho(t-1-\tau)}\right) \\
&\quad \le u(x - ct, t) \\
&\quad \le \min\left\{\left(1 + \delta e^{-\rho t}\right) U\left(x + \xi_0 + z_0 - \delta\sigma e^{-\rho t}\right), K\right\}.
\end{aligned} \tag{4.4}$$

Consequently, for all $t \ge 1$,

$$1 - \delta e^{-\rho(t-1-\tau)} \le \inf_{\mathbb{R}} \frac{u(\cdot - ct, t)}{U(\cdot + \xi_0 - z_0)}, \ \sup_{\mathbb{R}} \frac{u(\cdot - ct, t)}{U(\cdot + \xi_0 + z_0)} \le 1 + \delta e^{-\rho t}. \tag{4.5}$$

Proof In view of (4.3), there is $u(x + 2 - c(1+\tau+s), 1+\tau+s) \ge U(x+\xi_0)$ for any $x \le \xi_1(1)$, and hence, $u(x+2, 1+\tau+s) \ge U(x + c(1+\tau+s) + \xi_0)$ for all $x \le \xi_1(1) - c(1+\tau)$ and $s \in [-\tau, 0]$.

Since $\liminf_{x\to+\infty} \varphi(x, 0) > 0$, there exist $\delta_1 > 0$ and $x_4 > 0$ such that $\varphi(x, 0) > \delta_1$ for all $x > x_4$. Fix a positive integer $N > x_4 - [\xi_1(1) - c(1+\tau)]$. If $x \ge \xi_1(1) - c(1+\tau)$, then $x + N > x_4$. Then Theorem 3.3 implies that

$$\begin{aligned}
u(x+2, 1+\tau+s) &\ge \Theta(N, 1+\tau+s) \int_{x+2+N}^{x+3+N} \varphi(y, 0)\, dy \\
&\ge \frac{\delta_1}{\sqrt{4\pi d(1+\tau+s)}} \exp\left(-L_1(1+\tau+s) - \frac{(N+1)^2}{4d(1+\tau+s)}\right) \\
&\ge \frac{\delta_1}{\sqrt{4\pi d(1+\tau)}} \exp\left(-L_1(1+\tau) - \frac{(N+1)^2}{4d}\right) \\
&\ge (1-\delta) K
\end{aligned}$$

for any $x \ge \xi_1(1) - c(1+\tau)$, $s \in [-\tau, 0]$ and some $0 < \delta < 1$. Thus, for $\rho > 0$ with $\delta e^{\rho\tau} < \frac{1+\delta}{2}$, we have

$$\begin{aligned}
u(x+2, 1+\tau+s) &\ge (1-\delta) U(x + c(1+\tau+s) + \xi_0) \\
&\ge \left(1 - \delta e^{-\rho s}\right) U\left(x + c(1+\tau+s) + \xi_0 - \delta\sigma e^{\rho\tau} + \delta\sigma e^{-\rho s}\right)
\end{aligned}$$

for any $x \in \mathbb{R}$ and $s \in [-\tau, 0]$. Consequently, using Lemma 3.7 and Theorem 3.3, we have

$$u(x+2, 1+\tau+t) \geq \left(1-\delta e^{-\rho t}\right) U\left(x+c(1+\tau+t)+\xi_0-\delta\sigma e^{\rho\tau}+\delta\sigma e^{-\rho t}\right)$$

for any $x \in \mathbb{R}$ and $t \geq -\tau$, namely,

$$u(x-c(1+\tau+t), 1+\tau+t) \geq \left(1-\delta e^{-\rho t}\right) U\left(x-2+\xi_0-\delta\sigma e^{\rho\tau}+\delta\sigma e^{-\rho t}\right).$$

Let $\bar{t} = 1+\tau+t$ and denote the variable $\bar{t}$ still by t. Then for all $x \in \mathbb{R}$ and $t \geq 1$, we have

$$u(x-ct, t) \geq \left(1-\delta e^{-\rho(t-1-\tau)}\right) U\left(x-2+\xi_0-\delta\sigma e^{\rho\tau}+\delta\sigma e^{-\rho(t-1-\tau)}\right). \quad (4.6)$$

Again, in view of (4.3), there is $u(x-2-cs, s) < U(x+\xi_0)$ for all $x \leq \xi_1(1)$, and hence, $u(x-2, s) < U(x+cs+\xi_0)$ for all $x \leq \xi_1(1)$ and $s \in [-\tau, 0]$. Also, for δ given in the above estimate and sufficiently large $x_5 > 0$ satisfying $U(\xi_1(1)-c\tau+x_5+\xi_0) \geq \frac{K}{1+\delta}$, we have $u(x-2, s) \leq K \leq (1+\delta) U(x+cs+x_5+\xi_0)$ for any $x \geq \xi_1(1)$ and $s \in [-\tau, 0]$. Thus, for all $x \in \mathbb{R}$ and $s \in [-\tau, 0]$, we have

$$\begin{aligned} u(x-2, s) &\leq (1+\delta) U(x+cs+x_5+\xi_0) \\ &\leq \left(1+\delta e^{-\rho s}\right) U\left(x+cs+x_5+\xi_0+\delta\sigma e^{\rho\tau}-\delta\sigma e^{\rho s}\right). \end{aligned}$$

Hence, $u(x-2, s) \leq \min\left\{\left(1+\delta e^{-\rho s}\right) U(x+cs+x_5+\xi_0+\delta\sigma e^{\rho\tau}-\delta\sigma e^{\rho s}), K\right\}$. Obviously, Lemma 3.7 and Theorem 3.3 yield

$$u(x-2, t) \leq \min\left\{\left(1+\delta e^{-\rho t}\right) U\left(x+ct+x_5+\xi_0+\delta\sigma e^{\rho\tau}-\delta\sigma e^{\rho t}\right), K\right\}$$

for any $x \in \mathbb{R}$ and $t \geq -\tau$. Therefore,

$$u(x-ct, t) \leq \min\left\{\left(1+\delta e^{-\rho t}\right) U\left(x+2+x_5+\xi_0+\delta\sigma e^{\rho\tau}-\delta\sigma e^{\rho t}\right), K\right\}. \quad (4.7)$$

Now, letting $z_0 = 2+x_5+\delta\sigma e^{\rho\tau}$, then (4.4) follows from (4.6) and (4.7), and (4.5) is a direct consequence of (4.4). The proof is complete. □

Lemma 4.4 *There exists $M_0 > 0$ such that for any $\epsilon \in (0, \delta]$ and $\xi \geq M_0 + \xi_0$,*

$$(1-\epsilon) U\left(\xi+3\epsilon\sigma e^{\rho\tau}\right) \leq U(\xi) \leq (1+\epsilon) U\left(\xi-3\epsilon\sigma e^{\rho\tau}\right). \quad (4.8)$$

Proof Note that

$$\frac{d}{ds}\left\{(1+s) U\left(\xi-3s\sigma e^{\rho\tau}\right)\right\} = U\left(\xi-3s\sigma e^{\rho\tau}\right) - 3\sigma e^{\rho\tau}(1+s) U'\left(\xi-3s\sigma e^{\rho\tau}\right).$$

Since $\lim_{\xi\to+\infty} U'(\xi) = 0$, there exists $M_0 > 0$ such that $U(\xi) - 6\sigma e^{\rho\tau} U'(\xi) > 0$ for any $\xi \geq M_0+\xi_0-3\sigma e^{\rho\tau}$. Thus, $\frac{d}{ds}\{(1+s) U(\xi-3s\sigma e^{\rho\tau})\} > 0$ for any $s \in [-\delta, \delta]$ and $\xi \geq M_0+\xi_0$. This completes the proof. □

