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Travelling waves for the Gross-Pitaevskii equation I

by

Fabrice BETHUEL and Jean-Claude SAUT

Université de Paris-Sud,
Laboratoire d'Analyse Numérique et EDP,
Département de Mathématiques, Bât. 425,
URA CNRS 760,
91405 Orsay Cedex, France

ABSTRACT. – We consider the two-dimensional nonlinear Schrödinger equation with repulsion $iv_t = \Delta v + v(1 - |v|^2)$ subject to the boundary condition $v \to 1$, $|x| \to \infty$. We establish the existence of travelling wave solutions $v(x,t) = \tilde{v}(x_1 - ct, x_2)$, $c > 0$, for sufficiently small values of c and exhibit their vortex structure. © Elsevier, Paris

Key words: Nonlinear Schrödinger equation, travelling waves, vortices, Ginzburg-Landau functional, dark solitons.

RÉSUMÉ. – On considère l'équation de Schrödinger bidimensionnelle avec répulsion $iv_t = \Delta v + v(1 - |v|^2)$, avec la condition aux limites $v \to 1$, $|x| \to \infty$. On montre l'existence de solutions ondes progressives $v(x,t) = \tilde{v}(x_1 - ct, x_2)$, $c > 0$, pour des valeurs petites de c et on exhibe leur structure de vortex. © Elsevier, Paris

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I. INTRODUCTION

I.1. Statement of the results

In this paper, we will focus on the following non-linear Schrödinger equation, for a complex valued function $v(x,t)$ defined on $\mathbb{R}^2 \times \mathbb{R}$ (x stands for the spatial variable, t being time)

$$-i\ \frac{\partial v}{\partial t} + \Delta v + v(1-|v|^2) = 0 \text{ on } \mathbb{R}^2 \times \mathbb{R} \tag{1}$$

where $\Delta = \sum_{i=1}^{2} \frac{\partial^2}{\partial x_i^2}$. Here x_1 and x_2 denote cartesian coordinates on $\mathbb{R}^2$, $x = (x_1, x_2)$.

This equation is often termed Gross-Pitaevskii equation and appears in various areas of Physics : non linear optics, fluid dynamics, superfluidity, Bose condensation ... (see for instance the pioneering papers [GR], [JR], [JPR] and also [KR1], [KR2] for references).

At least on a formal level, equation (1) is hamiltonian. The conserved hamiltonian is a "Ginzburg-Landau type" energy, namely

$$E(v) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla v|^2 + \frac{1}{4}\int_{\mathbb{R}^2} (1-|v|^2)^2 .$$

We will only be interested in finite energy solutions here. In view of the form of E, and the potential $V(v) = (1-|v|^2)^2$, natural boundary condition at infinity on $\mathbb{R}^2$ are

$$v(x) \longrightarrow 1, \qquad as \qquad |x| \longrightarrow +\infty . \tag{2}$$

It is easy to prove (see Appendix A) that the Cauchy problem for (1) supplemented by (2) is globally well-posed.

The question we are going to investigate in this paper is the existence problem for travelling wave solutions to (1). Let $c > 0$ be some positive number. We are looking for solutions v to (1) of the form

$$v(x,t) = \tilde{v}(x_1 - ct, x_2), \qquad x = (x_1, x_2) \tag{3}$$

where $\tilde{v}$ is a function defined on $\mathbb{R}^2$. The parameter $c > 0$ represents the speed of the travelling wave. If v is a solution to (1), then the equation for $\tilde{v}$ is

$$-ic\ \frac{\partial \tilde{v}}{\partial x_1} = \Delta \tilde{v} + \tilde{v}(1-|\tilde{v}|^2) \text{ on } \mathbb{R}^2 \tag{4}$$

and conversely any solution to (4) yields a travelling wave solution to (1).

Equation (4) has been studied in a number of places, where the existence problem and the dynamical stability of finite energy solutions to (4) was addressed both on a numerical and a formal level (see the works of Roberts and coworkers ([GR], [JR], [JPR], Pismen and Rubinstein [PR], Pismen and Nepomnyashchy [PN], Kuznetsov and Rasmussen [KR1], [KR2]). They established (at least formally) that travelling waves with finite energy should exist for small values of the parameter c, whereas for large values, they should not. More precisely, the velocity c of the "soliton" should lie between 0 and the minimum phase velocity, coinciding with the sound velocity ($c_s = \sqrt{2}$ in our scaled version). In the limit $c \to 0$, one gets formally two widely separated parallel vortex filaments with opposite circulation while for c closed to c_s the travelling wave solutions should be, after scaling, close to the $2D$ lumps of the Kadomtsev-Petviashvili (KP1) equation. Our aim in this paper, is to give a rigourous proof of the first part of the previous statement. More precisely, we will establish the following

THEOREM 1. – *There exists some constant $c_0 > 0$, such that, for $0 < c < c_0$, equation (4) has a non-constant finite energy solution $\tilde{v}$. Moreover, there exist constants Λ_0 and Λ_1 such that*

$$2\pi \,|\log c| + \Lambda_0 \leq E(\tilde{v}) \leq 2\pi \,|\log c| + \Lambda_1 . \tag{5}$$

The above quoted papers actually tell us more about the structure of the solutions which are constructed. The solutions are expected to have two vortices, *i.e.* two points where $\tilde{v}(x) = 0$. Around each of the vortices the winding number should be $+1$ and -1 respectively. The distance between the two vortices should be of order c^{-1} in the asymptotic limit $c \to 0$. We will show in the course of the paper how much can be proved in this direction (see Proposition VI.7 for a precise statement).

Most of this paper will be devoted to the proof of Theorem 1. Subsections I.2 to I.7 are devoted to explaining the strategy of the proof of Theorem 1. The precise content of the rest of the paper will be presented in subsection I.8.

The nonexistence of travelling waves when $c > c_s = \sqrt{2}$ seems to be linked to the absence of embedded eigenvalues for the linearization of (4) around $\tilde{v} = 1$. More precisely, setting $\tilde{v} = V + 1$, $V = f + ig$, (4) writes

$$L\begin{pmatrix} f \\ g \end{pmatrix} = L_0 \begin{pmatrix} f \\ g \end{pmatrix} + \begin{pmatrix} A(x) & B(x) \\ B(x) & D(x) \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \tag{6}$$

where

$$L_0 = \begin{pmatrix} -\Delta + 2 & +c\partial_{x_1} \\ -c\partial_{x_1} & -\Delta \end{pmatrix}.$$

$$A(x) = 3f + f^2 + g^2, \quad B(x) = g, \quad D(x) = f^2 + g^2.$$

It is easily checked that the continuous spectrum of L_0 (viewed as an unbounded operator in $L^2(\mathbb{R}^2)^2$) is $[0, \infty)$ if $c < \sqrt{2}$ and $\left[-\frac{(c^2-2)^2}{4}, +\infty\right)$ if $c > \sqrt{2}$. The continuous spectrum of L is the same, if f, g decay sufficiently fast as $|x| \to \infty$.

Hence, when $c > \sqrt{2}$, the existence of a non trivial solution of (4) would imply that L has the eigenvalue 0 embedded in the continuous spectrum, which is unlikely to occur, if V decays sufficiently fast as $|x| \to +\infty$.

On the other hand another argument in favor of nonexistence of travelling waves when $c > \sqrt{2}$ is that the mountain pass argument used in the proof of Theorem 1 clearly does not work in this case.

As we will see below, equation (4) is variational. We shall use a mountain-pass theorem to establish Theorem 1.

In order to define the variational formulation, it will be convenient to introduce a change of scale. This will allow us to make use of technics introduced in the study of asymptotic limits of solutions to the Ginzburg-Landau equation (see, for instance [BBH], and references therein).

I.2. A rescaling

In Theorem 1, the small parameter is the speed c. In order to be consistent with works on Ginzburg-Landau functionals (for instance [BBH], we will change the notation and set

$$\varepsilon \equiv c.$$

Next, we change variables and set

$$\tilde{x} = cx = \varepsilon x, \qquad u(\tilde{x}) = \tilde{v}(x), \text{ with } x = \frac{\tilde{x}}{\varepsilon}.$$

In the new variables, equation (4) reads

$$-i\,\frac{\partial u}{\partial \tilde{x}_1} = \Delta u + \frac{1}{\varepsilon^2} u(1 - |u|^2).$$

In the sequel, we will drop the tilda on the coordinates, *i.e.* write

$$-i\,\frac{\partial u}{\partial x_1} = \Delta u + \frac{1}{\varepsilon^2} u(1 - |u|^2).$$

The hamiltonian is now

$$E_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_{\mathbb{R}^2} (1 - |u|^2)^2 \equiv \int_{\mathbb{R}^2} e_\varepsilon(u)$$

which is precisely the energy studied in [BBH] (see also Struwe [Str], Lin [L]). Note that

$$E_\varepsilon(u) = E(\tilde{v}).$$

As we will show in the course of the proof (see Proposition 5) the solution we will obtain satisfies a bound of the form

$$E_\varepsilon(u) \leq 2\pi|\log\varepsilon| + \Lambda_1 \tag{7}$$

for some absolute constant Λ_1. Following the analysis of [BBH], [BR], [Str], we have the following useful result, which we will use in many places.

PROPOSITION 1. – *Let C_0 be any arbitrary constant, and let $0 < \beta < 1$. Let $f \in L^2(\mathbb{R}^2)$, $v \in H^1_{\rm loc}(\mathbb{R}^2)$ be maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that*

$$-\Delta v = \frac{1}{\varepsilon^2}v(1-|v|^2) + f \quad \textit{for} \quad 0 < \varepsilon < 1 \tag{8}$$

$$E_\varepsilon(v) \leq C_0|\log\varepsilon| \tag{9}$$

$$|f|_{L^2(\mathbb{R}^2)} \leq C_0|\log\varepsilon|\varepsilon^{-\beta}. \tag{10}$$

Then v is continuous on $\mathbb{R}^2$, and there exist constants $N > 0$, and $\lambda > 0$, depending only on C_0 and β, ℓ points $a_1, \ldots, a_\ell$ in $\mathbb{R}^2$, such that

$$\ell \leq N \tag{11}$$

$$|v(x)| \geq \frac{1}{2}, \qquad \forall x \in \mathbb{R}^2 \setminus \bigcup_{i=1}^{\ell} B(a_i, \lambda\varepsilon). \tag{12}$$

We will term the points $a_1, \ldots, a_\ell$ the vortices of our map : in their neighborhood, $|v|$ might be small, or even might vanish. Since $|v|$ is larger than one-half on the boundary $\partial B(a_i, \lambda\varepsilon)$ the map $\frac{v}{|v|}$ has some winding number on $\partial B(a_i, \lambda\varepsilon)$. We will call it the degree of v at a_i.

Note that for any solution to (6) satisfying the bound (7), Proposition 1 applies and we may define its vortices. Actually (7) is more precise than (9), and in this case, we may prove that v has at most two vortices with winding number $+1$ or -1, all other vortices having degree zero. In order to do so, we have however to redefine slightly the notion of vortices, and enlarge the radius around them. More precisely, we have

PROPOSITION 2. – *Let Λ, C_0 be arbitrary constants, and let $0 < \beta < 1$. Assume that f and v satisfy the conditions of Proposition 1, that is (8), (9) and (10). Assume moreover that*

$$E_\varepsilon(v) \leq 2\pi|\log\varepsilon| + \Lambda. \tag{13}$$

Then there exist constants μ *and* $\tilde{\mu}$ *in* $(0,1)$, *a number* $N \in \mathbb{N}$, *depending only on* β, Λ *and* C_0, *such that there exist* $\rho > 0$, $\overline{\mu} > 0$, ℓ *points* $a_1, \ldots, a_\ell$ *in* $\mathbb{R}^2$ *such that*

$$\varepsilon^{\mu} \le \rho \le \varepsilon^{\overline{\mu}}, \qquad 0 < \tilde{\mu} < \overline{\mu} \le \mu < 1 \tag{14}$$

$$|a_i - a_j| \ge \varepsilon^{\overline{\mu}/2}, \qquad \forall i \neq j \text{ in } \{1, \ldots, \ell\} \tag{15}$$

$$|v(x)| \ge \frac{1}{2} \quad \text{on} \quad \mathbb{R}^2 \setminus \bigcup_{i=1}^{\ell} B(a_i, \rho) \tag{16}$$

$$\ell \le N \tag{17}$$

$$|v(x)| \ge 1 - \frac{2}{|\log \varepsilon|^2} \quad \text{on} \quad \partial B(a_i, \rho). \tag{18}$$

There exist at most two vortices in the collection $\{a_1, \ldots, a_\ell\}$ *such that*

$$\deg a_i \equiv \deg \left(\frac{v}{|v|}, \partial B(a_i, \rho) \right) \neq 0.$$

Call a_1 *and* a_2 *these vortices. Then we have*

$$\deg a_1 = - \deg a_2$$
$$|\deg a_1| = 1.$$

Finally we have also

$$\int_{\partial B(a_i, \rho)} e_\varepsilon(v) \le \frac{\pi}{\rho}(1 + K(\varepsilon)) \quad \text{if} \quad i = 1, 2 \tag{19}$$

$$\int_{\partial B(a_i, \rho)} e_\varepsilon(v) \le \frac{\pi}{\rho} K(\varepsilon) \text{ for } i > 2, \tag{20}$$

where $K(\varepsilon)$ *depends only on* ε, *and* $K(\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

The proof is essentially the same as the proof of Theorem 5 in [AB] (relying on a construction introduced in a preliminary version of [BBH]). For sake of completeness we will give a proof in Appendix B.

As a consequence of Proposition 2, in view of the bound (5), the solution we construct in Theorem 1 will essentially have two vortices. An important question is to determine the distance of these vortices. We will turn to this question later. (See Proposition VI.7).

I.3. Variational formulation

As already noticed in the references quoted above, equation (6) is variational. To be more specific, let us introduce the space V of maps definined by

$$V = \left\{ v : \mathbb{R}^2 \to \mathbb{C}, \text{ such that } v - 1 \in H^1(\mathbb{R}^2) \right\}.$$

V is an affine space modelled on the Hilbert space $H^1(\mathbb{R}^2)$ equipped with the standard norm

$$||v||^2_{H^1(\mathbb{R}^2)} = ||v||^2_{L^2(\mathbb{R}^2)} + ||\nabla v||^2_{L^2(\mathbb{R}^2)}.$$

Next, for $v \in V$ set

$$L(v) = \frac{1}{2} \int_{\mathbb{R}^2} (iv_{x_1}, v - 1)$$

and

$$F_\varepsilon(v) = E_\varepsilon(v) - L(v).$$

Note that by the embedding $H^1(\mathbb{R}^2) \subset L^4(\mathbb{R}^2)$, E_ε is well defined on $H^1(\mathbb{R}^2)$.

Then, we have

PROPOSITION 3. – *The functionals E_ε, L and F_ε are C^1 on V. Moreover any critical point of F_ε is a solution to (4).*

The proof of Proposition 3 is rather standard and we omit it.

Note that the space V has been introduced for the convenience of the variational formulation. By no means do we claim that the solution constructed in Theorem 1 belongs to V (actually it is conjectured that it does **not**). However, the space V is well-fitted for implementing our mountain-pass argument. This will yields us so called Palais-Smale sequences (*i.e.* approximate solutions to (4)). We will then prove convergence to a solution to (4) on compact sets.

The variational formulation we introduced above is still not very satisfactory for our purposes. One important flaw is that F is not bounded below (see the construction in Section I.5) : this gives serious troubles in order to make Palais-Smale sequences converge. In order to overcome this difficulty, we modify slightly the functional, introducing a cut-off for L.

Let $K > 0$ be some large number, to be determined later (and remaining fixed troughout the paper). Consider a function φ_K, smooth from $\mathbb{R}$ to $\mathbb{R}$ having the following properties

$$\begin{array}{ll} \varphi_K(s) = s & \text{if} \quad |s| \leq K \\ \varphi_K(s) = 2K & \text{if} \quad s \geq 2K \\ \varphi_K(s) = -2K & \text{if} \quad s \leq -2K, \end{array}$$

and finally

$$0 \leq \varphi'_K(s) \leq \frac{4}{K}.$$

Our perturbed functional will then be given by

$$\overline{F}_{K,\varepsilon} = E_\varepsilon - \varphi_K(L(u)).$$

(We will simply write $\overline{F}_\varepsilon$ or $\overline{F}$ and φ, when no confusion is possible). We then verify that $\overline{F}$ is of class C^1 on V and that

$$\langle d\overline{F}(v), \Phi\rangle = \langle dE(v), \Phi\rangle - \varphi'(L(v))\langle dL(v), \Phi\rangle, \quad \forall \Phi \in V$$

so that critical points for $\overline{F}$ verify

$$-i\varphi'(L(v))\frac{\partial v}{\partial x_1} = \Delta v + \frac{1}{\varepsilon^2}v(1-|v|^2). \tag{21}$$

Therefore a solution to (21) is a critical point of F provided $\varphi'(L(v)) = 1$, which is in particular the case if

$$L(v) \leq K. \tag{22}$$

Actually, our choice for K will be

$$K = 4\pi\, d_1, \tag{23}$$

where d_1 is another constant, which will be introduced in Section I.6 (the mountain-pass argument), and which will be determined later.

With this choice, we may work with $\overline{F}_\varepsilon$ instead of F_ε, and still obtain solutions for our initial problem.

I.4. Geometrical interpretation of L

As we will see for maps having nice vortices, the functional L is very simply related to the location of the vortices. To get a feeling for that property, consider a very simple model situation of a smooth map v having two nice vortices, of degree $+1$ and -1 respectively, denoted P and N respectively. In coordinates, we write

$$P = (P_1, P_2), \qquad N = (N_1, N_2).$$

We assume moreover that v satisfies the following five properties
- v is smooth on $\mathbb{R}^2$,
- $|v| = 1$ outside $B(P, \varepsilon) \cap B(N, \varepsilon)$, $[|P - N| \geq \varepsilon]$,
- $\deg(v, \partial B(P, \varepsilon)) = 1$, $\deg(v, \partial B(N, \varepsilon)) = -1$,
- $|\nabla v| \leq \frac{4}{\varepsilon}$,
- $v = 1$, outside some large ball $B(R)$.

Though very special, the above situation bears some resemblance with the result of Proposition 2. Next, we estimate $L(v)$. We have, by Fubini's theorem

$$(24) \qquad \int_{\mathbb{R}^2} (iv_x, v - 1) = \int_{\mathbb{R}} ds \left[\int_{x_2 = s} [iv_{x_1}, v - 1] \, dx_1 \right].$$

By our assumption $|v| = 1$ on the line $x_2 = s$ provided $s \notin [P_2 - \varepsilon, P_2 + \varepsilon] \cup [N_2 - \varepsilon, N_2 + \varepsilon]$. In that case, the integral

$$\int_{x_2 = s} (iv_{x_1}, v - 1)$$

is a topological number, equal to -2π if $s \in [N_2 + \varepsilon, P_2 - \varepsilon]$ and to zero if $s \in]-\infty, N_2 - \varepsilon] \cup [P_2 + \varepsilon, +\infty[$. Going back to (23), a simple computation shows that

$$|L(v) - \pi(P_2 - N_2)| \longrightarrow 0 \qquad \text{as} \qquad \varepsilon \to 0,$$

so that L is approximately $\pi(P_2 - N_2)$. In view of our later analysis we will present next a slightly more canonical way of constructing a map with two vortices (in the spirit of [BBH]). This will in particular be useful for our mountain-pass argument.

Let f be any smooth function from $\mathbb{R}$ to $\mathbb{R}^+$ such that

$$f(t) = 0 \text{ if } t \leq \frac{1}{2}$$

$$f(t) = 1 \text{ if } t \geq 1,$$

and $|f'| \leq 4$.

Let $d \geq 0$, and set $P = \left(0, -\frac{d}{2}\right)$, $N = \left(0, +\frac{d}{2}\right)$. Let $\varepsilon > 0$, and define the map v_d^ε from $\mathbb{R}^2$ to $\mathbb{R}^2$ by

$$v_d^\varepsilon(z) = f\left(\frac{|z-P|}{\varepsilon}\right) f\left(\frac{|z-N|}{\varepsilon}\right) \frac{z-P}{|z-P|} \left(\frac{z-N}{|z-N|}\right)^{-1}.$$

Clearly the map v_d^ε has two vortices (provided $d > 2\varepsilon$) of opposite degree $+1$ and -1, located at P and N, and $|v_d^\varepsilon(z)| = 1$, provided $\operatorname{dist}(z, P) \geq \varepsilon$ and $\operatorname{dist}(z, N) \geq \varepsilon$. Similar estimates as above yield

LEMMA 1. – *We have, for fixed* $d > 0$, $v_d^\varepsilon \in V$, *and for* $0 < \varepsilon < 1$

$$E_\varepsilon(v_d^\varepsilon) = 2\pi \left| \log \frac{\varepsilon}{d} \right| + C_0 + R_\varepsilon^1$$

$$L(v_d^\varepsilon) = \pi d + R_\varepsilon^2$$

where $|R_\varepsilon^1| \leq C \left|\frac{\varepsilon}{d}\right|$, $|R_\varepsilon^2| \leq C\varepsilon$ *provided* $d \geq 8\varepsilon$, *and* C_0 *is some constant.*

In order to use calculus of variation, we need next to extend the notion of vortices to arbitrary maps in V. This is the aim of the next Section.

I.5. Vortices for maps in V

The argument is borrowed from [AB] and is of a slightly indirect nature. Starting from a map v in V, we wish to show that v has "essential" vortices, as described in the previous sections. In order to do so, we have to impose an E_ε–energy bound on v, as (13). However this is not enough to avoid dipole configuration on a small scale. To get rid of these, basically unnecessary, details on a small scale, we introduce a regularization. Set

$$h = \varepsilon^{1/4},$$

which represents the scale of regularization. Consider next the functional G_h defined on V, by

$$G_h(w) = E_\varepsilon(w) + \frac{1}{2h^2} \int_{\mathbb{R}^2} |w - v|^2.$$

Clearly G_h is well defined, and C^1 on V. We have

LEMMA 2.

$$\operatorname*{Inf}_{w \in V} G_h(w)$$

is achieved, by some map $v_h \in V$, *which satisfies the equation*

$$\frac{v_h - v}{h^2} = \Delta v_h + \frac{1}{\varepsilon^2} v_h (1 - |v_h|^2). \tag{25}$$

Moreover

$$|L(v) - L(v_h)| \leq 2\varepsilon^{1/4} E_\varepsilon(v) \tag{26}$$

and

$$E_\varepsilon(v_h) + \int_{\mathbb{R}^2} \frac{|v_h - v|^2}{h^2} \leq E_\varepsilon(v). \tag{27}$$

The proof of Lemma 2 will be given in Appendix C.

