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Traveling Waves in a Convolution Model for Phase Transitions

Peter W. Bates, Paul C. Fife,
Xiaofeng Ren & Xuefeng Wang

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Abstract

The existence, uniqueness, stability and regularity properties of traveling-wave solutions of a bistable nonlinear integrodifferential equation are established, as well as their global asymptotic stability in the case of zero-velocity continuous waves. This equation is a direct analog of the more familiar bistable nonlinear diffusion equation, and shares many of its properties. It governs gradient flows for free-energy functionals with general nonlocal interaction integrals penalizing spatial nonuniformity.

1. Introduction

Consider the following evolution problem for functions $u(x,t)$ defined on $\mathbb{R} \times \mathbb{R}^+$:

$$u_t = J * u - u - f(u) \tag{1.1}$$

where the kernel J of the convolution $J * u(x) = \int_{-\infty}^{\infty} J(x-y)u(y)dy$ is nonnegative, even, with unit integral, and the function f is bistable. The analysis to follow uses some properties that the linear operator A, defined by $Au = J * u - u$, shares with the Laplacian, such as a form of maximum principle. One can also see that A is a nonpositive operator on $L^2(\mathbb{R})$ by taking Fourier transforms since $\hat{J}(s) \equiv \int_{-\infty}^{\infty} e^{isx} J(x)dx$ is real and bounded by 1.

Thus, we see that (1.1) is a nonlocal analog of the usual bistable reaction-diffusion equation

$$u_t = \Delta u - f(u).$$

As such, (1.1) as well as this equation may model a variety of physical and biological phenomena involving media with properties varying in space. The possible advantages of (1.1) lie in the fact that much more general types of interactions between states at nearby locations in the medium can be accounted for.

Integrodifferential equations with many of the properties of (1.1) have been derived and studied from the point of view of certain continuum limits of dynamic Ising models [DGP, DOPT1–3, KS1–3]. For an excellent review, see [S].

Our principal motivation for studying (1.1) lies in the fact that it is a gradient flow for a natural generalization of the usual Ginzburg-Landau functional for an order parameter describing the state of a solid material. Let $u(x,t)$ be such an order parameter, representing the state at position x and time t. We call the states $u = \pm 1$ "pure"; they may represent two different orientations of a perfect crystal, for example, or different variants of a given crystal. Values of u between -1 and $+1$ then may represent disordered states intermediate between the pure states. Thus u^2 rather than u would be a measure of the order.

We postulate a Helmholtz free-energy functional of the form

$$E(u) = \frac{1}{4}\iint_{\mathbb{R}^2} J(x-y)(u(x)-u(y))^2 dx\,dy + \int_{\mathbb{R}} F(u(x))\;dx, \tag{1.2}$$

where $J(r) = J(-r) \geqq 0$ is a measure of the energy resulting from $u(x)$ being different from $u(x \pm r)$, and F is a double-well function having (not necessarily equal) minima at ± 1, representing the bulk energy density of a state with u constant. This can be considered as the continuum limit of an analogous free-energy functional on a one-dimensional discrete lattice.

Note that the first term in (1.2) penalizes spatially inhomogeneous materials while the second term penalizes states which take values other than ± 1. If we consider the L^2-gradient flow associated with (1.2), we expect to observe the effects of any competition between (1) the attraction of the material to one or the other of the pure states $u = \pm 1$ due to F, depending on which domain of attraction it is in at a given location, and (2) the propensity to become homogeneous. The competition arises because a spatial variation in the domain of attraction in which the material lies means the attraction property induces inhomogeneity.

A short calculation reveals that if $\int_{\mathbb{R}} J = 1$, then the equation representing the L^2-gradient flow is (1.1), where $*$ is convolution and $f = F'$ is as pictured in Figure 1, below.

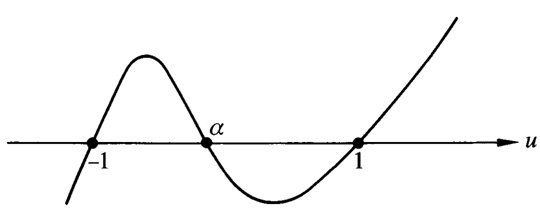

Figure 1

If the minima of F have equal depth, i.e., if $\int_{-1}^{1} f(u)du = 0$, as would be the case when the two pure states are variants of a single crystalline structure, then it is conceivable that a stable state u exists taking values near -1 on an interval of the form $(-\infty, a]$ and values near 1 on an interval of the form $[b, \infty)$ with a simple transition occurring on $[a, b]$. On the other hand, if one pure state has higher energy than the other, say $F(1) > F(-1)$, due for example to intrinsic stress from a distribution of dislocations, then an initial state of the type mentioned above could evolve to decrease its energy on a large interval $[-k, k]$ by simply moving to the right. One may expect the gradient flow to produce a stable traveling wave in this case. Such a solution may be thought of as a planar front propagating into the region of higher energy, hence the one-dimensional character of the solution.

Such fronts are well known in the more familiar case when the interaction (first) term indicated in (1.2) is replaced by $\frac{1}{2}\int_{\mathbb{R}}(u_x(x))^2dx$; then in place of (1.1), we have $u_t = u_{xx} - f(u)$. When it is replaced by an integral of a quadratic form in the first and higher derivatives of u, see [GJ] and in the case of stationary fronts, [PT, BR]. Finally, for results concerning traveling waves for systems obtained when temperature is dynamically coupled to order parameter, see [CN] or [BFGJ1] for the case when the integrand of the analogous interaction integral in E is $(u_x)^2$ or a higher-order expression, respectively.

We are unaware of any results giving traveling waves for (1.1) or for (1.1) coupled with a heat equation. However, Ermentrout & McLeod [EM] and De Masi and others [DGP,OT] have shown the existence of traveling waves for other nonlinear integrodifferential equations. In the latter case, the equation is

$$u_t = \tanh\{\beta(J * u + h)\} - u,$$

where $\beta > 1$ and h are constants. Stability results were also given. This equation arises from a statistical mechanics approach to phase transitions. In [P], Penrose made a comparison between spatially discrete versions of this equation and (1.1) for a particular function f.

Concerning (1.1) we make the following assumptions which will be in force for the remainder of the paper:

(H1) $J \in C^1(\mathbb{R}),\ J(s) = J(-s) \geqq 0$ for all s, $\int_{\mathbb{R}} J = 1$, $\int_{\mathbb{R}} J(y)|y|dy < \infty$, $J' \in L^1(\mathbb{R})$,

(H2) $f \in C^2(\mathbb{R}), f(\pm 1) = 0 < f'(\pm 1)$, f has only one zero, α, in $(-1, 1)$, and no zeros outside $[-1, 1]$.

We do not require (as in the last cited papers) that J have compact support.

We seek solutions to (1.1) of the form $u(x, t) = \hat{u}(x - ct)$ for some velocity c, with u having limits ± 1 at $x = \pm\infty$. Thus, making the change of variables $\xi = x - ct$, we seek a function $\hat{u}(\xi)$ and a constant c satisfying

$$J * \hat{u} - \hat{u} + c\hat{u}' - f(\hat{u}) = 0 \text{ on } \mathbb{R}, \tag{1.3}$$

$$\hat{u}(\pm\infty) = \pm 1. \tag{1.4}$$

Our approach, inspired by a similar idea of Ermentrout, McLeod and Xie in [EM, MX], is to embed (1.3) in a family of equations parametrized by $\theta \in [0,1]$. When $\theta = 0$, the equation is already known to have a (traveling-wave) solution; when $\theta = 1$, we have (1.3). A continuation argument allows us to pass in increments from 0 to 1 in θ, obtaining existence for all values in the process. From now on we drop the caret and simply write u. The family we choose is

$$\theta(J * u - u) + (1-\theta)u'' + cu' - f(u) = 0 \text{ on } \mathbb{R}. \tag{1.5}$$

Unlike previous applications of this method, our artificial problems (those with $\theta < 1$) are therefore of an essentially different type from (1.3); a second-order derivative term is introduced, which degenerates to the desired problem as $\theta \to 1$.

When $\theta = 0$ it is well known that (1.5) has a solution (u, c) that is unique (up to shifts in the independent variable) such that (1.4) holds. Furthermore, $u' > 0$ on $\mathbb{R}$. The Implicit Function Theorem is used to obtain the same conclusion for all $\theta \in [0,1)$. Using the weak formulation of the equation allows us to pass to the limit as $\theta \to 1$, having first shown that the speeds $c = c_\theta$ are bounded. This is done in Section 2.

In Section 3 we examine the regularity properties of solutions, observing that waves with nonzero speed are smooth while (under an additional hypothesis on f) monotone stationary waves have at most one point of discontinuity and are smooth elsewhere. We also characterize nonlinearities f which give rise to discontinuous stationary waves and at the same time determine the jump in the solution.

In Section 4 we prove uniqueness of the solution (u, c), up to translation in ξ, and establish a nonlinear stability result, showing that solutions to (1.1) with certain initial functions, u_0, remain close to traveling waves for all time.

Finally, in Section 5 we establish the global asymptotic stability of continuous stationary waves (see Theorem 5.1 and 5.5).

2. Existence of Weak Solutions

Under the assumptions (H1), (H2) we will establish the existence of traveling or stationary waves through a series of lemmas.

Lemma 2.1. *Let* $\theta \in [0,1)$ *and let* u *satisfy* (1.4) *and* (1.5). *Then* $u(\xi) \in (-1,1)$ *for all* $\xi \in \mathbb{R}$.

Proof. First, it is clear that any L^∞ solution of (1.5) is of class C^3. If u has a global maximum at ξ_0 with $u(\xi_0) \geqq 1$, then $u(\xi) \leqq u(\xi_0)$ for all $\xi \in \mathbb{R}$, and $\int_{\mathbb{R}} J = 1$ implies $(J * u - u)(\xi_0) \leqq 0$. Since u is not constant, ξ_0 can be chosen so that $J(\xi - \xi_0) \neq 0$ for some ξ with $u(\xi) < u(\xi_0)$; this shows that the inequality is strict. That the other terms in (1.5) are nonpositive when evaluated at ξ_0 provides a contradiction. A similar argument shows that $u(\xi) > -1$ for all ξ. $\square$

Now suppose that (u_0, c_0) is a solution to (1.4) and (1.5) for some $\theta_0 \in [0,1)$ and suppose that $u_0' > 0$ on $\mathbb{R}$. We shall use the Implicit Function Theorem to obtain a solution for $\theta > \theta_0$. We take perturbations in the space

$$X_0 = \{\text{uniformly continuous functions on } \mathbb{R} \text{ which vanish at } \pm\infty\}.$$