Lemma 4.5 *Let z and M be any given positive constants and $w^{\pm}$ be the solutions of the equation*

$$\frac{\partial w}{\partial t} = d\Delta w + f(w(x,t), (h*w)(x,t))$$

on $\mathbb{R} \times (0, +\infty)$ with initial values

$$\begin{aligned} w^{+}(x, s) = {} & U(x+cs+\xi_0+z)\,\zeta(x+cs+M) + U(x+cs+\xi_0+2z) \\ & \times [1-\zeta(x+cs+M)] \end{aligned} \quad (4.9)$$

and

$$w^{-}(x,s) = U(x+cs+\xi_0 - z)\,\zeta(x+cs+M) + U(x+cs+\xi_0-2z) \times [1-\zeta(x+cs+M)] \tag{4.10}$$

for $x \in \mathbb{R}$ *and* $s \in [-\tau, 0]$*, respectively, where* $\zeta(y) = \min\{\max\{0,-y\},1\}$ *for all* $y \in \mathbb{R}$*. Then there exists an* $\epsilon \in \left(0, \min\left\{\frac{\delta}{2}, ze^{-\rho\tau}/(3\sigma)\right\}\right)$ *such that for any* $x \in [-M, +\infty)$*,*

$$w^{+}(x - c(1+\tau+s), 1+\tau+s) \le (1+\epsilon)\, U\left(x+\xi_0+2z-3\epsilon\sigma e^{\rho\tau}\right), \tag{4.11}$$

$$w^{-}(x - c(1+\tau+s), 1+\tau+s) \ge (1-\epsilon)\, U\left(x+\xi_0+2z+3\epsilon\sigma e^{\rho\tau}\right). \tag{4.12}$$

Proof We only consider w^+, a similar argument can be used for w^-. In view of $w^+(\cdot,s) \le U(\cdot+cs+\xi_0+2z)$ on $\mathbb{R}$ and $w^+(\cdot,s) < U(\cdot+cs+\xi_0+2z)$ on $(-\infty,-M-1]$, by Theorem 3.3, we have $w^+(x-c(1+\tau+s), 1+\tau+s) < U(\cdot+\xi_0+2z)$ for any $x \in \mathbb{R}$ and $s \in [-\tau, 0]$. Since w^+ and U are continuous, there exists $\epsilon \in \left(0, \min\left\{\frac{\delta}{2}, ze^{-\rho\tau}/(3\sigma)\right\}\right)$ such that

$$w^{+}(x-c(1+\tau+s), 1+\tau+s) \le U\left(x+\xi_0+2z-3\epsilon\sigma e^{\rho\tau}\right)$$

on the interval $[-M, M_0-2z]$, where $M_0 > 0$ is defined as in Lemma 4.4 which asserts that $U(\cdot+\xi_0) \le (1+\epsilon)\,U(\cdot+\xi_0-3\epsilon\sigma e^{\rho\tau})$ on $[M_0,+\infty)$. Hence, we also have

$$\begin{aligned} w^{+}(\cdot - c(1+\tau+s), 1+\tau+s) &< U(\cdot+\xi_0+2z) \\ &\le (1+\epsilon)\,U\left(\cdot+\xi_0+2z-3\epsilon\sigma e^{\rho\tau}\right) \end{aligned}$$

on $[M_0-2z,+\infty)$. Therefore, (4.11) holds. The proof is complete. □

Proof of Theorem 4.1 Define

$$z^{+} := \inf\left\{z \,\middle|\, z \in A^{+}\right\},\ A^{+} := \left\{z \ge 0 \,\middle|\, \limsup_{t\to+\infty} \sup_{\mathbb{R}} \frac{u(\cdot-ct,t)}{U(\cdot+\xi_0+2z)} \le 1\right\} \tag{4.13}$$

and

$$z^{-} := \inf\left\{z \,\middle|\, z \in A^{-}\right\},\ A^{-} := \left\{z \ge 0 \,\middle|\, \liminf_{t\to+\infty} \inf_{\mathbb{R}} \frac{u(\cdot-ct,t)}{U(\cdot+\xi_0-2z)} \ge 1\right\}. \tag{4.14}$$

In view of (4.5), we see that $\frac{1}{2}z_0 \in A^{\pm}$. Hence, z^+ and z^- are well defined and $z^{\pm} \in \left[0, \frac{1}{2}z_0\right]$. Furthermore, as $\lim_{\epsilon\to 0} \frac{U(\cdot+\epsilon)}{U(\cdot)} = 1$ uniformly on $\mathbb{R}$, we see that $z^{\pm} \in A^{\pm}$ and $A^{\pm} = \left[z^{\pm}, +\infty\right)$. Thus, to complete the proof, it is sufficient to show that $z^+ = z^- = 0$.

First we prove that $z^+ = 0$ by a contradiction argument. For the contrary, suppose that $z^+ > 0$. We fix $z = z^+$ and $M = -\xi_1\left(z^+/2\right) + z^+$, and denote by ϵ the resulting constant in Lemma 4.5. Since $z^+ \in A^+$, $\limsup_{t\to+\infty}\sup_{\mathbb{R}} \frac{u(\cdot-ct,t)}{U(\cdot+\xi_0+2z^+)} \le 1$. It then follows that there exists $T \ge 0$ such that $\sup_{\mathbb{R}} \frac{u(\cdot-c(T+s),T+s)}{U(\cdot+\xi_0+2z^+)} \le 1+\bar{\epsilon}/K$ for any $s \in [-\tau, 0]$, where $\bar{\epsilon} = \epsilon U(-M+\xi_0-3\epsilon\sigma e^{\rho\tau})\,e^{-\mu(1+\tau)}$. From (4.9), we have

$$w^{+}(\cdot,s) = U\left(\cdot+cs+\xi_0+2z^{+}\right)$$

on $[-M-cs,+\infty)$. Thus, we have $u(\cdot-cT, T+s) \le U\left(\cdot+cs+\xi_0+2z^+\right)+\bar{\epsilon} = w^+(\cdot,s)+\bar{\epsilon}$ on $[-M-cs,+\infty)$.

On $(-\infty, -M - cs] = \left(-\infty, \xi_1\left(\frac{z^+}{2}\right) - z^+ - cs\right]$, by (4.3) and the definition of $w^+(\cdot, s)$ in (4.9), we have $u(\cdot - cT, T + s) \leq U(\cdot + cs + \xi_0 + z^+) \leq w^+(\cdot, s)$. Thus, for any $s \in [-\tau, 0]$, $u(\cdot - cT, T + s) \leq w^+(\cdot, s) + \bar{\epsilon}$ on $\mathbb{R}$. By virtue of Lemma 3.6, we have

$$\begin{aligned} u(\cdot - cT, T + 1 + \tau + s) &< w^+(\cdot, 1 + \tau + s) + \bar{\epsilon}e^{\mu(1+\tau)} \\ &= w^+(\cdot, 1 + \tau + s) + \epsilon U\left(-M + \xi_0 - 3\epsilon\sigma e^{\rho\tau}\right) \text{ on } \mathbb{R}. \end{aligned}$$

Therefore, it follows from (4.11) that

$$\begin{aligned} &u(\cdot - c(T + 1 + \tau + s), T + 1 + \tau + s) \\ &\leq w^+(\cdot - c(1 + \tau + s), 1 + \tau + s) + \epsilon U\left(-M + \xi_0 - 3\epsilon\sigma e^{\rho\tau}\right) \\ &\leq (1 + \epsilon) U\left(\cdot + \xi_0 + 2z^+ - 3\epsilon\sigma e^{\rho\tau}\right) + \epsilon U\left(-M + \xi_0 - 3\epsilon\sigma e^{\rho\tau}\right) \\ &\leq (1 + 2\epsilon) U\left(\cdot + \xi_0 + 2z^+ - 3\epsilon\sigma e^{\rho\tau}\right) \text{ on } [-M, +\infty). \end{aligned}$$