Note that as in [AB], we do not claim uniqueness for v_h. However, most of the information carried by the vortices of v will be preserved for v_h, provided the bound (13) holds. Therefore assume that

$$E_\varepsilon(v) \leq 2\pi |\log \varepsilon| + \Lambda$$

and set

$$f_h = \frac{v_h - v}{h^2} = \varepsilon^{-1/2}(v_h - v).$$

In view of (26) we have for small ε,

$$|f_h|_{L^2(\mathbb{R}^2)} \leq \left[3\pi(|\log \varepsilon|)\right]^{1/2} \varepsilon^{-1/4}.$$

Hence we may apply Proposition 2 to v_h, f_h with $\beta = \frac{1}{4}$. Thus, we deduce

LEMMA 3. – *There exist constants* μ *and* $\tilde{\mu}$ *in* $(0, 1)$, *and a constant* $N \in \mathbb{N}^*$, *depending only on* Λ *such that there are* ℓ *points* $a_1, \ldots, a_\ell$ *in* $\mathbb{R}^2$, $\rho > 0$, $\overline{\mu} > 0$, *such that*

$$\varepsilon^{\mu} \leq \rho \leq \varepsilon^{\overline{\mu}}, \qquad 0 < \tilde{\mu} < \overline{\mu} \leq \mu < 1, \qquad \forall\ 0 < \varepsilon < \frac{1}{2}$$

$$|a_i - a_j| \geq \varepsilon^{\overline{\mu}/2} \quad \textit{if} \quad a_i \neq a_j$$

$$\ell \leq N$$

$$|v_h(x)| \geq \frac{1}{2} \quad \textit{on} \quad \mathbb{R}^2 \setminus \bigcup_{i=1}^{\ell} B(a_i, \rho)$$

$$\sum_{i=1}^{\ell} |d_i| \in \{0,2\} \text{ where } d_i = \deg a_i$$

and such that (18), (19) and (20) (if $\Sigma|d_i| = 2$*) hold. Moreover, we have*

$$|L(v_h) - \pi(a_{1,2} - a_{2,2})| \leq K(\varepsilon), \tag{28}$$

where $K(\varepsilon)$ *depends only on* Λ *and* ε*, and* $K(\varepsilon)\to 0$ *as* $\varepsilon \to 0$.

In Lemma 3, we have used the notation

$$a_i = (a_{i,1}, a_{i,2})$$

in coordinates.

Of course, the map v_h might well have no vortices a_1, a_2 (*i.e.*.may well have no vortices of degree $+1$ or -1, that is only vortices of degree zero, or not vortices at all : take for instance $v = 1$, so that $v_h = 1$). In this case we have made the following convention : $a_1 = a_2$ are arbitrary points and we set $\deg a_i = (-1)^i$ by pure convention. Note however, that if v_h has a vortex of degree 1, it has to have another vortex of opposite degree, since $E_\varepsilon(v_h)$ is finite.

The only point in Lemma 3 which is not a consequence of Proposition 2 is (28). The proof will be given in Appendix C. An important consequence of (28) and (26) is

LEMMA 4.

$$|L(v) - \pi(a_{1,2} - a_{2,2})| \leq K(\varepsilon) \tag{29}$$

where $K(\varepsilon)$ *depends only on* Λ *and* ε*, and* $K(\varepsilon)\to 0$ *as* $\varepsilon \to 0$.

To conclude this Section, an important problem is to determine some continuity properties of the map which assigns to v its vortices a_1 and a_2. This problem has already been considered in [AB]. Since we cannot expect continuity (the vortices are only defined modulo a certain regularization, *i.e.* introducing a lengthscale), the notion of $\eta-$almost continuity was introduced in [AB]. We recall it.

DEFINITION 1. – *Let* F *and* G *be two metric spaces and* $\eta \geq 0$ *be a given constant. Let* f *be a map from* F *to* G*. We say that* f *is* $\eta-$*almost continuous at a point* $u_0 \in F$ *if and only if, for any* $\theta > 0$*, there exists* $\delta > 0$*, such that if* $d(u, u_0) \leq \delta$*, then*

$$d(f(u), f(u_0)) \leq \eta + \theta.$$

Next consider for given $\Lambda \in \mathbb{R}$

$$\tilde{V} = \Big\{v \in V, E_\varepsilon(v) \leq 2\pi(|\log\varepsilon| + \Lambda)\Big\}$$

and the map ψ from $\tilde{V}$ to $\mathbb{R}^2 \times \mathbb{R}^2$ defined by

$$u \longmapsto (a_1, a_2)$$

considered as a configuration space of charged particles (see [AB] for more precision). Then we have

PROPOSITION 4. – *The map ψ is η-almost continuous for*

$$\eta = c\varepsilon^{\overline{\mu}}$$

where c is a positive constant.

I.6. The mountain-pass argument

Here we will describe how to obtain critical point for $\overline{F}$. The first step will be to establish the existence of Palais-Smale sequences v_n for $\overline{F}$, with a lower bound on $|L(v_n)|$. The later estimate will be crucial in order to prove that the sequence does not converge to a trivial ($u \equiv 1$) solution. In order to obtain such a bound, we will use a variant of the mountain-pass theorem (Ambrosetti and Rabinowitz), due to Ghoussoub and Preiss ([GP]), based on Ekeland's variational principle ([E]). Let us recall that Theorem.

THEOREM 2 ([GP]). – *Let X be a Banach space, and ϕ be a C^1 functional on X. Let $\mathcal{C}_0$ and $\mathcal{C}_1$ be two non-empty closed and disjoint subsets of X, and consider the set $\mathcal{P}$ of paths joining $\mathcal{C}_0$ and $\mathcal{C}_1$ i.e.*

$$\mathcal{P} = \Big\{p \in C^0([0,1], X), p(0) \in \mathcal{C}_0, p(1) \in \mathcal{C}_1\Big\}.$$

Set

$$c = \operatorname*{Inf}_{p\in\mathcal{P}} \Big(\operatorname*{Max}_{s\in[0,1]} \phi(p(s))\Big).$$

Assume that there exists a closed subset M of X such that

$$M^c = M \cap \{x \in X, \phi(x) \geq c\}$$

separates $\mathcal{C}_0$ and $\mathcal{C}_1$, that is $\mathcal{C}_0$ and $\mathcal{C}_1$ are included in two disjoint connected components of $X \setminus M^c$. Then, there exists a sequence x_n in X such that

$$d(x_n, M) \longrightarrow 0 \quad as \quad n \longrightarrow +\infty$$

$$\text{(P.S)} \qquad \begin{cases} \phi(x_n) & \longrightarrow c \\ ||d\phi(x_n)||_{X^*} & \longrightarrow 0. \end{cases}$$

Note that the usual mountain-pass theorem corresponds to the case $M = X$. Recall that a sequence x_n satisfying (PS) is called a Palais-Smale sequence for ϕ, at the level c.

We will apply Theorem 2 to our problem. We take

$$X = V$$

(which is an affine space : this obviously makes no difference), and

$$\phi = \overline{F}_\varepsilon .$$

Next we have to define the sets $\mathcal{C}_0$ and $\mathcal{C}_1$. We set for $0 < \varepsilon < \frac{1}{2}$

$$\text{(30)} \qquad \mathcal{C}_0^\varepsilon = \left\{ v \in V,\, |L(v)| \le \frac{\pi}{8},\, E_\varepsilon(v) \le 2\pi\,|\log\varepsilon| + \check{\Lambda} \right\}$$

and

(31)

$$\mathcal{C}_1^\varepsilon = \left\{ v \in V,\, |L(v) - \pi d_1| \le \frac{\pi}{8},\, E_\varepsilon(v) \le 2\pi\,|\log\varepsilon| + 2\pi\,\log d_1 + \hat{\Lambda} \right\}$$

where $d_1 > 4$, $\check{\Lambda}$ and $\hat{\Lambda}$ are three constants which will be fixed, throughout the paper, but determined later.

We have

LEMMA 5. – *$\mathcal{C}_0^\varepsilon$ and $\mathcal{C}_1^\varepsilon$ are closed subsets, which are disjoint. $\mathcal{C}_1^\varepsilon$ is not empty, provided $\hat{\Lambda} \ge \hat{\Lambda}_0$, where $\hat{\Lambda}_0$ is some absolute constant, and $0 < \varepsilon < \frac{1}{2}$.*

Proof. – The first assertion is obvious. For the second assertion we have, assuming $d_1 \ge 4$,

$$|L(v_{d_1}^\varepsilon) - \pi\, d_1| \le o(1)$$

and by Lemma 1

$$E_\varepsilon(v_{d_1}^\varepsilon) = 2\pi\,|\log\varepsilon| + 2\pi\,\log d_1 + C_0 + R(\varepsilon)$$

so that

$$v_{d_1}^\varepsilon \in \mathcal{C}_1^\varepsilon$$

provided $\hat{\Lambda} \geq \hat{\Lambda}_0 \equiv C_0 + \sup_{0 \leq \varepsilon \leq \frac{1}{2}} R(\varepsilon)$.

This completes the proof of the Lemma.

We will actually determine the values of $\hat{\Lambda}$ and $\check{\Lambda}$ as follows. We set

$$\hat{\Lambda} = \text{Max}(\Lambda_1 + 3\pi, \hat{\Lambda}_0) + 1 \tag{32}$$

$$\check{\Lambda} = 2\pi(\log 2 - 1) + 2\gamma_0 + 1\,. \tag{33}$$

Here Λ_1 is an absolute constant appearing in Proposition 5 (see Lemma II.1). Similarly γ_0 is an absolute constant, introduced in [BBH], p.43 and appearing in Lemma II.2.

With this choice of $\hat{\Lambda}$ and $\check{\Lambda}$, we determine the value of d_1. For $v \in \mathcal{C}_\varepsilon^1$ we observe that

$$F_\varepsilon(v) = E_\varepsilon(v) - L(v) \leq 2\pi\,|\log\varepsilon| + 2\pi\,\log d_1 - \pi\,d_1 + \frac{\pi}{8} + \hat{\Lambda}\,.$$

We determine d_1 so that $d_1 \geq 4$ and

$$2\pi\,\log d_1 - \pi\,d_1 + \frac{\pi}{8} + \hat{\Lambda} < 2\pi(\log 2 - 1) + 2\gamma_0 - 1\,. \tag{34}$$

The choices of $\check{\Lambda}$, $\hat{\Lambda}$, and d_1 ensure us that

$$\text{Max}_{u \in \mathcal{C}_1^\varepsilon} \overline{F}_\varepsilon(u) < c_\varepsilon\,, \qquad \text{Max}_{u \in \mathcal{C}_0^\varepsilon} \overline{F}_\varepsilon(u) < c_\varepsilon$$

if ε is sufficiently small.

Finally, we set

$$\mathcal{P}_\varepsilon = \left\{p \in C^0([0,1], V), p(0) \in \mathcal{C}_0^\varepsilon, p(1) \in \mathcal{C}_1^\varepsilon\right\},$$

$$c_\varepsilon = \text{Inf}_{p \in \mathcal{P}_\varepsilon} \left(\text{Max}_{s \in [0,1]} \overline{F}_\varepsilon(p(s))\right)$$

and

$$\overline{F}_\varepsilon^{c_\varepsilon} = \{v \in V, \overline{F}_\varepsilon(v) \leq c_\varepsilon\}\,.$$

PROPOSITION 5. – *There are two constants* Λ_0 *and* Λ_1 *such that, if* $0 < \varepsilon < 1$, *then*

$$2\pi\,|\log\varepsilon| + \Lambda_0 \leq c_\varepsilon \leq 2\pi\,|\log\varepsilon| + \Lambda_1\,. \tag{35}$$

This fact already shows that if d_1 is choosen sufficiently large, then $V \setminus \overline{F}_\varepsilon^{c_\varepsilon}$ separates C_0^ε and C_1^ε, and provides us, thanks to Theorem 2 (with

$M = V$), a Palais-Smale sequence for $\overline{F}_\varepsilon$. In order to prove that we may choose a Palais-Smale sequence which does not converge (weakly) to a constant solution, we define M as

$$M = \{v \in V, \pi \check{d}_0 \leq |L(v)| \leq \pi \hat{d}_0\}$$

where $\check{d}_0$ and $\hat{d}_0$ are fixed numbers, such that

$$\frac{1}{4} < \check{d}_0 < 2 < \hat{d}_0 < 4.$$

Then we have

PROPOSITION 6. – *There exists $\varepsilon_1 > 0$ depending on $\check{d}_0$ and $\hat{d}_0$, such that*

$$N^\varepsilon = V \setminus M^{c_\varepsilon} = \{u \in V, \overline{F}_\varepsilon(v) < c_\varepsilon\} \cup \{V \setminus M\}$$

separates C_0^ε and C_1^ε, if $0 < \varepsilon < \varepsilon_1$.

As a consequence of Theorem 2, Propositions 5 and 6, we finally may assert the following

PROPOSITION 7. – *There exists a sequence of maps u_n in V such that*

$$|L(u_n)| \geq \pi \check{d}_0 + o(1) \tag{36}$$

$$\overline{F}_\varepsilon(u_n) \leq 2\pi |\log \varepsilon| + \Lambda_1 + o(1) \tag{37}$$

where $o(1) \to 0$ as $n \to +\infty$ and

$$||d\overline{F}_\varepsilon(u_n)||_{H^{-1}} \to 0. \tag{38}$$

I.7. End of the proof of Theorem 1

In order to complete the proof of Theorem 1, we need to show that the sequence u_n converges to a non trivial solution (6). Since the functional F_ε is invariant under translations, we first have to get rid of this invariance. We have

PROPOSITION 8. – *Let $\Lambda \geq 0$ be a constant, and let v_n be a sequence of maps in V such that*

$$E_\varepsilon(v_n) \leq 2\pi |\log \varepsilon| + \Lambda \tag{39}$$

and

$$4\pi \geq |L(v_n)| \geq \frac{\pi}{8}. \tag{40}$$

Then, there exists $\overline{\varepsilon} > 0$, depending only on Λ, such that if $0 < \varepsilon < \overline{\varepsilon}$, there exists a sequence of points $(b_n)_{n \in \mathbb{N}}$ in $\mathbb{R}^2$ such that a subsequence of the *sequence* $\tilde{v}_n = v_n(\cdot - b_n)$ *converges strongly in* $H^1(K)$, *for any compact subset* $K \subset \mathbb{R}^2$, *to a map* v, *which is not constant.*

Combining Proposition 7 and Proposition 8, Theorem 1 follows easily. More precise properties of the solution u will be given at the end of the paper (see Section VI).

This paper is organized as follows. The background analysis in Sections I.2 to I.5 (and which are adaptations from [BBH] and [AB]), will be exposed in a separate Appendix. In the next Section (Section II) we will prove Proposition 5. Section III is devoted to the proof of Proposition 6. The key ingredients are Lemma III.4 and III.5, which deal with constructions of continuous paths in M^{c_ε}. Section IV is devoted to the proof of Proposition 8. The proof of Theorem 1 will be completed in Section V. Further properties of the solution will be given in Section VI, in particular the fact that $v(x) \to 1$ as $|x| \to +\infty$. The global well-posedness of the Cauchy problem is established in Appendix A while Appendix B is devoted to the technical proofs of Propositions 1 and 2. Finally, Lemmas 2 and 3 are proved in Appendix C. We postpone to a subsequent paper the study of the three-dimensional case as well of further properties of the travelling waves. After this work was completed we have been aware of the paper [CJ] where the dynamics of vortices as $\varepsilon \to 0$ is studied, for the space-periodic problem associated to (1).

II. PROOF OF PROPOSITION 5

Since Proposition 5 contains both an upper bound and a lower bound for c_ε, we are going to divide the proof into two separate parts. We start with the easy one.

LEMMA II.1. – *There exists an absolute constant* $\Lambda_1 \in \mathbb{R}$, *such that for* $0 < \varepsilon < \frac{1}{2}$, *we have*

$$c_\varepsilon \leq 2\pi |\log \varepsilon| + \Lambda_1 . \tag{II.1}$$

Proof of Lemma II.1. – It suffices to exhibit a path on which the maximal value of $\dot{F}_\varepsilon$ is less than $2\pi|\log\varepsilon| + \Lambda_1$. Take for that purpose $p_0 \in \mathcal{P}_\varepsilon$ defined by, for $s \in [0,1]$

$$p_0(s) = v_d^\varepsilon, \text{ with } d = d_1 s \text{ (recall that } d_1 < 4, \text{ see Section 1.6)}$$

(since $v_0^\varepsilon \in \mathcal{C}_0^\varepsilon$ and $v_{d_1}^\varepsilon \in \mathcal{C}_1^\varepsilon$, p_0 is a path in $\mathcal{P}_\varepsilon$), where the map v_d^ε was constructed in Section I.4. From Lemma 1, we then deduce that, for $s \geq 8\varepsilon$

$$F_\varepsilon(p_0(s)) \leq \Big(2\pi \log \frac{d}{\varepsilon} - \pi d\Big) + C_0 + C\,\frac{\varepsilon}{s}.$$

Hence,

$$F_\varepsilon(p_0(s)) \leq 2\pi \log \frac{1}{\varepsilon} + (2\pi \log d - \pi d) + C. \tag{II.2}$$

Consider the one variable function ψ from $\mathbb{R}^+$ to $\mathbb{R}$ defined by

$$\psi(d) = 2\pi \log d - \pi d.$$

This function achieves its maximum at the point

$$d_0 = 2, \qquad \psi(d_0) = 2\pi(\log 2 - 1)$$

so that we obtain the estimate

$$\operatorname{Max} F_\varepsilon(p_0(s)) \leq 2\pi\,|\log \varepsilon| + 2\pi(\log 2 - 1) + C.$$

Choosing K in the definition of $\overline{F}$ such that

$$K = 4\pi d_1 \qquad \text{(see Section I.3)},$$

we obtain similarly

$$\operatorname*{Max}_{s\in[0,1]} \overline{F}_\varepsilon(p_0(s)) \leq 2\pi\,|\log \varepsilon| + 2\pi(\log 2 - 1) + C,$$

which yields the conclusion (II.1).

The lower bound for c_ε, and actually a much more precise asymptotic analysis, will be a consequence of our next result. First we introduce some notation. Set (as in [BBH], p.42), for $R > 0$, $D_R = \{z \in \mathbb{R}^2, |z| < R\}$, $H_R = \big\{u \in H^1(D_R, \mathbb{R}^2), u(z) = \frac{z}{|z|} \text{ on } \partial D_R\big\}$ and, for $\varepsilon > 0$,

$$I(\varepsilon, R) = \operatorname*{Min}_{u\in H_R} \Big\{\frac{1}{2}\int_{D_R} |\nabla u|^2 + \frac{1}{4\varepsilon^2}(1 - |u|^2)^2\Big\}.$$

Set also

$$I(\varepsilon) \equiv I(\varepsilon, 1).$$

Note that, by scaling

$$I(\varepsilon, R) = I\Big(\frac{\varepsilon}{R}\Big).$$

It follows from the analysis in [BBH], that the function $t \to I(t) + \pi \log t$ is increasing and has a limit as $t \to 0$. Set

(II.3) $$\gamma_0 = \lim_{t \to 0} [I(t) + \pi \log t].$$

We shall need the following result

LEMMA II.2. – *Let $1 > \rho > \varepsilon^{\mu}$, and let w be a map from $B(\rho)$ to $\mathbb{R}^2$ such that*

$$|w| \geq 1 - \frac{2}{|\log \varepsilon|^2} \quad on\ \partial B(\rho),$$

$$\int_{\partial B(\rho)} \frac{1}{2} |w_\tau|^2 + \frac{1}{4\varepsilon^2}(1 - |w|^2)^2 \leq \frac{\pi}{\rho}(1 + K(\varepsilon)),$$

where τ denotes a unit tangent to $\partial B(\rho)$, oriented properly, and where $K(\varepsilon) \to 0$ as $\varepsilon \to 0$. Assume also

$$\deg\Big(\frac{w}{|w|}, \partial B(\rho)\Big) = \pm 1.$$

Then we have

$$\int_{B(\rho)} e_\varepsilon(w) \geq \pi \log \frac{\rho}{\varepsilon} + \gamma_0 + K_1(\varepsilon),$$

where $K_1(\varepsilon)$ depends only on ε and $K_1(\varepsilon) \to 0$ as $\varepsilon \to 0$.

We postpone the proof of Lemma II.2, and turn to the central result of this section.

PROPOSITION II.2. – *Let $\Lambda \in \mathbb{R}$ be a given constant, and let v be a map in V such that*

$$E_\varepsilon(v) \leq 2\pi |\log \varepsilon| + \Lambda.$$

Let v_h be the regularized map obtained from v by Lemma 2. Let a_1 and a_2 be the vortices of v_h of degree $(-1)^i$ given by Lemma 3 and set $d = |a_1 - a_2|$. Assume $d \neq 0$, so that

$$d \geq \varepsilon^{\overline{\mu}/2}.$$

Then we have

$$E_\varepsilon(v) \geq E_\varepsilon(v_h) \geq 2\pi \log \frac{d}{\varepsilon} + 2\gamma_0 + K(\varepsilon),$$

where $K(\varepsilon)$ depends only on Λ and ε, and $K(\varepsilon)\to 0$ as $\varepsilon\to 0$.

Proof of Proposition II.2. – The proof goes back to techniques introduced in [BBH] and [AB]. Let ρ, a_i, μ and $\overline{\mu}$ be as in Lemma 3. Set

$$\Omega_\rho = \mathbb{R}^2 \setminus \bigcup_{i=1}^{\ell} B(a_i, \rho).$$

We are going to bound first $\int_{\Omega_\rho} |\nabla v_h|^2$, then $\int_{\mathbb{R}^2\setminus\Omega_\rho} |\nabla v_h|^2$.

Step 1. – Estimates for $\int_{\Omega_\rho} |\nabla v_h|^2$.

Since $|v_h| \geq \frac{1}{2}$, we may consider on Ω_ρ the map

$$\overline{v} = \frac{v_h}{|v_h|} \quad \text{(the bar is not a complex conjugation !)}$$

Clearly $\overline{v} : \Omega_\rho \to S^1$. Moreover

$$\begin{aligned} \deg(\overline{v}, \partial B(a_i\rho)) &= (-1)^i \qquad \text{for } i = 1, 2 \\ &= 0 \qquad\qquad \text{for } i > 2. \end{aligned}$$

Let us estimate first $\int_{\Omega_\rho} |\nabla \overline{v}|^2$. We claim that

$$\int_{\Omega_\rho} |\nabla \overline{v}|^2 \geq 2\pi \log \frac{d}{\varepsilon} + K_1(\varepsilon), \tag{II.4}$$

where $K_1(\varepsilon)\to 0$ as $\varepsilon\to 0$.