Let $L = L(u_0, c_0; \theta_0)$ be the linear operator defined in X_0 by

$$\text{dom } L = X_2 \equiv \{v \in X_0 : v'' \in X_0\},$$

$$Lv = \theta_0(J * v - v) + (1-\theta_0)v'' + c_0 v' - f'(u_0)v. \tag{2.1}$$

Lemma 2.2. *L has* 0 *as a simple eigenvalue.*

Proof. The result is known for $\theta_0 = 0$, so we assume that $\theta_0 > 0$. Clearly, $p \equiv u_0'$ is an eigenfunction of L with corresponding eigenvalue 0, so the only question is simplicity. Suppose that ϕ is another eigenfunction with eigenvalue 0, and assume that ϕ takes on positive values at some points. We shall show that p and ϕ are linearly dependent by considering the family of eigenfunctions

$$\phi_\beta \equiv p + \beta\phi, \quad \beta \in \mathbb{R}.$$

Let $\bar\beta = \sup\{\beta < 0 : \phi_\beta(\xi) < 0 \text{ for some } \xi\}$. Then $\bar\beta$ is well-defined since ϕ is positive at some points. Recall that $p > 0$ on $\mathbb{R}$. For $\beta < \bar\beta$, let ξ_β be a point where ϕ_β achieves its minimum, so that $(J * \phi_\beta - \phi_\beta)(\xi_\beta) > 0$, $\phi_\beta''(\xi_\beta) \geqq 0 = \phi_\beta'(\xi_\beta)$. It follows that ξ_β lies in that interval on which $f'(u_0)$ is negative; that interval is bounded since $u_0(\xi) \to \pm 1$ as $\xi \to \pm\infty$. Now take the limit $\beta \nearrow \bar\beta$ along a sequence such that ξ_β converges to some $\bar\xi$ and observe that $\phi_{\bar\beta}(\bar\xi) = 0 \leqq \phi_{\bar\beta}(\xi)$ for all $\xi \in \mathbb{R}$. It follows that $0 \leqq (J * \phi_{\bar\beta} - \phi_{\bar\beta})(\bar\xi) = (1 - 1/\theta_0)\phi_{\bar\beta}''(\bar\xi) \leqq 0$ and so $J * \phi_{\bar\beta} - \phi_{\bar\beta} = 0$ at $\bar\xi$. A short computation shows that if $[a,b] \subset \text{supp}(J)$, then $\phi_{\bar\beta}(\xi) = \phi_{\bar\beta}(\bar\xi)$ for $\xi \in [\bar\xi - b, \bar\xi - a] \cup [\bar\xi + a, \bar\xi + b]$, and then an induction argument shows that $\phi_{\bar\beta}$ must be constant, namely, zero. Hence, p and ϕ are linearly dependent. $\square$

The formal adjoint of L is given by

$$L^* v = \theta_0(J * v - v) + (1-\theta_0)v'' - c_0 v' - f'(u_0)v,$$

and it is easy to show that 0 is a simple eigenvalue of L^* also. Moreover, 0 is isolated since that would be true if the first term in the expression for L^* were missing, and one can show that this term does not change the essential spectrum. Let ϕ^* be the corresponding eigenfunction; then by the Fredholm Alternative, for $g \in X_0$, $Lv = g$ has a solution in X_2 if and only if $\int_{\mathbb{R}} g\phi^* = 0$. We can now give the continuation result:

Lemma 2.3. *With θ_0, u_0 and c_0 as above, there exists $\eta > 0$ such that for $\theta \in [\theta_0, \theta_0 + \eta)$, problem* (1.4), (1.5) *has a solution* (u, c).

Proof. Without loss of generality, we may assume $u_0(0) = 0$. For $w = (v, c) \in X_2 \times \mathbb{R}$ and $\theta \in \mathbb{R}$ define

$$G(w,\theta) \equiv (\theta(J * (u_0 + v) - (u_0 + v)) + (1-\theta)(u_0 + v)'' + (c_0 + c)(u_0 + v)' \\ - f(u_0 + v), v(0))$$

so that $G\colon (X_2 \times \mathbb{R}) \times \mathbb{R} \to X_0 \times \mathbb{R}$ is of class C^1. We have $G(0, \theta_0) = (0, 0)$ and

$$DG \equiv \frac{\partial G}{\partial w}(0, \theta_0) = \begin{bmatrix} L & u_0' \\ \delta & 0 \end{bmatrix},$$

where $\delta v \equiv v(0)$.

If we can show that $DG : X_2 \times \mathbb{R} \to X_0 \times \mathbb{R}$ is invertible, then the lemma would follow from the Implicit Function Theorem. To that end, let $h \in X_0$ and $b \in \mathbb{R}$ and consider

$$DG(v, c) = (h, b),$$

that is,

$$Lv + cu_0' = h, \tag{2.2}$$

$$v(0) = b. \tag{2.3}$$

As we observed above, (2.2) is solvable if and only if

$$c \int_{\mathbb{R}} u_0' \phi^* = \int_{\mathbb{R}} h\phi^*.$$

This determines c, since the simplicity of the eigenvalue 0 of L ensures that the integral on the left is not zero. With this value of c the solution to (2.2) is determined up to an additive term $\gamma u_0'$, where $\gamma \in \mathbb{R}$. Now (2.3) is satisfied by a unique choice of γ since $u_0'(0) > 0$. Thus, DG is invertible and the lemma is proved. □

In order to continue to the whole interval $\theta \in [0, 1)$, and for other considerations, it is necessary to show that for all $\theta \in [\theta_0, \theta_0 + \eta)$ the solution u obtained in the previous lemma is strictly increasing. We have

Lemma 2.4. *Let $\theta \in [\theta_0, \theta_0 + \eta)$ and (u, c) be the solution given above. Then $u'(\xi) > 0$ for all $\xi \in \mathbb{R}$.*

Proof. First, we show that $u'(\xi) \geqq 0$. Suppose not. Then

$$\bar{\theta} \equiv \inf\{\theta > \theta_0 \colon u'(\xi) < 0 \text{ for some } \xi \in \mathbb{R}\}$$

is well defined. Note that we have suppressed the θ-dependence of u and c. Since the set over which this infimum is taken is open, there exists a decreasing sequence $\theta_n \searrow \bar{\theta}$ on which $p_\theta(\xi) \equiv u'(\xi)$ has a negative minimum at some point ξ_θ. Also,

$$\theta(J * p_\theta - p_\theta) + (1-\theta)p_\theta'' + cp_\theta' - f'(u)p_\theta = 0 \tag{2.4}$$

so that at $\xi_\theta, f'(u)p_\theta \geqq 0$ and therefore ξ_θ is bounded, independent of θ. Take a limit along a sequence such that $\theta \searrow \bar{\theta}$ and $\xi_\theta \to \bar{\xi}$. Then at $\theta = \bar{\theta}$ and $\xi = \bar{\xi}$, with $\bar{p} = p_{\bar{\theta}}$ we have

$$\begin{aligned} 0 &= \bar{\theta}(J * \bar{p} - \bar{p}) + (1 - \bar{\theta})\bar{p}'' + c\bar{p}' - f'(u)\bar{p} \\ &= \bar{\theta}(J * \bar{p} - \bar{p}) + (1 - \bar{\theta})\bar{p}'' \\ &\geqq \bar{\theta}(J * \bar{p} - \bar{p}) \geqq 0. \end{aligned}$$

As in the proof of Lemma 2.2, we can show that $\bar{p}$ is constant, namely 0. This is impossible, since $\bar{p} = u'$ and u is not constant. Thus, for all $\theta \in [\theta_0, \theta_0 + \eta)$ we have $u' \geqq 0$. The preceding integro-differential inequality shows that u' cannot achieve a minimum value of 0 and so the lemma is proved. □

Remark. Eventually we will pass to the limit $\theta \to 1$ and obtain a solution u to (1.3) which is monotone. By the inequality above with $\theta = 1$, one can see that u' can never be zero. This observation will be used later in Section 4.

We wish to continue the solution branch to $\theta \in [0, 1)$. To do this we must establish some a priori bounds.

Lemma 2.5. *Suppose that for* $\theta \in [0, \bar{\theta})$ *there exists a solution* (u_θ, c_θ) *to* (1.4), (1.5) *and that* $\bar{\theta} < 1$. *Then* $\{u_\theta : \theta \in [0, \bar{\theta})\}$ *is bounded in* C^3.

Proof. Fix $\theta \in [0, \bar{\theta})$ and denote (u_θ, c_θ) by (u, c). The previous lemma shows that $u' > 0$ on $\mathbb{R}$, and since $u'(\xi) \to 0$ as $|\xi| \to \infty$, u' has a maximum value. At the point ξ where that occurs we have $u'' = 0$ and so

$$\|cu'\|_\infty = f(u(\xi)) - \theta(J * u - u)(\xi) \leqq K \tag{2.5}$$

for some K, independent of θ, by Lemma 2.1.

Returning to equation (1.5), since $\theta < \bar{\theta} < 1$, we have

$$\|u''\|_\infty \leqq 2K/(1 - \bar{\theta}),$$

and a standard interpolation argument yields

$$\|u'\|_\infty \leqq 2 + K/(1 - \bar{\theta}).$$

The bound in C^3 follows from differentiating equation (1.5). □

Remark. The family $\{u_\theta : 1 - \delta < \theta < 1\}$ is bounded in C^2 for some $\delta > 0$ provided that $1 + f'(u) > 0$ and $f \in C^2$. To see this, note that $p = u'$ satisfies the variational equation (2.4) and that this may be rewritten as

$$\theta J' * u + (1 - \theta)p'' + cp' - (\theta + f'(u))p = 0. \tag{2.6}$$

At the point ξ_0 where p achieves its (positive) maximum one has

$$(\theta + f'(u))p \leqq \theta J' * u$$

and so $p = u'$ is uniformly bounded for θ near 1. Differentiating equation (2.6) and using the same argument allows us to obtain uniform bounds on u'' as desired.

To pass to the limit $\theta \nearrow \bar{\theta}$ in the equation, we must also bound c_θ.

Lemma 2.6. *Under the hypotheses of Lemma 2.5, $\{c_\theta : \theta \in [0, \bar{\theta})\}$ is bounded.*

Proof. Suppose, on the contrary, that this set is unbounded. Then there would exist a sequence $\{\theta_n\}$ with $c_n \equiv c_{\theta_n} \to \pm\infty$ as $n \to \infty$. Writing $u_n \equiv u_{\theta_n}$, we see from (2.5) that

$$\|u_n'\|_\infty \to 0 \text{ as } n \to \infty. \tag{2.7}$$

We now assert that for any $\varepsilon > 0$ and closed interval $U \subset (-1, 1)$ of positive length there exists ξ_n such that $u_n(\xi_n) \in U$ and $|u_n''(\xi_n)| < \varepsilon$. If this were not the case, there would exist such an interval U and a number $\varepsilon > 0$ such that $|u_n''| \geqq \varepsilon$ on the interval $[a_n, b_n]$, where $u_n([a_n, b_n]) = U$. For definiteness assume $u_n'' \geqq \varepsilon$ on $[a_n, b_n]$. Then

$$2\|u_n'\|_\infty \geqq u_n'(b_n) - u_n'(a_n) \geqq \varepsilon(b_n - a_n),$$

and by the Mean Value Theorem, the length of U, is

$$|U| = u_n(b_n) - u_n(a_n) \leqq \|u_n'\|_\infty(b_n - a_n).$$

It follows that $2\|u_n'\|_\infty^2 \geqq \varepsilon\,|U|$ contradicting (2.7), thus establishing the assertion.

Now pick $\eta > 0$ and small and let U be such that $f(u) < -\eta$ for all $u \in U$ in the case that $c_n \to +\infty$ and such that $f(u) > \eta$ for all $u \in U$ in the case that $c_n \to -\infty$. Take $\varepsilon = \eta/2$ and take $\{\xi_n\}$ to be the sequence given by the assertion above. Then (1.5) with $\theta = \theta_n, c = c_n$ and $u = u_n$ evaluated at ξ_n gives

$$\eta < |c_n u_n' - f(u_n)| < \varepsilon + |J * u_n - u_n|. \tag{2.8}$$

The first inequality relies upon $u_n' > 0$. Finally, since $||u_n'||_\infty \to 0$ and since $\int_{\mathbb{R}} J(y)|y|dy < \infty$, we see that

$$J * u_n - u_n \to 0 \text{ uniformly as } n \to \infty,$$

so that (2.8) gives a contradiction, proving the lemma. □

We are now prepared to obtain a solution to (1.3) and (1.4). It is easiest to begin by considering *weak solutions*, i.e., functions $u \in L^\infty(\mathbb{R})$ and constants c satisfying

$$\int_{\mathbb{R}} [J * u - u - f(u)]\phi - c\int_{\mathbb{R}} u\phi' = 0 \quad \text{for all } \phi \in C_0^\infty(\mathbb{R}). \tag*{$(1.3)_{\mathrm{w}}$}$$

Correspondingly, the weak formulation of (1.5) is

$$\int_{\mathbb{R}} [\theta(J * u - u) - f(u)]\phi + (1-\theta)\int_{\mathbb{R}} u\phi'' - c\int_{\mathbb{R}} u\phi' = 0 \text{ for all } \phi \in C_0^\infty(\mathbb{R}). \tag*{$(1.5)_{\mathrm{w}}$}$$

Theorem 2.7. *There exists a solution (u, c) to $(1.3)_{\mathrm{w}}$ satisfying (1.4).*

Proof. Lemma 2.3 gave a solution, u_θ, to (1.4) and (1.5) for each $\theta \in [0, \bar{\theta})$ for some $\bar{\theta} \in (0, 1]$. Furthermore, $u_\theta' > 0$ on $\mathbb{R}$ by Lemma 2.4.