Again, by (4.3) and $3\epsilon\sigma e^{\rho\tau} \leq z^+$, we have $u(\cdot - c(T + 1 + \tau + s), T + 1 + \tau + s) \leq U(\cdot + \xi_0 + z^+) \leq U\left(\cdot + \xi_0 + 2z^+ - 3\epsilon\sigma e^{\rho\tau}\right)$ on $(-\infty, -M]$. Thus,

$$\begin{aligned} &u(\cdot - c(T + 1 + \tau + s), T + 1 + \tau + s) \\ &\leq (1 + 2\epsilon) U\left(\cdot + \xi_0 - 3\epsilon\sigma e^{\rho\tau}\right) \\ &\leq \left(1 + 2\epsilon e^{-\rho s}\right) U\left(\cdot + \xi_0 + 2z^+ - \epsilon\sigma - 2\epsilon\sigma e^{\rho s}\right) \text{ on } \mathbb{R}. \end{aligned}$$

Therefore, for any $s \in [-\tau, 0]$,

$$\begin{aligned} &u(\cdot - c(T + 1 + \tau + s), T + 1 + \tau + s) \\ &\leq \min\left\{\left(1 + 2\epsilon e^{-\rho s}\right) U\left(\cdot + \xi_0 + 2z^+ - \epsilon\sigma - 2\epsilon\sigma e^{\rho s}\right), K\right\} \text{ on } \mathbb{R}. \end{aligned}$$

Then, Lemma 3.7 and Theorem 3.3 give that

$$\begin{aligned} &u(\cdot - c(T + 1 + \tau + t), T + 1 + \tau + t) \\ &\leq \min\left\{\left(1 + 2\epsilon e^{-\rho t}\right) U\left(\cdot + \xi_0 + 2z^+ - \epsilon\sigma - 2\epsilon\sigma e^{\rho t}\right), K\right\} \end{aligned}$$

for any $x \in \mathbb{R}$ and $t \geq 0$, which implies that

$$\limsup_{t\to+\infty} \sup_{\mathbb{R}} \frac{u(\cdot - ct, t)}{U\left(\cdot + \xi_0 + 2z^+ - \epsilon\sigma\right)} \leq 1.$$

That is, $z^+ - \epsilon\sigma/2 \in A^+$. But this contradicts the definition of z^+. This contradiction shows that $z^+ = 0$.

In a similar manner, we can show that $z^- = 0$. The proof is complete. □

In the remainder of this section, we show the uniqueness of traveling wave fronts of (1.5). Our method is to firstly establish the exact asymptotic behavior of the profile $U(\xi)$ as $\xi \to -\infty$ by using the approach developed by Carr and Chmaj [8] for a nonlocal reaction–diffusion equation, and then obtain the uniqueness by using Theorem 4.1. For the sake of simplicity, we first provide a technical lemma about the asymptotic behavior of a positive decreasing function, which is given by Carr and Chmaj ([8], Proposition 2.3) and is important to prove our result. Also, we consider the nonexistence of traveling wave fronts.

Lemma 4.6 *Let $\ell(\lambda) = \int_0^\infty u(\xi)e^{-\lambda\xi}d\xi$ with $u(\xi)$ being a positive decreasing function. Assume that ℓ has the representation*

$$\ell(\lambda) = \frac{E(\lambda)}{(\lambda + \alpha)^{k+1}},$$

where $k > -1$ and E is analytic in the strip $-\alpha \leq Re\lambda < 0$. Then

$$\lim_{\xi \to +\infty} \frac{u(\xi)}{\xi^k e^{-\alpha\xi}} = \frac{E(-\alpha)}{\Gamma(\alpha+1)}.$$

Lemma 4.7 *Assume that $\widetilde{U}(\xi)$ is a traveling wave front of (2.1) satisfying $0 < \widetilde{U}(\xi) < K$ and (1.6). Then $\lim_{\xi\to\pm\infty} \widetilde{U}'(\xi) = 0$.*

The proof is similar to that of ([47], Proposition 2.1) and is omitted.

Theorem 4.8 *Assume that (A1), (A2), (G1) and (G2) hold and that $\widetilde{U}(\xi)$ is a traveling wave front of (1.5) with speed $c \geq c^*$ satisfying (1.6) and $0 < \widetilde{U}(\xi) < K$. Further suppose that $\lambda_0 > \sqrt{\partial_1 f(0,0)}/\sqrt{d}$ if (G2)(ii) and $\partial_2 f(0,0) = 0$ hold at the same time. Then for $c > c^*$, $\lim_{\xi\to-\infty} \widetilde{U}(\xi)e^{-\lambda_1(c)\xi}$ exists, and for $c = c^*$, $\lim_{\xi\to-\infty} \widetilde{U}(\xi)\xi^{-1}e^{-\lambda^*\xi}$ exists. Moreover, for c with $0 < c < c^*$, there is no traveling wave front with speed c of (1.5) between 0 and K satisfying (1.6).*

Proof Define $\widetilde{V}(\xi) = \int_{-\infty}^{\infty} h(z)\widetilde{U}(\xi - c\tau - z)\,dz$. Let $\varpi_1 = \partial_1 f(0,0) + \partial_2 f(0,0)$ and $\varpi_2 = \partial_2 f(0,0) - \partial_1 f(0,0)$. Since $\varpi_1 = \partial_1 f(0,0) + \partial_2 f(0,0) > 0$, by Taylor's expansion, there exists $\xi' < 0$ such that for any $\xi < \xi'$,

$$\frac{\varpi_1}{4}\left(\widetilde{U}(\xi) + \widetilde{V}(\xi)\right) > M\left(\widetilde{U}^2(\xi) + 2\widetilde{U}(\xi)\widetilde{V}(\xi) + \widetilde{V}^2(\xi)\right),$$

where M is given in Lemma 2.4. Then for any $\xi < \xi'$,

$$\begin{aligned}
-d\widetilde{U}''(\xi) + c\widetilde{U}'(\xi) &= f\left(\widetilde{U}(\xi), \widetilde{V}(\xi)\right) \\
&\geq \partial_1 f(0,0)\,\widetilde{U}(\xi) + \partial_2 f(0,0)\,\widetilde{V}(\xi) - M\left(\widetilde{U}^2(\xi)\right. \\
&\quad \left. + 2\widetilde{U}(\xi)\widetilde{V}(\xi) + \widetilde{V}^2(\xi)\right) \\
&= \frac{\varpi_1}{4}\widetilde{U}(\xi) + \frac{\varpi_2}{2}\left(\widetilde{V}(\xi) - \widetilde{U}(\xi)\right) + \frac{\varpi_1}{4}\widetilde{V}(\xi) + \frac{\varpi_1}{4}\left(\widetilde{U}(\xi) + \widetilde{V}(\xi)\right) \\
&\quad - M\left(\widetilde{U}^2(\xi) + 2\widetilde{U}(\xi)\widetilde{V}(\xi) + \widetilde{V}^2(\xi)\right) \\
&\geq \frac{\varpi_1}{4}\widetilde{U}(\xi) + \frac{\varpi_2}{2}\left(\widetilde{V}(\xi) - \widetilde{U}(\xi)\right) + \frac{\varpi_1}{4}\widetilde{V}(\xi).
\end{aligned} \tag{4.15}$$