Proof of the claim (II.4). – Following [BBH], chap. 1, we have, by Hodge-de-Rham decomposition

$$\begin{aligned} (i\overline{v}, \overline{v}_{x_1}) &= H_{x_1} - \Phi_{x_2} \\ (i\overline{v}, \overline{v}_{x_2}) &= H_{x_2} + \Phi_{x_1} \end{aligned} \tag{II.5}$$

where

$$\Phi(z) = -(\log|z - a_1|) + \log(|z - a_2|), \quad \forall z \in \mathbb{R}^2,$$

so that

$$-\Delta\Phi = 2\pi(-\delta_{a_1} + \delta_{a_2}), \text{ on } \mathbb{R}^2,$$

and where H is a function defined (up to an additive constant) on Ω_ρ. We deduce from (II.5) that

$$\text{(II.6)} \quad \int_{\Omega_\rho} |\nabla \overline{v}|^2 = \int_{\Omega_\rho} |\nabla \Phi|^2 + \int_{\Omega_\rho} |\nabla H|^2 + \int_{\Omega_\rho} (\Phi_{x_1} H_{x_2} - \Phi_{x_2} H_{x_1}).$$

We observe that the last term is the integral of a Jacobian, *i.e.*

$$\Phi_{x_1} H_{x_2} - \Phi_{x_2} H_{x_1} = \frac{\partial}{\partial x_1}(\Phi H_{x_2}) - \frac{\partial}{\partial x_2}(\Phi H_{x_1})$$

so that

$$\text{(II.7)} \qquad \int_{\Omega_\rho} (\Phi_{x_1} H_{x_2} - \Phi_{x_2} H_{x_1}) = \sum_{i=1}^{\ell} \int_{\partial B(a_i,\rho)} \Phi H_\tau ,$$

where again τ denotes a unit tangent to $\partial B(a_i, \rho)$, oriented properly. We have, for any constant Φ_i :

$$\int_{\partial B(a_i,\rho)} \Phi H_\tau = \int_{\partial B(a_i,\rho)} (\Phi - \Phi_i) H_\tau$$

so that

$$\text{(II.8)} \quad \left| \int_{\Omega_\rho} \Phi_{x_1} H_{x_2} - \Phi_{x_2} H_{x_1} \right| \leq \sum_{i=1}^{\ell} \sup_{\partial B(a_i,\rho)} |\Phi - \Phi_i| \int_{\partial B(a_i,\rho)} |\nabla H| .$$

Choose next

$$\begin{aligned} \Phi_i &= \Phi(a_i), \text{ for } i > 2 \qquad\qquad \text{and} \\ \Phi_1 &= -(\log|a_1| - \log|a_1 - a_2|) \\ \Phi_2 &= -(\log|a_2 - a_1| - \log|a_2|), \end{aligned}$$

so that, since

$$|a_i - a_j| \geq \varepsilon^{\overline{\mu}/2}$$

$$\text{(II.9)} \qquad \sup_{\partial B(a_i,\rho)} |\Phi - \Phi_i| \leq 2\varepsilon^{\overline{\mu}/2}, \qquad \forall i \in \{1, \ldots, \ell\}.$$

On the other hand, we have

$$\text{(II.10)} \qquad \begin{aligned} \int_{\partial B(a_i,\rho)} |\nabla H| &\leq 2 \int_{\partial B(a_i,\rho)} (|\nabla \Phi| + |\nabla \overline{v}|) \\ &\leq C, \end{aligned}$$

where C is some constant, independent of ε. For the last inequality, we may invoke (19) and (20). Combining (II.8), (II.9) and (II.10), we deduce that

$$\left| \int \Phi_{x_1} H_{x_2} - \Phi_{x_2} H_{x_1} \right| \leq C \varepsilon^{\frac{\overline{\mu}}{2}} \to 0 \text{ as } \varepsilon \to 0.$$

Hence, by (II.6)

$$\int_{\Omega_\rho} |\nabla \overline{v}|^2 \geq \int_{\Omega_\rho} |\nabla \Phi|^2 - C \varepsilon^{\overline{\mu}/2}. \tag{II.11}$$

We clearly have

$$\int_{\mathbb{R}^2 \setminus \bigcup_{i=1}^{2} B(a_i, \rho)} |\nabla \Phi|^2 = 2\pi \log \frac{d}{\rho} + C \varepsilon^{\overline{\mu}/2}. \tag{II.12}$$

On the other hand, for $i \neq 1, 2$

$$\int_{B(a_i, \rho)} |\nabla \Phi|^2 \leq 2\pi \varepsilon^{-\overline{\mu}} \rho^2 \leq 2\pi \varepsilon^{\overline{\mu}}, \tag{II.13}$$

so that the claim (II.4) follows. Next, using (II.4) we are going to show that

$$\int_{\Omega_\rho} |\nabla v_h|^2 \geq 2\pi \log \frac{d}{\rho} + K_2(\varepsilon) \tag{II.14}$$

where $K_2(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Proof of (II.14). – Set

$$\mu = |v_h|.$$

Then, we have

$$|\nabla v_h|^2 = |\nabla \mu|^2 + \mu^2 |\nabla \overline{v}|^2,$$

and hence

$$\begin{aligned} \int_{\Omega_\rho} |\nabla v_h|^2 &\geq \int_{\Omega_\rho} \mu^2 |\nabla \Phi|^2 + \int_{\Omega_\rho} \mu^2 (\Phi_{x_1} H_{x_2} - \Phi_{x_2} H_{x_1}) \\ &\geq \int_{\Omega_\rho} |\nabla \Phi|^2 + I_1 + I_2 \end{aligned} \tag{II.15}$$

where

$$I_1 = \int_{\Omega_\rho} (1-\mu^2)|\nabla\Phi|^2$$

$$I_2 = 2\int_{\Omega_\rho} (1-\mu^2)|\nabla\Phi||\nabla H|.$$

In view of the definition of Φ we have

$$|\nabla\Phi|(z) \leq \frac{2}{|z|} \leq \frac{2}{|\rho|} \leq 2\varepsilon^{-\mu} \text{ on } \Omega_\rho.$$

Hence

$$\text{(II.16)} \qquad \begin{aligned} |I_1| &\leq ||(1-\mu^2)||_{L^2(\mathbb{R}^2)} ||\nabla\Phi||^{1/2}_{L^4(\Omega_\rho)} \\ &\leq C\,\varepsilon|\log\varepsilon|^{1/2}\rho^{-1} \\ &\leq C\,\varepsilon^{1-\mu}|\log\varepsilon|^{1/2} \to 0, \text{ as } \varepsilon\to 0. \end{aligned}$$

Similarly, we have

$$\text{(II.17)} \qquad \begin{aligned} |I_2| &\leq ||(1-\mu^2)|\nabla\Phi|||_{L^2(\Omega_\rho)} ||\nabla H||_{L^2(\Omega_\rho)} \\ &\leq \frac{1}{\rho}||(1-\mu^2)||_{L^2(\mathbb{R}^2)} ||\nabla H||_{L^2(\Omega_\rho)} \\ &\leq C\,\varepsilon|\log\varepsilon|\rho^{-1} \leq C\,\varepsilon^{1-\mu}|\log\varepsilon| \to 0 \text{ as } \varepsilon\to 0. \end{aligned}$$

Inequality (II.14) then follows, combining (II.15), (II.16), (II.17) and (II.12), (II.13).

Step 2. – Estimates for $\int_{\Omega_\rho} e_\varepsilon(v_h)$.

For $i = 1, 2$, we have

$$|v_h(x)| \geq 1 - \frac{2}{|\log\varepsilon|^2} \text{ on } \partial B(a_i,\rho),$$

and

$$\int_{\partial B(a_i,\rho)} e_\varepsilon(v_h) \leq \frac{\pi}{\rho}(1+K(\varepsilon)),$$

by (19). Moreover

$$\deg\left(\frac{v_h}{|v_h|}, \partial B(a_i,\rho)\right) = (-1)^i.$$

Hence we may invoke Lemma II.2 to assert that

$$\text{(II.18)} \qquad \sum_{i=1}^{2} \int_{B(a_i,\rho)} e_\varepsilon(v_h) \geq 2\pi \log\frac{\rho}{\varepsilon} + 2\gamma_0 + K_3(\varepsilon)$$

where $K_3(\varepsilon)\to 0$ as $\varepsilon\to 0$.

Step 3. *Proof of Proposition* II.2 *completed.* – Combining (II.14) and (II.18) we easily complete the proof of Proposition II.2.

Next, we give the proof of Lemma II.2, which has been postponed until now.

Proof of Lemma II.2. – On $\partial B(\rho)$, we may write, assuming for instance that $\deg\left(\frac{w}{|w|}, \partial B(\rho)\right) = +1$,

$$w(z) = |w|(z)\exp(i(\theta+\varphi(\theta))), \quad \text{for } \theta\in[0,2\pi]$$

where $\varphi\in H^1([0,2\pi])$, with $\varphi(0)=\varphi(2\pi)$. Set

$$\overline{w} = \frac{w}{|w|} \quad \text{on } \partial B(\rho).$$

We have

$$|\overline{w}_\theta|^2 = (1+\varphi_\theta)^2 \quad \text{on } \partial B(\rho).$$

On the other hand, we have

$$\text{(II.19)} \qquad \int_{\partial B(\rho)} |w_\theta|^2 = \int_{\partial B(\rho)} (|\mu_\theta|^2 + \mu^2|\overline{w}_\theta|^2)$$

where $\mu=|w|$. Since $\mu > 1-\frac{2}{|\log\varepsilon|^2}$ on $\partial B(\rho)$. We deduce that

$$\text{(II.20)} \qquad \int_{\partial B(\rho)} |w_\theta|^2 \geq \int_{\partial B(\rho)} (|\mu_\theta|^2 + |\overline{w}_\theta|^2) + K_1(\varepsilon)$$

where $K_1(\varepsilon)\to 0$ as $\varepsilon\to 0$. Therefore, since

$$\int_{\partial B(\rho)} \left(|w_\theta|^2 + |\mu_\theta|^2 + \frac{\rho}{\varepsilon^2}(1-|w|^2)^2\right) \leq 2\pi(1+K(\varepsilon))$$

we deduce from (II.19), (II.20), (II.21) that

$$\text{(II.21)} \qquad \int_{\partial B(\rho)} \left(|\mu_\theta|^2 + |\varphi_\theta|^2 + \frac{\rho}{\varepsilon^2}(1-|w|^2)^2\right) \leq K_2(\varepsilon)$$

where $K_2(\varepsilon)\to 0$ as $\varepsilon\to 0$.

We consider next the ball $B(2\rho)$. We claim that we may construct a map $\tilde{w}$ on $B(2\rho)$ such that

$$\tilde{w} = w \text{ on } B(\rho) \tag{II.22}$$

$$\tilde{w}(z) = \exp i(\theta + \varphi(0)) \text{ on } \partial B(2\rho) \tag{II.23}$$

and

$$\left| \int_{B(2\rho)\setminus B(\rho)} e_\varepsilon(\tilde{w}) - \pi \log 2 \right| \leq K_3(\varepsilon), \tag{II.24}$$

where $K_3(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Proof of the claim. – Set

$$\tilde{w}(z) = \eta(z)\breve{w}(z) \text{ on } B(2\rho) \setminus B(\rho)$$

where $\breve{w}(z)$ is S^1–valued defined by

$$\breve{w}(z) = \exp(i\theta + \varphi(\theta)\Big(\frac{2\rho - |z|}{|\rho|}\Big) + \varphi(0)\Big(\frac{|z| - \rho}{\rho}\Big)$$

and where $\eta(z)$ is real-valued, defined by

$$\eta(z) = |w|\Big(\frac{z}{|z|}\rho\Big)\Big(\frac{2\rho - |z|}{\rho}\Big) + \frac{|z| - \rho}{\rho}$$

so that $\tilde{w}$ verifies the boundary conditions (II.22) and (II.23). We have

$$|\nabla \breve{w}|^2 = \frac{2\pi}{|z|} + \frac{|\varphi'(\theta)|^2}{|z|^2}\Big(\frac{2\rho - |z|}{\rho}\Big) + |\varphi(\theta) - \varphi(0)|^2 \frac{1}{|z|}.$$

Since

$$\sup_{\theta \in [0, 2\pi]} |\varphi(0) - \varphi(\theta)| \leq \int_0^{2\pi} |\varphi'(\theta)| d\theta \leq [K_2(\varepsilon)]^{1/2} (2\pi)^{1/2}$$

we verify that

$$\left| \int_{B(2\rho)\setminus B(\rho)} |\nabla \breve{w}|^2 - 2\pi \log 2 \right| \leq K_4(\varepsilon)$$

where $K_4(\varepsilon) \to 0$ as $\varepsilon \to 0$. Similar estimates for η yield the conclusion (II.24).

Proof of Lemma II.2 *completed.* – In view of the definition of I we have

$$\int_{B(2\rho)} e_\varepsilon(\tilde{w}) \geq I(\varepsilon, 2\rho) = I\Big(\frac{\varepsilon}{2\rho}\Big)$$

so that by (II.24)

$$\int_{B(\rho)} e_\varepsilon(w) \geq I\Big(\frac{\varepsilon}{2\rho}\Big) - \pi \log 2 - K_3(\varepsilon).$$

The result follows from (II.3).

As a consequence of Proposition II.2 and the η–almost continuity, we obtain the following

PROPOSITION II.3. – *We have*

$$c_\varepsilon \geq 2\pi |\log \varepsilon| + 2\gamma_0 + 2\pi(\log 2 - 1) + K(\varepsilon) \tag{II.25}$$

where $K(\varepsilon)$ *depends only on* ε *and* $K(\varepsilon)\to 0$ *as* $\varepsilon\to 0$.

Proof of Proposition II.3. – First note, that for any map v in V, verifying a bound of the form (7), we have

$$\pi |d(v)| \geq |L(v)| + o(1) \tag{II.26}$$

where, throughout the lemma $o(1)$ denotes some function depending only on ε, such that $o(1)\to 0$ as $\varepsilon\to 0$. Here $d(v)$ represents the distance between the vortices of the map v_h. Inequality (II.26) is then an easy consequence of Lemma 3 of the introduction.

In view of the η–almost continuity of the map Φ, it clearly turns out that the function $d : v\to d(v)$ is also η–almost continuous (for the same η given by Proposition 4 of the Introduction). Consider next a path $p \in \mathcal{P}_\varepsilon$, that is a continuous map p from $[0, 1]$ to $\tilde{V}$ such that

$$p(0) \in \mathcal{C}_0^\varepsilon$$
$$p(1) \in \mathcal{C}_1^\varepsilon.$$

Recall that the value of d_1 is determined by (34) ; in particular

$$d_1 \geq 4. \tag{II.27}$$

We claim that there is some $s_0 \in [0, 1]$ such that (if ε is sufficiently small)

$$|d(p(s_0)) - 2| \leq 3\eta. \tag{II.28}$$

Indeed, by Proposition IV.1 of [AB] there exists some continuous map $\tilde{f}$ from $[0,1]$ to $\mathbb{R}$ such that

$$|d(p(s)) - \tilde{f}(s)| \leq 3\eta, \qquad \forall s \in [0,1]. \tag{II.29}$$

On the other hand, we have

$$|d(p(0)| \leq o(1)$$
$$|d(p(1)) - 4| \leq o(1)$$

so that

$$\begin{cases} |\tilde{f}(0)| \leq o(1) \\ |\tilde{f}(1) - 4| \leq o(1). \end{cases}$$

It then follows by the intermediate value theorem that there is some $s_0 \in [0,1]$ such that

$$f(s_0) = 2,$$

and (II.28) then can be deduced from (II.29).

Applying Proposition II.2, we obtain

$$E_\varepsilon(p(s_0)) \geq 2\pi \log\left(\frac{d(p_0(s))}{\varepsilon}\right) + 2\gamma_0 + R_\varepsilon.$$

By (II.28), this yields

$$E_\varepsilon(p(s_0)) \geq 2\pi \log \frac{1}{\varepsilon} + 2\pi \log 2 + 2\gamma_0 + o(1).$$

On the other hand, by (II.26), we have

$$-L(p(s_0)) \geq -\pi\, d(p(s_0)) + o(1).$$

Hence, by (II.28) again

$$\overline{F}_\varepsilon(p(s_0)) \geq F_\varepsilon(p(s_0) \geq 2\pi \log\left(\frac{1}{\varepsilon}\right) + 2\pi(\log 2 - 1) + o(1),$$

which yields (II.25).

Proof of Proposition 5 completed. – It suffices to combine Proposition II.1 and Proposition II.2. Actually, with slightly more work, we may prove the following

PROPOSITION 5 bis. – *We have*

$$c_\varepsilon = 2\pi \log \frac{1}{\varepsilon} + 2\pi(\log 2 - 1) + 2\gamma_0 + o(1)$$

where $o(1) \to 0$ *as* $\varepsilon \to 0$.

The proof of Proposition 5 bis relies on a slight refinement of Proposition II.1.

III. PROOF OF PROPOSITION 6

The proof of Proposition 6, is the key ingredient in the proof of Theorem 1 : it is also the most technical part of this paper. The proof will be split into two parts, Proposition III.1 and Proposition III.2, which describe paths in the level set

$$\overline{F}_\varepsilon^c = \{v \in V, \overline{F}_\varepsilon(v) \leq c\} \tag{III.1}$$

for $c \leq c_\varepsilon$, joining $\mathcal{C}_1^\varepsilon$ and $\mathcal{C}_0^\varepsilon$. These paths will be constructed using (partially) heat-flow techniques. This will first be described in the next subsection.

III.1. Heat-flows for $\overline{F}_\varepsilon$

Let $v \in V$. We consider the heat-flow for $\overline{F}_\varepsilon$, defined by

(III.2)
$$\frac{\partial}{\partial t}v(t,\cdot) - i\varphi'(L(v(t,\cdot)))\frac{\partial v(t,\cdot)}{\partial x_1} = \Delta v(t,\cdot) + \frac{1}{\varepsilon^2}v(t,\cdot)(1-|v(t,\cdot)|^2)$$

$$v(0,\cdot) = v \tag{III.3}$$

where $v(t,x)$ is defined for $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^2$, and where

$$\Delta v(t,\cdot) = \sum_{i=1}^{2} \frac{\partial^2}{\partial x_1^2} v(t,\cdot).$$

In the sequence, we will often make use of the notation

$$v_t(x) = v(t,x), \qquad \forall t \in \mathbb{R}^+, \quad \forall x \in \mathbb{R}^2$$

so that (III.3) can be rephrased as

$$v_0(x) = v(x).$$

The following existence result can be established.

LEMMA III.1. – *Let $v \in V$. Then there exists a solution $v(\ ,\cdot)$ to (III.2) and (III.3), which is smooth for all $t > 0$, and such that we have $v_t \in V$,* ($\forall t \geq 0$). *Moreover, we have the inequality*

$$\int_0^T \int_{\mathbb{R}^2} \left|\frac{\partial v}{\partial t}\right|^2 + \overline{F}_\varepsilon(v_T(\cdot)) \leq \overline{F}_\varepsilon(v_0). \tag{III.4}$$

The proof of Lemma III.1 is standard and we will omit it. ■

Next we assume that $v \in V$, and verifies a bound of the form

$$E_\varepsilon(v) \le 2\pi \log \frac{1}{\varepsilon} + \Lambda, \tag{III.5}$$

where Λ is fixed and given by

$$\Lambda = \overline{\Lambda} + 2\pi \log d_1 + 2\gamma_0, \tag{III.6}$$

for instance. We have

LEMMA III.2. – *Let v be in V, and assume that v satisfies (III.5). Then for $0 < \varepsilon < \frac{1}{2}$, and any $t \in [0, \varepsilon^{1/4}]$, we have*

$$\int_{\mathbb{R}^2} |v - v_t|^2 \le C |\log \varepsilon| \varepsilon^{1/4} \tag{III.7}$$

and

$$|L(v) - L(v_t)| \le C |\log \varepsilon| \varepsilon^{1/4}. \tag{III.8}$$

Proof. – Write for $x \in \mathbb{R}$

$$v_t(x) - v(x) = \int_0^t \frac{\partial v}{\partial t}(s, x)\, ds$$

so that

$$\begin{aligned} |v - v_t|^2(x) &\le \left(\int_0^t \left| \frac{\partial v}{\partial t}(s,x) \right|^2 ds \right) t \\ &\le \left(\int_0^{\varepsilon^{1/4}} \left| \frac{\partial v}{\partial t} \right|^2 (s,x)\, ds \right) \varepsilon^{1/4} \end{aligned}$$

and inequality (III.7) follows by integration. Inequality (III.8) can be derived as was inequality (26) in Lemma 2. Next, we have

LEMMA III.2. – *Let v be in V, and assume that v satisfies (III.5). Then for $0 < \varepsilon < \frac{1}{2}$, there exists $t_1 \in [0, \varepsilon^{1/4}]$ such that*

$$\int_{\mathbb{R}^2} \left| \frac{\partial}{\partial t} v(t_1, \cdot) \right|^2 \le C |\log \varepsilon| \varepsilon^{-1/4} \tag{III.9}$$

where C is some absolute constant.

Proof. – Applying (III.4) for $T = \varepsilon^{1/4}$, we have

$$\int_0^{\varepsilon^{1/4}} dt\left(\int_{\mathbb{R}^2}\left|\frac{\partial v}{\partial t}(t,x)\right|^2 dx\right) \leq \overline{F}_\varepsilon(v) - \overline{F}_\varepsilon(v_{\varepsilon^{1/4}}).$$

From the definition of $\overline{F}_\varepsilon$, we deduce (with our choice $K = 4\pi d_1$) that

$$E_\varepsilon(v) - 8\pi d_1 \leq \overline{F}_\varepsilon(v) \leq E_\varepsilon(v) + \pi d_1, \qquad \forall v \in V.$$

Therefore

$$\begin{aligned}\overline{F}_\varepsilon(v) - \overline{F}_\varepsilon(v_{\varepsilon^{1/4}}) &\leq \overline{F}_\varepsilon(v) + 8\pi d_1 \\ &\leq 2\pi|\log\varepsilon| + \Lambda + 9\pi d_1.\end{aligned}$$

The conclusion then follows easily applying Fubini's theorem.

Observe next that $v_1 \equiv v_{t_1}$ satisfies the equation

$$-\Delta v_1 = \frac{1}{\varepsilon^2} v_1(1 - |v_1|^2) + f_1 \text{ on } \mathbb{R}^2 \tag{III.10}$$

where

$$f_1 = -\frac{\partial v}{\partial t}(t_1, \cdot) + \varphi'(L(v_1))\frac{\partial v_1}{\partial x_1}.$$

Hence, we verify that, for $0 < \varepsilon < \frac{1}{2}$

$$|f_1|_{L^2(\mathbb{R}^2)} \leq C(\log\varepsilon)^{1/2}\varepsilon^{-1/8}. \tag{III.11}$$

Hence, we may apply Proposition 2 to v_1. This yields

LEMMA III.3. – *There are constants μ and $\tilde{\mu}$ in $(0,1)$, N in $\mathbb{N}^*$, independent of ε, and ℓ points $a_1, \ldots, a_\ell$ in $\mathbb{R}^2$, and $\rho > 0$, $\overline{\mu} > 0$, such that*

$$\varepsilon^\mu \leq \rho \leq \varepsilon^{\overline{\mu}}, \quad \forall 0 < \varepsilon < \frac{1}{2}, \quad 0 < \tilde{\mu} < \overline{\mu} \leq \mu < 1 \tag{III.12}$$

$$|a_i - a_j| \geq \varepsilon^{\overline{\mu}/2} \quad \textit{if} \quad a_i \neq a_j \tag{III.13}$$

$$\ell \leq N \tag{III.14}$$

$$|v_1(x)| \geq \frac{1}{2} \quad \textit{on} \quad \mathbb{R}^2 \setminus \bigcup_{i=1}^{\ell} B(a_i, \rho) \tag{III.15}$$

$$\begin{aligned}&\deg a_i = 0, \quad i \in \{3, \ldots, \ell\} \\ &\deg a_i = (-1)^i \quad \textit{for} \quad i = 1, 2,\end{aligned} \tag{III.16}$$

and

$$\int_{\partial B(a_i,\rho)} e_\varepsilon(v_1) \leq \frac{\pi}{\rho}(1 + K(\varepsilon)) \quad \textit{if} \quad i = 1, 2 \tag{III.17}$$

$$\int_{\partial B(a_i,\rho)} e_\varepsilon(v_1) \leq \frac{\pi}{\rho}(K(\varepsilon)) \qquad \textit{if} \quad i > 2 \tag{III.18}$$

where $K(\varepsilon)\to 0$ *as* $\varepsilon\to 0$.