If $\bar{\theta} < 1$, then along a sequence $\theta_n \nearrow \bar{\theta}$, by Lemmas 2.5 and 2.6, we may pass to the limit in (1.5), thereby obtaining a smooth solution $(\bar{u}, \bar{c})$ to (1.5) for $\theta = \bar{\theta}$. Clearly, this solution satisfies $\bar{u}' \geqq 0$ but the proof of Lemma 2.4 again shows that $\bar{u}' > 0$ if $\bar{u}$ satisfies (1.4). Once we show that $\bar{u}$ satisfies (1.4), Lemma 2.3 again may be applied, showing that solutions exist for $\theta \in [0, 1)$.

The remark following Lemma 2.5 shows the same is true for $\bar{\theta} = 1$ provided that $1 + f'(u) > 0$ and f is sufficiently smooth. But if this inequality is false, we do not have bounds on the derivatives of u_θ, so we cannot pass to the limit in (1.5). We do have bounds on $||u_\theta||_\infty$ and c_θ, by Lemmas 2.1 and 2.6, and since u_θ is increasing for each θ, we may take a sequence of $\theta \nearrow 1$ so that u_θ converges pointwise to a function, again called $\bar{u}$, and c_θ converges to $\bar{c}$. Lebesgue's Theorem applied in $(1.5)_{\mathrm{w}}$ shows that $(\bar{u}, \bar{c})$ satisfies $(1.3)_{\mathrm{w}}$. As we shall see in the next section, $\bar{u}$ satisfies (1.3) if $\bar{c} \neq 0$ and if $\bar{c} = 0$

$$J * \bar{u} - \bar{u} - f(\bar{u}) = 0 \text{ a.e.} \tag{2.9}$$

We now show that $\bar{u}$ satisfies (1.4). The same argument holds for either of the cases $\bar{\theta} < 1$ or $\bar{\theta} = 1$. Because $\bar{u}$ is bounded and monotone, it has limits as $\xi \to \pm\infty$, and using the dominated convergence theorem in the convolution term we see from (1.5) or (2.9) that these limits are zeros of f.

Suppose that $\bar{c} \geqq 0$. Recall the intermediate zero α from (H2). Take $\bar{\alpha} \in (\alpha, 1)$ and translate u_θ so that $u_\theta(0) = \bar{\alpha}$ for each θ. We still may take a sequence of $\theta \nearrow \bar{\theta}$, a subsequence of the original one, so that u_θ converges pointwise to some $\bar{u}$. Since c is independent of translations, we still have $c_\theta \to \bar{c}$. Then $\lim_{\xi\to\infty} \bar{u}(\xi) = 1$ and $\lim_{\xi\to-\infty} \bar{u}(\xi) \in \{\alpha, -1\}$. If $\lim_{\xi\to-\infty} \bar{u}(\xi) = -1$, then we are done. So from now on assume $\lim_{\xi\to-\infty} \bar{u}(\xi) = \alpha$. Then $f(\bar{u}(\xi)) < 0$ on $\mathbb{R}$.

If $\bar{\theta} < 1$, then by the above discussion, $\bar{u}$ is C^2 smooth and satisfies (1.5). So

$$\begin{aligned} 0 > \int_{-R}^{R} f(\bar{u}(\xi))\, d\xi &= \int_{-R}^{R} [\bar{\theta}(J * \bar{u} - \bar{u}) + (1 - \bar{\theta})\bar{u}'' + \bar{c}\bar{u}']\, d\xi \\ &\geqq \bar{\theta} \int_{-R}^{R} (J * \bar{u} - \bar{u})\, d\xi + (1 - \bar{\theta})(\bar{u}'(R) - \bar{u}'(-R)), \end{aligned} \tag{2.10}$$

but

$$\begin{aligned} \int_{-R}^{R} (J * \bar{u} - \bar{u}) d\xi &= \int_{-R}^{R} \int_{-\infty}^{\infty} J(y)[\bar{u}(\xi - y) - \bar{u}(\xi)]\, dy\, d\xi \\ &= -\int_{-R}^{R} \int_{-\infty}^{\infty} J(y) \int_{0}^{1} \bar{u}'(\xi - ty) y\, dt\, dy\, d\xi \\ &= -\int_{-\infty}^{\infty} y\, J(y) \int_{0}^{1} \left(\int_{-R}^{R} \bar{u}'(\xi - ty) d\xi \right) dt\, dy \end{aligned}$$

$$= -\int_{-\infty}^{\infty} y\,J(y)\int_0^1 (\bar{u}(R-ty)-\bar{u}(-R-ty))\;dt\,dy$$

$$\to -(1-\alpha)\int_{-\infty}^{\infty} yJ(y)\;dy = 0$$

as $R\to\infty$ by Fubini's Theorem and Lebesgue's Theorem. Now sending $R\to\infty$ in (2.10), and noting that by (1.5) $\bar{u}'(x)\to 0$ as $|x|\to\infty$, we reach a contradiction.

If $\bar{\theta}=1$, then $\bar{u}$ satisfies (1.3) or (2.9). So $J*\bar{u}-\bar{u}<0$ a.e. in $\mathbb{R}$. By Lebesgue's Theorem and the calculation following (2.10), we have

$$0 > \int_{-R}^{R} (J*\bar{u}-\bar{u})\;d\xi = \lim_{\theta\nearrow 1}\int_{-R}^{R} (J*u_\theta-u_\theta)\;d\xi$$

$$= -\int_{-\infty}^{\infty} y\,J(y)\int_0^1 (\bar{u}(R-ty)-\bar{u}(-R-ty))\;dt\,dy \to 0$$

as $R\to\infty$. Again a contradiction.

If $\bar{c}<0$, a similar argument is used taking $\bar{\alpha}\in(-1,\alpha)$. This completes the proof of the theorem. □

We note that if (u,c) is a solution to (1.3) with $u\in C^1(\mathbb{R})$, then the speed is given by

$$c = \int_{-1}^{1} f(u)\;du\Big/\int_{-\infty}^{\infty} (u')^2 d\xi. \tag{2.11}$$

This follows by multiplying (1.3) by u' and integrating, observing that

$$\int_{-\infty}^{\infty} u'(J*u-u)\;d\xi = 0.$$

3. Regularity and Discontinuous Solutions

We now examine the regularity of the weak solution guaranteed by Theorem 2.7. For this we define $g(u)=u+f(u)$, and for simplicity assume that g has at most three u-intervals of monotonicity:

(H3) $\qquad g'>0$ on $[-1,\beta)\cup(\gamma,1]$, $\quad g'<0$ on (β,γ)

for some $\beta\leqq\gamma$. Recall from (H2) that $g(\pm 1)=\pm 1$. Note that $g'>0$ on $[-1,1]$ is allowed by having $\beta=\gamma=$ any value in that interval. In this case it is also possible that $g'(\beta)=0$ with $g'>0$ elsewhere.

If the number $k\in\{g(u):u\in[-1,\beta]\}\cap\{g(u):u\in[\gamma,1]\}$ (always non-empty), we define $\hat{g}_k(u)$ to be the continuous nondecreasing function obtained by modifying g to be the constant value k between the ascending branches of $\hat{g}_k$ (see Figure 2). Note that if g is monotone, then $\beta = \gamma = k$ can be chosen to be any number in [−1,1], and $\hat{g}_k(u) = g(u)$ for all u.

Theorem 3.1. *Assume that* (H1)–(H3) *hold and suppose that f is of class C^r for some $r \geqq 1$. Let (u, c) be the weak solution, guaranteed to exist by Theorem 2.7. Recall that it is monotone.*

(a) *If $c \neq 0$, then u satisfies* (1.3) *and is of class C^{r+1}.*
(b) *If $c = 0$, there exists a value of k with $\int_{-1}^{1} \hat{g}_k(u)du = 0$. Conversely, if there exists a value of k for which this integral condition is satisfied, then there is a solution with $c = 0$.*
(c) *In any case, (u, c) is a classical solution. If $c = 0$, the solution has at most one point of discontinuity, which with no loss of generality we take to be at $x = 0$. This is a jump discontinuity and the jump occurs between the minimum and maximum values of u for which g equals the value of k referred to in* (*b*) (*see Figure 3*). In fact, *u is C^r on $(-\infty, 0) \cup (0, \infty)$. If $g' > 0$ on $[-1, 1]$ (hence $\beta = \gamma$), then u is in C^r on $(-\infty, \infty)$.*

Remark. If (H3) is violated and $g(u)$ has a finite number of maximal intervals of monotonicity, then most of the conclusions of Theorem 3.1 remain true.

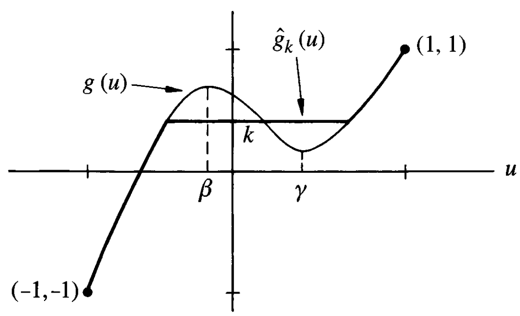

Figure 2

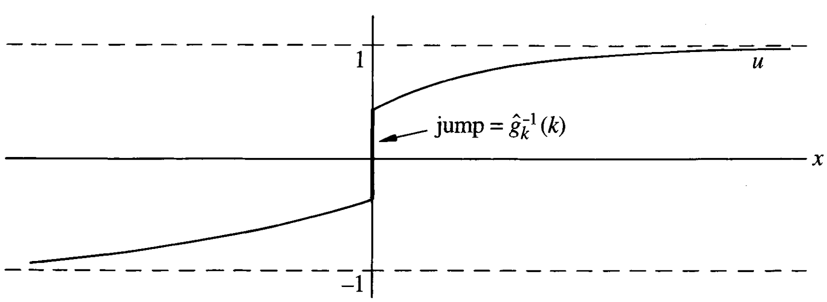

Figure 3

The difference is that when $c = 0$, $u(x)$ may have several points of discontinuity and, if discontinuous, typically is not unique. Uniqueness for the solutions under (H3) is proved in the next section.

Proof. If $c \neq 0$, then

$$c \int_{\mathbb{R}} u\phi' = \int_{\mathbb{R}} [J * u - u - f(u)]\phi \quad \text{for all } \phi \in C_0^\infty(\mathbb{R})$$

implies that $u \in W^{1,\infty}(\mathbb{R})$. A bootstrap argument then shows that u has the required regularity indicated in part (a) of the theorem.

If $c = 0$, then

$$\int_{\mathbb{R}} [J * u - u - f(u)]\phi = 0 \quad \text{for all } \phi \in C_0^\infty(\mathbb{R}),$$

so that

$$u + f(u) = J * u \text{ a.e. on } \mathbb{R}. \tag{3.1}$$

Furthermore, $u(x)$ satisfies (1.4) and is nondecreasing on $\mathbb{R}$; therefore it has only jump discontinuities. Hence $g(u(x)) = u(x) + f(u(x))$ has only jump discontinuities. Since $J * u(x)$ is continuous and increasing, it follows from (3.1) that the jumps which $g(u(x))$ undergoes all have magnitude zero. Thus $u(x)$ can be redefined at discrete points, if necessary, to make $g(u)$ continuous and increasing. It is now a classical solution, since (3.1) holds everywhere. It also follows that $u(x)$ has only one point of discontinuity, jumps between the two ascending branches of g, and has the regularity claimed in part (c).