Now we show that for any $\xi \in \mathbb{R}$, $\widetilde{U}(\xi)$ is integrable on $(-\infty, \xi]$ and there exists $\gamma > 0$ such that $\sup_{\xi\in\mathbb{R}} \widetilde{U}(\xi)e^{-\gamma\xi} < +\infty$. By Fubini's theorem and Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}
\int_y^\xi \left(\widetilde{V}(s) - \widetilde{U}(s)\right) ds &= \int_y^\xi \int_{-\infty}^{\infty} h(z)\left(\widetilde{U}(s - c\tau - z) - \widetilde{U}(s)\right) dz\,ds \\
&= -\int_y^\xi \int_{-\infty}^{\infty} (c\tau + z)\,h(z) \int_0^1 \widetilde{U}'(s - t(c\tau + z))\,dt\,dz\,ds
\end{aligned}$$

$$= -\int_{-\infty}^{\infty} (c\tau + z)\, h(z) \int_0^1 \int_y^{\xi} \widetilde{U}'(s - t(c\tau + z))\, ds\, dt\, dz$$

$$= -\int_{-\infty}^{\infty} (c\tau + z)\, h(z) \int_0^1 \left[\widetilde{U}(\xi - t(c\tau + z)) - \widetilde{U}(y - t(c\tau + z))\right] dt\, dz$$

$$\to -\int_{-\infty}^{\infty} (c\tau + z)\, h(z) \int_0^1 \widetilde{U}(\xi - t(c\tau + z))\, dt\, dz$$

as $y \to -\infty$. Since $\lim_{\xi \to -\infty} \widetilde{U}'(\xi) = 0$ by Lemma 4.7, integrating (4.15) from $-\infty$ to ξ, we have for any $\xi < \xi'$ that

$$-d\widetilde{U}'(\xi) + c\widetilde{U}(\xi) + \frac{\varpi_2}{2} \int_{-\infty}^{\infty} (c\tau + z)\, h(z) \int_0^1 \widetilde{U}(\xi - t(c\tau + z))\, dt\, dz$$

$$\geq \frac{\varpi_1}{4} \int_{-\infty}^{\xi} \widetilde{U}(s)\, ds + \frac{\varpi_1}{4} \int_{-\infty}^{\xi} \widetilde{V}(s)\, ds,$$

which implies that $\widetilde{U}(\xi)$ and $\widetilde{V}(\xi)$ are integrable on $(-\infty, \xi]$.

Now we define a function $\widetilde{W}(\xi) = \int_{-\infty}^{\xi} \widetilde{U}(s)\, ds$, which is increasing and satisfies $\lim_{\xi \to -\infty} \widetilde{W}(\xi) = 0$ and $\widetilde{W}(\xi) \leq \widetilde{W}(0) + K\xi$ for $\xi \geq 0$. Obviously,

$$\int_{-\infty}^{\xi} \widetilde{V}(s)\, ds = \int_{-\infty}^{\xi} \int_{-\infty}^{\infty} h(z)\, \widetilde{U}(s - c\tau - z)\, dz\, ds$$

$$= \lim_{y \to -\infty} \int_y^{\xi} \int_{-\infty}^{\infty} h(z)\, \widetilde{U}(s - c\tau - z)\, dz\, ds$$

$$= \lim_{y \to -\infty} \int_{-\infty}^{\infty} h(z) \int_y^{\xi} \widetilde{U}(s - c\tau - z)\, ds\, dz$$

$$= \int_{-\infty}^{\infty} h(z) \int_{-\infty}^{\xi} \widetilde{U}(s - c\tau - z)\, ds\, dz = \int_{-\infty}^{\infty} h(z)\, \widetilde{W}(\xi - c\tau - z)\, dz.$$

Integrating (4.15) from $-\infty$ to ξ with $\xi < \xi'$, we get

$$-d\widetilde{U}'(\xi) + c\widetilde{U}(\xi) \geq \frac{\varpi_1}{4}\widetilde{W}(\xi) + \frac{\varpi_2}{2}\left(\int_{-\infty}^{\infty} h(z)\, \widetilde{W}(\xi - c\tau - z)\, dz - \widetilde{W}(\xi)\right)$$

$$+ \frac{\varpi_1}{4} \int_{-\infty}^{\infty} h(z)\, \widetilde{W}(\xi - c\tau - z)\, dz. \tag{4.16}$$

Note that

$$\int_y^\xi \int_{-\infty}^\infty h(z)\left[\widetilde{W}(s-c\tau-z)-\widetilde{W}(s)\right]dzds$$
$$=-\int_y^\xi \int_{-\infty}^\infty (c\tau+z)\,h(z)\int_0^1 \widetilde{W}'(s-t(c\tau+z))\,dtdzds$$
$$=-\int_{-\infty}^\infty (c\tau+z)\,h(z)\int_0^1 \left[\widetilde{W}(\xi-t(c\tau+z))-\widetilde{W}(y-t(c\tau+z))\right]dtdz$$
$$\to -\int_{-\infty}^\infty (c\tau+z)\,h(z)\int_0^1 \widetilde{W}(\xi-t(c\tau+z))\,dtdz \text{ as } y\to-\infty,$$

then, for any $\xi<\xi'$, (4.16) implies that

$$-d\widetilde{U}(\xi)+c\widetilde{W}(\xi)+\frac{\varpi_2}{2}\int_{-\infty}^\infty (c\tau+z)\,h(z)\int_0^1 \widetilde{W}(\xi-t(c\tau+z))\,dtdz$$
$$\geq \frac{\varpi_1}{4}\int_{-\infty}^\xi \widetilde{W}(s)\,ds+\frac{\varpi_1}{4}\int_{-\infty}^\xi\int_{-\infty}^\infty h(z)\,\widetilde{W}(s-c\tau-z)\,dzds,$$

which means that $\widetilde{W}(\xi)$ and $\int_{-\infty}^\infty h(z)\,\widetilde{W}(\xi-c\tau-z)\,dz$ are integrable on $(-\infty,\xi]$.

Since $\widetilde{W}(\xi)$ is increasing, for any $z\in\mathbb{R}$, we have

$$(c\tau+z)\,h(z)\,\widetilde{W}(\xi-(c\tau+z))$$
$$\leq (c\tau+z)\,h(z)\int_0^1 \widetilde{W}(\xi-t(c\tau+z))\,dt\leq (c\tau+z)\,h(z)\,\widetilde{W}(\xi).$$

Thus, if $\varpi_2=\partial_2 f(0,0)-\partial_1 f(0,0)\geq 0$, then

$$-d\widetilde{U}(\xi)+c\widetilde{W}(\xi)+\frac{c\tau\varpi_2}{2}\widetilde{W}(\xi)=-d\widetilde{U}(\xi)+c\widetilde{W}(\xi)+\frac{\varpi_2}{2}\int_{-\infty}^\infty (c\tau+z)\,h(z)\,\widetilde{W}(\xi)dz$$
$$\geq \frac{\varpi_1}{4}\int_{-\infty}^\xi \widetilde{W}(s)\,ds. \tag{4.17}$$

If $\varpi_2=\partial_2 f(0,0)-\partial_1 f(0,0)<0$, that is, $\partial_1 f(0,0)>\partial_2 f(0,0)\geq 0$ (see (A2)), then, for sufficiently small ξ, we have $-d\widetilde{U}''(\xi)+c\widetilde{U}'(\xi)=f\left(\widetilde{U}(\xi),\widetilde{V}(\xi)\right)\geq f\left(\widetilde{U}(\xi),0\right)\geq \frac{1}{2}\partial_1 f(0,0)\,\widetilde{U}(\xi)$. Thus,

$$-d\widetilde{U}(\xi)+c\widetilde{W}(\xi)\geq \frac{\varpi_1}{4}\int_{-\infty}^\xi \widetilde{W}(s)\,ds. \tag{4.18}$$

Now, for sufficiently small ξ and any $r > 0$, we have

$$c\widetilde{W}(\xi) + \frac{c\tau}{2}\left|\varpi_2\right|\widetilde{W}(\xi) \geq \frac{\varpi_1}{4}\int_{-\infty}^{0}\widetilde{W}(\xi+s)\,ds \geq \frac{\varpi_1}{4}\int_{-r}^{0}\widetilde{W}(\xi+s)\,ds \geq \frac{\varpi_1}{4}r\widetilde{W}(\xi-r).$$