Set $\Omega_\rho = \mathbb{R}^2 \setminus \bigcup_{i=1}^{\ell} B(a_i,\rho)$. We may consider on Ω_ρ the map

$$\overline{v}_1 = \frac{v_1}{|v_1|}$$

which is S^1-valued. As in the proof of Proposition II.2, we may write

$$\text{(III.19)}\qquad \begin{aligned} (i\overline{v}_1, \overline{v}_{1_{x_1}}) &= H_{1_{x_1}} - \Phi_{x_2} \\ (i\overline{v}_1, \overline{v}_{1_{x_2}}) &= H_{1_{x_2}} + \Phi_{x_1} \end{aligned}$$

where

$$\Phi(z) = -\log(|z-a_1|) + \log(|z-a_2|).$$

Arguing as in the proof of Lemma II.4, inequality (II.21), we may prove, using (III.17) and (III.18)

LEMMA III.4. – *We have*

$$\text{(III.20)}\qquad \sum_{i=1}^{\ell} \int_{\partial B(a_i,\rho)} |\nabla H_1|^2 + \big|\nabla |v_1|\big|^2 + \frac{1}{\varepsilon^2}(1-|v_1|^2)^2 \le \frac{K(\varepsilon)}{\rho},$$

where $K(\varepsilon)$ *depends only on* ε, *and* $K(\varepsilon)\to 0$, *as* $\varepsilon\to 0$.

Similarly, arguing as in the proof of Proposition II.2, we have

LEMMA III.5. – *We have*

$$\text{(III.21)}\qquad \int_{\Omega_\rho} |\nabla v_1|^2 \ge \int_{\Omega_\rho} |\nabla\Phi|^2 + \int_{\Omega_\rho} \eta^2 |\nabla H_1|^2 + K(\varepsilon)$$

where $K(\varepsilon)\to 0$ *as* $\varepsilon\to 0$.

Our next aim will be to construct a deformation of H_1, which decrease $\overline{F}_\varepsilon$ together with H_1.

III.2. A deformation for the phase term H

Since H_1 is defined up to an additive constant, we may always ajust this constant so that

$$\text{(III.22)}\qquad \overline{v}_1(z) = \left(\frac{z-a_1}{|z-a_1|}\right)^{-1} \frac{z-a_2}{|z-a_2|} \exp iH_1(z), \quad \text{on } \Omega_\rho$$

so that H_1 represents somehow a phase term. We are going to construct, on Ω_ρ a flow for F_ε, based on deforming only the phase term H_1, but not the location of the vortices, nor the modulus v_1, and keeping H_1 fixed on $\partial\Omega_\rho$. For that purpose, we notice that

$$\int_{\Omega_\rho} e_\varepsilon(v_1) = I_1 + I_2 + I_3 + I_4$$

where

$$I_1 = \int_{\Omega_\rho} |\nabla\eta|^2, \text{ where } \eta = |v_1|$$

$$I_2 = \int_{\Omega_\rho} |\nabla\Phi|^2 \eta^2$$

$$I_3 = \int_{\Omega_\rho} \eta^2 \big(H_{1_{x_2}} \Phi_{x_1} - H_{1_{x_1}} \Phi_{x_2}\big)$$

and

$$I_4 = \int_{\Omega_\rho} \eta^2 |\nabla H_1|^2 .$$

The function H enters only in the last two terms I_3 and I_4. Similarly we may write

$$\int_{\Omega_\rho} \big(iv_{1_{x_1}}, v_1 - 1\big) = \int_{\Omega_\rho} (\eta^2 - 1) H_{1_{x_1}} + I_5 ,$$

where I_5 does not depend on H_1. We therefore consider the functional, for real valued maps H on Ω_ρ

(III.23)
$$W(H) = \frac{1}{2}\int_{\Omega_\rho} \eta^2 |\nabla H|^2 + \eta^2 \big(H_{x_2}\Phi_{x_1} - H_{x_1}\Phi_{x_2}\big) - (\eta^2 - 1) H_{x_1} .$$

In order to define a well-posed evolution equation for H which decreases both W and the L^2−norm of the gradient of H, we will restrict ourselves to a bounded domain. Therefore, we choose some $\tilde{R} > 0$, such that

$$\tilde{R} > 2 \sup_{i\in\{1,\dots,\ell\}} |a_i - a_1|$$

(III.24)
$$\int_{|z-a_1|>\frac{\tilde{R}}{2}} |v_1 - 1|^2 \leq \varepsilon^4$$

(III.25)
$$\int_{|z-a_1|=\tilde{R}} |\nabla H_1|^2 + |\nabla\eta|^2 + \frac{1}{4\varepsilon^2}(1 - |\eta|^2)^2 \leq \frac{\varepsilon^4}{\tilde{R}}$$

and

(III.26)
$$|\eta| \geq 1 - \frac{2}{|\log \varepsilon|^2} \qquad \text{on } \partial B(a_1, \tilde{R}).$$

For (III.26), we may argue as in the proof of Proposition 2, using the coarea formula. We consider the bounded domain

$$\tilde{\Omega} = \Omega_\rho \cap B(a_1, \tilde{R})$$

and the functional

$$\tilde{W}(H) = \frac{1}{2} \int_{\tilde{\Omega}} \eta^2 |\nabla H|^2 + \eta^2 \big(H_{x_2} \Phi_{x_1} - H_{x_1} \Phi_{x_2}\big) - (\eta^2 - 1) H_{x_1}.$$

Next, we define the functional space for H. Set

$$\Sigma = \left\{ \begin{array}{ll} \varphi \in H^1(\tilde{\Omega}),\ \varphi = \text{Cte} = C_i & \text{on } \partial B(a_i, \rho) \\ \qquad\qquad \varphi = 0 & \text{on } \partial B(a_1, R) \end{array} \right\}$$

(note that the constant value of φ on $\partial B(a_i, \rho)$ is not prescribed) and

$$\tilde{\Sigma} = \big\{ H \in H^1(\Omega) / \exists \varphi \in \Sigma,\, H = \varphi + H_1 \big\}.$$

$\tilde{\Sigma}$ will be the functional space on which we will consider the functional $\tilde{W}$. Set

$$h = (h_1, h_2) = \frac{1}{2} \big(\eta^2 \Phi_{x_2} - (\eta^2 - 1), \eta^2 \Phi_{x_1}\big).$$

The variational formulation for the heat flow for $\tilde{W}$ on $\tilde{\Sigma}$ writes :

(III.27)
$$\int_{\mathbb{R}^+} \left(\int_{\tilde{\Omega}} \varphi \frac{\partial H(t,\cdot)}{\partial t} + \eta^2 \nabla H(t,\cdot) \nabla \varphi(t,\cdot) + h \nabla \varphi(t,\cdot)\, dx \right) dt = 0$$

$\forall \varphi \in C_c^\infty(]0, +\infty[\times\overline{\tilde{\Omega}}, \mathbb{R})$, such that $\varphi(t,\cdot) \in \Sigma$, $\forall t \in \mathbb{R}^+$. We supplement this equation with initial and boundary conditions

(III.28)
$$H(0, z) = H_1(z) \text{ on } \tilde{\Omega}$$

(III.29)
$$H(t,\cdot) \in \tilde{\Sigma}, \qquad \forall t \in \mathbb{R}^+.$$

We have the following existence result.

LEMMA III.6. – *There exists a solution $H(\cdot,\cdot)$ on $\mathbb{R}^+ \times \tilde{\Omega}$ to (III.27), (III.28), (III.29), which satisfies : $\forall T > 0$*

$$\int_0^T dt\left(\int_{\tilde{\Omega}} \left|\frac{\partial H}{\partial t}\right|^2 dx\right) + W(H(T)) \leq W(H(0)). \tag{III.30}$$

Moreover, as $t \to +\infty$, $H(t,\cdot)$ converges strongly in $H^1(\Omega)$ to H_2, where H_2 is the unique solution in variational form to

$$\int_{\tilde{\Omega}} \eta^2 \nabla H_2 \nabla \varphi + h \nabla \varphi = 0, \qquad \forall \varphi \in \Sigma. \tag{III.31}$$

The proof of Lemma III.6, is standard and will be omitted. The main property of the solution H_2, which will be of interest for us is the following

LEMMA III.7. – *We have*

$$\int_{\tilde{\Omega}} |\nabla H_2|^2 \leq K(\varepsilon), \tag{III.32}$$

where $K(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Proof. – Set, for $i = 1, \ldots, \ell$

$$c_i = \frac{1}{2\pi\rho}\int_{\partial B(a_i,\rho)} H_2\, ds,$$

the mean value of H_2 on $\partial B(a_i,\rho)$. Consider the solution $\overline{H}_2$ to

$$\begin{aligned} \Delta \overline{H}_2 &= 0 && \text{on } \tilde{\Omega} \\ \overline{H}_2 &= H_2 - c_i && \text{on } \partial B(a_i,\rho) \\ \overline{H}_2 &= H_2 && \text{on } \partial B(a_1,\tilde{R}). \end{aligned}$$

Since

$$(\overline{H}_2)_\tau = (H_1)_\tau \text{ on } \partial B(a_i,\rho),$$

we deduce from Lemma III.4 that

$$\operatorname{Max}_{\tilde{\Omega}} |\overline{H}_2| \leq K_1(\varepsilon) \tag{III.33}$$

and

$$\int_{\tilde{\Omega}} |\nabla \overline{H}_2|^2 \leq K_1(\varepsilon), \tag{III.34}$$

where $K_1(\varepsilon)$ depends only on ε, and $K_1(\varepsilon) \to 0$ as $\varepsilon \to 0$. We have

$$H_2 - \overline{H}_2 \in \Sigma$$

so that we may use $\psi = H_2 - \overline{H}_2$ as a test function in (III.31). This yields

$$\int_{\tilde{\Omega}} \eta^2 \nabla H_2 \cdot \nabla(H_2 - \overline{H}_2) + h \nabla(H_2 - \overline{H}_2) = 0. \tag{III.35}$$

We have from the definition of h,

$$\int_{\tilde{\Omega}} h \nabla \psi = \frac{1}{2} \int_{\tilde{\Omega}} \eta^2 \big(\psi_{x_1} \Phi_{x_2} - \psi_{x_2} \Phi_{x_1}\big) - (\eta^2 - 1) \psi_{x_1}.$$

From the Jacobian property, and since $\psi = \text{Cte}$ on $\partial B(a_i, \rho)$ we see that

$$\int_{\tilde{\Omega}} \psi_{x_1} \Phi_{x_2} - \psi_{x_2} \Phi_{x_1} = 0.$$

Hence

$$\int_{\tilde{\Omega}} h \nabla \psi = \frac{1}{2} \int_{\tilde{\Omega}} (\eta^2 - 1) \big[\psi_{x_1} \Phi_{x_2} - \psi_{x_2} \Phi_{x_1} - \psi_{x_1}\big]$$

and, since $|\nabla \Phi| \leq \frac{2}{\rho}$

$$\left| \int_{\tilde{\Omega}} h \nabla \psi \right| \leq \frac{C}{\rho} \int_{\tilde{\Omega}} |(\eta^2 - 1)| \nabla \psi|.$$

Therefore

$$\begin{aligned} \left| \int_{\tilde{\Omega}} h \nabla \psi \right| &\leq C\, \varepsilon^{1-\mu} |\log \varepsilon|^{1/2} ||\nabla \psi||_{L^2} \\ &\leq C\, \varepsilon^{1-\mu} |\log \varepsilon| \big(||\nabla H_2||_{L^2} + ||\nabla \overline{H}_2||_{L^2}\big). \end{aligned}$$

Going back to (III.35), we obtain

$$\begin{aligned} \int_{\tilde{\Omega}} |\nabla H_2|^2 \leq C\big(&||\nabla H_2||_{L^2} ||\nabla \overline{H}_2||_{L^2}\big) \\ &+ \varepsilon^{1-\mu} |\log \varepsilon|^{1/2} \big(||\nabla H_2||_{L^2} + ||\nabla \overline{H}||_{L^2}\big). \end{aligned} \tag{III.36}$$

From the fact that (by (III.30))

$$W(H_2) \leq W(H_1)$$

we deduce that

$$\int_{\tilde{\Omega}} |\nabla H_2|^2 \leq C(\log \varepsilon|)$$

which combined with (III.36) yields (III.32), and completes the proof of the Lemma.

For $t \in [0, +\infty[$, we will define the map $w_t \in V$, by

$$w_t = v_1 \quad \text{on} \quad \mathbb{R}^2 \setminus B(a_1, \tilde{R})$$

$$w_t = |\eta| \Big(\frac{z - a_1}{|z - a_1|} \Big)^{-1} \frac{z - a_2}{|z - a_2|} \exp iH(t, z) \text{ on } \tilde{\Omega}$$

and

$$w_t = v_1 \exp i\theta_i(t) \text{ on } B(a_i, \rho)$$

where

$$\theta_i(t) \equiv H(t, z) - H_1(z) \text{ on } \partial B(a_i, \rho)$$

(which is a number by definition of $\tilde{\Sigma}$). We easily verify that w_t is a continuous path in V, and in view of the definition of $\tilde{W}$, we clearly have

$$\text{(III.37)} \qquad \frac{d}{dt} F(w_t) \leq 0.$$

Moreover as $t \to +\infty$, w_t converges strongly in $H^1(\mathbb{R}^2)$ to the map v_2 defined by

$$v_2 = v_1 \text{ on } \mathbb{R}^2 \setminus B(a_1, \tilde{R})$$
$$v_2 = |\eta| \Big(\frac{z - a_1}{|z - a_1|} \Big)^{-1} \frac{z - a_2}{|z - a_2|} \exp iH_2(z) \text{ on } \tilde{\Omega}$$
$$v_2 = v_1 \exp i\overline{\theta}_i \text{ on } B(a_i, \rho)$$

where $\overline{\theta}_i = H_1 - H_2$ on $B(a_i, \rho)$.

Finally, since the vortices do not move during the flow

LEMMA III.7. – *We have* $\forall t \geq 0$,

$$|L(w_t) - L(v_1)| \leq K(\varepsilon)$$
$$|L(v_2) - L(v_1)| \leq K(\varepsilon),$$

where $K(\varepsilon)$ *depends only on* ε, *and* $K(\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

The proof is similar to the proof of Lemma 3. Therefore, we omit it.

III.3. Deformation for the modulus η

Our next purpose is to deform η so that this decreases F and, at the end of the deformation, we obtain a function η_3 such that

$$\frac{1}{4\varepsilon^2} \int_{\tilde{\Omega}} (1 - |\eta_3|^2)^2 = o(1),$$

where $o(1) \to 0$ as $\varepsilon \to 0$. This fact will be crucial in the proofs of Propositions III.1 and III.2. The ideas are similar to that developped in the previous subsection. In view of the form of $\overline{F}_\varepsilon$, we define the functional $\mathcal{Z}$ for real valued function ξ on $\tilde{\Omega}$

$$\mathcal{Z}(\xi) = \frac{1}{2}\int_{\tilde{\Omega}} |\nabla\xi|^2 + |\xi|^2|\nabla\overline{v}_2|^2 + (|\xi|^2 - 1)\overline{v}_{2x_1} + \frac{1}{2\varepsilon^2}(1 - |\xi|^2)^2$$

where $\overline{v}_2 = \frac{v_2}{|v_2|}$ on $\tilde{\Omega}$.

By Cauchy-Schwarz inequality, we verify that

$$\mathcal{Z}(\xi) \geq C, \qquad \forall\xi, \quad \forall 0 < \varepsilon < \frac{1}{2}, \tag{III.38}$$

where C is some universal constant independent of ξ and ε.

The deformation will be performed in two steps.

Step 1. – We deform η to a map η_1 such that

$$0 \leq \eta_1 \leq 2 \text{ on } \tilde{\Omega}. \tag{III.39}$$

Step 2. – Using (III.39), we are able to deform η_1 to a map η_3 having the desired property.

Note that the first step has only a technical purpose. For $t \in [0, 1]$, we define the map η_t in the following way. Set

$$\begin{aligned} \eta_t(z) &= \eta(z) & &\text{if } \eta(z) \leq 2, \text{ for } z \in \tilde{\Omega}, \ t \in [0,1] \\ \eta_t(z) &= \eta(z)((1-t) + 2t) & &\text{if } \eta(z) \geq 2, \text{ for } z \in \tilde{\Omega}, \ t \in [0,1]. \end{aligned}$$

On $\partial\tilde{\Omega}$, one easily verifies, thanks to (III.20), (III.25), that

$$\eta(z) \leq 2 \text{ on } \partial\tilde{\Omega}$$

so that

$$\eta_t(z) = \eta(z), \quad \forall z \in \partial\tilde{\Omega}.$$

Next we have to verify that

$$\mathcal{Z}(\eta_t) \leq \mathcal{Z}(\eta), \quad \forall t \in [0, 1]. \tag{III.40}$$

To that aim, we note that

$$\eta_t^2|v_{2_{x_1}}|^2 + \eta_t^2\overline{v}_{2_{x_1}} = \eta_t^2\left(\left[\overline{v}_{2_{x_1}} + \frac{1}{2}\right]^2 - \frac{1}{4}\right)$$

so that (III.40) follows easily. Finally we notice that η_1 satisfies (III.39), and this completes the first step.

Turning to the second step of our construction, we introduce the flow for $\mathcal{Z}$ which writes

$$\text{(III.41)} \qquad \frac{\partial\eta(t,\cdot)}{\partial t} - \Delta\eta(t,\cdot) + \xi\left(|\nabla\overline{v}_1|^2 + \overline{v}_{1_{x_1}}\right) = \frac{1}{\varepsilon^2}\xi(1-\xi^2)$$

equation (III.41) with initial value (for $t = 1$) and boundary conditions

$$\text{(III.42)} \qquad \eta(1,z) = \eta_1(z), \quad \forall z \in \tilde{\Omega}$$

$$\text{(III.43)} \qquad \eta(t,z) = \eta(z), \quad \text{on } \partial\tilde{\Omega}, \quad \forall t \in [1,+\infty[.$$

We have

LEMMA III.8. – *There exists a solution $\eta(\cdot,\cdot)$ to (III.41), (III.42) and (III.43), defined on $[1,+\infty[\times\tilde{\Omega}$, such that*

$$\text{(III.44)} \qquad 0 \le \eta(t,z) \le 2, \quad \forall t \ge 1, \ \forall z \in \tilde{\Omega}$$

$$\text{(III.45)} \qquad \int_1^T \left(\int_{\tilde{\Omega}} \left|\frac{\partial\xi}{\partial t}\right|^2\right) dt + \mathcal{Z}(\xi(T)) \le \mathcal{Z}(\eta), \quad \forall T \ge 1.$$

Moreover as $t \to +\infty$ (up to a subsequence), $\xi(t,\cdot)$ converges strongly in $\tilde{\Omega}$ to a solution η_3 to

$$\text{(III.46)} \qquad -\Delta\eta_3 + \eta_3\left(|\nabla\overline{v}_2|^2 + \overline{v}_{2_{x_1}}\right) = \frac{1}{\varepsilon^2}\eta_3(1-\eta_3^2)$$

$$\text{(III.47)} \qquad \eta_3(z) = \eta_1(z) \ \textit{on} \ \partial\tilde{\Omega}.$$

The proof is standard and therefore we omit it. Next, studying η_3 we obtain

LEMMA III.9. – *We have*

$$\frac{1}{2}\int_{\Omega} |\nabla\eta_3|^2 + \frac{1}{4\varepsilon^2}(1-\eta_3^2)^2 \le K(\varepsilon)$$

where $K(\varepsilon)$ depends only on ε and $K(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Proof. – Arguing as in the proof of Lemma II.2, we may construct some function $\check{\eta}_3$ on $\tilde{\Omega}$ such that

$$\check{\eta}_3 = \eta_3 \quad \text{on } \partial\tilde{\Omega}$$

$$\check{\eta}_3 \equiv 1 \quad \text{on } \check{\Omega} \equiv B\Big(a_i, \frac{\tilde{R}}{2}\Big) \setminus \bigcup_{i=1}^{\ell} B(a_i, 2\rho)$$

$$1 - \frac{2}{|\log \varepsilon|^2} \leq \check{\eta}_3 \leq 1 \text{ on } \tilde{\Omega},$$

and

$$\text{(III.48)} \qquad \int_{\tilde{\Omega}} |\nabla \check{\eta}_3|^2 + \frac{1}{4\varepsilon^2}(1 - \check{\eta}_3^2)^2 \leq K_1(\varepsilon)$$

where $K_1(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Multiplying (III.46) by $\eta_3 - \check{\eta}_3$ we obtain, after integration by parts

$$\int_{\tilde{\Omega}} \nabla \eta_3 \cdot \nabla(\eta_3 - \check{\eta}_3) + (\eta_3 - \check{\eta}_3)\big(|\nabla \overline{v}_2| + v_{2_{x_1}}\big) = \int_{\Omega} \frac{1}{\varepsilon^2} \eta_3(1 - \eta_3^2)(\eta_3 - \check{\eta}_3)$$

so that

$$\text{(III.49)} \qquad I(\eta_3) = \int_{\tilde{\Omega}} |\nabla \eta_3|^2 + \frac{1}{\varepsilon^2}(1 - \eta_3^2)(1 - \eta_3)\eta_3 = I_1 + I_2 + I_3$$

where

$$I_1 = \int_{\tilde{\Omega}} \nabla \eta_3 \cdot \nabla \check{\eta}_3$$

$$I_2 = \int_{\tilde{\Omega}} (\check{\eta}_3 - \eta_3)\big(|\nabla \overline{v}_2| + v_{2_{x_1}}\big)$$

and

$$I_3 = \frac{1}{\varepsilon^2} \int_{\tilde{\Omega}} \eta_3(1 - \eta_3^2)(\check{\eta}_3 - 1).$$

For I_1, we write

$$\text{(III.50)} \qquad |I_1| \leq ||\nabla \eta_3||_{L^2} ||\nabla \check{\eta}_3||_{L^2} \leq ||\nabla \check{\eta}_3||_{L^2(\tilde{\Omega})} K_1(\varepsilon)^{1/2}.$$

For I_2, we write, by the definition of v_2

$$|\nabla \overline{v}_2|^2 \leq 2\big(|\nabla \Phi|^2 + |\nabla H_2|^2\big) \text{ on } \tilde{\Omega}$$

$$|I_2| \leq J_1 + J_2 + J_3$$

where

$$J_1 = 4 \int_{\tilde{\Omega}} \big(|\eta_3 - 1| + |\check{\eta}_3 - 1|\big)\big(|\nabla \Phi|^2 + |\nabla \Phi|\big)$$

and

$$J_2 = \int_{\tilde{\Omega}} \big(|\eta_3 - 1| + |\check{\eta}_3 - 1|\big)|\nabla H|^2$$

$$J_3 = \int_{\tilde{\Omega}} |\eta_3 - 1| + \big(|\check{\eta}_3 - 1|\big)|\nabla H|.$$

For J_1, we note that, since

$$|\nabla\Phi| \leq \frac{2}{|z-a_1|} + \frac{2}{|z-a_2|},$$

that $||\nabla\Phi||_{L^4}^4 \leq \frac{C}{\rho^2}$, so that

$$\begin{aligned}|J_1| &\leq C\big(||\eta_3 - 1||_{L^2} + ||\check{\eta}_3 - 1||_{L^2}\big)\big(||\nabla\Phi||_{L^2} + ||\nabla\Phi||_{L^4}^2\big)\\ &\leq C\,\varepsilon|\log\varepsilon|^{1/2}\cdot\varepsilon^{-\mu} = C\,\varepsilon^{1-\mu}|\log\varepsilon|.\end{aligned}$$

For J_2, we write, since

$$\sup|\eta_3 - 1| + |\check{\eta}_3 - 1| \leq 4$$

$$|J_2| \leq 4\int|\nabla H_2|^2 \leq 4K(\varepsilon).$$

For J_3, we write

$$\begin{aligned}|J_3| &\leq \Big(|\check{\eta}_3 - 1|_{L^2(\tilde{\Omega})} + |\eta_3 - 1|_{L^2(\tilde{\Omega})}\Big)\Big(||\nabla H_2||_{L^2(\tilde{\Omega})}\Big)\\ &\leq C|\log\varepsilon|\varepsilon^{1/2}.\end{aligned}$$

Hence, combining the previous estimates, we see that

$$|I_2| \leq K_2(\varepsilon), \tag{III.51}$$

where $K_2(\varepsilon)\to 0$ as $\varepsilon\to 0$.