Only part (b) remains to be established. Suppose that $c = 0$, and $\beta < \gamma$. Let k be the value of g at which the discontinuity in u occurs, as indicated above. We show first that k satisfies the integral condition in part (b).

Multiply (3.1) by u', integrate the product over $(-\infty, 0)$ and over $(0, \infty)$, and add the results to get

$$\int_{-1}^{u(0^-)} g(u)\, du + \int_{u(0^+)}^{1} g(u)\, du = \int_{-\infty}^{0} u'J * u + \int_{0}^{\infty} u'J * u.$$

To evaluate this, we set $v(x) = u(x) - S(x)$, where $S(x) = \pm 1$ is the sign of x. Thus $v(\pm\infty) = 0$. Then

$$\begin{aligned}
&\int_{-\infty}^{0} u'(x)J * u(x)\, dx \\
&\quad = \int_{-\infty}^{0} v'(x)J * (v + S) = v(0^-)J * u(0^-) - \int_{-\infty}^{0} v(x)J' * (v + S)\, dx \\
&\quad = (u(0^-) + 1)J * u(0) - \int_{-\infty}^{0} v(x)J' * v(x)\, dx - 2\int_{-\infty}^{0} v(x)J(x)\, dx,
\end{aligned}$$

since $J' * S(x) = 2J(x)$. Similarly,

$$\int_0^\infty u'(x) J * u(x)\, dx = (-u(0^+) + 1) J * u(0)$$
$$- \int_0^\infty v(x) J' * v(x)\, dx - 2 \int_0^\infty v(x) J(x)\, dx.$$

The sum of these last two integrals equals

$$(u(0^-) - u(0^+)) J * u(0) + 2J * u(0) - \int_{-\infty}^{\infty} v(x) J' * v(x) dx - 2J * u(0).$$

.

By (3.1), by the assumption that $f'(\pm 1) \neq 0$ and by the arguments around (2.10), it is easy to show that $v \in L_1$ on the line. Hence, $v(x)v(y)J'(x-y)$ is in L_1 on the plane, and we may use Fubini's theorem to deduce that

$$\int_{-\infty}^{\infty} v(x) J' * v(x)\, dx = \iint v(x) v(y) J'(x-y)\, dx\, dy = 0,$$

since $J'(s)$ is odd. Since $J * u(0) = g(u(0^-)) = k$ by (3.1), we obtain

$$\int_{-1}^{1} \hat{g}_k(u)\, du = \int_{-1}^{u(0^-)} g(u)\, du + \int_{u(0^+)}^{1} g(u) du + k(u(0^+) - u(0^-)) = 0.$$

This shows that there is a value of k satisfying the integral condition in (b), and that the jump occurs at that value of g.

If $\beta = \gamma$, the argument simplifies. The choice of k is arbitrary, but we still obtain that $\int_{-1}^{1} \hat{g}_k(u) du = \int_{-1}^{1} g(u) du = 0$.

Finally, the converse part of (b) is established by the following lemma, whose proof will complete the proof of the theorem.

Lemma 3.2. *Assume that* (H1)–(H3) *hold and that* $\int_{-1}^{1} \hat{g}_k(u) du = 0$ *for some value* k. *Then there exists a monotone solution* $u \in C^1(\mathbb{R} \backslash \{0\})$ *to*

$$\begin{aligned} J * u - g(u) = 0 \quad & \textit{for} \ \ x \neq 0, \\ u(\pm\infty) = \pm 1, & \\ u(0^-) = u^-, \quad u(0^+) = u^+, & \end{aligned} \tag{3.2}$$

where $\hat{g}_k^{-1}(k) = [u^-, u^+]$.

Proof. We consider only the case $\beta < \gamma$, as the proof can then be easily extended to the case $\beta = \gamma$. Recall that $g \in C^2$. Let $g_n(u), a_n, b_n$ satisfy

$$a_n \uparrow u^-, b_n \downarrow u^+ \quad \text{as } n \to \infty,$$

$$g_n(u) = g(u), \quad u \in [-1, a_n] \cup [b_n, 1],$$

$$g_n \in C^1(-1,1), \quad g_n'(u) > 0,$$

$$\int_{-1}^{1} g_n(u)\, du = 0.$$

The construction of the functions g_n is possible, due to (H3) and the vanishing integral of $\hat{g}_k$. Moreover, it follows that

$$g_n(u) \to \hat{g}_k(u) \quad \text{uniformly as } n \to \infty. \tag{3.3}$$

We have shown that if $g'(u) > 0$, then the solution given by Theorem 2.7 is of class C^2. Therefore for each n, there exist smooth monotone profiles $u_n(x)$ satisfying (1.4) and

$$J * u_n - g_n(u_n) = 0, \quad u_n(0) = a_n. \tag{3.4}$$

Let δ_n be defined by

$$u_n(\delta_n) = b_n > a_n. \tag{3.5}$$

By Helly's Theorem, there exists a subsequence of the functions u_n (still denoted by u_n) converging pointwise to a monotone increasing function $\bar{u}(x)$ as $n \to \infty$. By the Dominated Convergence Theorem, $J * u_n \to J * \bar{u}$, also pointwise, so that in fact by (3.3) and (3.4),

$$J * \bar{u} - \hat{g}_k(\bar{u}) = 0. \tag{3.6}$$

It is easy to see that $\bar{u}$ satisfies (1.4) by employing the argument used in Theorem 2.7.

We now show that $\bar{u}$ satisfies (3.2), which will complete the proof.

For $\xi < 0$, it follows from the monotonicity of u_n and (3.4) that $u_n(\xi) < a_n$, hence by passing to the limit, $\bar{u}(\xi) \leqq u^-$. Therefore

$$\bar{u}(0-) \equiv \underline{a} \leqq u^-. \tag{3.7}$$

Now let

$$\delta = \liminf \{\delta_n\}.$$

If $\delta > 0$, then $\bar{u}(\xi)$ satisfies

$$J * \bar{u}(\xi) = \hat{g}_k(u^-) = k = \text{ const} \tag{3.8}$$

for $\xi \in [0, \delta]$. But $\bar{u}$ is not identically constant, since it satisfies (1.4). It follows that $J * \bar{u}(\xi)$ is strictly increasing in ξ, which contradicts (3.8). Therefore $\delta = 0$, and by taking another subsequence, we may assume that $\delta_n \to 0$.

We therefore have, by an argument similar to the one leading to (3.7), that

$$\bar{u}(0+) \equiv \bar{b} \geqq u^+. \tag{3.9}$$

We know from (3.6) that $\hat{g}_k(\bar{u}(\xi))$ is continuous, so that $g(\underline{a}) = g(\bar{b})$. This together with $\underline{a} \leqq u^-$, $\bar{b} \geqq u^+$ and the definition of $\hat{g}_k$ shows that

$\underline{a} = u^-$, $\bar{b} = u^+$ hence (3.2). This completes the proof of the lemma and of the theorem. □

To summarize the results of this section, if we assume (H1)–(H3) and if (u, c) is a solution to $(1.3)_w$ and (1.4) with u monotone, then $c \neq 0$ implies u is smooth, it satisfies (1.3) and

$$c = \int_{-1}^{1} f \bigg/ \int_{-\infty}^{\infty} (u')^2.$$

If u is not continuous then it has a single jump discontinuity and is otherwise smooth, $c = 0$ and $\int_{-1}^{1} \hat{g}_k = 0$ for some k. Furthermore, this last condition is sufficient for the existence of a discontinuous solution with $c = 0$. When the results in this section are combined with the uniqueness result in the next section, we obtain

(i) $c = 0$ if and only if there exists a value k satisfying the integral condition in Theorem 3.1(b), and
(ii) the profile $u(\xi)$ is discontinuous if and only if there is such a value of k and $\beta < \gamma$.

Remark. If (H3) holds with $\beta = \gamma$ and if $g'(\beta) = 0$, then one can show that $c = 0$ and u is continuous, but not continuously differentiable provided that $\int_{-1}^{1} g = 0$. The behavior of u near $u = \beta$ can be read off from the equation $J * u = g(u)$ and the fact that $J * u$ is continuously differentiable with positive derivative.

4. Uniqueness

Because they may have independent interest, we consider solutions (u, c) to (1.5) for $\theta \in [0, 1]$ with u satisfying (1.4). We normalize by requiring $u(0) = 0$ if u is continuous and if not, then by requiring that u have its jump at 0. As we remarked following Lemma 2.4, $u' > 0$ on $\mathbb{R}$. This applies even if u is discontinuous, in which case right- and left-handed derivatives are taken at points of discontinuity.

Theorem 4.1. *Assume that* (H1)–(H3) *hold. Let* (u, c) *be the solution to* (1.4) *and* (1.5) *given in Section 2 and let* $(v, \bar{c})$ *be another solution with* v *having at most isolated discontinuities. Then* $c = \bar{c}$. *If* u *or* v *is continuous or if* v *is also monotone*, *then*, *up to translation*, $u = v$.

Remark. From the proof of this theorem, we will see that (i) If $0 \leqq \theta < 1$, assumption (H3) is not needed. (ii) In case $\theta = 1$, without assuming (H3), the velocity c is unique. (iii) Without assuming (H3), for any fixed nonzero c, there exists at most one u so that (u, c) satisfies (1.4) and (1.5). (iv) In the case of discontinuous waves, uniqueness within the class of monotone solutions is all that can be expected since Xinfu Chen has pointed out that nonmonotone stationary solutions to (1.4) and (1.5) exist in certain cases. (v) In case $\theta = 1$ and $c = 0$, the uniqueness of waves actually holds in the class of solutions each having only one discontinuity, jumping upward.

Proof of Theorem 4.1. Our proof is based on ideas from [ABC] (see also [FM]) where horizontal and vertical translates of u are used to construct sub- and super-solutions, trapping v. Suppose that $c \neq \bar{c}$. Assume that $c > \bar{c}$ and take the case that $c \neq 0$ so that u is smooth and $u' > 0$ on $\mathbb{R}$. If v has discontinuities, we assume that these are isolated, therefore countable, and that they occur at $\xi = \xi_j, j = 1, 2, \ldots$. Note that this situation can occur only when $\theta = 1$ and $\bar{c} = 0$, and so $c > 0$.

Choose $\delta > 0$ such that

$$f'(p) \geqq 2\delta \text{ when } \|p| - 1| \leqq \delta. \tag{4.1}$$

Define, for some $\mu \in (0, \delta/2)$,

$$A(t) = \mu e^{-\delta t}$$

and choose $M > 0$ such that

$$\|u(\xi)| - 1| \leqq \tfrac{1}{2}\,\delta \text{ for } |\xi| \geqq M - 1. \tag{4.2}$$

Choose $K > 0$ such that

$$u'(\xi) > K \text{ on } [-M - 1, M + 1], \tag{4.3}$$

and define

$$B(t) = \mu\bar{\delta}[1 - e^{-\delta t}]/K,$$

where

$$\bar{\delta} = 1 - \min\{f'(p) : -1 \leqq p \leqq 2\}/\delta.$$

We further restrict μ so that

$$\mu < K/\bar{\delta}.$$

Define

$$\tilde{u}(\xi, t) = u(\xi + B(t) + (\bar{c} - c)t) + A(t), \quad \tilde{v}(\xi, t) = v(\xi - z),$$

where $z > z_0 \equiv \inf\{z : u(\xi) + A(0) > v(\xi - z) \text{ for all } \xi \in \mathbb{R}\}$ is fixed.