Thus, there exist $r_0 > 0$ and some θ with $0 < \theta < 1$ such that $\widetilde{W}(\xi - r_0) \leq \theta\widetilde{W}(\xi)$. Let $e(\xi) = \widetilde{W}(\xi)\,e^{-\gamma\xi}$, where $\gamma = \frac{1}{r_0}\ln\frac{1}{\theta} < \bar{\lambda}$, where $\bar{\lambda}$ is defined in Sect. 2. Then

$$e(\xi - r_0) = \widetilde{W}(\xi - r_0)\,e^{-\gamma(\xi - r_0)} = \frac{1}{\theta}\widetilde{W}(\xi - r_0)\,e^{-\gamma\xi} \leq \widetilde{W}(\xi)e^{-\gamma\xi} = e(\xi).$$

In view of $\lim_{\xi\to+\infty}\widetilde{W}(\xi)\,e^{-\gamma\xi} = 0$, then $\sup_{\xi\in\mathbb{R}}\left\{\widetilde{W}(\xi)\,e^{-\gamma\xi}\right\} < +\infty$. Moreover,

$$\begin{aligned} e^{-\gamma\xi}\int_{-\infty}^{\xi}\widetilde{W}(s)\,ds &= e^{-\gamma\xi}\int_{-\infty}^{0}\widetilde{W}(\xi+s)\,ds \\ &= \int_{-\infty}^{0}\widetilde{W}(\xi+s)\,e^{-\gamma(\xi+s)}e^{\gamma s}ds \leq \sup_{z\in\mathbb{R}}\left\{\widetilde{W}(z)\,e^{-\gamma z}\right\}\frac{1}{\gamma}. \end{aligned}$$

From (4.17) and (4.18), we have $\sup_{\xi\in\mathbb{R}}\left\{\widetilde{U}(\xi)\,e^{-\gamma\xi}\right\} < +\infty$.

Next we prove that $\lim_{\xi\to-\infty}e^{-\lambda_1(c)\xi}\widetilde{U}(\xi)$ exists. For λ with $0 <\mathrm{Re}\lambda < \gamma$, we define a two-sided Laplace transform of $\widetilde{U}$ by

$$\ell(\lambda) \equiv \int_{-\infty}^{\infty}e^{-\lambda\xi}\widetilde{U}(\xi)d\xi.$$

Note that for $\xi \geq 0$, $h(y)\,\widetilde{U}(\xi - c\tau - y)\,e^{-\mathrm{Re}\lambda\xi} < Kh(y)\,e^{-\mathrm{Re}\lambda\xi}$ and for $\xi < 0$,

$$\begin{aligned} h(y)\,\widetilde{U}(\xi - c\tau - y)\,e^{-\mathrm{Re}\lambda\xi} &= h(y)\,\widetilde{U}(\xi - c\tau - y)\,e^{-\gamma(\xi - c\tau - y)}e^{-\gamma c\tau}e^{-\gamma y}e^{(\gamma - \mathrm{Re}\lambda)\xi} \\ &\leq e^{-\gamma c\tau}\widetilde{M}h(y)\,e^{-\gamma y}e^{(\gamma - \mathrm{Re}\lambda)\xi}, \end{aligned}$$

where $\widetilde{M} = \sup_{\xi\in\mathbb{R}}\left\{\widetilde{U}(\xi)\,e^{-\gamma\xi}\right\}$, then $h(y)\,\widetilde{U}(\xi - c\tau - y)\,e^{-\mathrm{Re}\lambda\xi}$ is integrable on $\mathbb{R}^2$. Since $e^{-i\mathrm{Im}\lambda\xi}$ is bounded and $h(y)\,\widetilde{U}(\xi - c\tau - y)\,e^{-\lambda\xi}$ is integrable on $\mathbb{R}^2$, by Fubini's theorem, we have

$$\begin{aligned} \int_{-\infty}^{\infty}e^{-\lambda\xi}\widetilde{V}(\xi)d\xi &= \int_{-\infty}^{\infty}e^{-\lambda\xi}\int_{-\infty}^{\infty}h(y)\,\widetilde{U}(\xi - c\tau - y)\,dyd\xi \\ &= \int_{-\infty}^{\infty}h(y)\,e^{-\lambda(c\tau+y)}\int_{-\infty}^{\infty}e^{-\lambda(\xi - c\tau - y)}\widetilde{U}(\xi - c\tau - y)\,d\xi dy \\ &= \ell(\lambda)\,e^{-\lambda c\tau}\int_{-\infty}^{\infty}h(y)\,e^{-\lambda y}dy \\ &= \ell(\lambda)\,e^{-\lambda c\tau}G(\lambda). \end{aligned}$$

Since $d\widetilde{U}''(\xi) - c\widetilde{U}'(\xi) + \partial_1 f(0,0)\widetilde{U}(\xi) + \partial_2 f(0,0)\widetilde{V}(\xi) = \partial_1 f(0,0)\widetilde{U}(\xi) + \partial_2 f(0,0)$ $\widetilde{V}(\xi) - f(\widetilde{U}(\xi), \widetilde{V}(\xi))$, we have

$$\Delta(\lambda, c)\,\ell(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda\xi}\left[\partial_1 f(0,0)\,\widetilde{U}(\xi) + \partial_2 f(0,0)\,\widetilde{V}(\xi) - f\left(\widetilde{U}(\xi), \widetilde{V}(\xi)\right)\right] d\xi. \tag{4.19}$$

By $\lim_{\xi\to-\infty}\widetilde{U}(\xi) = 0$ and $\lim_{\xi\to-\infty}\widetilde{V}(\xi) = 0$, we have

$$\partial_1 f(0,0)\,\widetilde{U}(\xi) + \partial_2 f(0,0)\,\widetilde{V}(\xi) - f\left(\widetilde{U}(x), \widetilde{V}(\xi)\right) = O\left(\widetilde{U}^2(\xi) + \widetilde{V}^2(\xi)\right)$$

as $\xi \to -\infty$. Since it is not difficult to prove that $\sup_{\xi\in\mathbb{R}}\left\{\widetilde{V}(\xi)\,e^{-\gamma\xi}\right\} < +\infty$, the right-hand side of (4.19) is defined for λ with $0 < \mathrm{Re}\lambda < 2\gamma$. Now we use a property of Laplace transforms ([45], p. 58). Since $\widetilde{U}(x) > 0$, there exists a real η such that $\ell(\lambda)$ is analytic for $0 < \mathrm{Re}\lambda < \eta$ and $\ell(\lambda)$ has a singularity at $\lambda = \eta$. Hence, for $c > c^*$, $\ell(\lambda)$ is defined for $\mathrm{Re}\lambda < \lambda_1(c)$.

Using (4.19), we conclude that, for $0 < c < c^*$, there is no traveling wave front of (1.5) bounded between 0 and K. We argue by contradiction. Since $\Delta(\lambda, c)$ has no real zeroes, $\ell(\lambda)$ is defined for all λ such that $\mathrm{Re}\lambda > 0$. Also, (4.19) can be written as

$$\int_{-\infty}^{\infty} e^{-\lambda\xi}\left[\Delta(\lambda, c)\,\widetilde{U}(\xi) - \partial_1 f(0,0)\,\widetilde{U}(\xi) - \partial_2 f(0,0)\,\widetilde{V}(\xi) + f\left(\widetilde{U}(\xi), \widetilde{V}(\xi)\right)\right] d\xi = 0.$$

By $\Delta(\lambda, c) \to +\infty$ as $\lambda \to +\infty$, we have a contradiction.