For I_3 we write

$$|I_3| \leq \left(\frac{1}{\varepsilon^2}\int_{\tilde{\Omega}}\eta_3(1-\eta_3^2)^2\right)^{1/2}\left(\frac{1}{\varepsilon^2}\int_{\tilde{\Omega}}(1-\eta_3)^2\right)^{1/2}$$

so that by (III.48)

$$|I_3| \leq \left[\frac{1}{\varepsilon^2}\int_{\tilde{\Omega}}\eta_3(1-\eta_3^2)^2\right]^{1/2}\left[K_1(\varepsilon)\right]^{1/2}. \tag{III.52}$$

Combining (III.52), (III.53) and (III.54) we obtain

$$[I(\eta_3)] \leq \big(I(\eta_3)^{1/2} + 1\big)\,K_3(\varepsilon) \tag{III.53}$$

where $K_3(\varepsilon)\to 0$ as $\varepsilon\to 0$.

We deduce from (III.53) that

$$\left| \int_{\tilde{\Omega}} |\nabla \eta_3|^2 + \frac{1}{\varepsilon^2} \eta_3 (1 - \eta_3)^2 \right| \leq I(\eta_3) \leq K_4(\varepsilon) \tag{III.54}$$

where $K_4(\varepsilon) \to 0$ as $\varepsilon \to 0$.

We are going to show that

$$\eta_3 \geq \frac{1}{2} \text{ on } \tilde{\Omega}, \tag{III.55}$$

which clearly yields the conclusion. Since η_3 verifies equation (III.46), which is of the form (8), we have by Lemma B3 of Appendix B

$$|\eta_3(x) - \eta_3(y)| \leq C\, \varepsilon^\gamma |x - y|^\gamma$$

for some universal constants $C > 0$, $\gamma \in (0, 1)$. Next assume that (III.55) is not true. Then, by continuity, there is some $x_0 \in \tilde{\Omega}$ such that

$$\eta_3(x_0) = \frac{1}{2}$$

then

$$\frac{1}{4} \leq \eta_3(x) \leq \frac{3}{4}$$

on the ball $B\big(x_0, \frac{\varepsilon}{16C}\big)$ so that

$$\frac{1}{\varepsilon^2} \int_{B\left(x_0, \frac{\varepsilon}{16C}\right)} \eta_3 (1 - \eta_3)^2 \geq \frac{1}{16^4 C}$$

contradicting (III.55) for small ε. This completes the proof of the Lemma.

III.4. Path in $\overline{F}_\varepsilon^c$

At this stage, let us sum up the results of the previous subsections in the following Lemma.

LEMMA III.9. – *Let $v \in V$, and assume that v verifies (III.5). Then there exists a continuous path $\tilde{p}$ from $[0, 1]$ to V, such that*

$$\tilde{p}(0) = v$$

and $\overline{F}_\varepsilon(\tilde{p}(s))$ is non increasing. Moreover, we have

$$|L(\tilde{p}(s)) - L(v)| \leq K(\varepsilon),$$

where $K(\varepsilon)$ *depends only on* ε *and* $K(\varepsilon)\to 0$ *as* $\varepsilon\to 0$. *The map* $v_3 = \tilde{p}(1)$ *has the following properties : there exist* ℓ *points in* $\mathbb{R}^2$, $a_1, a_2, \ldots, a_\ell$, *and* $\rho > 0$ *such that*

$$\ell \leq N, \qquad \varepsilon^{\mu} \leq \rho \leq \varepsilon^{\overline{\mu}/2}, \tag{III.56}$$

$$|v_3| \geq 1 - \frac{2}{|\log \varepsilon|^2} \textit{ on } \partial B(a_i, \rho), \tag{III.57}$$

$$|v_3| \geq \frac{1}{2} \textit{ on } \Omega = \mathbb{R}^2 \setminus \bigcup_{i=1}^{\ell} B(a_i, \rho), \tag{III.58}$$

$$\deg a_i = (-1)^i \textit{ if } i = 1, 2 \ , \ \deg a_i = 0 \textit{ otherwise}, \tag{III.59}$$

$$\int_{\partial B(a_i,\rho)} e_\varepsilon(v_3) \leq \frac{\pi}{\rho} \left(|\deg a_i| + K(\varepsilon) \right), \tag{III.60}$$

$$\int_\Omega \left|\nabla |v_3|\right|^2 + \frac{1}{\varepsilon^2} (1 - |v_3|^2)^2 \leq \frac{\pi K(\varepsilon)}{\rho}, \tag{III.61}$$

$$\int_\Omega \left|\nabla |v_3|\right|^2 + \frac{1}{\varepsilon^2} (1 - |v_3|^2)^2 \leq K(\varepsilon). \tag{III.62}$$

Remark. – The constructions of subsections III.2 and III.3 have mainly been introduced to obtain (III.61).

Next, we will try to construct a path connecting v_3 to $\mathcal{C}_0^\varepsilon$ or $\mathcal{C}_1^\varepsilon$, on which the energy $\overline{F}_\varepsilon$ is decreasing. However, when c is less than the mountain-pass value c_ε, we cannot connect indifferently to $\mathcal{C}_0^\varepsilon$ or $\mathcal{C}_1^\varepsilon$. Roughly speaking if

$$L(v) < 2\pi \text{ we may connect to } \mathcal{C}_0^\varepsilon$$
$$L(v) > 2\pi \text{ we may connect to } \mathcal{C}_1^\varepsilon.$$

A more precise statement is given in the two following propositions.

PROPOSITION III.1. – *Let* δ *be a number such that*

$$2 < \delta < 3 \tag{III.63}$$

and v be a map in V, satisfying (III.5)

$$L(v) = \pi\delta, \tag{III.64}$$

and

$$c \equiv \overline{F}_\varepsilon(v) \leq c_\varepsilon. \tag{III.65}$$

Then there exists $\varepsilon_0 > 0$ depending only on δ, such that for $\varepsilon < \varepsilon_0$, there exists a continuous path $\hat{p} : [0,1] \to V$ such that

$$\hat{p}(0) = v$$
$$\hat{p}(1) \in \mathcal{C}_1^\varepsilon$$

and moreover $F_\varepsilon(\hat{p}(s))$ is decreasing on $[0,1]$, and

$$L(\hat{p}(s)) \geq \pi\delta + K(\varepsilon) \tag{III.66}$$

where $K(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Similarly, we have

PROPOSITION III.2. – *Let δ be a number such that*

$$\frac{1}{4} < \delta < 2, \tag{III.67}$$

and v be a map satisfying (III.5),

$$L(v) = \pi\delta \tag{III.68}$$

and

$$c \equiv \overline{F}_\varepsilon(v) \leq c_\varepsilon. \tag{III.69}$$

Then there exists a continuous path $\check{p} : [0,1] \to V$ such that

$$\check{p}(0) = v$$
$$\check{p}(1) \in \mathcal{C}_0^\varepsilon$$

and moreover $F_\varepsilon(p(s))$ is non-increasing on $[0,1]$, and

$$L(\check{p}(s)) \leq \pi\delta + K(\varepsilon), \tag{III.70}$$

where $K(\varepsilon) \to 0$ as $\varepsilon \to 0$.

The proofs are based on scaling arguments. For instance, for Proposition III.2, we scaled up the map v_3 on Ω by a factor $\lambda > 1$. However the region $B(a_i, \rho)$ is just translated, and the technical part of the proof consists in joining the two regions. Let us now be more precise.

Proof of Proposition III.1. – In view of Lemma III.9 it suffices to construct a deformation of v_3 to a map in $\mathcal{C}^1_\varepsilon$. For that purpose let $\lambda > 1$, and consider the sets

$$\Omega_1^\lambda \equiv \lambda\Omega = \left\{z \in \mathbb{R}^2, \, \frac{z}{\lambda} \in \Omega\right\}$$

$$\Omega_2^\lambda = \bigcup_{i=1}^{\ell} B(\lambda a_i, \rho)$$

and

$$\Omega_3^\lambda = \bigcup_{i=1}^{\ell} \mathcal{C}_i^\lambda \quad \text{where } \mathcal{C}_i^\lambda = B(\lambda a_i, \lambda\rho) \setminus B(\lambda a_i, \rho).$$

Note that the sets Ω_j^λ are disjoint and that $\mathbb{R}^2 = \bigcup_{j=1}^{3} \overline{\Omega_j^\lambda}$. On Ω_1^λ we define a map w_λ, scaling up v_3, that is

$$w_\lambda(z) = v_3\Big(\frac{z}{\lambda}\Big), \qquad \forall z \in \Omega_1^\lambda.$$

On Ω_2^λ we deduce w_λ from v_3 by a translation

$$w_\lambda(z) = v_3(z + (\lambda - 1)a_i), \qquad \forall z \in B(\lambda a_i, \rho),$$

and finally on Ω_3^λ we define w_λ as a radial extension of the previous boundary values, that is

$$w_\lambda(z) = v_3\Big(\frac{\lambda^{-1}z - a_i}{|\lambda^{-1}z - a_i|}\rho + a_i\Big) \text{ on } \mathcal{C}_i^\lambda.$$

In view of the definition of w_λ, w_λ belongs to V for any $\lambda \geq 1$, and

$$w_1 = v_3.$$

Moreover, the function $\lambda \rightarrow w_\lambda$ is continuous from $[1, +\infty[$ to V. Our next task will be to show that $F_\varepsilon(w_\lambda)$ is non increasing.

To that aim, set for $j = 1, 2, 3$

$$I_j^\lambda = \int_{\Omega_j^\lambda} e_\varepsilon(w_\lambda) - \ell(w_\lambda)$$

where we have used the notation

$$\ell(w)(z) = \frac{1}{2}(iw_{x_1}, w-1) \text{ for any } w \in V, \quad \forall z \in \mathbb{R}^2.$$

For I_1^λ, we have by scaling

$$I_1^\lambda = I_1^1 + \frac{\lambda^2-1}{4\varepsilon^2}\int_\Omega (1-|v_3|^2)^2 - (\lambda-1)\int_\Omega \ell(v_3).$$

Since on Ω_2^λ, w_λ is deduced from v_3 by translation, we have

$$I_2^\lambda = I_2^1, \qquad \forall \lambda \geq 1.$$

On Ω_3^λ we have

$$\begin{aligned}\int_{\Omega_3^\lambda} |\nabla w_\lambda|^2 &= \sum_{i=1}^{\ell}\int_{\tilde\rho}^{\lambda\tilde\rho}\left(\int_{\partial B(a_i,\rho)}\left|\frac{\partial v_3}{\partial r_i}\right|^2 d\sigma\right)\frac{\rho}{r}\,dr \\ &= \rho(\log\lambda)\sum_{i=1}^{\ell}\int_{\partial B(a_i,\rho)}\left|\frac{\partial v_3}{\partial r_i}\right|^2 d\sigma \text{ where } r_i = |z-a_i|.\end{aligned}$$

Similarly

$$\int_{\Omega_3^\lambda}(1-|w_\lambda|^2)^2 = \frac{1}{2}\rho(\lambda^2-1)\sum_{i=1}^{\ell}\left(\int_{\partial B(a_i,\rho)}(1-|v_3|^2)^2\,d\sigma\right)$$

and

$$\int_{\Omega_3^\lambda}\ell(w) = (\lambda-1)\rho\left(\int_{\partial B(a_i,\rho)}\ell(v_3)\,d\sigma\right).$$

Combining the previous identities, we are led to

(III.71)

$$F_\varepsilon(w_\lambda) = F_\varepsilon(v_3) - (\lambda-1)\,L(v_3) + R_1(\lambda) + R_2(\lambda) + R_3(\lambda) + R_4(\lambda) + R_5(\lambda)$$

where

$$\begin{aligned}R_1(\lambda) &= (\lambda-1)\sum_{i=1}^{\ell}\int_{B(a_i,\rho)}\ell(v_3)\,d\sigma \\ R_2(\lambda) &= \left(\frac{\lambda^2-1}{4\varepsilon^2}\right)\int_\Omega (1-|v_3|^2)^2 \\ R_3(\lambda) &= (\lambda^2-1)\,\rho\sum_{i=1}^{\ell}\left(\frac{1}{4\varepsilon^2}\int_{\partial B(a_i,\rho)}(1-|v_3|^2)^2\,d\sigma\right) \\ R_4(\lambda) &= (\lambda-1)\sum_{i=1}^{\ell}\left(\int_{\partial B(a_i,\rho)}\ell(v_3)\,d\sigma\right) \\ R_5(\lambda) &= \rho\log\lambda\sum_{i=1}^{\ell}\left(\int_{\partial B(a_i,\rho)}\left|\frac{\partial v_3}{\partial\sigma}\right|^2 d\sigma\right).\end{aligned}$$

We are going to estimate each of these terms separately. For R_1 we write, since $0 \leq |v_3| \leq 2$,

$$|R_1(\lambda)| \leq 3|\lambda - 1| \sum_{i=1}^{\ell} \int_{B(a_i,\rho)} |\nabla v_3| \leq 3|\lambda - 1| \, ||\nabla v_3||_{L^2(\mathbb{R}^2)} \pi^{1/2} \rho .$$

Hence

$$|R_1(\lambda)| \leq C|\lambda - 1| \varepsilon^{\overline{\mu}} |\log \varepsilon|^{1/2}.$$

For $R_2(\lambda)$, we use (III.62) so that

$$|R_2(\lambda)| \leq (\lambda - 1)(\lambda + 1) K(\varepsilon).$$

For R_3, we use (III.63) so that

$$|R_3(\lambda)| \leq 2(\lambda - 1)(\lambda + 1) K(\varepsilon).$$

For R_4, we have, since $0 < |v_3| < 2$, $\forall i = 1, 2, \ldots, \ell$

$$\left| \int_{\partial B(a_i,\rho)} \left(i v_{3_{x_1}}, v_3 - 1 \right) \right| \leq 3 \int_{\partial B(a_i,\rho)} |\nabla v_3| \leq C,$$

where C is some constant which is independent of ε : the last inequality is a consequence of Cauchy-Schwarz inequality together with (III.62). This yields

$$|R_4| \leq C|\lambda - 1| \rho \leq C(\lambda - 1) \varepsilon^{\overline{\mu}}.$$

Finally, for R_5, we have by (III.62)

$$|R_5(\lambda)| \leq 2\pi \log \lambda (1 + K_2(\varepsilon))$$

where $K_2(\varepsilon) \to 0$ as $\varepsilon \to 0$. Going back to (III.71) and combining our previous estimates we obtain, $\forall \lambda \geq 1$

(III.72)
$$\left| F_\varepsilon(w_\lambda) - \left[F_\varepsilon(w) + 2\pi \log \lambda - (\lambda - 1) L(v_3) \right] \right| \leq |\lambda - 1| (\lambda + 1) K_3(\varepsilon)$$

where $K_3(\varepsilon) \to 0$ as $\varepsilon \to 0$. For $L(w_\lambda)$, we have similarly

(III.73) $$|L(w_\lambda) - \lambda L(v_3)| \leq |\lambda - 1| K(\varepsilon).$$

In view of the definition of $\mathcal{C}_1^\varepsilon$ and (III.73) choose $\lambda_1 \in \mathbb{R}$, so that

$$\lambda_1 \delta = d_1$$

(clearly $\lambda_1 > 1$ since $\delta < 3$ and $d_1 > 4$). Recall that d_1 is a constant entering in the definition of $\mathcal{C}_1^\varepsilon$ and is determined by (34). Since

$$|L(v_3) - \pi\delta| \leq K(\varepsilon),$$

and since $\delta > 2$, we see that if ε is sufficiently small

$$L(v_3) > 2\pi,$$

so that

$$\frac{d}{d\lambda}\big(2\pi\log\lambda - (\lambda-1)L(v_3)\big) \leq (2\pi - L(v_3)) < 0.$$

Hence, we deduce from (III.72), that if ε is sufficiently small, then

$$\text{(III.74)} \qquad \frac{d}{d\lambda}F_\varepsilon(w_\lambda) < 0, \quad \text{for } \lambda \in [1, \lambda_1],$$

hence $F_\varepsilon(w_\lambda)$ is non increasing.

Next we are going to verify that $w_{\lambda_1} \in \mathcal{C}_1^\varepsilon$. By (III.73) we have

$$|L(w_{\lambda_1}) - \pi d_1| \leq (|\lambda_1 - 1|)K(\varepsilon).$$

Hence in order to prove that $w_{\lambda_1} \in \mathcal{C}_1^\varepsilon$, it suffices to show that

$$\text{(III.75)} \qquad E_\varepsilon(w_{\lambda_1}) \leq 2\pi|\log\varepsilon| + 2\pi\log d_1 + \overline{\Lambda}.$$

To establish (III.75), note that by (III.72)

$$\begin{aligned} F_\varepsilon(w_{\lambda_1}) &\leq F_\varepsilon(v_3) + 2\pi\log\lambda_1 - (\lambda_1 - 1)L(v_3) + K_4(\varepsilon) \\ &\leq C_\varepsilon + 2\pi\log\lambda_1 - (\lambda_1-1)L(v_3) + K_4(\varepsilon), \end{aligned}$$

where $K_4(\varepsilon)\to 0$ as $\varepsilon\to 0$. Hence

$$E_\varepsilon(w_{\lambda_1}) \leq C_3 + 2\pi\log\lambda_1 - \lambda_1 L(v_3) + K_4(\varepsilon).$$

In view of Proposition 5 bis, we deduce that

$$E_\varepsilon(w_{\lambda_1}) \leq 2\pi\log\frac{1}{\varepsilon} + 2\pi\log 2\lambda_1 - 2\pi - \lambda_1 L(v_3) + 2\gamma_0 + K_5(\varepsilon)$$

where $K_5(\varepsilon)\to 0$ as $\varepsilon\to 0$. Hence, since $\lambda_1\delta = d_1$, and $\delta > 2$

$$E_\varepsilon(w_{\lambda_1}) \leq 2\pi\log\frac{1}{\varepsilon} + 2\pi\log d_1 + 2\gamma_0 + K_5(\varepsilon).$$

In view of the definition of $\hat{\Lambda}$, we have

$$2\gamma_0 < \hat{\Lambda}$$

so that

$$E_\varepsilon(w_{\lambda_1}) < 2\pi \log \frac{1}{\varepsilon} + 2\pi \log d_1 + \Lambda$$

if ε is sufficiently small. Hence $w_{\lambda_1} \in \mathcal{C}_1^\varepsilon$.

We are able to define now the path $\hat{p}$. Set, for $s \in \left[0, \frac{1}{2}\right]$

$$\hat{p}(s) = \tilde{p}(2s) \text{ for } s \in \left[0, \frac{1}{2}\right]$$

$$\text{and} \quad \hat{p}(s) = w_\lambda, \text{ where } \lambda = (2s-1)(\lambda_1 - 1) + 1.$$

It is then easy to see that the path $\hat{p}$ has the desired properties.

Proof of Proposition III.2. – The proof follows basically similar ideas. We will construct a deformation of v_3 to a map in $\mathcal{C}_0^\varepsilon$. We set

$$\mathcal{A}_i = B\big(a_i, \varepsilon^{\overline{\mu}/2}\big) \setminus B(a_i, \rho) \text{ for } i = 1, \ldots, \ell$$

and

$$\check{\Omega} = \mathbb{R}^2 \setminus \bigcup_{i=1}^{\ell} B\big(a_i, \varepsilon^{\overline{\mu}/2}\big).$$

In view of the definition of v_3, we have

$$v_3(z) = \eta_3 \Big(\frac{z - a_1}{|z - a_1|}\Big)^{-1} \Big(\frac{z - a_2}{|z - a_2|}\Big) \exp iH_2(z) \text{ on } \Omega. \tag{III.76}$$

Hence, we deduce from Lemma III.7 and Lemma III.8 that

$$\sum_{i=1}^{\ell} \Big(\int_{\mathcal{A}_i} \Big|\frac{\partial v_3}{\partial r_i}\Big|^2\Big) \leq K_1(\varepsilon), \tag{III.77}$$

where K_1 depends only on ε, and $K_1(\varepsilon) \to 0$ as $\varepsilon \to 0$, and where

$$r_i = |z - a_i| \text{ on } \mathcal{A}_i.$$

We set, as in Proposition III.1, for $0 < \lambda < 1$

$$\check{\Omega}_1^\lambda = \frac{1}{\lambda}\check{\Omega} = \Big\{z \in \mathbb{R}^2, \frac{z}{\lambda} \in \check{\Omega}\Big\}$$

$$\check{\Omega}_2^\lambda = \bigcup_{i=1}^{\ell} B(\lambda a_i, \rho)$$

$$\text{and} \quad \Omega_3^\lambda = \bigcup_{i=1}^{\ell} \mathcal{A}_i^\lambda, \text{ where}$$

$$\mathcal{A}_i^\lambda = B(\lambda a_i, \lambda \varepsilon^{\overline{\mu}/2}) \setminus B(\lambda a_i, \rho).$$

For $0 < \lambda < 1$, we construct a map w_λ in the following way. On $\check{\Omega}_1^\lambda$, we set

$$w_\lambda(z) = v_3\Big(\frac{z}{\lambda}\Big), \qquad \forall z \in \Omega_1^\lambda.$$

On $\check{\Omega}_2^\lambda$, we set

$$w_\lambda(z) = v_3(z + (\lambda - 1)a_i) \text{ on } B(\lambda a_i, \rho).$$

Finally, on $\check{\Omega}_3^\lambda$, we set

$$w_\lambda(z) = v_3(\psi_\lambda(z)), \text{ if } z \in \mathcal{A}_i^\lambda.$$

Here the map ψ_λ is a diffeomorphism of $\mathcal{A}_i^\lambda$ onto $\mathcal{A}_i^1 = B\big(a_i, \varepsilon^{\overline{\mu}/2}\big) \setminus B(a_i, \rho)$, defined by

$$\psi(z) = f_\lambda(|z - \lambda a_i|) \ \frac{z - \lambda a_i}{|z - \lambda a_i|} + \lambda a_i,$$

where f_λ interpolates linearily,

$$\begin{aligned} &f_\lambda(\rho) = \rho \\ &f_\lambda\big(\varepsilon^{\overline{\mu}/2}\big) = \lambda \varepsilon^{\overline{\mu}/2} \end{aligned}$$

so that

$$f_\lambda(r) = a_\lambda r + b_\lambda,$$

with

$$a_\lambda = \frac{\lambda \varepsilon^{\overline{\mu}/2} - \rho}{\varepsilon^{\overline{\mu}/2} - \rho} = \lambda\Big(1 + \frac{\lambda - 1}{\lambda} \ \frac{\rho}{\varepsilon^{\overline{\mu}/2} - \rho}\Big)$$

and

$$b_\lambda = \rho(1 - a_\lambda) = (1 - \lambda)\rho\Big(1 + \frac{\rho}{\varepsilon^{\overline{\mu}/2} - \rho}\Big).$$

Next, we compare $F_\varepsilon(w_\lambda)$ with $F_\varepsilon(v_3)$. As in the proof of Proposition III.1, we set, for $j = 1, 2, 3$

$$I_j^\lambda = \int_{\Omega_j^\lambda} e_\varepsilon(w_\lambda) - \ell(w_\lambda)$$

so that $F_\varepsilon(w_\lambda) = \sum\limits_{j=1}^{3} I_j^\lambda$.