Then $w \equiv \tilde{u} - \tilde{v}$ satisfies

$$\begin{aligned} w(\xi, 0) > 0 \text{ on } \mathbb{R}, \quad w(\pm\infty, t) = A(t) \text{ for } t \geqq 0,& \\ w_t - \theta(J * w - w) - (1 - \theta)w_{\xi\xi} - \bar{c}w_\xi = u'B' + A' + u'(\bar{c} - c)& \\ - \theta[J * (u - v) - (u - v)] - (1 - \theta)(u - v)'' - \bar{c}u' + \bar{c}v'& \\ = u'B' + A' - f(u) + f(v),& \end{aligned} \tag{4.4}$$

for $(\xi, t) \in \mathbb{R} \times [0, \infty)$, except possibly on the lines $\xi - z = \xi_j$, $j = 1, \ldots,$ in the case that v has jump discontinuities. In (4.4) u and u' are evaluated at $\xi + B(t) + (\bar{c} - c)t$ while v is evaluated at $\xi - z$. Suppose that for some $\xi_0 \in \mathbb{R}$ and $t_0 > 0$ we have

$$w(\xi_0, t_0) = 0 \leqq w(\xi, t) \text{ for all } t \in (0, t_0), \xi \in \mathbb{R}.$$

Note that if ξ_0 is a point of discontinuity of v, then $\bar{c} = 0$ and $\theta = 1$ and no derivatives of v are required. In any case, at (ξ_0, t_0) we have, since $J * w \geqq 0$,

$$0 \geqq u'B' + A' - f(u) + f(u + A) \equiv E(\xi_0, t_0). \tag{4.5}$$

We consider two cases:

Case I. If $|\xi_0 + (\bar{c} - c)t_0| \leqq M$, then $E > \mu e^{-\delta t_0}(\delta\bar{\delta} - \delta + f'(d))$ (4.6)

for some $d = d(\xi_0, t_0) \in [-1, 2]$, by (4.3), the definitions of A and B, and the Mean Value Theorem. The definition of $\bar{\delta}$ shows that (4.5) and (4.6) are incompatible.

Case II. If $|\xi_0 + (\bar{c} - c)t_0| > M$, then $|\xi_0 + B(t_0) + (\bar{c} - c)t_0| \geqq M - 1$,

since $\mu \leqq K/\bar{\delta}$. Hence, $E > [-\delta + f'(d)]A(t_0)$ for some $d = d(\xi_0, t_0)$ satisfying $||d| - 1| \leqq \delta$, by (4.2) since $A \leqq \delta/2$. Consequently $E > 0$, by (4.1), again contradicting (4.5).

Therefore, $w(\xi, t) > 0$ for all $t \geqq 0$ and $\xi \in \mathbb{R}$.

Choose $\xi = -\mu\bar{\delta}/K + (c - \bar{c})t$ and let $t \to \infty$ to get

$$0 \leqq u(0) - v(\infty) = -1$$

since $c > \bar{c}$. This contradiction shows that $c \leqq \bar{c}$. If $c < \bar{c}$, a similar argument may be given using translates of u and v so that $w(\xi, 0) < 0$. The result is that $c = \bar{c}$. The case that $c = 0$ can be treated similarly, using right- and left-handed derivatives if u is discontinuous at a point.

If u is continuous, $c = 0$ and $\theta = 1$, it follows from (1.5) that the function $g(s) = f(s) + s$ is continuous and monotone as a function of s, so that every solution must be continuous. This eliminates the possibility that u is continuous but v is discontinuous. Therefore in any case it suffices to assume that v is either continuous or monotone, and we do that from now on. A further analysis of (1.5) in the case $c = 0$ and $\theta = 1$ now shows that either u and v are both continuous, or both discontinuous with the same range.

This implies that v has at most one point of discontinuity. Since the above analysis could have been carried out with any translate of v, there is no loss in now taking a particular translation. Regardless of whether u and v are continuous, we may select a point $\bar{\xi}$ such that $u(\bar{\xi})$ lies in their range and then require $v(\bar{\xi}) = u(\bar{\xi})$.

Continuing with our original definition of w, we have (now that $c = \bar{c}$)

$$u(\xi + B(t)) + A(t) > v(\xi - z) \text{ for all } \xi \in \mathbb{R} \text{ and } t \geqq 0. \tag{4.7}$$

This is because the analysis with $c \neq \bar{c}$ which yielded $w > 0$ may still be done with $c = \bar{c}$, taking into account the possibility that u is discontinuous, in which case the necessary left- and right-handed derivatives exist. A question might arise about the possibility of w changing sign without becoming zero in the case that u and v are both discontinuous. First, if u or v is discontinuous, we include the right- or left-hand limits according to our needs. Second, we see that $v(\xi - z)$ does not evolve with t while $\tilde{u}(\xi, t)$ is monotone in ξ and as t

increases its graph descends and moves to the left. Consequently, $\tilde{u}$ and v must touch if they are to cross.

Letting $t \to \infty$ in (4.7) yields

$$u(\xi + \mu\bar{\delta}/K) \geqq v(\xi - z) \quad \text{for all } \xi \in \mathbb{R},$$

and hence, letting $z \searrow z_0$ yields

$$u(\xi) \geqq v(\xi - (z_0 + \mu\bar{\delta}/K)) \quad \text{for all } \xi \in \mathbb{R}. \tag{4.8}$$

Because of (4.8) we can find a minimal $\bar{z} \geqq 0$ such that

$$u(\xi) \geqq v(\xi - \bar{z}) \quad \text{for all } \xi \in \mathbb{R}. \tag{4.9}$$

We assert that if $u \not\equiv v$, then the inequality in (4.9) is strict, and consequently $\bar{z} > 0$ since $u(\bar{\xi}) = v(\bar{\xi})$. If for some ξ_0 equality were to hold, then at $\xi_0, \bar{w}(\xi) \equiv u(\xi) - v(\xi - \bar{z})$ would satisfy

$$\begin{aligned} 0 \geqq \theta(J * \bar{w}) + (1-\theta)\bar{w}'' &= \theta(J * u - u) + (1-\theta)u'' + cu' \\ &\quad -[\theta(J * v - v) + (1-\theta)v'' + cv'] \\ &= f(u) - f(v) = 0. \end{aligned} \tag{4.10}$$

If $\theta \neq 0$, then we have $\bar{w}(\xi) \geqq 0,\ \bar{w}(\xi_0) = J * w(\xi_0) = 0$ and from this it follows that $\bar{w} \equiv 0$. If $\theta = 0$, then the Maximum Principle implies $\bar{w} \equiv 0$. This establishes the assertion. Recall that if v is discontinuous, then $c = 0$ and $\theta = 1$ and so no derivatives are present in (4.10).

For the moment suppose that at least one of u and v is continuous. For $\eta > 0$ define

$$z(\eta) = \inf\{z : u(\xi) \geqq v(\xi - z) - \eta \quad \text{for all } \xi \in \mathbb{R}\}.$$

Note that $z(\eta) < \bar{z}$ and that $\lim_{\eta \searrow 0} z(\eta) = \bar{z}$. Fix $N > 0$. We assert that there exists $\eta_N > 0$ such that for all $\eta \in (0, \eta_N]$,

$$u(\xi) > v(\xi - z(\eta)) - \eta \quad \text{for } |\xi| \leqq N. \tag{4.11}$$

If not, then there would exist a sequence $\{\xi_n\} \subset [-N, N]$ and $\eta_n \searrow 0$ with ξ_n converging to some $\bar{\xi}$ as $n \to \infty$, and $u(\xi_n) = v(\xi_n - z(\eta_n)) - \eta_n$. Taking the limit as $n \to \infty$ gives

$$u(\bar{\xi}) = v(\bar{\xi} - \bar{z})$$

contradicting our previously established assertion. Now let $A(t) = \mu e^{-\delta t}$ with the further restriction $\mu < \eta_M$, where M is from (4.2), and with δ from (4.1). Since $z(\mu) < \bar{z}$, we may take $\varepsilon > 0$ such that $2\varepsilon < \bar{z} - z(\mu)$. Let $\hat{w}(\xi, t) \equiv u(\xi) + A(t) - v(\xi - (\bar{z} - \varepsilon))$. Then $\hat{w}(\xi, 0) > 0$ and if for some $t_0 > 0$ and $\xi \in \mathbb{R}, \hat{w}(\xi_0, t_0) = 0 < \hat{w}(\xi, t)$ for all $t < t_0$ and $\xi \in \mathbb{R}$, then at (ξ_0, t_0)

$$\begin{aligned} 0 &\geqq \hat{w}_t - \theta(J * \hat{w} - \hat{w}) - (1-\theta)\hat{w}_{\xi\xi} - c\hat{w}_\xi \\ &= A'(t_0) + f(v) - f(u) = [f'(d) - \delta]A(t_0) \end{aligned} \tag{4.12}$$

for some $d \in [u(\xi_0), u(\xi_0) + A(t_0)]$. Because $u(\xi_0) = v(\xi_0 - (\bar{z} - \varepsilon)) - \mu e^{-\delta t_0}$, it follows that $z(\mu e^{-\delta t_0}) = \bar{z}_0 - \varepsilon$ and since $\mu e^{-\delta t_0} < \eta_M$, (4.11) implies that

$|\xi_0| > M$ and hence $||d| - 1| \leqq \delta$. Consequently $f'(d) - \delta > 0$ by (4.1), contradicting (4.12).

We have shown that $\hat{w}(\xi, t) > 0$ for all $t > 0$ and $\xi \in \mathbb{R}$. Taking the limit as $t \to \infty$ gives

$$u(\xi) \geqq v(\xi - (\bar{z} - \varepsilon)) \quad \text{for all } \xi \in \mathbb{R},$$

contradicting the minimality of $\bar{z}$ and proving that $u \equiv v$.

Finally, suppose that both u and v are discontinuous, u having a jump at $\xi = 0$ and v having its jump at some point ξ_1. Recall that u is monotone by construction and v is monotone by assumption. The above analysis for the case with u or v continuous also applies unless both $u(\xi)$ and $v(\xi - \bar{z})$ have their jumps at the same point, that is, $\xi_1 = -\bar{z}$. In this case, lowering v slightly will not allow it to be translated to the left while remaining below u. However, we may now use an argument similar to that which produced (4.9) to show that $v(\xi) \geqq u(\xi - \hat{z})$ for some minimal $\hat{z}$. Then one can again show that $v(\xi)$ and $u(\xi - \hat{z})$ have their jumps at the same point, that is, $\hat{z} = \xi_1$. This implies that $u(\xi) \geqq v(\xi + \xi_1) \geqq u(\xi)$ for all $\xi \in \mathbb{R}$, completing the proof. □

Now consider the evolution given by the generalized version of (1.1), namely,

$$v_t = \theta(J * v - v) + (1 - \theta)v_{xx} - f(v) \quad \text{for } t > 0 \text{ and } x \in \mathbb{R}, \tag{4.13}$$

$$v(x, 0) = v_0(x) \quad \text{for } x \in \mathbb{R},$$

where $\theta \in [0, 1]$ is fixed. It is easy to see that (4.13) generates a semiflow (or flow, when $\theta = 1$) on $BC(\mathbb{R})$, the space of bounded continuous functions on $\mathbb{R}$. If u is the traveling-wave solution to (4.13), say $u(x, t) = U(x - ct)$ for some constant c and increasing function U with $U(0) = 0$, then we claim that u is stable in the following sense.

Theorem 4.2. *Let v be the solution to* (4.13) *with $v_0 \in BC(\mathbb{R})$. Suppose that*

$$\|v_0\|_\infty \leqq 1, \ \liminf_{x \to \infty} v_0(x) > \alpha, \ \limsup_{x \to -\infty} v_0(x) < \alpha. \tag{4.14}$$

Then

(a) *There exist constants $x_1, x_2, \mu > 0$ and $\delta > 0$ such that*

$$U(x - x_1 - ct) - \mu e^{-\delta t} < v(x, t) < U(x - x_2 - ct) + \mu e^{-\delta t} \tag{4.15}$$

for all $x \in \mathbb{R}$ and $t > 0$.