From now on we study the case $c \geq c^*$. We rewrite (4.19) as

$$\begin{aligned}\int_{-\infty}^{0} \widetilde{U}(\xi)e^{-\lambda\xi}d\xi &= \frac{\int_{-\infty}^{\infty} e^{-\lambda\xi}\left[\partial_1 f(0,0)\,\widetilde{U}(\xi) + \partial_2 f(0,0)\,\widetilde{V}(\xi) - f\left(\widetilde{U}(\xi), \widetilde{V}(\xi)\right)\right] d\xi}{\Delta(\lambda, c)} \\ &\quad - \int_{0}^{\infty} \widetilde{U}(\xi)e^{-\lambda\xi}d\xi.\end{aligned}$$

Note that $\int_0^\infty \widetilde{U}(\xi)e^{-\lambda\xi}d\xi$ is analytic for $\mathrm{Re}\lambda > 0$. Also, the equation $\Delta(\lambda, c) = 0$ does not have any zero with $\mathrm{Re}\lambda = \lambda_1(c)$ other than $\lambda = \lambda_1(c)$. In fact, let $\lambda = \lambda_1(c) + i\beta$, then $\Delta(\lambda, c) = 0$ implies

$$\begin{aligned}& d\lambda_1^2(c) + \partial_1 f(0,0) \\ &\quad + \partial_2 f(0,0)\,e^{-\lambda_1(c)c\tau}\int_{-\infty}^{\infty} h(y)\,e^{-\lambda_1(c)y}\left[\cos\beta c\tau\cos\beta y - \sin\beta c\tau\sin\beta y\right]dy \\ &= d\beta^2 + c\lambda_1(c)\end{aligned} \tag{4.20}$$

and

$$2d\beta - c\beta - \partial_2 f\,(0,0)\,e^{-\lambda_1(c)c\tau}\int_{-\infty}^{\infty} h\,(y)\,e^{-\lambda_1(c)y}\,[\sin\beta c\tau\cos\beta y + \cos\beta c\tau\sin\beta y]\,dy = 0.$$

By using $\Delta\,(\lambda_1\,(c)\,,c) = 0$, then (4.20) can be rewrite as

$$-d\beta^2 = \partial_2 f\,(0,0)\,e^{-\lambda_1(c)c\tau}\int_{-\infty}^{\infty} h\,(y)\,e^{-\lambda_1(c)y}\left[2\left(\sin\frac{\beta c\tau}{2}\right)^2 + 2\left(\sin\frac{\beta y}{2}\right)^2 - 4\left(\sin\frac{\beta c\tau}{2}\right)^2\left(\sin\frac{\beta y}{2}\right)^2 + \sin\beta c\tau\sin\beta y\right]dy.$$

Since

$$\begin{aligned}
&2\left(\sin\frac{\beta c\tau}{2}\right)^2 + 2\left(\sin\frac{\beta y}{2}\right)^2 - 4\left(\sin\frac{\beta c\tau}{2}\right)^2\left(\sin\frac{\beta y}{2}\right)^2 + \sin\beta c\tau\sin\beta y\\
&= 2\left(\sin\frac{\beta c\tau}{2}\right)^2\left(\cos\frac{\beta y}{2}\right)^2 + 2\left(\cos\frac{\beta c\tau}{2}\right)^2\left(\sin\frac{\beta y}{2}\right)^2 + \sin\beta c\tau\sin\beta y\\
&\geq 4\left|\sin\frac{\beta c\tau}{2}\cos\frac{\beta c\tau}{2}\sin\frac{\beta y}{2}\cos\frac{\beta y}{2}\right| + \sin\beta c\tau\sin\beta y\\
&= |\sin\beta c\tau\sin\beta y| + \sin\beta c\tau\sin\beta y\\
&\geq 0,
\end{aligned}$$

we have $-d\beta^2 \geq 0$, which implies $\beta = 0$.

There are two cases to be considered:

(i) $\widetilde{U}\,(\xi)$ is increasing for small ξ; and
(ii) $\widetilde{U}\,(\xi)$ is not monotone for small ξ.

If case (i) holds, then we can choose a translation of $\widetilde{U}$, which is monotone for $\xi < 0$. Let $u(\xi) = \widetilde{U}(-\xi)$. Then Lemma 4.6 implies that our conclusion holds. If case (ii) holds, let

$$p = \frac{-c + \sqrt{c^2 + 4d\,(1 + L_1)}}{2d} \text{ and } \overline{U}(\xi) = \widetilde{U}(\xi)e^{p\xi}.$$

Then,

$$-d\overline{U}''(\xi) + (c + 2dp)\,\overline{U}'(\xi) = \left[(1 + L_1)\,\widetilde{U}(\xi) + f\left(\widetilde{U}(\xi), \widetilde{V}(\xi)\right)\right]e^{p\xi} > 0$$

for any $\xi \in \mathbb{R}$. We assert that $\overline{U}'(\xi) > 0$ for any $\xi \in \mathbb{R}$.

First, for any $\xi \in \mathbb{R}$ and $s > 0$, $\overline{U}\,(\xi - s) \leq \overline{U}(\xi)$. By a contradiction argument, if there exist ξ_1 and $s_1 > 0$ such that $\overline{U}\,(\xi_1 - s_1) > \overline{U}\,(\xi_1)$, then there exists $\xi_2 \in (\xi_1 - s_1, +\infty)$ such that

$$\overline{U}\,(\xi_2) = \min_{\xi\in[\xi_1 - s_1, +\infty)} \overline{U}(\xi),\ \overline{U}'\,(\xi_2) = 0$$

and $\overline{U}''\,(\xi_2) \geq 0$. Hence,

$$0 \geq -d\overline{U}''\,(\xi_2) + (c + 2dp)\,\overline{U}'\,(\xi_2) = \left[(1 + L_1)\,\widetilde{U}\,(\xi_2) + f\left(\widetilde{U}\,(\xi_2)\,,\widetilde{V}\,(\xi_2)\right)\right]e^{p\xi_2} > 0,$$

which is a contradiction and hence $\overline{U}'(\xi) \geq 0$ for any $\xi \in \mathbb{R}$.

Second, $\overline{U}'(\xi) > 0$ for any $\xi \in \mathbb{R}$. In fact, if there exists $\xi_3 \in \mathbb{R}$ such that $\overline{U}'(\xi_3) = 0$, then by $\overline{U}'(\xi) \geq 0$ for any $\xi \in \mathbb{R}$, $\overline{U}''(\xi_3) = 0$. Consequently,

$$0 = -d\overline{U}''(\xi_3) + (c + 2dp)\,\overline{U}'(\xi_3) = \left[\left(1 + \left|f_1'(0,0)\right|\right)\widetilde{U}(\xi_3) \right.$$
$$\left. + f\left(\widetilde{U}(\xi_3), \widetilde{V}(\xi_3)\right)\right] e^{p\xi_3} > 0,$$

which is also a contradiction. Therefore, for any $\xi \in \mathbb{R}$, $\overline{U}'(\xi) > 0$.

Let $\overline{\ell}(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda\xi}\overline{U}(\xi)d\xi$. Note that $\overline{\ell}(\lambda) = \ell(\lambda - p)$, we apply Lemma 4.6 to $\overline{U}(\xi)$ to see that $\lim_{\xi\to-\infty} \frac{\overline{U}(\xi)}{e^{(p+\lambda_1(c))\xi}}$ exists for $c > c^*$ and $\lim_{\xi\to-\infty} \frac{\overline{U}(\xi)}{\xi e^{(p+\lambda^*)\xi}}$ exists for $c = c^*$.