For I_1^λ, using similar arguments as in the proof of Proposition III.1 we may verify that

$$\text{(III.78)} \qquad \Big| I_1^\lambda - \big[I_1^\lambda - (\lambda - 1) L(v_3) \big] \Big| \le (1 - \lambda) K_1(\varepsilon)$$

where $K_1(\varepsilon) \to 0$ as $\varepsilon \to 0$. For I_2^λ we have

$$\text{(III.79)} \qquad I_2^\lambda = I_2^1 .$$

The computation of I_3^λ is a little more involved. For $i = 1, \dots, \ell$, let (r_i, θ_i) (resp. $r_i^\lambda, \theta_i^\lambda$) be polar coordinates associated to the origin a_i (resp. λa_i). We have

$$|\nabla w_\lambda|^2 = \left| \frac{\partial w_\lambda}{\partial r_i^\lambda} \right|^2 + |r_i^\lambda|^2 \left| \frac{\partial w_\lambda}{\partial \theta_i^\lambda} \right|^2 \text{ on } \mathcal{A}_i^\lambda .$$

Set

$$J_r^\lambda = \frac{1}{2} \sum_{i=1}^{\ell} \int_{\mathcal{A}_i^\lambda} \left| \frac{\partial w_\lambda}{\partial r_i^\lambda} \right|^2$$

$$J_\theta^\lambda = \frac{1}{2} \sum_{i=1}^{\ell} \int_{\mathcal{A}_i^\lambda} \left| \frac{\partial w_\lambda}{\partial \theta_i^\lambda} \right|^2$$

$$J_v^\lambda = \frac{1}{4} \sum_{i=1}^{\ell} \int_{\mathcal{A}_i^\lambda} \frac{1}{\varepsilon^2} (1 - |w_\lambda|^2)^2$$

and

$$J_\ell^\lambda = \frac{1}{2} \sum_{i=1}^{\ell} \int_{\mathcal{A}_i^\lambda} \ell(w_\lambda) .$$

We are going to estimate each of these terms. Note that, for J_r^λ

$$\frac{\partial w_\lambda}{\partial r_i^\lambda} = \frac{\partial v_3(\psi_\lambda(z))}{\partial r_i^\lambda} = \frac{\partial \psi_\lambda(z)}{\partial r_i^\lambda} \cdot \frac{\partial v_3(\psi_\lambda(z))}{\partial r_i}$$

so that, by change of variables in the integrals, we have

$$|J_r^\lambda - J_r^1| \le C(\lambda - 1) J_r^1, \quad \forall \lambda \in \left[\frac{1}{32}, 1 \right]$$

so that, by (III.76)

$$\text{(III.80)} \qquad |J_r^\lambda - J_r^1| \le C(1 - \lambda) K_1(\varepsilon), \quad \forall \lambda \in \left[\frac{1}{32}, 1 \right]$$

similar computations show that

(III.81) $$|J_\ell^\lambda - J_\ell^1| + |J_v^\lambda - J_v^1| \le C(1-\lambda)K_2(\varepsilon), \quad \forall \lambda \in \left[\frac{1}{32}, 1\right]$$

where $K_2(\varepsilon) \to 0$ as $\varepsilon \to 0$. Here we use for J_v^λ, Lemma III.9.

For J_θ^λ, we set for $r \in [\rho, \varepsilon^{\overline{\mu}/2}]$

$$h_i(r) = \int_{\partial B(a_i, r)} \left|\frac{\partial v_3}{\partial \theta i}\right|^2 d\sigma$$

so that, in view of (III.77) and Lemmas III.7 and III.8, we may write

$$h_i(r) = 2\pi\,|\deg a_i| + g_i \ \text{ for } i = 1, 2, \ldots, \ell$$

with

(III.82) $$\left|\int_\rho^{\varepsilon^{\mu/2}} \frac{g_i(r)}{r}\,dr\right| \le K_3(\varepsilon)$$

where $K_3(\varepsilon) \to 0$ as $\varepsilon \to 0$. A simple computation then shows that

$$J_\theta^\lambda = \frac{1}{2} \sum_{i=1} \int_\rho^{\varepsilon^{\overline{\mu}/2}} h_i(r)\left(\frac{1}{r + c_\lambda}\right) dr$$

with

$$c_\lambda = \frac{b_\lambda}{a_\lambda}.$$

Hence we have

(III.83)
$$J_\theta^\lambda - J_\theta^1 = 2\pi \int_\rho^{\varepsilon^{\overline{\mu}/2}} \left(\frac{1}{r + c_\lambda} - \frac{1}{r}\right) dr + \sum_{i=1}^{\ell} \int_\rho^{\varepsilon^{\overline{\mu}/2}} -\frac{g_i(r)}{r}\,\frac{c_\lambda}{r + c_\lambda}\,dr.$$

From (III.82), we deduce that

(III.84) $$\left|\int_\rho^{\varepsilon^{\overline{\mu}/2}} \frac{g_i(r)}{r}\,\frac{c_\lambda}{r + c_\lambda}\,dr\right| \le (1-\lambda)K_4(\varepsilon), \quad \forall \lambda \in \left[\frac{1}{32}, 1\right]$$

where $K_4(\varepsilon) \to 0$ as $\varepsilon \to 0$. On the other hand

$$\int_\rho^{\varepsilon^{\overline{\mu}/2}} \left(\frac{1}{r + c_\lambda} - \frac{1}{r}\right) dr = \log\ \frac{\varepsilon^{\overline{\mu}/2} + c_\lambda}{\varepsilon^{\overline{\mu}/2}} - \log\ \frac{\rho + c_\lambda}{\rho}.$$

Hence since $c_\lambda \simeq \frac{(\lambda-1)\rho}{\lambda}$, we deduce

(III.85)
$$\left| \int_\rho^{\varepsilon^{\overline{\mu}/2}} \left(\frac{1}{r+c_\lambda} - \frac{1}{r} \right) dr + \log\lambda \right| \leq (1-\lambda) K_5(\varepsilon), \quad \forall \lambda \in \left[\frac{1}{32}, 1\right].$$

Combining estimates (III.78), (III.79), (III.80), (III.81), (III.83), (III.84) and (III.85), we deduce

(III.86)
$$\left| F_\varepsilon(w_\lambda) - F_\varepsilon(v_3) + 2\pi \log\lambda - \pi(\lambda-1)\delta \right| \leq (1-\lambda) K_6(\varepsilon), \quad \forall \lambda \in \left[\frac{1}{32}, 1\right]$$

where $K_6(\varepsilon) \to 0$ as $\varepsilon \to 0$. Since $\delta < 2$, we have

$$\frac{d}{d\lambda}(2\pi \log\lambda - \pi(\lambda-1)\delta) \geq 2\pi - \pi\delta > 0$$

so that $F_\varepsilon(w_\lambda)$ is increasing on $\left[\frac{1}{32}, 1\right]$, if ε is sufficiently small, *i.e.*

(III.87)
$$\frac{d}{d\lambda} F_\varepsilon(w_\lambda) > 0 \text{ on } \left[\frac{1}{32}, 1\right].$$

On the other hand, we easily verify that

$$|L(w_\lambda) - \pi\lambda\delta| \leq K_7(\varepsilon)$$

where $K_7(\varepsilon) \to 0$ as $\varepsilon \to 0$. In particular

$$\left| L\left(w_{\frac{1}{32}}\right) \right| < \frac{\pi}{8}$$

if ε is sufficiently small. In order to prove that $w_{\frac{1}{32}}$ belongs to $\mathcal{C}_0^\varepsilon$ it suffices to show that

$$E_\varepsilon\left(w_{\frac{1}{32}}\right) \leq 2\pi |\log\varepsilon| + \hat{\Lambda}.$$

We have, since

$$F_\varepsilon\left(w_{\frac{1}{32}}\right) \leq F_\varepsilon(v_3) + 2\pi \log\frac{1}{32} - \pi\left(\frac{1}{32} - 1\right)\delta + K_6(\varepsilon)$$

that

$$\begin{aligned} E_\varepsilon\left(w_{\frac{1}{32}}\right) &\leq C_\varepsilon + 2\pi \log\frac{1}{32} - \pi\delta + K_6(\varepsilon) \\ &\leq 2\pi |\log\varepsilon| + 2\pi(\log z - 1) - \frac{\pi}{4} + 2\gamma_0 - \log 32 + K_6(\varepsilon) \\ &\leq 2\pi |\log\varepsilon| + \hat{\Lambda} \end{aligned}$$

by the definition of $\hat{\Lambda}$.

We complete the proof of Proposition III.2 as the proof of Proposition III.1. We construct the path $\check{p} : [0,1] \to V$ by setting

$$\check{p}(s) = \tilde{p}(2s) \qquad \text{if } s \in \left[0, \frac{1}{2}\right]$$
$$\check{p}(s) = w_\lambda(s) \qquad \text{for } \lambda(s) = -\frac{31}{16} s + \frac{63}{32}.$$

It is then easy to see that $\check{p}$ has the desired properties.

Using Propositions III.1 and III.2, we may now proceed to the proof of Proposition 6.

III.6. Proof of Proposition 6 completed

Recall that $\check{d}_0$ and $\hat{d}_0$ are two numbers such that

$$\frac{1}{4} < \check{d}_0 < 2 < \hat{d}_0 < 3$$

and that

$$N^\varepsilon = N_1^\varepsilon \cup N_2^\varepsilon$$

with

$$N_1^\varepsilon = \{u \in V, \overline{F}_\varepsilon(v) < c_\varepsilon\}$$
$$N_2^\varepsilon = \left\{u \in V, |L(v)| < \pi \check{d}_0 \text{ or } |L(v)| > \pi \hat{d}_0\right\}.$$

In order to prove Proposition 6 we will argue by contradiction, and assume that N^ε does not separate $\mathcal{C}_0^\varepsilon$ and $\mathcal{C}_1^\varepsilon$. That means that we will assume that there exists a continuous path $p : [0,1] \to V$ such that

$$p(0) \in \mathcal{C}_0^\varepsilon \tag{III.88}$$
$$p(1) \in \mathcal{C}_1^\varepsilon \tag{III.89}$$
$$p(s) \in N^\varepsilon, \quad \forall s \in [0,1]. \tag{III.90}$$

Since L is continuous on V and since p is a continuous path in V, the function ψ defined by

$$\psi(0) = L(p(s))$$

is continuous from $[0,1]$ to $\mathbb{R}$. On the other hand, since $p(0) \in \mathcal{C}_0^\varepsilon$ (resp. $p(1) \in \mathcal{C}_1^\varepsilon$) we see that

$$|\psi(0)| = |L(p(0)| \leq \frac{\pi}{8} \text{ (resp. } |\psi(1)| > 3\pi). \tag{III.91}$$

Now let $\eta > 0$ be a positive number such that

$$\frac{1}{4} < \check{d}_0 < \check{d}_0 + \eta < 2 < \hat{d}_0 - \eta < \hat{d}_0 < 3.$$

In view of (III.91), we see that the sets

$$\check{\mathcal{K}} = \left\{ s \in [0,1],\, L(p(s)) \leq \pi(\check{d}_0 + \eta) \right\}$$
$$\hat{\mathcal{K}} = \left\{ s \in [0,1],\, L(p(s)) \geq \pi(\hat{d}_0 - \eta) \right\}$$

are not empty ($0 \in \check{\mathcal{K}}$, $1 \in \hat{\mathcal{K}}$), and closed by the continuity of ψ. Set

$$\check{s} = \mathrm{Max}\left\{ s \in \check{\mathcal{K}} \right\}$$
$$\hat{s} = \mathrm{Min}\left\{ s \in \hat{\mathcal{K}} \right\},$$

so that

$$0 < \check{s} < \hat{s} < 1$$

and

$$L(p(\check{s})) = \pi(\check{d}_0 + \eta),\;\; L(p(\hat{s})) = \pi(\hat{d}_0 - \eta).$$

Moreover, we have

$$\text{(III.92)} \qquad \pi(\check{d}_0 + \eta) \leq L(p(s)) \leq \pi(\hat{d}_0 - \eta), \quad \forall s \in [\check{s}, \hat{s}].$$

Since $p(s) \in N_1^\varepsilon \cup N_2^\varepsilon$, $\forall s \in [\check{s}, \hat{s}]$, and since by (III.92), $p(s) \notin N_2^\varepsilon$ we deduce that $p(s) \in N_1^\varepsilon$, $\forall s \in [\check{s}, \hat{s}]$, that is

$$\text{(III.93)} \qquad \mathrm{Max}\left\{ F_\varepsilon(p(s)),\, s \in [\check{s}, \hat{s}] \right\} = c < c_\varepsilon.$$

Set $\check{v} = p(\check{s})$ (resp. $\hat{v} = p(\hat{s})$). We have

$$L(\hat{v}) = \pi(\check{d}_0 + \eta) > 2\pi, \;\; F_\varepsilon(\hat{v}) \leq c < c_\varepsilon$$
$$L(\check{v}) = \pi(\hat{d}_0 - \eta) < 2\pi, \;\; F_\varepsilon(\check{v}) \leq c < c_\varepsilon.$$

Therefore, we may apply Proposition III.1 to $\hat{v}$, and Proposition III.2 to $\check{v}$. Hence, if ε is sufficiently small, there exists a continuous path $\hat{p}$ from $[\hat{s}, 1]$ to V such that

$$\hat{p}(\hat{s}) = \hat{v}$$
$$\hat{p}(1) \in \mathcal{C}_1^\varepsilon$$

and

$$\overline{F}_\varepsilon(p(s)) \leq F_\varepsilon(\hat{v}) \leq c < c_\varepsilon, \quad \forall s \in [\hat{s}, 1].$$

Similarly, if ε is sufficiently small, there exists a path $\check{p} : [0, \check{s}] \to V$ such that

$$\check{p}(\check{s}) = \check{v}$$
$$\check{p}(0) \in \mathcal{C}_0^\varepsilon$$
$$F_\varepsilon(p(s)) \leq F_\varepsilon(\check{v}) \leq c < c_\varepsilon, \quad \forall s \in [0, \check{s}].$$

Define now the continuous path $\overline{p}$ (for ε small enough) by

$$\overline{p}(s) = \check{p}(s) \text{ if } s \in [0, \check{s}]$$
$$\overline{p}(s) = p(s) \text{ if } s \in [\check{s}, \hat{s}]$$
$$\overline{p}(s) = \hat{p}(s) \text{ if } s \in [\hat{s}, 1].$$

We verify that

$$\overline{p} \in \mathcal{P}_\varepsilon$$

and that

$$\underset{s\in[0,1]}{\text{Max}} \overline{F}_\varepsilon(p(s)) = c < c_\varepsilon.$$

This contradicts the definition of c_ε and completes the proof, by contradiction, of Proposition 6.

IV. WEAK CONVERGENCE PROPERTIES

This section is mainly devoted to the proof of Proposition 8. Hence, we consider a sequence v_n in V, with the following properties

(IV.1) $$E_\varepsilon(v_n) \leq 2\pi|\log\varepsilon| + \Lambda, \qquad \forall n \in \mathbb{N}$$

and

(IV.2) $$\frac{\pi}{8} \leq |L(v_n)| \leq 4\pi,$$

where Λ is some absolute constant, independent of ε. Fo $n \in \mathbb{N}$, we denote by v_n^h the regularized map associated to v_n, *i.e.* a minimizing map for G_h^n defined by

$$G_h^n(w) = E_\varepsilon(w) + \frac{1}{2h^2}\int_{\mathbb{R}^2} |w - v_n|^2.$$

We also denote by ℓ_n the number of vortices for v_n^h, and for $i = 1, \dots, \ell_n$ the vortices a_i^n of v_n^h defined by Lemma 3. We denote ρ_n the corresponding radius, verifying in particular

$$\varepsilon^{\mu} \leq \rho_n \leq \varepsilon^{\overline{\mu}}. \tag{IV.3}$$

In view of assumption (IV.2), and Lemma 3, we see that v_n^h must have two vortices a_1^n and a_2^n with degree -1 and 1 respectively. All other vortices have degree zero. In view of (28) and (IV.2), we have, if ε is sufficiently small

$$|a_1^n - a_2^n| \geq \frac{1}{16}. \tag{IV.4}$$

On the other hand, by Proposition II.2 and assumption (IV.2), we see that if ε is sufficiently small, then

$$|a_1^n - a_2^n| \leq C, \tag{IV.5}$$

for some constant C independent of ε, depending only on Λ.

Since the problem is invariant by translations, we reduce this invariance by a change of origin. For that purpose, we define maps $\tilde{v}_n$, $\tilde{v}_n^h$, by setting

$$\tilde{v}_n(z) = v_n(z - a_1^n)$$
$$\tilde{v}_n^h(z) = v_n^h(z - a_1^n).$$

We also set

$$\tilde{a}_i^n = a_i^n - a_1^n$$

so that, $\tilde{v}_n^h$ has vortices $0, \tilde{a}_2^n, \dots, \tilde{a}_{\ell_n}^n$. Since by Lemma 3, the number ℓ_n of vortices is bounded independently of n, we may assume, passing possibly to a subsequence, that

$$\ell_n = \mathrm{Cte} = \ell.$$

Similarly, since by (IV.5) the sequence $\tilde{a}_2^n$ is bounded, we may assume, passing possibly to another subsequence, that

$$\tilde{a}_2^n \longrightarrow \tilde{a}_2, \quad \text{as } n \to +\infty, \tag{IV.6}$$

where $\tilde{a}_2$ is a point in $\mathbb{R}^2$.

Finally, passing possibly to another subsequence, we may assume that there exist maps $\tilde{v}$ and $\tilde{v}^h$ in $H^1_{\rm loc}(\mathbb{R}^2)$, such that

$$\text{(IV.7)} \qquad \tilde{v}_n \rightharpoonup \tilde{v}, \quad \text{weakly in } H^1(K), \ \text{ as } n \to +\infty$$

$$\text{(IV.8)} \qquad \tilde{v}_n^h \rightharpoonup \tilde{v}^h, \quad \text{weakly in } H^1(K), \ \text{ as } n \to +\infty,$$

for any compact set $K \subset \mathbb{R}^2$. Note that by lower-semicontinuity we have

$$\text{(IV.9)} \qquad E_\varepsilon(\tilde{v}) \leq \liminf_{n \to +\infty} E_\varepsilon(v_n)$$

and

$$\text{(IV.10)} \qquad E_\varepsilon(\tilde{v}^h) \leq \liminf_{n \to +\infty} E_\varepsilon(v_n^h).$$

Note however that we do not claim that $\tilde{v} \in V$, since we have no control on the L^2–norm of $v_n - 1$. Actually, as we will see $\tilde{v}$ may not be in V.

In order to prove Proposition 8, we have mainly to show that $\tilde{v}$ is not a constant map. For that purpose, we will use the notion of minimal connection for charged vortices, introduced by H. Brezis, J.-M. Coron and E. Lieb in [BCL], and already used in a similar context in [AB]. Let $R > 0$, fixed so that

$$R \geq 10|\tilde{a}_1 - \tilde{a}_2| = 10|\tilde{a}_2|.$$

We may take $R = 10C$, where C is the constant in inequality (IV.5). Next consider the set T_R of real-valued functions defined on $B(R)$ by

$$T_R = \Big\{\xi \in C^\infty(B(R)), \xi = 0 \text{ on } \partial B(R), ||\nabla\xi||_{L^\infty(B(R))} \leq 1\Big\}.$$

Finally, for $w \in H^1(B(R), \mathbb{R}^2)$, set

$$D(w) = \frac{1}{2\pi} \sup\left\{ \int_{B(R)} (iw, w_{x_1}) \cdot \xi_{x_2} - (iw, w_{x_2})\xi_{x_1}, \xi \in T_R \right\}.$$

The functional D has the following continuity property :

LEMMA IV.1. – *Let w_n be a sequence in $H^1(B(R), \mathbb{R}^2)$, such that w_n converges weakly in $H^1(B(R))$ to some map w. Then, we have*

$$\text{(IV.11)} \qquad D(w_n) \longrightarrow D(w) \quad \textit{as } n \to +\infty.$$

Proof. – The proof is an easy consequence of the fact that the embedding $H^1(B(R)) \hookrightarrow L^2(B(R))$ is compact. ■

Next consider a map w in $H^1_{\rm loc}(\mathbb{R}^2)$, such that

$$E_\varepsilon(w) \leq 2\pi\,|\log\varepsilon| + \Lambda\,.$$

Let w^h be a regularized map for w, *i.e.* a minimizer for

$$G_h(u) = E_\varepsilon(u) + \frac{1}{2h^2}\int_{\mathbb{R}^2} |u-w|^2\,.$$

We have

LEMMA IV.2. – *For w and w^h as above,*

$$|D(w^h) - D(w)| \leq K(\varepsilon)\,,$$

where $K(\varepsilon)\to 0$ as $\varepsilon\to 0$.

Proof. – For $\xi \in T_R$, we estimate

$$I(\xi) = \int_{B(R)} \Big[(iw, w_{x_1}) - (iw^h, w^h_{x_1})\Big]\,\xi_{x_2} - \Big[(iw, w_{x_2}) - (iw^h, w^h_{x_2})\Big]\,\xi_{x_1}\,.$$

We write

$$I = I_1 + I_2$$

where

$$I_1(\xi) = \int_{B(R)} \Big(i(w-w^h), w_{x_1}\Big)\,\xi_{x_2} - \Big(i(w-w^h), w_{x_2}\Big)\,\xi_{x_1}\,,$$

and

$$I_2(\xi) = \int_{B(R)} \big(iw^h, w_{x_1} - w^h_{x_1}\big)\,\xi_{x_2} - \big(iw^h, w_{x_2} - w^h_{x_2}\big)\,\xi_{x_1}\,.$$

For I_1, we have by Cauchy-Schwarz inequality

$$\begin{aligned} |I_1(\xi)| &\leq |\nabla\xi|_{L^\infty}\,|w-w^h|_{L^2(\mathbb{R}^2)}\,||\nabla(w^h)||_{L^2} \\ &\leq C h\,|\log\varepsilon| \leq C\,\varepsilon^{1/4}\,|\log\varepsilon|\to 0 \quad \text{as } \varepsilon\to 0\,. \end{aligned}$$

For I_2, we integrate by parts, and obtain

$$I_2(\xi) = -\int_{B(R)} \big(iw^h_{x_1}, w-w^h\big)\,\xi_{x_2} - \big(iw^h_{x_2}, w-w^h\big)\,\xi_{x_1}\,.$$

Hence, we deduce as above that

$$|I_2(\xi)| \leq C\, \varepsilon^{1/4} |\log \varepsilon|,$$

and hence

$$|I(\xi)| \leq C\, \varepsilon^{1/4} |\log \varepsilon|.$$

The conclusion then follows easily. ■

The next lemma relates the functional D to the position of the vortices (in the spirit of [BCL]).