(b) *There is a function $\omega(\varepsilon)$, defined for small $\varepsilon > 0$, such that* $\lim_{\varepsilon \to 0} \omega(\varepsilon) = 0$ *with the property that if v_0 satisfies* (4.15) *at $t = 0$ with $\mu = \varepsilon$ and with $x_2 - x_1 \leqq \varepsilon$, then for all $x \in \mathbb{R}$ and $t \geqq 0$*

$$U(x - x_1 - \omega(\varepsilon) - ct) - \varepsilon e^{-\delta t} < v(x, t) < U(x - x_2 + \omega(\varepsilon) - ct) + \varepsilon e^{-\delta t}. \tag{4.16}$$

The proof of (a) uses sub- and super-solutions exactly as in the proof of Theorem 4.1 but the only restriction needed on μ is that $0 < \mu < 1$. Condition (4.14) allows us to find x_1^*, x_2^* and such a μ so that

$$U(x - x_1^*) - \mu < v_0(x) < U(x - x_2^*) + \mu.$$

Then $B_1(t)$ and $B_2(t)$ are chosen so that $B_j(0) = -x_j^*$, $\lim_{t\to\infty}(B_j(t) + ct)$ exists and such that $U(x + B_1(t)) - \mu e^{-\delta t} < v(x,t) < U(x + B_2(t)) + \mu e^{-\delta t}$ for all $x \in \mathbb{R}$ and $t > 0$, where δ is as in (4.1). This last requirement is accomplished, as before, by considering the cases where $v(x,t)$ touches the sub- or super-solution at values near ± 1 or away from ± 1. The details are omitted. Part (b) follows from the proof of part (a) by noting that B_j can be chosen so that $|B_j(t) + ct| \leqq o(\varepsilon)$.

5. Asymptotic stability

We conclude by showing that when U is a stationary-wave solution to (1.1) satisfying the boundary condition (1.4), it is globally asymptotically stable in various senses, up to translation.

Consider the Cauchy problem for (1.1) with initial condition

$$u(x,0) = \phi(x), \quad -1 \leqq \phi \leqq 1. \tag{5.1}$$

Our first result deals with the asymptotic stability of smooth stationary waves. Recall that (1.1) has a smooth stationary wave U if and only if

$$\int_{-1}^{1} f(u)\, du = 0, \; g'(u) > 0 \quad \text{for } u \in [-1,1], \tag{5.2}$$

where $g(u) = u + f(u)$.

Theorem 5.1. *Suppose* (5.2) *holds and the initial value* ϕ *satisfies*

$$\limsup_{x\to-\infty} \phi(x) < \alpha < \liminf_{x\to\infty} \phi(x). \tag{5.3}$$

If ϕ *is continuous on* $\mathbb{R}$, *then the solution* $u(x,t)$ *of the Cauchy problem converges to a unique shift of* U *in the* $L^\infty(\mathbb{R})$ *norm as* $t \to \infty$. *Furthermore, the convergence is exponentially fast as* $t \to \infty$.

The basic idea in the proof of the first part of the theorem comes from [FM]. First we prove that $u(x,t)$ converges to a shift of U in the $L^\infty(\mathbb{R})$ norm along a sequence $t_n \to \infty$, by using the Lyapunov functional method. Then the first part of Theorem 5.1 follows from the stability result in Theorem 4.2(b). To obtain the exponential convergence, we study the spectrum of the linear operator L obtained by linearizing the right-hand side of (1.1) at the shift of U (or equivalently at U). We show in the L^∞ setting that 0 is a simple eigenvalue of L and the rest of the spectrum lies in the left half-plane, bounded away from the imaginary axis. Then the exponential convergence in the L^∞ norm follows from the first part of Theorem 5.1 and standard theory. We start with

Lemma 5.2. *Suppose that $g'(u) = 1 + f'(u) > 0$ on $[-1, 1]$. If the initial value ϕ is continuous in $\mathbb{R}$, then the modulus of continuity of $u(x, t)$ in x is bounded in any bounded interval $[a, b]$, uniformly in $t \geqq 0$.*

Proof. For every small constant h, let

$$(\delta u)(x, t) = u(x + h, t) - u(x, t).$$

By (1.1) and (5.1),

$$\begin{gathered}(\delta u)_t = \int_{\mathbb{R}^1} (J(x + h - y) - J(x - y))u(y, t)\, dy - g'(\bar{u})(\delta u), \\ (\delta u)(0, t) = (\delta\phi)(x)\end{gathered} \tag{5.4}$$

where $\bar{u}$ is between $u(x, t)$ and $u(x + h, t)$. Observe that since $-1 \leqq u \leqq 1$,

$$\left| \int_{\mathbb{R}} (J(x + h - y) - J(x - y))\, u(y, t)\, dy \right| \leqq \int_{\mathbb{R}} |J(y + h) - J(y)|\, dy \equiv \varepsilon(h)$$

with $\varepsilon(h) \to 0$ as $h \to 0$. Set $l = \inf_{u \in [-1,1]} g'(u) > 0$.

Let $v(t)$ be the solution of

$$v'(t) = \varepsilon(h) - lv(t), \quad v(0) = \|\delta\phi\|_{L^\infty[a,b]}. \tag{5.5}$$

Then

$$v(t) = e^{-lt}\|\delta\phi\|_{L^\infty[a,b]} + \frac{\varepsilon(h)}{l}(1 - e^{-lt}) > 0.$$

A simple comparison between (5.4) and (5.5) yields that for $x \in [a, b]$,

$$|(\delta u)(x, t)| \leqq v(t) \leqq \|\delta\phi\|_{L^\infty[a,b]} + \frac{\varepsilon(h)}{l} \to 0$$

as $h \to 0$. This completes the proof of Lemma 5.2. $\square$

Proof of Theorem 5.1. To prove the first part of Theorem 5.1, we note that by Lemma 5.2, the Arzelà-Ascoli Theorem and (4.15) (with $c = 0$), for any sequence $t'_n \to \infty$, there exists a subsequence $t_n \to \infty$ such that

$$u(x, t_n) \to \text{ some } u_\infty(x) \text{ in } L^\infty(\mathbb{R}) \tag{5.6}$$

as $t_n \to \infty$.

Let η be a C^∞ function defined on $[0, \infty)$ such that $\eta(x) = 1$ for $x \in [0, \frac{1}{2}]$ and $= 0$ for $x \geqq 1$. Define

$$W(x, t) = \begin{cases} u(x, t), & |x| \leqq t, \\ u(x, t)\eta(x - t) + (1 - \eta(x - t)), & x \geqq t, \\ u(x, t)\eta(-x - t) - (1 - \eta(-x - t)), & x \leqq -t. \end{cases} \tag{5.7}$$

Then $W(x, t) \equiv 1$ for $x \geqq t + 1$ and $W(x, t) \equiv -1$ for $x \leqq -t - 1$. Note also that $-1 \leqq W(x, t) \leqq 1$, $W(x, t) \geqq u(x, t)$ for $x \geqq 0$, and $W(x, t) \leqq u(x, t)$ for $x \leqq 0$.

Let $F(u) = \int_{-1}^{u} f(s)ds$. Then $F(-1) = F(1) = F'(-1) = F'(1) = 0$. Define an energy functional associated with (1.1) by

$$V(t) = \frac{1}{2}\int_{\mathbb{R}} (J * W - W)W(x,t)\ dx - \int_{\mathbb{R}} F(W(x,t))\ dx.$$

Now we show that $V(t)$ is well-defined and in fact bounded in $t \geqq 0$. We first tackle the last term:

$$\int_{\mathbb{R}} F(W(x,t))\ dx = \int_0^\infty + \int_{-\infty}^0 = \int_0^{t+1} F(W(x,t))\ dx + \int_{-t-1}^0 F(W(x,t))\ dx.$$

From now on, let C be a generic positive constant which may vary from line to line. By the fact that $|F(u)| \leqq C(1-u)(1+u)$ for $u \in [-1,1]$ and by (4.15), we have

$$\begin{aligned}\left|\int_0^{t+1} F(W(x,t))\ dx\right| &\leqq C\int_0^{t+1} (1 - W(x,t))\ dx \leqq C\int_0^{t+1} (1-u(x,t))\ dx \\ &\leqq C\int_0^{t+1} (1 - U(x-x_1) + \mu e^{-\delta t})\ dx = C(t+1)e^{-\delta t} \\ &\quad + C\int_0^{t+1} (1 - U(x - x_1))\ dx. \qquad (5.8)\end{aligned}$$

On the other hand, using the equation satisfied by U and the fact that $f'(1) > 0$, we see that there exist positive constants C and R such that

$$-(J * U)(x) + U(x) \geqq C(1 - U(x))$$

for $x \geqq R$. Integrating this inequality on $[R, \infty)$ and using the calculation immediately following (2.10), we have the first of the following two inequalities:

$$\int_0^\infty (1 - U(x))\ dx < \infty, \quad \int_{-\infty}^0 (1 + U(x))\ dx < \infty \qquad (5.9)$$

(the second inequality follows similarly). Thus $|\int_0^{t+1} F(W)dx|$ is bounded in $t \geqq 0$. Similarly, this is true for $|\int_{-t-1}^0 F(W)dx|$. We have proved that the second term in the definition of $V(t)$ is bounded in $t \geqq 0$.

Now we come back to the first term in the definition of $V(t)$:

$$\begin{aligned}\left|\int_{\mathbb{R}} (J * W - W)W(x,t)\ dx\right| &\leqq \int_{\mathbb{R}} |J * W - W|(x,t)\ dx \\ = \left(\int_0^t + \int_{-t}^0 + \int_t^\infty + \int_{-\infty}^{-t}\right)&|J * W - W|(x,t)\ dx \equiv I_1 + I_2 + I_3 + I_4. \qquad (5.10)\end{aligned}$$

$$I_1 \leqq \int_0^t dx \int_0^\infty J(x-y)|W(y,t)-W(x,t)|\ dy$$
$$+ 2\int_0^t dx \int_{-\infty}^0 J(x-y)\ dy \equiv I_{11} + 2I_{12}.$$
$$I_{12} = \int_0^t dx \int_{-\infty}^{-x} J(y)\ dy = \int_{-t}^0 \int_0^{-y} J(y)\ dx\ dy + \int_{-\infty}^{-t} \int_0^t J(y)\ dx\ dy$$
$$\leqq \int_{-t}^0 |y| J(y)\ dy + \int_{-\infty}^{-t} |y| J(y)\ dy < \infty$$

by assumption (H1). By (4.15),

$$I_{11} \leqq \int_0^t dx \int_0^\infty J(x-y)(1-W(y,t))\ dy + \int_0^t dx \int_0^\infty J(x-y)(1-W(x,t))\ dy$$
$$\leqq \int_0^t dx \int_0^\infty J(x-y)(1-U(y-x_1)+\mu e^{-\delta t})\ dy$$
$$+ \int_0^t (1-U(x-x_1)+\mu e^{-\delta t})\ dx$$
$$\leqq \int_0^\infty (1-U(y-x_1))\,dy + 2\mu t e^{-\delta t} + \int_0^\infty (1-U(x-x_1))\,dx,$$

which is bounded by (5.9). Thus I_1 and similarly I_2 are bounded for $t \geqq 0$.