Thus, the proof is complete. □

Corollary 4.9 *Assume that (A1), (A2), (A3), (G1) and (G2) hold. Further suppose that* $\lambda_0 > \sqrt{\partial_1 f(0,0)}/\sqrt{d}$ *if (G2)(ii) and* $\partial_2 f(0,0) = 0$ *hold at the same time. For any* $c > c^*$, *if* $\widetilde{U}(x+ct)$ *is a traveling wave front of (1.5) satisfying (1.6) and* $0 < \widetilde{U}(\xi) < K$, *then* $\widetilde{U}(\xi) = U(\xi + \xi_0)$, *where* $U(x+ct)$ *is the traveling wave front with wave speed c given in Theorem 2.6 and* $\xi_0 \in \mathbb{R}$ *is a constant.*

Proof Fix $\xi = x + ct$. In view of Theorem 4.8, there exists $\rho_0 \in \mathbb{R}$ such that $\lim_{\xi\to-\infty} \frac{\widetilde{U}(\xi)}{e^{(\lambda_1(c))\xi}} = \rho_0$. By (4.2), there are

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} \left| \frac{\widetilde{U}(x+ct)}{U(x+ct+\xi_0)} - 1 \right| = 0.$$

Then for any $\varepsilon > 0$, let $t \to +\infty$, we have $\left|\frac{\widetilde{U}(\xi)}{U(\xi+\xi_0)} - 1\right| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that $\frac{\widetilde{U}(\xi)}{U(\xi+\xi_0)} - 1 = 0$, that is $\widetilde{U}(\xi) = U(\xi + \xi_0)$. This completes the proof. □

5 Applications

In this section, we shall apply our results developed in Sects. 2–4 to some specific biological and epidemiological models.

5.1 An Age-Structured Population Model

Consider the age-structured reaction diffusion model of a single species [3]:

$$\frac{\partial u_m}{\partial t} = d_m \frac{\partial^2 u_m}{\partial x^2} + a_0 e^{-\gamma\tau} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi d_i \tau}} e^{\frac{-(x-y)^2}{4d_i\tau}} u_m(y, t-\tau)\,dy - b_0 u_m^2, \qquad (5.1)$$

where u_m denotes the number of mature members of a single species population, the delay τ is the time taken from birth to maturity, $b_0 u_m^2$ represents death of matures, the remaining delayed term is the adult recruitment, which represents the rate of leaving the immature and entering the mature class, and d_i is the constant diffusion rate of the immature species. Obviously, (5.1) has two equilibria $u_m \equiv 0$ and $u_m \equiv \frac{a_0}{b_0} e^{-\gamma\tau}$. By using a perturbation argument together with Fredholm orthogonality theory, Al-Omari and Gourley [3] proved that there exists $c_* > 0$ such that for every $c > c_*$ there exists a traveling wave front for (5.1) with speed c when d_i is sufficiently *small*. By taking $\varepsilon = \tau$ and the kernel function $g(x,t)$ with the form (1.2)(iv) in Ai [1], then for any $c \geq 2\sqrt{a_0}$, we know that for sufficient *small* $\tau > 0$, (5.1) has an unique traveling wave front with wave speed c and satisfying some additional conditions, see ([1], Theorem 1.2). For sufficiently small $\tau > 0$, the existence of traveling wave fronts of (5.1) can also be obtained from Theorem 1.2 of Ou and Wu [34]. Following Theorems 4.2 and 4.3 of Thieme and Zhao [42], there exists $c^* > 0$ such that for $c \geq c^*$, (5.1) has a monotone traveling wave front with speed c, and (5.1) admits no traveling wave front connecting 0 and $\frac{a_0}{b_0}e^{-\gamma\tau}$ with speed $c \in (0, c^*)$. In particular, the traveling wave front with speed $c > c^*$ is unique up to a translation in the class of monotone solutions and c^* is the spreading speed.

Let $h(y) = \frac{1}{\sqrt{4\pi d_i \tau}} \exp\left\{\frac{-y^2}{4d_i\tau}\right\}$ and $f(u, v) = a_0 e^{-\gamma\tau} v - b_0 u^2$. Then (A1), (A2), (G1) and (G2) hold. We now verify that (A3) holds. Taking $\delta \in (0, 1)$ and $a = a(\delta) = (1-\delta)b_0$. Then for $\theta \in (0, \delta]$,

$$
\begin{aligned}
&(1-\theta) f(u, v) - f((1-\theta) u, (1-\theta) v) \\
&\quad = (1-\theta)\left[a_0 e^{-\gamma\tau} v - b_0 u^2\right] - (1-\theta)\left[a_0 e^{-\gamma\tau} v - b_0 (1-\theta) u^2\right] \\
&\quad = -(1-\theta) b_0 \theta u^2 \leq -(1-\delta) b_0 \theta u^2 = -a\theta u^2,
\end{aligned}
$$

which implies that (A3) holds if we take $\alpha = 2$ and $\beta = 0$. Also, we can prove that any traveling wave front of (5.1) has an upper bound $\frac{a_0}{b_0}e^{-\gamma\tau}$. Thus, we have the following result.

Theorem 5.1 *There exists $c^* > 0$ such that for every $c \geq c^*$,* (5.1) *has a traveling wave front which is increasing and satisfies* (1.6) *with $K = \frac{a_0}{b_0}e^{-\gamma\tau}$. Furthermore, for $c > c^*$, such a traveling wave front is unique (up to a translation) and asymptotically stable with phase shift in the sense under Theorem 4.1, and* (5.1) *has no nonnegative traveling wave front with speed $c \in (0, c^*)$.*

Remark 5.2 Al-Omari and Gourley [3] assumed that $d_i > 0$ is sufficiently small and $c > c^*$, where d_i depends on the size of τ. Our result is independent of the size of d_i. On the other hand, if $d_i = 0$, that is to say, the immatures are immobile, then $h(x) = \delta(x)$. In this case, our result still holds. Though the existence of traveling wave fronts can be derived from the results in Ai [1] and Ou and Wu [34], it is only valid for sufficient small delay $\tau > 0$. In [42], there are no results about the stability of traveling wave fronts of (5.1) and the uniqueness of traveling wave fronts of (5.1) in [42] is only valid for monotone waves. However, our result on the uniqueness of traveling wave fronts of (5.1) in Theorem 5.1 is valid not only for monotone waves but also for nonmonotone waves. Thus, Theorem 5.1 improves and complements the results in [3,42].

5.2 A Vector Disease Model

Consider a vector-disease model with nonlocal delay [36]

$$
\frac{\partial u}{\partial t} = d\Delta u - a_0 u + b_0 [1-u] \int_{-\infty}^{t} \int_{-\infty}^{\infty} F(x, y, t, s) u(y, s)\, dy ds, \tag{5.2}
$$

where $x \in \mathbb{R}$ and $t > 0$, d is the diffusion constant of the infectious host, and $u(x, t)$ is the normalized spatial density of the infectious host at time t in x. If $b_0 > a_0$, then (5.2) has two equilibria 0 and $(b_0 - a_0)/b_0$. Ruan and Xiao [36] considered the existence of traveling waves that connect 0 and $\frac{b_0 - a_0}{b_0}$ for the two cases: (i) without delay, i.e., $F(x, y, t, s) = \delta(x-y)\delta(t-s)$; (ii) with local delay, i.e., $F(x, y, s, t) = \delta(x-y)\frac{t-s}{\tau^2}e^{-\frac{(t-s)^2}{\tau}}$, τ is sufficiently *small*, where the geometric singular perturbation theory of Fenichel [18] was used to obtain a traveling wave front when τ is sufficiently small. Under the assumptions that $F(x, y, t, s) = F(x-y, t-s)$, $F(x, t) = F(-x, t)$, $\int_0^\infty \int_{-\infty}^\infty F(y, s)dyds = 1$ and $\int_0^\infty \int_{-\infty}^\infty F(y, s)e^{\lambda(y-cs)}dyds < \infty$ for all $c \geq 0$ and $\lambda \geq 0$, Zhao and Xiao [50] further showed that there exists a constant c^* such that for $c \geq c^*$, (5.2) has a monotone traveling wave front with speed c, and (5.2) admits no traveling wave front connecting 0 and $\frac{b_0-a_0}{b_0}$ with speed $c \in (0, c^*)$. In particular, c^* is the spreading speed.