LEMMA IV.3. – *Let w and w^h be as above, and for $i = 1, \dots, \ell$, consider the vortices a_i for w^h. Assume that $\deg(a_i) = (-1)^i$, for $i = 1, 2$ and zero otherwise and*

$$a_1 = 0.$$

Then

$$\big|D(w^h) - |a_2|\big| \leq K_0(\varepsilon)$$

where $K_0(\varepsilon)$ depends only on ε and $K_0(\varepsilon) \to 0$ as $\varepsilon \to 0$. Hence

$$\big|D(w) - |a_2|\big| \leq K(\varepsilon)$$

where $K(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Proof. – Let $f : \mathbb{R}^* \to \mathbb{R}$ be a smooth function such that

$$f(t) = 0, \qquad if\ 0 \leq t \leq \frac{1}{4}$$

$$f(t) = 1, \qquad if\ t \geq \frac{1}{2}$$

$$\text{and} \qquad |f'|_{L^\infty} \leq 8, \qquad f' \geq 0.$$

Consider next the map $\tilde{w}$ defined on

$$\hat{w} = f(|w|^h)\, \frac{w^h}{|w|^h},$$

so that $\hat{w} = \frac{w^h}{|w|^h}$, on $\Omega = \mathbb{R}^2 \setminus \bigcup_{i=1}^{\ell} B(a_i, \rho)$, and is hence S^1-valued on Ω. In order to prove the Lemma, we will first compare $D(w^h)$ with $D(\tilde{w})$, and then show that $D(\tilde{w})$ is close to $|a_2|$.

Step 1. – We have

$$|D(\hat{w}) - D(w^h)| \leq K_1(\varepsilon),$$

where K_1 depends only on ε, and $K_1(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Proof. – The proof is similar to the proof of Lemma IV.2. It suffices to establish that

$$\text{(IV.12)} \qquad |\hat{w} - w^h|_{L^2} \leq C\, \varepsilon |\log \varepsilon|^{1/2}$$

and then, to argue as in the proof of Lemma IV.2. In order to establish (IV.12) notice, that for any $z \in \mathbb{R}^2$

$$|\hat{w} - w^h|(z) = \big|f(|w|^h) - |w_h|\big|.$$

In view of the definition of f, we hence deduce that

$$|\hat{w} - w^h| \leq 2\big||w^h| - 1\big|.$$

It follows that

$$|\hat{w} - w^h|_{L^2(\mathbb{R}^2)} \leq \big||w^h| - 1\big|_{L^2(\mathbb{R}^2)} \leq C\, \varepsilon |\log \varepsilon|^{1/2},$$

which proves (IV.12). This completes the proof of the first step.

Step 2. – Estimates for $D(\hat{w})$.

We write

$$I(\xi) = \int_{B(R)} (i\hat{w}, \hat{w}_{x_1})\, \xi_{x_2} - (i\hat{w}, \hat{w}_{x_2})\, \xi_{x_1} = A_1 + A_2$$

where

$$A_i = \int_{W_i} \xi_{x_2}(i\hat{w}, \hat{w}_{x_1}) - \xi_{x_1}(i\hat{w}, \hat{w}_{x_2}), \text{ for } i = 1, 2$$

where $W_1 = \overset{\ell}{\underset{i=1}{\cup}} B(a_i, \rho)$ and $W_2 = B(R) \setminus \overset{\ell}{\underset{i=1}{\cup}} B(a_i, \rho)$.

For A_1, we have, by Cauchy-Schwarz inequality

$$\begin{aligned} |A_1| &\leq |\nabla \xi|_{L^\infty} ||\nabla \hat{w}||_{L^2(\mathbb{R}^2)} |\ell| \operatorname{meas}(B(a_i, \rho))^{1/2} \\ &\leq C\, \rho |\log \varepsilon| \leq C\, \varepsilon^{\overline{\mu}/2} |\log \varepsilon| \to 0 \quad \text{as } \varepsilon \to 0. \end{aligned}$$

For A_2, we notice, that since $\hat{w}$ is S^1-valued on W_2, that

$$\frac{\partial}{\partial x_2}\left(i\hat{w},\hat{w}_{x_1}\right)-\frac{\partial}{\partial x_1}\left(i\hat{w},\hat{w}_{x_2}\right)=0.$$

Hence integration by parts, yields for A_2

$$\begin{aligned} A_2 &= \sum_{i=1}^{\ell}\int_{\partial B(a_i,\rho)\cap B(R)}(i\hat{w},\hat{w}_\tau)\,\xi\,d\sigma \\ &= \sum_{i=1}^{\ell}\int_{\partial B(a_i,\rho)}(i\hat{w},\hat{w}_\tau)\,\xi\,d\sigma \end{aligned}$$

where, we have set $\xi\equiv 0$ on $\mathbb{R}^2\setminus B(R)$. Hence

$$A_2=\sum_{i=1}^{\ell}\left(\int_{\partial B(a_i,\rho)}(i\hat{w},\hat{w}_\tau)\,d\sigma\right)\xi(a_i)+\sum_{i=1}^{\ell}\int_{\partial B(a_i,\rho)}(iw,w_\tau)\,(\xi-\xi(a_i))\,d\sigma.$$

By degree theory,

$$\int_{\partial B(a_i,\rho)}(i\hat{w},\hat{w}_\tau)\,d\sigma=2\pi\,\deg(a_i).$$

On the other hand, since $|\nabla\xi|\leq 1$, we have

$$|\xi(z)-\xi(a_i)|\leq\rho,\quad\forall z\in\partial B(a_i,\rho).$$

Hence, for $i=1,\ldots,\ell$

$$\begin{aligned} \left|\int_{\partial B(a_i,\rho)}(i\hat{w},\hat{w}_\tau)(\xi-\xi(a_i))\right|d\sigma &\leq \rho\int_{\partial B(a_i,\rho)}|\nabla\hat{w}|\,d\sigma \\ &\leq \rho\,C\leq C\,\varepsilon^{\overline{\mu}/2} \end{aligned}$$

where we have used Cauchy-Schwarz inequality together with (19) et (20). This yields finally, since $\deg(a_i)=(-1)^i$ for $i=1,2$, and zero otherwise

$$\left|A_2-\left(2\pi(\xi(a_2)-\xi(a_1))\right)\right|\leq C\,\varepsilon^{\overline{\mu}/2}$$

and finally

$$\left|I(\xi)-2\pi(\xi(a_2)-\xi(a_1))\right|\leq C\,\varepsilon^{\overline{\mu}/2}\log\varepsilon.$$

In particular, since $|\nabla\xi| \leq 1$, and $a_1 = 0$, we obtain, taking

$$|I(\xi)| \leq 2\pi |a_2| + K_2(\varepsilon)$$

where $K_2(\varepsilon) = C\,\varepsilon^{\overline{\mu}/2}\,|\log\varepsilon|$ tends to zero as $\varepsilon \to 0$. Taking the supremum over all ξ

$$\text{(IV.13)} \qquad |D(\hat{w})| \leq |a_2| + K_2(\varepsilon).$$

We claim, that actually

$$\text{(IV.14)} \qquad \big|D(\hat{w}) - |a_2|\big| \leq K_2(\varepsilon).$$

It suffices to construct a map ξ such that

$$\begin{aligned} &\xi(0) = 0 \\ &\xi(a_2) = 1 \\ &\xi(z) = 0, \ \text{ on } \partial B(R) \end{aligned}$$

and

$$|\nabla\xi|_{L^\infty} \leq 1.$$

It is easy to see that $\xi(z) \equiv \operatorname{Inf}\{\operatorname{dist}(z, \partial B(R), |z - a_i|\}$ is a suitable choice.

Step 3. Proof of Lemma IV.3. – Combining step 1 with (IV.14) we easily deduce the first estimate of the Lemma. The second estimate can then be obtained using Lemma IV.2.

We are now in position to complete the proof of Proposition 8.

Proof of Proposition 8 *completed.* – We have mainly to prove that $\hat{v}$ is not a constant map. For that purpose, we are going to prove that

$$\text{(IV.15)} \qquad D(\hat{v}) \neq 0,$$

which will clearly yield the desired result. To that aim we first notice that

$$\pi\,|\hat{a}_2^n| \geq L(v_n) + K(\varepsilon), \quad \forall n \in \mathbb{N},$$

where $K(\varepsilon) \to 0$ as $\varepsilon \to 0$. It follows in view of Lemma IV.3 that

$$\begin{aligned} D(\hat{v}^n) &\geq \frac{1}{\pi}(L(v_n) + K(\varepsilon)) \\ &\geq \frac{1}{8} + K(\varepsilon). \end{aligned}$$

Hence, by Lemma IV.2, we deduce that

$$D(\hat{v}) \geq \frac{1}{8} + K(\varepsilon)$$

which shows, that, if ε is sufficiently small

$$D(\hat{v}) \geq \frac{1}{16},$$

and establishes (IV.15).

Remark. – We have actually shown that

$$D(\hat{v}) \geq \frac{1}{\pi} \liminf_{n\to+\infty} L(v_n) + K(\varepsilon) \quad \text{(IV.16)}$$

where $K(\varepsilon)$ depends only on ε and $K(\varepsilon)\to 0$ as $\varepsilon\to 0$. Similarly one has

$$D(\hat{v}) \leq \frac{1}{\pi} \liminf_{n\to+\infty} |L(v_n)| + K(\varepsilon). \quad \text{(IV.17)}$$

V. PROOF OF THEOREM 1

Recall, that in Proposition 6 we have shown that N^ε separates $\mathcal{C}_0^\varepsilon$ and $\mathcal{C}_1^\varepsilon$. As a consequence of the Ghoussoub-Preiss variant of the mountain-pass theorem (*i.e.* Theorem 2), we deduce directly Proposition 7, that is, for any $\check{d}_0$, $\hat{d}_0$, satisfying

$$\frac{1}{4} < \check{d}_0 < 2 < \hat{d}_0 < 4, \quad \text{(V.1)}$$

there exists $\varepsilon_1 > 0$, such that, for $\varepsilon < \varepsilon_1$, there is a sequence $(v_n)_{n\in\mathbb{N}}$ of maps in V, such that

$$\pi\,\check{d}_0 < L(v_n) < \hat{d}_0\,\pi \quad \text{(V.2)}$$

$$\overline{F}_\varepsilon(v_n) \longrightarrow c_\varepsilon \quad \text{(V.3)}$$

and

$$||d\overline{F}_\varepsilon(v_n)|| \longrightarrow 0. \quad \text{(V.4)}$$

Consider next the map $\tilde{v}_n$ as defined in Section IV. Up to a subsequence, still denoted $\tilde{v}_n$, $\tilde{v}_n$ converges weakly in $H^1(K)$ for any compact subset K of $\mathbb{R}^2$, to some map u, belonging to $H^1_{\text{loc}}(\mathbb{R}^2)$. Moreover we have

$$\begin{aligned} D(u) &\geq \frac{1}{\pi} \liminf_{n\to+\infty} L(v_n) + K(\varepsilon) \\ &\geq \check{d}_0 + K(\varepsilon) \end{aligned}$$

where $K(\varepsilon)$ depends only on ε and tends to zero as $\varepsilon\to0$.

Changing ε_1 possibly to a smaller value, we see that for $\varepsilon < \varepsilon_1$, then

$$D(u) > \frac{1}{4}. \tag{V.5}$$

Hence u is not a constant map. It remains to prove that u is a solution to equation (6). This is actually an easy consequence of (V.4). Indeed, let $r > 0$, and consider the ball $B(r)$. Then V.4 implies that

$$i\frac{\partial \tilde{v}_n}{\partial x_1} = \Delta \tilde{v}_n + \frac{1}{\varepsilon^2}\tilde{v}_n(1 - |\tilde{v}_n|^2) + f_n \text{ on } B(r) \tag{V.6}$$

where $f_n\to0$ in $H^{-1}(B(r))$. Hence, since the equation is actually subcritical, it implies compactness of solutions, that is

$$\tilde{v}_n \longrightarrow u \quad \text{strongly in} \quad H^1(B(r)) \tag{V.7}$$

and u verifies

$$-i\frac{\partial u}{\partial x_1} = \Delta u + \frac{1}{\varepsilon^2}u(1 - |u|^2) \text{ on } B(r). \tag{V.8}$$

Since r was arbitrary, we see that u solves (6). Since u is not constant, this completes the existence of solutions, as asserted in the Theorem. Inequality (5) will be proved in the next Section.

VI. PROPERTIES OF SOLUTIONS

VI.1. Decay of $v(x)$ as $|x|\to\infty$

This subsection is devoted to showing that the solution v obtained in Theorem 1 converges pointwise to 1 as $|x|\to+\infty$. Recall (Proposition B.2) that v is bounded. Moreover

LEMMA VI.1. – $|v(x)| \to 1$ *as* $|x| \to \infty$

$$\nabla v \in L^p(\mathbb{R}^2), \quad 2 \leq p \leq +\infty.$$

Proof. – Since v is bounded, Lemma B.3 implies that $g = (1 - |v|^2)^2$ is Hölder continuous on every ball of radius ε, with a constant which depends only on ε. Therefore g is uniformly continuous, and integrable (since $E_\varepsilon(v) < +\infty$). Thus $g(x) \to 0$ as $|x| \to +\infty$. The assertion on ∇v results from the standard elliptic estimates. ■

Actually Lemma VI.1 is valid for any arbitrary finite energy solution w of

$$-icw_{x_1} - \Delta w = w(1 - |w|^2) \text{ in } \mathbb{R}^2. \tag{VI.1}$$

Let $w(x) = \rho(x) e^{i\theta(x)}$, so that

$$E(w) = \frac{1}{2} \int_{\mathbb{R}^2} \left\{ [\rho^2 |\nabla \theta|^2 + |\nabla \rho|^2] + \frac{1}{2}(1 - \rho^2)^2 \right\}.$$

■

The next lemma precises the behavior of ρ as $|x| \to +\infty$.

LEMMA VI.2. – *Let* $\varphi = 1 - \rho^2$. *Then*

$$\varphi \in H^2(\mathbb{R}^2)$$
$$1 - \rho \in L^2(\mathbb{R}^2).$$

Proof. – By a straightforward computation, one obtains

$$\begin{aligned} -\Delta \varphi + 2\varphi &= 2\varphi^2 + 2|\nabla w|^2 - 2c \operatorname{Im}(w \overline{w}_{x_1}) \\ &= 2\varphi^2 + 2|\nabla w|^2 + 2c\rho^2 \theta_{x_1}. \end{aligned}$$

Using $E(w) < +\infty$ and Lemma VI.1, the assertion results from the L^2 regularity of $-\Delta + 2$. With some extra work, we could in fact show that $\varphi \in L^p(\mathbb{R}^2)$, $\forall p > 1$ (see [BS]). ■

Next we turn to the behavior of the phase $\theta(x)$ of w.

LEMMA VI.3. – $\theta(x) \to 0$ *as* $|x| \to +\infty$.

Proof. – One readily obtains from (VI.1) that

$$\operatorname{div}(\rho^2 \nabla \theta) = -\frac{c}{2}(\rho^2)_{x_1}. \tag{VI.2}$$

By the Hodge decomposition,

$$\rho^2\nabla\theta = \nabla\Phi + \nabla^\perp\psi,\ \ \nabla\Phi,\ \ \nabla^\perp\psi \in L^2(\mathbb{R}^2), \tag{VI.3}$$

so that

$$\Delta\Phi = -\frac{c}{2}(\rho^2)_{x_1} = +\frac{c}{2}\varphi_{x_1} \tag{VI.4}$$

and

$$\Delta\Psi = (\rho^2)_{x_2}\theta_{x_1} - (\rho^2)_{x_1}\theta_{x_2}. \tag{VI.5}$$

Thus $\nabla\Phi \in L^p(\mathbb{R}^2)$, $1 < p < +\infty$. On the other hand, we deduce from [CLMS] that the right hand side of (VI.5) belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$ and thus (see [H], chap. III), $\nabla\Psi$ belongs to the Lorentz space $L^{2,1}(\mathbb{R}^2)$. Finally, $\nabla\theta \in L^{2,1}(\Omega_\delta)$, $\forall\delta > 0$, where $\Omega_\delta = \{x \in \mathbb{R}^2,\ \rho \geq \delta\}$.

Therefore (see for instance [H]), θ is continuous and bounded on Ω_δ, $\forall\delta > 0$. One can then prove that in fact $\theta(x)$ tends to some constant which can be normalized to 0 as $|x| \to +\infty$. This will be detailled in a subsequent paper ([BS]) and will lead to :

PROPOSITION VI.4. – *$v(x) \to 1$ as $|x| \to +\infty$*

VI.2. Some identities

Next we establish various identities of Pohojaev type. Those identities were formally derived in [JPR].

PROPOSITION VI.5. – *Let u be any finite energy solution of*

$$-icu_{x_1} - \Delta u - u(1 - |u|^2) = 0 \text{ on } \mathbb{R}^2 \tag{VI.6}$$

such that there exists $q \geq 2$ such that

$$u - 1 \in L^q(\mathbb{R}^2),\quad u_{x_1} \in L^{q'}(\mathbb{R}^2),\quad \frac{1}{q} + \frac{1}{q} = 1. \tag{VI.7}$$

Then, if $p \underset{\text{def}}{=} \frac{1}{2i}\int_{\mathbb{R}^2}[(\overline{u} - 1)u_{x_1} - (u - 1)\overline{u}_{x_1}]$, where $\overline{u}$ is the complex conjugate of u, one has

$$\begin{aligned} E &= \frac{1}{2}\int_{\mathbb{R}^2}\left[|\nabla u|^2 + \frac{1}{2}(1 - |u|^2)^2\right] \\ &= -\frac{1}{2}pc + \frac{1}{4}\int_{\mathbb{R}^2}(1 - |u|^2)\,|1 - u|^2 \end{aligned} \tag{VI.8}$$

$$-cp = \int_{\mathbb{R}^2} (1 - |u|^2)^2. \tag{VI.9}$$

Proof. – (VI.8) is readily obtained by multiplying (VI.6) by $\overline{u} - 1$ and integrating the real part.

To get (VI.9), we first multiply (VI.1) by $x_2 \frac{\partial \overline{u}}{\partial x_2}$ we integrate the real part (as usual this can be justified by a truncation process on x_2). After a few integrations by parts one gets

$$-cp = \int_{\mathbb{R}^2} \left[|u_{x_1}|^2 - |u_{x_2}|^2\right] + \frac{1}{2} \int_{\mathbb{R}^2} (1 - |u|^2)^2. \tag{VI.10}$$

Now we multiply (VI.7) by $x_1 \frac{\partial \overline{u}}{\partial x_1}$ and integrate the real part to get

$$\int_{\mathbb{R}^2} \left[|u_{x_1}|^2 - |u_{x_2}|^2\right] = \frac{1}{2} \int_{\mathbb{R}^2} (1 - |u|^2)^2, \tag{VI.11}$$

combining (VI.10) and (VI.11) yields (VI.8).

Note that (VI.11) implies also that

$$E = \int_{\mathbb{R}^2} |u_{x_1}|^2. \tag{VI.12}$$

COROLLARY. – *(VI.6) has no non trivial finite energy solution whenever* $c = 0$.

Remark that the solutions obtained by Brezis-Merle-Rivière [BMR] have infinite energy.

VI.3 Qualitative properties of the mountain-pass solution

We will now turn to the proof of inequalities (5) and of further properties of the solution.

In view of the semi-continuity of E_ε we have

LEMMA VI.6. – *The map* u *verifies*

$$E_\varepsilon(u) \leq c_\varepsilon. \tag{VI.13}$$

The proof is straightforward. Note that Lemma VI.6 yields the upper bound in inequality (5).

The next Proposition displays the vortex structure of the solution and thus justifies rigorously the numerical solution obtained in [JR], [JPR]. It shows in particular that, in the physical coordinates, the distance between the two vortices is of order $c^{-1} = \varepsilon^{-1}$ as $\varepsilon \to 0$.

PROPOSITION VI.7. – *The solution u obtained is smooth. Moreover if $\varepsilon < \varepsilon_1$, there exists exactly two points a_1 and a_2 in $\mathbb{R}^2$, and a radius ρ such that*

$$\varepsilon^{\mu} \leq \rho \leq \varepsilon^{\overline{\mu}/2} \tag{VI.14}$$

$$|u(x)| \geq \frac{1}{2} \quad on \quad \mathbb{R}^2 \setminus \bigcup_{i=1}^{2} B(a_i, \rho) \tag{VI.15}$$

$$\deg(a_i) = (-1)^i. \tag{VI.16}$$

Moreover, we have

$$\big||a_1 - a_2| - 2\big| \leq K(\varepsilon) \tag{VI.17}$$

and

$$|a_{1,1} - a_{2,1}| \leq K(\varepsilon)$$

where $K(\varepsilon)$ depends only on ε, and $K(\varepsilon)\to 0$ as $\varepsilon\to 0$.

Proof. – For the first assertion (namely smoothness), we refer to standard elliptic theory. Next we turn to the study of the vortices of u. In view of the bound (VI.6) and equation (6), we may apply Proposition 2 to u. This yields ℓ points $a_1, \ldots, a_\ell$ in $\mathbb{R}^2$, and a radius $\rho > 0$, such that (14) to (20) holds. In order to establish the Lemma we have first to show that :

i) u has no degree zero vortex,

ii) u has two vortices say a_1, a_2 of degree $+1$ and -1 respectively.

Step 1. *Proof of i).* – Suppose that u has a vortex of degree zero a_i *i.e.*

$$\deg(a_i) = 0.$$

By (20), we therefore have

$$\int_{\partial B(a_i,\rho)} e_\varepsilon(v) \leq \frac{\pi}{\rho} K(\varepsilon)$$

where $K(\varepsilon)\to 0$ as $\varepsilon\to 0$. By Lemma of Appendix 2, this yields

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_{B(a_i,\rho)} &(1-|u|^2)^2 \\ &\leq K(\varepsilon) + \left|\int_{B(a_i,\rho)} ic\frac{\partial u}{\partial x_1}(x-a_1)\frac{\partial u}{\partial x_1}\right| \\ &\leq K(\varepsilon) + \rho\int_{B(a_i,\rho)} |\nabla u|^2 \leq K(\varepsilon) + \varepsilon^{\overline{\mu}}|\log\varepsilon| \to 0 \text{ as } \varepsilon\to 0.\end{aligned}$$

By Lemma B.4, this yields

$$|u| \geq \frac{1}{2} \quad \text{on} \quad B(a_i, \rho)$$

contradicting the fact that a_i is a vortex.

Step 2. Proof of ii). – We have shown that

$$D(u) \geq \check{d}_0 + K(\varepsilon) > \frac{1}{4}.$$

This shows that u has two vortices of degree $+1$ and -1 respectively.

Assertion (VI.17) then easily follows from Lemma IV.3, (IV.16) and (IV.17).

APPENDIX A

THE CAUCHY PROBLEM

We consider the Cauchy problem associated to (1)

$$\text{(A.1)} \qquad \begin{cases} i\,\dfrac{\partial v}{\partial t} = \Delta v + v(1-|v|^2) \ \text{ on } \mathbb{R}^d \times \mathbb{R}, \ \ d = 2, 3 \\ v \to 1 \ \text{ as } |x| \to +\infty \\ v(x,0) = v_0(x). \end{cases}$$

To reduce (A.1) to a standard NLS problem, we set $v = 1 + u$. Then (A.1) writes

$$\text{(A.2)} \qquad \begin{cases} i\,\dfrac{\partial u}{\partial t} = \Delta u + F(u) = 0 \\ u(x,0) = u_0(x), \end{cases}$$

where

$$F(u) = (1+u)(|u|^2 + 2\,\mathrm{Re}\,u) = \frac{1}{2}\frac{\partial}{\partial \overline{u}} H(u, \overline{u})$$

$$H(u, \overline{u}) = |u|^4 + 4\,\mathrm{Re}\,u\,|u|^2 + 2(\mathrm{Re}(u^2) + |u|^2 = (|u|^2 + 2\,\mathrm{Re}\,u)^2.$$

The next theorem proves that (A.1) is globally well-posed, as expected for a nonlinear Schrödinger equation with repulsion.