We now proceed to show that $I_3 + I_4$ in (5.10) is bounded and in fact converges to 0 as $t \to \infty$. By estimates similar to those for I_{12} and I_{11}, we have

$$I_3 \leqq \int_t^\infty dx \int_0^\infty J(x-y)|W(y,t)-W(x,t)|\ dy + 2\int_t^\infty dx \int_{-\infty}^0 J(x-y)\ dy$$
$$\leqq \int_t^\infty dx \int_0^\infty J(x-y)(1-W(y,t))\ dy$$
$$+ \int_t^\infty dx \int_0^\infty J(x-y)(1-W(x,t))\ dy + 2\int_{-\infty}^{-t} |y| J(y)\ dy$$

$$
\begin{aligned}
&= \int_t^\infty dx \int_0^{t+1} J(x-y)(1-W(y,t))\ dy \\
&\quad + \int_t^{t+1} dx \int_0^\infty J(x-y)(1-W(x,t))\ dy + 2\int_{-\infty}^{-t} |y|\,J(y)\ dy \\
&\leqq \int_t^\infty dx \int_0^{t+1} J(x-y)(1-U(y-x_1))\ dy + \mu(t+1)e^{-\delta t} \\
&\quad + (1-U(t-x_1)+\mu e^{-\delta t})\ + 2\int_{-\infty}^{-t} |y|J(y)\ dy.
\end{aligned}
$$

The first term in the last expression is not larger than

$$
\begin{aligned}
&2\int_t^\infty dx \int_0^{t/2} J(x-y)\ dy + \int_t^\infty dx \int_{t/2}^{t+1} J(x-y)\ dy\big(1-U\big(\tfrac{1}{2}t-x_1\big)\big) \\
&\quad \leqq 2\frac{t}{2}\int_{t/2}^\infty J(x)\ dx + \int_t^\infty dx \int_{-x}^{t+1-x} J(y)\ dy\ \big(1-U\big(\tfrac{1}{2}t-x_1\big)\big) \\
&\quad \leqq 2\int_{t/2}^\infty xJ(x)\ dx + C(1+\int_{\mathbb{R}} |y|J(y)\ dy)\big(1-U\big(\tfrac{1}{2}t-x_1\big)\big).
\end{aligned}
$$

By the above estimates for I_3, we conclude that

$$
I_3 \text{ is bounded and converges to 0 as } t \to \infty. \text{ The same is true for } I_4. \tag{5.11}
$$

We have thus proved that the energy $V(t)$ is bounded for $t \geqq 0$.

Now we differentiate $V(t)$ to obtain

$$
V'(t) = \int_{\mathbb{R}} (J*W-W)W_t(x,t)\ dx - \int_{\mathbb{R}} f(W)W_t(x,t)\ dx.
$$

(That we can change the order of differentiation and integration follows from Lebesgue's Theorem and the facts that W_t is bounded and the L^1 norm of $J*W-W$ and $f(W)$ are bounded for $t \geqq 0$. See the proof of the boundedness of $V(t)$.) We write

$$
V'(t) - Q(t) = \int_{\mathbb{R}} (J*W-W-f(W))(W_t - J*W+W+f(W))\ dx \equiv P(t) \tag{5.12}
$$

where

$$Q(t) = \int_{\mathbb{R}} (J * W - W - f(W))^2 dx.$$

We proceed to show that $P(t) \to 0$ as $t \to \infty$. Using (1.1) and (5.7), we obtain

$$\begin{aligned} P(t) &\leqq C \int_{|x| \leqq t} |W_t - J * W + W + f(W)|\ dx + C \int_{|x| \geqq t} |J * W - W - f(W)|\ dx \\ &\leqq C \int_{|x| \leqq t} |J * (W - u)|\ dx + C \int_{|x| \geqq t} |J * W - W|\ dx + C \int_{|x| \geqq t} |f(W)|\ dx \\ &\equiv C(J_1 + J_2 + J_3). \end{aligned}$$

By (5.11), $J_2 = I_3 + I_4 \to 0$ as $t \to \infty$. Observe that there exists a positive constant C such that $|f(u)| \leqq C(1-u)(1+u)$ for $u \in [-1, 1]$. So

$$\begin{aligned} J_3 &= \int_t^{t+1} |f(W)|\ dx + \int_{-t-1}^{-t} |f(W)|\ dx \\ &\leqq C \int_t^{t+1} (1 - W(x,t))\ dx + C \int_{-t-1}^{-t} (1 + W(x,t))\ dx \\ &\leqq C \int_t^{t+1} (1 - U(x - x_1) + \mu e^{-\delta t})\ dx + C \int_{-t-1}^{-t} (1 + U(x - x_2) + \mu e^{-\delta t})\ dx \\ &= C(1 - U(t - x_1) + \mu e^{-\delta t}) + C(1 + U(-t - x_2) + \mu e^{-\delta t}) \end{aligned}$$

which converges to 0 as $t \to \infty$. (The second inequality follows from (4.15).) Also

$$\begin{aligned} J_1 &\leqq \int_{-t}^{t} dx \int_{|y| \geqq t} J(x-y)|W(y,t) - u(y,t)|\ dy \\ &\leqq \int_{-t}^{t} dx \int_t^{\infty} J(x-y)(1 - u(y,t))\ dy + \int_{-t}^{t} dx \int_{-\infty}^{-t} J(x-y)(1 + u(y,t))\ dy \\ &\equiv J_{11} + J_{12}, \end{aligned}$$

$$\begin{aligned} J_{11} &\leqq \int_{-t}^{t} dx \int_t^{\infty} J(x-y)(1 - U(y - x_1) + \mu e^{-\delta t})\ dy \\ &\leqq \int_{-t}^{t} dx \left(\int_t^{2t} + \int_{2t}^{\infty} \right) J(x-y)(1 - U(y - x_1))\ dy + 2\mu t e^{-\delta t} \end{aligned}$$

$$\leqq \int_{-t}^{t} dx \int_{t}^{\alpha t} J(x-y)\, dy\, (1 - U(t - x_1))$$

$$+ \int_{-\infty}^{-t} J(x)dx \int_{2t}^{\infty} (1 - U(y - x_1))\, dy + 2\mu t e^{-\delta t}$$

$$\leqq C \int_{0}^{\infty} |y| J(y)\, dy\, (1 - U(t - x_1))$$

$$+ \int_{-\infty}^{-t} J(x)\, dx \int_{2t}^{\infty} (1 - U(y - x_1))\, dy + 2\mu t e^{-\delta t},$$

which converges to 0 as $t \to \infty$. Similarly, we can show that this is true for J_{12}.

We have shown that

$$V'(t) - Q(t) = P(t) \to 0 \quad \text{as} \ \ t \to \infty. \tag{5.13}$$

Since $V(t)$ is bounded, there exists a sequence $t'_n \to \infty$ such that $V'(t'_n) \to 0$. From this, (5.13), (5.6) and Fatou's Lemma, we deduce that there exists a subsequence $t_n \to \infty$ such that

$$u(x, t_n) \to \ \text{some } u_\infty \text{ in the } L^\infty \text{ norm} \tag{5.14}$$

where u_∞ is a stationary wave of (1.1), satisfying (4.15) without the exponential terms. By the uniqueness result in Section 4, $u_\infty(x) \equiv U(x + x_0)$ for some $x_0 \in \mathbb{R}$. Combining (5.14) with (b) of Theorem 4.2, we have that as $t \to \infty$,

$$u(x, t) \to U(x + x_0) \text{ in the } L^\infty \text{ norm.} \tag{5.15}$$

The first part of Theorem 5.1 is proved.

We now proceed to prove the exponential convergence. To this end, we need to understand the spectrum of the following linear operator obtained by linearizing (1.1) at U:

$$L(v) = J * v - v - f'(U)v.$$

Lemma 5.3. *Suppose that* $g'(u) = 1 + f'(u) > 0$ *on* $[-1, 1]$. *Then* $L : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ *has simple eigenvalue* 0 *with an eigenfunction* U'. *The rest of the spectrum* (*which is compact*) *lies on the left half plane, bounded away from the imaginary axis.*

Now by (5.15), Lemma 5.1 and well-known results (see, e.g., Theorem 1.1 in Chapter 5 of [VVV]), we have the exponential convergence in L^∞, as claimed in Theorem 5.1. This finishes the proof of Theorem 5.1.

Proof of Lemma 5.3. It is obvious that U' is an eigenfunction of L with eigenvalue 0. To prove the simplicity, we use the argument in the proof of Lemma 2.2 as follows. Let $v \in \mathrm{Ker}(L)$, $v \not\equiv 0$. Then

$$J * v - v - f'(U)v = 0, \text{ and hence } v = \frac{J * v}{1 + f'(U)}.$$

So v is continuous in $\mathbb{R}$. Without loss of generality, assume that v is real-valued and is positive somewhere. Define $v_\beta = U' + \beta v, \quad \beta \in \mathbb{R}$; $\bar{\beta} = \sup\{\beta < 0 : v_\beta \text{ is negative somewhere}\}$. For $\beta < \bar{\beta}$, we assert that $\inf v_\beta$ is assumed at a point ξ_β, not at $\pm\infty$. To see this, we argue by contradiction. Assume that for some $\beta < \bar{\beta}$, there exists a sequence x_n approaching, say $+\infty$, as $n \to \infty$, such that $v_\beta(x_n) \to \inf v_\beta < 0$, so

$$\begin{aligned} 0 > f'(1)(\inf v_\beta) &= \lim_{n\to\infty} f'(U(x_n))v_\beta(x_n) \\ &= \lim_{n\to\infty}((J * v_\beta)(x_n) - v_\beta(x_n)) \\ &= \lim_{n\to\infty}[(J * v_\beta)(x_n) - \inf v_\beta) + (\inf v_\beta - v_\beta(x_n))] \geqq 0, \end{aligned}$$

which is a contradiction. This completes the proof of the assertion and this argument actually leads to the boundedness of the minimum point ξ_β of v_β as $\beta \to \bar{\beta}$. Now exactly as in the proof of Lemma 2.2, we have $v_\beta \equiv 0$ and hence the simplicity of the eigenvalue 0 of L.

To locate the rest of the spectrum of L, we first show that $L - \lambda I$ is a Fredholm operator with index 0, for any $\lambda \in S_\delta = \{\lambda \in \mathbb{C} | \text{ Re } \lambda \geqq -\delta\}$, where δ is a small positive constant satisfying

(1) $\inf(1 + f'(U(x))) > \delta$,

(2) $f'(1) + f'(-1) > \delta$.

We write

$$\begin{aligned} (L - \lambda)v &= J * v - (1 + f'(U) + \lambda)v \\ &= (1 + f'(U) + \lambda)\left[\left(\frac{1}{1 + f'(U) + \lambda} - \frac{1}{q(x)}\right)J * v + \left(\frac{1}{q(x)}J * v - v\right)\right] \\ &\equiv (1 + f'(U) + \lambda)[L_1 + L_2] \end{aligned}$$

where $q(x)$ is chosen to be a continuous function so that for $\lambda \in S_\delta$, $q(\pm\infty) = 1 + f'(\pm 1) + \lambda$ and $|q(x)|$ is bounded from below by a constant $C > 1$ (such a choice is possible because of (2) above). Note by (1), $|1 + f'(U) + \lambda|$ is bounded away from 0. By the Arzelà-Ascoli Theorem and the fact that

$$\frac{1}{1 + f'(U) + \lambda} - \frac{1}{q(x)} \to 0 \quad \text{as } |x| \to \infty,$$

it is easy to see that $L_1 : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ is a compact operator. Moreover, since $|q(x)| \geqq C > 1$, the first operator in the definition of L_2 has norm less than 1. Thus $L_2 : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ has bounded inverse. Therefore $L - \lambda I$ is a Fredholm operator with 0 index for $\lambda \in S_\delta$. This and the simplicity of the eigenvalue 0 of L imply that $L^\infty(\mathbb{R}) = \text{Range}(L) \oplus N$, where N is one-dimensional. On the other hand, for any $v \in L^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} U' L v = 0.$$

So we can take $N = \mathrm{Ker}(L)$ and hence $L^\infty(\mathbb{R}) = \mathrm{Range}(L) \oplus \mathrm{Ker}(L)$. Now $L : \mathrm{Range}(L) \to \mathrm{Range}(L)$ is one-to-one and onto, so it has a bounded inverse. In particular, for small $|\lambda|$, $(L - \lambda I)|_{\mathrm{Range(L)}}$ is invertible and hence if $\lambda \neq 0$, $Lv = \lambda v$ has no non-zero solution in L^∞. From this and the fact that $L - \lambda I$ is a Fredholm operator with zero index for $\lambda \in S_\delta$, we have that if $|\lambda| \neq 0$ is small, then λ is in the resolvant set of L.