It is easy to show that any traveling wave front of (5.2) has an upper bound $\frac{b_0-a_0}{b_0}$. Let $f(u, v) = -a_0u + b_0(1-u)v$. Then it is easy to verify that (A1), (A2) and (A3) hold. In fact, if we take $\delta \in (0, 1)$, $a = a(\delta) = (1-\delta)b_0 > 0$, $\alpha = \alpha(\delta) = 1$ and $\beta = \beta(\delta) = 1$, then for $\theta \in (0, \delta]$,

$$(1-\theta) f(u, v) - f((1-\theta) u, (1-\theta) v) = -(1-\theta)\theta uv \leq -(1-\delta)\theta uv = -a\theta uv.$$

Thus, if we choose $F(x, y, t, s) = h(x-y)\delta(t-s-\tau)$, where $h(x)$ satisfies (G1) and (G2), then we have the following result.

Theorem 5.3 *Assume $b_0 > a_0 \geq 0$ and $F(x, y, t, s) = h(x-y)\delta(t-s-\tau)$, where $h(x)$ satisfies (G1) and (G2). Then there exists $c^* > 0$ such that for every $c \geq c^*$, (5.2) has a traveling wave front which is increasing. Furthermore, for $c > c^*$, such a traveling wave front is unique (up to a translation) and asymptotically stable with phase shift in the sense under Theorem 4.1 and for $0 < c < c^*$, (5.2) has no nonnegative traveling wave fronts.*

Remark 5.4 In [50], there are no results on the uniqueness and stability of traveling wave fronts of (5.2). Obviously, Theorem 5.3 improves and complements the results in [36,50]. In particular, our results holds for the case $h(x) = \frac{1}{2\rho}e^{-\frac{|x|}{\rho}}$.

5.3 The Nicholson's Blowflies Equation with Nonlocal Delay

Consider the following diffusive Nicholson's Blowflies equation with nonlocal delay [24]:

$$\frac{\partial u}{\partial t} = d\Delta u - \tau u + b_0\tau((g * u)(x, t)) \exp[-(g * u)(x, t)], \tag{5.3}$$

where

$$(g * u)(x, t) = \int_{-\infty}^{t} \int_{-\infty}^{\infty} g(x, y, t, s)u(x, s)dyds$$

and $1 < b_0 \leq e$. It is easy to see that (5.3) has two equilibria 0 and $\ln b_0$. Let $f(u, v) = -\tau u + b_0\tau ve^{-v}$. Then (A1), (A2), and (A3) hold.

Theorem 5.5 *Assume that $g(x, y, t, s) = h(x-y)\delta(t-s-\tau_0)$ and $h(x)$ satisfies (G1) and (G2). Then there exists $c^* > 0$ such that for every $c > c^*$, (5.3) has a traveling wave front which is increasing. Furthermore, for $c > c^*$, such a traveling wave front is unique (up to a translation) and asymptotically stable with phase shift in the sense under Theorem 4.1 and for $0 < c < c^*$, (5.3) has no nonnegative traveling wave front.*

Recently, Li et al. [24] proved that any traveling wave front of (5.3) has an upper bound $\ln b_0$. Thus, Theorem 5.5 follows from Theorems 2.6, 4.1 and Corollary 4.9.

Remark 5.6 In [24], we established the existence of traveling wave fronts of (5.3) for several types of the kernel function g. Theorem 5.5 further shows that such a traveling wave front is unique up to translation and is asymptotically stable. We note that the existence of traveling wave fronts of (5.3) can also be obtained from the results in [1,34], but the time delay and spatial nonlocality, for example $\tau_0 > 0$ and $\rho > 0$ (see the next section), have to be sufficiently small. For the case $b_0 > e$, that is, ve^{-v} is nonmonotone on $v \in [0, \ln b_0]$, we refer to [16,17,29].

Remark 5.7 When $h(y) = \delta(y)$, $K = 1$, $\partial_1 f(0,0) + \partial_2 f(0,0) > 0$ and $\partial_2 f(u,v) \geq 0$ for $(u,v) \in [0,1]^2$, Schaaf [39] showed that for every $c > c^*$, (1.5) has a unique traveling wave front (up to a translation) and for sufficiently small τ, the traveling wave front is linearly stable with a weighted norm, and for $0 < c < c^*$, (1.5) has no traveling wave front bounded between 0 and 1. But his results are not valid for (5.1), (5.2) ($a_0 > 0$) and (5.3). Obviously, our results extend and complement that established by Schaaf [39].

6 The Effects of Delay and Nonlocality on the Spreading Speed

In this section, we shall consider the effects of the delay and nonlocality in (1.5) on the spreading speed c. In fact, Li et al. [24], Schaaf [39] and Zou [51] have considered the similar problem for special models. Here we shall consider the general Eq. 1.5. In particular, we shall see that the monotone condition $\partial_2 f(0,0) \geq 0$ plays an important role in this problem. From Theorem 2.6 and Corollary 4.9, we know that the c^* defined by Lemma 2.2 is the minimal wave speed of traveling wave fronts. From [27,42,48,50], we know that the minimal wave speed c^* coincides with the spreading speed.

6.1 The Effect of Delay on the Spreading Speed

For the sake of simplicity, we take $h(y) = \delta(y)$. Then $G(\lambda) = 1$ for any $\lambda > 0$ and hence,

$$\Delta(\lambda, c) = d\lambda^2 - c\lambda + \partial_1 f(0,0) + \partial_2 f(0,0)\, e^{-\lambda c\tau}, \ \lambda \in \mathbb{C}.$$

From $\Delta(\lambda^*, c^*) = 0$ and $\frac{\partial}{\partial\lambda}\Delta(\lambda, c^*)\big|_{\lambda=\lambda^*} = 0$, we see that $c^* = c^*(\tau)$ and $\lambda^* = \lambda^*(\tau)$ are differentiable functions with respect to τ. Furthermore, it is easy to see that

$$\frac{dc^*}{d\tau} = -\frac{c^*\partial_2 f(0,0)\, e^{-\lambda^* c^* \tau}}{1 + \tau\partial_2 f(0,0)\, e^{-\lambda^* c^* \tau}} \leq 0,$$

which implies that if $\partial_2 f(0,0) > 0$, then the delay will slow the spreading speed, and if $\partial_2 f(0,0) = 0$, then the delay is independent of the spreading speed.

6.2 The Effect of Nonlocality on the Spreading Speed

In order to show the effect of nonlocality of (1.5) on the spreading speed c^*, without loss of generality, we take $\tau = 0$ and $h(y) = \frac{1}{\sqrt{4\pi\rho}} e^{-\frac{y^2}{4\rho}}$. Then $G(\lambda) = e^{\rho\lambda^2}$ for any $\lambda > 0$ and

$$\Delta(\lambda, c) = d\lambda^2 - c\lambda + \partial_1 f(0,0) + \partial_2 f(0,0)\, e^{\rho\lambda^2}, \ \lambda \in \mathbb{C}.$$

Similarly, $c^* = c^*(\rho)$ and $\lambda^* = \lambda^*(\rho)$ are differentiable functions with respect to ρ and

$$\frac{dc^*}{d\rho} = \lambda^* \partial_2 f(0,0)\, e^{\rho\lambda^{*2}} \geq 0. \tag{6.1}$$

It is well-known that the constant ρ is a measure of the nonlocality of (1.5). Then (6.1) implies that if $\partial_2 f(0, 0) > 0$, then the nonlocality will increase the spreading speed and if $\partial_2 f(0, 0) = 0$, then the nonlocality is independent of the spreading speed.

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