Now we proceed to show that if $\mathrm{Re}\,\lambda \geqq 0$, $\lambda \neq 0$, then λ is in the resolvant set of L. Suppose otherwise. Then once again, since $L - \lambda I$ is a Fredholm operator with index 0, there exists $v \in L^\infty(\mathbb{R})$, $v \not\equiv 0$, such that $Lv = \lambda v$, i.e.,

$$J * v = (1 + f'(U) + \lambda)v. \tag{5.16}$$

Since $\mathrm{Re}\,\lambda \geqq 0$ and $1 + f'(U) > 0$, we have

$$J * |v| \geqq (1 + f'(U))|v|. \tag{5.17}$$

Define $v_\beta = U' + \beta|v|$, $\bar{\beta} = \sup\{\beta < 0 : \ v_\beta \text{ is negative somewhere}\}$. Then for $\beta < \bar{\beta} \leqq 0$,

$$J * v_\beta \leqq (1 + f'(U))v_\beta.$$

Now the proof of the simplicity of eigenvalue 0 of L again leads us to $v_{\bar{\beta}} \equiv \text{const.} = 0$. In particular, (5.17) is an equality. But this happens only if $\lambda = 0$, contradicting $\lambda \neq 0$. This completes the proof of Lemma 5.3. □

Finally we turn to the asymptotic stability of the monotone wave solution of (1.1) even when it is not smooth.

Theorem 5.4. *Suppose that* (H1)–(H3) *hold, that* (1.1) *has a non-smooth stationary monotone wave U, and that the initial value ϕ satisfies* (5.3) *and is a non-decreasing continuous function on $\mathbb{R}$. Then*

(1) *If the stationary wave U is continuous everywhere,* (*i.e., $\beta = \gamma$ in* (H3) *with $g'(\beta) = 0$*), *then the solution $u(x,t)$ of the Cauchy problem* (1.1) *and* (5.1) *converges to a unique shift of U in the $L^\infty(\mathbb{R})$ norm as $t \to \infty$.*
(2) *If U is discontinuous* (*i.e., if $\beta < \gamma$*) *at, say, $x = 0$, then $u(x,t)$ converges to a unique shift $U(x + x_0)$ of $U(x)$ pointwise on $\mathbb{R}$ as $t \to \infty$, and the convergence is uniform outside any fixed neighborhood of the point of discontinuity $x = -x_0$.*

Remark 5.5. In (2), $u(x,t)$ does not converge uniformly to $U(x + x_0)$ on $\mathbb{R}$ as $t \to \infty$ because $u(x,t)$ is continuous in x for each $t \geqq 0$. Unlike the smooth case, further studies are needed to obtain the convergence rates in (1) and (2). As pointed out in Section 4, if U is discontinuous one cannot relax the monotonicity condition on the initial value since there may exist nonmonotone discontinuous solutions. In the case of continuous U, for which $1 + f'(u) > 0$ on $[-1, 1]$ except at $u = \beta$, we can show that if ϕ is continuous (not necessarily monotone), then each sequence of $t_n' \to \infty$ contains a subsequence $t_n \to \infty$ such that $u(x, t_n) \to$ some $u_\infty(x)$ a.e. in $\mathbb{R}$, and u_∞ is a shift of U. The details of the last point, will be given at the end of this section.

Proof of Theorem 5.4. We assert that $u(x,t)$ is non-decreasing in x. To see this, for every h, define $v(x,t) = u(x+h,t) - u(x,t)$. Then v satisfies

$$\begin{aligned} v_t &= Av + c(x,t)v, \\ v(x,0) &= \phi(x+h) - \phi(x) \geqq 0 \end{aligned} \tag{5.18}$$

where $Av = J * v$, $c(x,t)$ is defined in the obvious way and is bounded on $\mathbb{R} \times \mathbb{R}^+$. The assertion follows if $v \geqq 0$. Without the loss of generality, assume that $c(x,t) \geqq 0$ (otherwise, consider $e^{Mt}v$ with M being large). The "variation-of-constants" formula corresponding to (5.18) is

$$\bar{v}(t) = e^{tA}v(0) + \int_0^t e^{(t-s)A}c(\cdot,s)\bar{v}(s)ds \equiv F(\bar{v}),$$

where $v(0)$ is the initial value in (15.18). Since A is order-preserving and $c \geqq 0$, F is also order-preserving. Now it is easy to show by the Contraction Mapping Theorem that F has a unique fixed point $\bar{v} \geqq 0$ in a ball B centered at $v(0)$ in $L^\infty(\mathbb{R} \times [0,T])$ for a small constant $T > 0$. On the other hand, v, the solution of (5.18), is a fixed point of F in B. So $v = \bar{v} \geqq 0$ in $\mathbb{R} \times [0,T]$. A ladder argument shows $v \geqq 0$ in $\mathbb{R} \times \mathbb{R}^+$. The assertion is proved.

Now since $u(x,t)$ is non-decreasing in x for every $t \geqq 0$, and since $-1 \leqq u \leqq 1$, for each sequence $t'_n \to \infty$, there exists a subsequence $t_n \to \infty$ such that for some nondecreasing function u_∞,

$$u(x,t) \to u_\infty(x) \tag{5.19}$$

pointwise on $\mathbb{R}$. Moreover, the convergence is uniform in any finite interval $[a,b]$ which contains no points of discontinuity of u_∞.

Next we define $W(x,t)$ and $F(u)$ as in the proof of Theorem 5.1. In this situation, $F(1)$ is not necessarily equal to 0. We define a new energy functional $V(t)$ by replacing $F(W)$ in the old one by $F(W) - \chi F(1)$, where $\chi(x) \equiv 0$ for $x < 0$, $\equiv 1$ for $x \geqq 0$. Then exactly as in the previous case, there exists a sequence $t_n \to \infty$ so that for some constant x_0,

$$u(x,t_n) \to U(x+x_0) \quad \text{for } x \neq -x_0; \tag{5.20}$$

furthermore, by the above discussion and (4.15), the convergence is uniform outside any fixed neighborhood of $-x_0$, and if U is continuous everywhere, it is uniform on the whole of $\mathbb{R}$. From this and (b) of Theorem 4.2, follows (1) of Theorem 5.4.

To finish the proof of the second part of Theorem 5.4, we need to take extra care, because we cannot apply (b) of Theorem 4.2. By the calculations in the proof of Theorem 4.1, it is easy to see that for any fixed $\varepsilon > 0$,

$$\underline{u}(x,t) = U(x + x_0 - \varepsilon - \xi(t)) - q(t)$$

is a subsolution of (1.1) where

$$q(t) = \varepsilon e^{-\delta t}, \quad \xi(t) = \frac{\delta + \|f'\|_{L^\infty[-1,1]}}{\delta K}(1 - e^{-\delta t}),$$

δ and K being given in (4.1) and (4.3). We wish to show that $u(x, t+t_n) \geqq \underline{u}(x,t)$ for $t \geqq 0$ if t_n is large. To this end, we need to show that for large t_n,

$$u(x,t_n) \geqq \underline{u}(x,0) = U(x+x_0-\varepsilon) - \varepsilon. \tag{5.21}$$

This is obviously true for x outside the interval $[-R,R]$, R being large, since the convergence in (5.20) is uniform outside $[-R,R]$. To see (5.21) for $x \in [-R,R]$, we argue by contradiction. So assume that there exists a sequence $x_n \in [-R,R]$ such that for large t_n,

$$u(x_n,t_n) < U(x_n+x_0-\varepsilon) - \varepsilon. \tag{5.22}$$

Without the loss of generality, assume that $x_n \to \bar{x}_1 \in [-R,R]$. Fix a constant $\bar{x}_2 < \bar{x}_1$. Then by (5.22) and the monotonicity of u in x,

$$u(\bar{x}_2,t_n) < U(x_n+x_0-\varepsilon) - \varepsilon.$$

Sending $n \to \infty$, we have by (5.20),

$$U^-(\bar{x}_2+x_0) \leqq U^+(\bar{x}_1+x_0-\varepsilon) - \varepsilon,$$

which in turn implies

$$U^-(\bar{x}_1+x_0) \leqq U^+(\bar{x}_1+x_0-\varepsilon) - \varepsilon,$$

where $U^{\pm}(x)$ represent right and left limits of U at x, respectively. This is clearly impossible.

We have shown that $u(x,t_n+t) \geqq \underline{u}(x,t)$ for large t_n, which implies that

$$U^-\left(x+x_0-\varepsilon-\frac{(\|f'\|_{L^\infty}+\delta)\varepsilon}{\delta K}\right) \leqq \liminf_{t\to\infty} u(x,t).$$

Sending $\varepsilon \to 0$, we deduce that $U^-(x+x_0) \leqq \liminf_{t\to\infty} u(x,t)$. Similarly, we can show that $\limsup_{t\to\infty} u(x,t) \leqq U^+(x+x_0)$. This completes the proof of Theorem 5.4 . □

Now we give the details for the last point in Remark 5.5. Suppose ϕ is continuous, satisfying (5.3) and $g'(u) > 0$ on $[-1,1]$ except at $u=\beta$ with $\int_{-1}^{1} g(u) = 0$, where $g(u) = u + f(u)$. We use the notation in the proofs of Theorems 5.4 and 5.1, in particular, $P(t)$ and $Q(t)$ as defined in (5.12). Under our present condition on ϕ, $V(t)$ is still bounded in $t \geqq 0$. Furthermore, by the estimates for $P(t)$ in the proof of Theorem 5.1, we have

$$\int_0^\infty P(t)\ dt < \infty,$$

provided that $\int_{-\infty}^{\infty} x^2 J(x) < \infty$. This and (5.12) imply that

$$\int_0^\infty Q(t)\ dt < \infty. \tag{5.23}$$

We claim $Q'(t)$ is bounded. To see this, observe

$$\begin{aligned}|Q'(t)| &\leqq 2\int_{\mathbb{R}} |J*W - W - f(W)||J*W_t - W_t - f'(W)W_t|\ dx\\ &\leqq C\int_{\mathbb{R}} |J*W - W - f(W)|\ dx\end{aligned}$$

which is bounded by the proof of the boundedness of $V(t)$. Thus $Q'(t)$ is bounded and combining this with (5.23), we see that as $t \to \infty$,

$$Q(t) = \|J * W - W - f(W)\|^2_{L^2(\mathbb{R})} \to 0. \tag{5.24}$$

We write

$$W_t = J * W - W - f(W) + e(x,t). \tag{5.25}$$

Then e is bounded, and $\|e(\cdot,t)\|_{L^1(\mathbb{R})} \to 0$ as $t \to \infty$ by the argument leading to (5.13). This, (5.25) and (5.24) imply that

$$u_t(\cdot,t) \to 0, \text{ in } L^2_{loc}(\mathbb{R}). \tag{5.26}$$

Now we write (1.1) as

$$g(u) = J * u - u_t. \tag{5.27}$$

Since $J * u(x,t)$ is equicontinuous in x, where t is regarded as a parameter, by the Arzelà-Ascoli Theorem, each sequence $t'_n \to \infty$ contains a subsequence $t_n \to \infty$ so that $(J * u)(x,t_n)$ converges pointwise on $\mathbb{R}$ as $t_n \to \infty$. Now (5.27), (5.26) and the fact that g has a continuous inverse imply that

$$u(x,t_n) \to \text{ some } u_\infty(x) \text{ a.e. in } \mathbb{R}, \tag{5.28}$$

after passing to another subsequence. From this, (5.26) and (5.27), we deduce that $g(u_\infty) = J * u_\infty$, i.e., u_∞ is a steady state of (1.1). By (4.15), $u_\infty(\pm\infty) = \pm 1$ and hence u_∞ is a stationary wave of (1.1). By the uniqueness, u_∞ is a shift of U (but might depend on the sequence t_n).

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Peter W. Bates
Department of Mathematics
Brigham Young University
Provo, Utah 84602

Paul C. Fife
Department of Mathematics
University of Utah
Salt Lake City, Utah 84112

Xiaofeng Ren
Department of Mathematics
Brigham Young University
Provo, Utah 84602

Xuefeng Wang
Department of Mathematics
Tulane University
New Orleans, Louisiana 70118

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