THEOREM A.1. – *Let $u_0 \in H^1(\mathbb{R}^d)$. There exists a unique solution $u \in C(\mathbb{R}; H^1(\mathbb{R}))$ of (A.2). Moreover the energy is conserved,*

$$\text{(A.3)} \quad E(t) = \int_{\mathbb{R}^d} |\nabla u(t))^2 + \frac{1}{2}\int_{\mathbb{R}^d} (|u(t)|^2 + 2\,\mathrm{Re}\,u(t))^2$$
$$= \int_{\mathbb{R}^d} |\nabla u_0|^2 + \frac{1}{2}\int_{\mathbb{R}^d} (|u_0|^2 + 2\,\mathrm{Re}\,u_0)^2, \quad \forall t \in \mathbb{R}.$$

Proof. – The local existence and uniqueness is a direct consequence of a result of Kato ([K1], [K2], Theorem II and II'). In order to prove the global existence we just need to derive a global a priori bound in $H^1(\mathbb{R}^d)$.

The following computations are formal but can be justified in the usual way (for instance by smoothing the initial data and constructing local H^2 solutions, see [K1], [K2]). We first multiply (A.2) by $\overline{u}_t$ and integrate the real part to get

$$\text{(A.4)} \quad \int_{\mathbb{R}^d} |\nabla u(t)|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^d} \left(|u(t)|^2 + 2\,\mathrm{Re}\,u(t)\right)^2 dx$$
$$= \int_{\mathbb{R}^d} |\nabla u_0|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^d} \left(|u_0|^2 + 2\,\mathrm{Re}\,u_0\right)^2 dx,$$

which gives a uniform a priori bound on $||\nabla u(t)||_{L^2}$.

Next we multiply (A.2) by $\overline{u}$ and integrate the imaginary part to get

(A.5)
$$\frac{1}{2}\,\frac{d}{dt}\int_{\mathbb{R}^d} |u(t)|^2\,dx \le \int_{\mathbb{R}^d} |u|\,|F(u)|\,dx$$
$$\le 2\int_{\mathbb{R}^d} |u(t)|^2\,dx + 3\int_{\mathbb{R}^d} |u(t)|^3\,dx + \int_{\mathbb{R}^d} |u(t)|^4\,dx.$$

If $d = 2$, we use the Gagliardo-Nirenberg inequalities

$$||u||_{L^3} \le C||\nabla u||_{L^2}^{1/3}\,||u||_{L^2}^{2/3}$$
$$||u||_{L^4} \le C||\nabla u||_{L^2}^{1/2}\,||u||_{L^2}^{1/2}$$

and the uniform bound on $||\nabla u(t)||_{L^2}$ to get from (A.5) and Gronwall's lemma

$$||u(t)||_{L^2} \le ||u_0||_{L^2}\,e^{ct}.$$

If $d = 3$, we use the Gagliardo-Nirenberg inequalities

$$||u||_{L^3} \le C||\nabla u||_{L^2}^{1/2}\,||u||_{L^2}^{1/2}$$
$$||u||_{L^4} \le C||\nabla u||_{L^2}^{3/4}\,||u||_{L^2}^{1/4},$$

and the uniform bound on $||\nabla u(t)||_{L^2}$ to get from (A.5)

$$\frac{d}{dt}\int_{\mathbb{R}^3}|u(t)|^2dx \leq C_1 + C_2\int_{\mathbb{R}^3}|u(t)|^2dx$$

which again leads to the desired L^2 bound.

APPENDIX B

The aim of this Appendix is to provide proofs for Propositions 1 and 2. We begin with some useful tools, which are adapted from [BBH].

LEMMA B.1. – *Let v and f satisfy (8), (9), (10). Let z_0 be a point in $\mathbb{R}^2$, and $r > 0$. Then we have*

$$\text{(B.1)}\quad \frac{1}{2\varepsilon^2}\int_{B(z_0,r)}(1-|v|^2)^2 \leq \frac{1}{2}\, r\int_{\partial B(z_0,r)} e_\varepsilon(v) + r\int_{B(z_0,r)}|f|\,|\nabla v|.$$

Proof. – The argument is similar to the proof of Theorem III.2 in [BBH], and relies on Pohojaev identity : it suffices to multiply equation (8), by the Pohojaev multiplier $(z-z_0)\cdot\nabla u$ and to integrate by part on $B(z_0,r)$. Then following [BBH], Section III) we obtain (B.1).

LEMMA B.2. – *Let v and f satisfy (8), (9), (10), and let $\alpha > 0$ be given, such that $0 < \beta < \alpha < 1$. There exists a constant $C_1 > 0$, depending only on α, β, and C_0 such that, for $0 < \varepsilon < 1$,*

$$\text{(B.2)}\qquad \frac{1}{\varepsilon^2}\int_{B(z_0,\varepsilon^\alpha)}(1-|v|^2)^2 \leq C_1\,.$$

Proof. – Translating the origin if necessary, we may assume that $z_0 = 0$. Since $E_\varepsilon(v) \leq C_0(|\log\varepsilon|)$, we have, for $\alpha_1 = \beta + \frac{\alpha-\beta}{2} = \frac{\alpha+\beta}{2}$

$$\int_{\varepsilon^\alpha}^{\varepsilon^{\alpha_1}} dr\left[\int_{\partial B(r)} e_\varepsilon(v)\right] \leq C_0(|\log\varepsilon|)$$

so that, we may assert, by Fubini's theorem, that there is some $r_0 \in (\varepsilon^\alpha, \varepsilon^{\alpha_1})$ such that

$$\text{(B.3)}\qquad r_0\int_{\partial B(r_0)} e_\varepsilon(v) \leq \frac{2C_0}{\alpha-\beta}\,.$$

Going back to Lemma B.1, we deduce therefore that

$$\frac{1}{2\varepsilon^2}\int_{B(r_0)}(1-|v|^2)^2 \leq \frac{C_0}{\alpha-\beta} + r_0\,||f||_{L^2}\,||\nabla v||_{L^2}$$
$$\leq \frac{C_0}{\alpha-\beta} + C_0'\varepsilon^{\alpha_1}\cdot\varepsilon^{-\beta}\,|\log\varepsilon|^{3/2}.$$

The conclusion then follows from the fact that $\alpha_1 > \beta$.

LEMMA B.3. – *Let v and f satisfy (8), (9), (10). Then there exist constants $0 < \gamma < 1$ and $C_2 > 0$ such that*

$$|v(z_0) - v(z_1)| \leq C_2\left|\frac{z_0 - z_1}{\varepsilon}\right|^{\gamma} \tag{B.4}$$

provided $|z_0 - z_1| \leq \varepsilon$.

Proof. – We may assume (translating if necessary the origin) that $z_0 = 0$. It is convenient to perform a change of scale. Therefore consider the maps, $\tilde{f}$ and $\tilde{v}$ defined by

$$\tilde{v}(z) = v(z\varepsilon), \qquad \tilde{f}(z) = f(z\varepsilon)$$

so that $\tilde{v}$ verifies

$$-\Delta\tilde{v} = \tilde{v}(1-|\tilde{v}|^2) + \varepsilon^2\tilde{f} \text{ on } \mathbb{R}^2. \tag{B.5}$$

Our aim is to bound $\tilde{v}$ in $W^{2,p}(B(2))$ for some exponent $p > 1$. This will then yield the result by Sobolev embedding. To that purpose, we consider the ball $B(3)$, and write,

$$\tilde{v} = \tilde{v}_1 + \tilde{v}_2 \quad \text{on} \quad B(3)$$

where

$$\begin{cases} -\Delta\tilde{v}_1 = 0 & \text{on } B(3) \\ \tilde{v}_1 = \tilde{v} & \text{on } \partial B(3) \end{cases} \tag{B.6}$$

and

$$\begin{cases} -\Delta\tilde{v}_2 = g & \text{on } B(3) \\ \tilde{v}_2 = 0 & \text{on } \partial B(3) \end{cases} \tag{B.7}$$

with

$$g \equiv \varepsilon^2\tilde{f} + \tilde{v}(1-|\tilde{v}|^2) \quad \text{on} \quad B(3).$$

For (B.7), we have by standard elliptic theory

$$|\tilde{v}_2|_{W^{2,p}} \leq C||g||_{L^p}$$

for any $1 < p < 2$. We are therefore going to estimate $||g||_{L^p}$ for a suitable p. First note that, for $1 < p < 2$

$$\int_{B(3)} |\tilde{f}|^p = \varepsilon^{-2} \int_{B(3\varepsilon)} |f|^p$$

so that

$$\int_{B(3)} |\varepsilon^2 \tilde{f}|^p = \varepsilon^{2(p-1)} \int_{B(3\varepsilon)} |f|^p$$

and

$$\text{(B.8)} \qquad \left(\int_{B(3)} |\varepsilon^2 \tilde{f}|^p\right)^{1/p} \leq C\,\varepsilon ||f||_{L^2(\mathbb{R}^2)} \leq C\,\varepsilon^{1-\beta} |\log \varepsilon| \to 0, \quad \varepsilon \to 0.$$

Next, observe that, since by Lemma B.2

$$\int_{B(z_0,\varepsilon^\alpha)} (1 - |\tilde{v}|^2)^2 \leq C_1$$

we deduce that

$$\text{(B.9)} \qquad \int_{B(3)} |\tilde{v}|^4 \leq C,$$

for some constant C, so that

$$\text{(B.10)} \qquad \int_{B(3)} |\tilde{v}(1 - |\tilde{v}|^2)|^{4/3} \leq C.$$

Hence, we will choose $p = \frac{4}{3}$. Combining (B.8) and (B.9), we easily verify that

$$||g||_{L^{4/3}(B(3))} \leq C,$$

where C is a constant depending only on α, β and C_0. As a consequence, we deduce that

$$\text{(B.11)} \qquad ||\tilde{v}_2||_{W^{2,4/3}(B(3))} \leq C$$

where C is independent of ε. Combining (B.11) with (B.9) we obtain

(B.12) $$||\tilde{v}_1||_{L^4(B(3))} \leq C.$$

Since $\tilde{v}_1$ is a harmonic function, it follows that

$$||\tilde{v}_1||_{W^{2,4/3}(B(2))} \leq C$$

and hence by (B.11)

$$||\tilde{v}||_{W^{2,4/3}(B(2))} \leq C.$$

By the Sobolev embedding $W^{2,4/3} \hookrightarrow C^{0,\gamma}$, for some $0 < \gamma < 1$ we deduce that

$$\left|\tilde{v}(z_1) - \tilde{v}(z_2)\right| \leq C|z_1 - z_2|^\gamma, \qquad \forall z_1, z_2 \text{ in } B(2).$$

This yields (B.4).

The next result generalizes Theorem III.4 of [BBH] to equation (8).

LEMMA B.4. – *Assume v and f verify (8), (9), (10). Then there exist positive constants λ_0 and η_0 depending only on C_0, and β such that, if for some $z_0 \in \mathbb{R}^2$, and some $\ell > 0$, such $\ell > 2\lambda_0\varepsilon$ we have*

(B.13) $$\frac{1}{\varepsilon^2}\int_{B(z_0,2\ell)} (1 - |v|^2)^2 \leq \eta_0$$

then

(B.14) $$|v(z)| \geq \frac{1}{2} \quad \textit{on} \quad B(z_0, \ell).$$

Proof. – Assume that there is a point z_1 in $B(z_0, \ell)$ such that

$$|v(z_1)| < \frac{1}{2}.$$

Then, by Lemma B.3, we have

$$|v(z)| < \frac{3}{4}, \quad \text{for} \quad z \in B(z_1, r)$$

with $r = C_3\varepsilon$, where $C_3 = \text{Inf}\left(1, \left(\frac{1}{4C_2}\right)^{1/\gamma}\right)$. It follows that we have

$$\frac{1}{\varepsilon^2}\int_{B(z_1,r)} (1 - |v|^2)^2 \geq \frac{C_3^2}{16}\pi.$$

Take $\lambda_0 = C_3$ and $\eta_0 = \frac{C_3^2}{32}\pi$. We then have

$$\frac{1}{\varepsilon^2}\int_{B(z_1,\ell)}(1-|v|^2)^2 > \eta_0,$$

contradicting (B.13). This completes the proof of the Lemma.

The next result is a first step towards locating the vortices (in the spirit of Proposition IV.4 of [BBH] or Proposition IV.2 of [BR]). It yields a local version (*i.e.* on the scale ε^α) of Proposition 1.

LEMMA B.5. – *Assume that v and f verify (8), (9), (10), and let $0 < \beta < \alpha < 1$. Let $z_0 \in \mathbb{R}^2$. There exists a constant $\overline{N}_\alpha \in \mathbb{N}$, depending only on α, β, C_0, and ℓ points $x_1, \dots, x_\ell$ in $B(x, \varepsilon^\alpha) \cap \Omega$, such that*

$$\ell \leq N_\alpha \tag{B.15}$$

and

$$|v(z)| \geq \frac{1}{2} \quad on \quad B(z_0, \varepsilon^\alpha) \setminus \bigcup_{i=1}^{\ell} B(z_i, \lambda_0\varepsilon) \tag{B.16}$$

(λ_0 being the constant in Lemma B.4).

Proof. – The proof relies on a covering argument. Consider a covering of the ball $B(z_0, \varepsilon^\alpha)$ by a collection of balls $B(z_i, \lambda_0\varepsilon)$, $i \in I$ such that

$$B\Big(z_i, \frac{\lambda_0\varepsilon}{8}\Big) \cap B\Big(z_j, \frac{\lambda_0\varepsilon}{8}\Big) = \emptyset, \quad \text{if} \quad i \neq j \tag{B.17}$$

and

$$B(z_0, \varepsilon^\alpha) \subset \bigcup_{i\in I} B(z_i, \lambda_0\varepsilon) \subset B(z_0, 2\varepsilon^\alpha). \tag{B.18}$$

Consider next the subset J of I defined, by

$$J = \left\{ i \in I,\ \frac{1}{\varepsilon^2}\int_{B(z_i, 2\lambda_0\varepsilon)}(1-|v|^2)^2 > \eta_0 \right\}.$$

By (B.17), (B.18) and Lemma B.2, we see that

$$\text{Card } J < C$$

where the constant C depends only on C_0, β and α. On the other hand, if some $z \in B(z_0, \varepsilon^\alpha)$ is such that

$$z \in B(z_i, \lambda_0\varepsilon) \quad \text{for some} \quad i \in I \setminus J,$$

then, by Lemma B.4, we have

$$|v(z)| \geq \frac{1}{2}.$$

Hence

$$|v(z)| \geq \frac{1}{2} \quad \text{if} \quad z \in B(z_0, \varepsilon^\alpha) \setminus \underset{i \in J}{\cup} B(z_i, \lambda_0 \varepsilon).$$

This establishes the Lemma.

Our next aim is to deduce, from the local behavior on regions of scale ε^α, a global behavior on $\mathbb{R}^2$. Here, we follow basically the approach of [BR]. we start again with a technical tool.

LEMMA B.6. – *Assume v and f verify (8), (9), (10). Let z_0 be some point in $\mathbb{R}^2$, and assume that*

$$|v(z_0)| \leq \frac{1}{2}.$$

Then there exists a constant C_α, depends only on α, β and C_0, such that

$$\int_{B(z_0, \varepsilon^\alpha)} e_\varepsilon(v) \geq C_\alpha |\log \varepsilon|. \tag{B.19}$$

Proof. – Set

$$A = |\log \varepsilon|^{-1} \int_{B(z_0, \varepsilon^\alpha)} e_\varepsilon(v).$$

Using the same argument as in the proof of Lemma B.2 (cf inequality (B.3)), we may find some $r_1 \in (\varepsilon^{2\alpha}, \varepsilon^\alpha)$ such that

$$r_1 \int_{\partial B(z_0, r_1)} e_\varepsilon(v) \leq \frac{A}{\alpha}.$$

By Lemma B.1 (Pohozaev inequality), it follows that

$$\begin{aligned} \frac{1}{\varepsilon^2} \int_{B(z_0, r_1)} (1 - |v|^2)^2 &\leq \frac{A}{\alpha} + r_1 ||f||_{L^2} ||\nabla v||_{L^2} \\ &\leq \frac{A}{\alpha} + C_0 \varepsilon^{\beta - \alpha} |\log \varepsilon|. \end{aligned}$$

Assume now by contradiction that A verifies

$$A \leq \frac{1}{2} \alpha \eta_0 \tag{B.20}$$

and if ε is sufficiently small, we obtain

$$\frac{1}{\varepsilon^2}\int_{B(z_0,r_1)}(1-|v|^2)^2 \leq \frac{3\eta_0}{4}$$

which implies, by Lemma B.4 that

$$|v(z_0)| \geq \frac{1}{2}.$$

A contradiction. Hence (B.20) cannot hold, *i.e.* we have

$$\int_{B(z_0,\varepsilon^\alpha)} e_\varepsilon(v) \geq \frac{1}{2}\alpha\eta_0\,|\log\varepsilon|.$$

This completes the proof.

We are now in position to "globalize" Lemma B.5, that is, to prove Proposition 1 of the introduction.

Proof of Proposition 1. – We follow very closely the proof of Theorem IV.1 of [BR]. We use again a covering argument. Consider a covering of $\mathbb{R}^2$ by balls $B(a_k,\varepsilon^\alpha)$, $k\in I$, such that

$$B(a_k, 8\varepsilon^\alpha)\cap B(a_\ell, 8\varepsilon^\alpha) = \phi, \quad k\neq \ell$$

and

$$\mathbb{R}^2 \subset \bigcup_{k\in I} B(a_k,\varepsilon^\alpha).$$

Consider the subset of I defined by

$$J = \left\{k\in I,\ \text{s.t.}\ \int_{B(a_k,2\varepsilon^\alpha)} e_\varepsilon(v) \geq C_\alpha\,|\log\varepsilon|\right\}.$$

Since

$$E_\varepsilon(v) \leq C_0(|\log\varepsilon|).$$

We clearly see that

$$\text{Card } J \leq C$$

where C is some constant depending only on C_0, α and β. Let now $z_0\in\mathbb{R}^2$ be a point such that

$$z_0 \in B(a_k,\varepsilon^\alpha)$$

for some $k \notin J$. Then, we have

$$\int_{B(a_k, 2\varepsilon^\alpha)} e_\varepsilon(v) < C_\alpha |\log \varepsilon|$$

so that

$$\int_{B(z_0, \varepsilon^\alpha)} e_\varepsilon(v) < C_\alpha |\log \varepsilon|.$$

It follows from Lemma B.6 that

$$|v(z_0)| \geq \frac{1}{2}.$$

Hence we have proved that

$$|v(z)| \geq \frac{1}{2}, \quad \forall z \in \mathbb{R}^2 \setminus \bigcup_{k \in J} B(a_k, \varepsilon^\alpha).$$

Since $\operatorname{Card} J$ is bounded independently of ε, it suffices to invoke Lemma B.5 to complete the proof of the Proposition.

We next proceed with the proof of Proposition 2. We closely follows a preliminary version (1992) of [BBH] : the ideas have also been presented in [AB]. We begin with a preliminary result, which turns out to be very useful (see Lemma VI.1 of [AB]).

LEMMA B.6. – *Assume that v and f satisfy (8), (9) and (10). For $\xi \in \mathbb{R}^+$, consider the set*

$$T(\xi) = \left\{ z \in \mathbb{R}^2, |v(z)| = \xi \right\}.$$

Then there exists some $\xi_0 \in \left(1 - \frac{2}{|\log \varepsilon|^2}, 1 - \frac{1}{|\log \varepsilon|^2}\right)$ such that

$$\mathcal{H}^1(T(\xi_0)) \leq C_0\, \varepsilon |\log \varepsilon|^5 \tag{B.21}$$

and $T(\xi_0)$ is an union of smooth curves. Here $\mathcal{H}^1$ denotes the 1-dimensional measure (i.e. the length).

Proof. – Set

$$A = \left\{ z \in \mathbb{R}^2, 1 - \frac{2}{|\log \varepsilon|^2} \leq |v(z)| \leq 1 - \frac{1}{|\log \varepsilon|^2} \right\}.$$

Since

$$\int_{\mathbb{R}^2} (1 - |v|^2)^2 \leq 4C_0(|\log \varepsilon|)\varepsilon^2,$$

we deduce that

$$\operatorname{meas} A \leq C_0 (|\log \varepsilon|)^5 \varepsilon^2 .$$

On the other hand, the coarea formula yields

$$\begin{aligned}\int_{1-\frac{2}{|\log \varepsilon|^2}}^{1-\frac{1}{|\log \varepsilon|^2}} \mathcal{H}^1(T(\xi))\, d\xi &\leq \int_A |\nabla |v|| \\ &\leq ||\nabla v||_{L^2(\mathbb{R}^2)} (\operatorname{meas} A)^{1/2} \\ &\leq C_0 (|\log \varepsilon|)^3 \varepsilon .\end{aligned}$$

The Lemma then follows from the mean value inequality.

Since the proof of Proposition 2 is rather lengthy, we will split it in several lemmas. We start with a purely combinatorial fact.

LEMMA B.7. – *Let $N \in \mathbb{N}^*$, and consider a collection of points $(a_i)_{i\in I}$, with*

$$\operatorname{Card} I \leq N . \tag{B.22}$$

Let $0 < \varepsilon < \frac{1}{2}$, and let $0 < \mu < 1$ be a fixed constant. Then there exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon \leq \varepsilon_0$, then there exist $k \in \mathbb{N}$ and a subset J of I such that

$$k \leq N + 1 \tag{B.23}$$

$$|a_i - a_j| \geq \varepsilon^{\mu^{k+1}}, \text{ for } i \neq j, \quad i \in J, \ j \in J \tag{B.24}$$

and

$$\bigcup_{i\in I\setminus J} \{a_i\} \subset \bigcup_{j\in J} B\big(a_j, 2\varepsilon^{\mu^k}\big) . \tag{B.25}$$

Proof of Lemma B.7. – The proof is by induction, using a finite number of steps (actually, at most N steps).

Step 1. – We have to distinguish two cases.

Case 1. – $|a_i - a_j| \geq \varepsilon^{\mu^2}$, $\forall i, j$ in I, such that $i \neq j$.

In this case the proof is complete with

$$k = 1, \quad I = J .$$

Case 2. – There exist two points in I, say $a_{\ell-1}$ and a_ℓ (if we write $I = \{1,\ldots,\ell\}$) such that

$$|a_{\ell-1} - a_\ell| \leq \varepsilon^{\mu^2}. \tag{B.26}$$

We eliminate the point a_ℓ from the collection, and set

$$J_1 = \{1,\ldots,\ell-1\}.$$

We then proceed to step 2.

Step 2. – Two cases may occur.

Case 1. – $|a_i - a_j| \geq \varepsilon^{\mu^3}$, $\forall i,j$ in J_1, $i \neq j$.

In this case, the proof is complete by (B.26), with

$$k = 2, \quad J = J_1.$$

Case 2. – There are two points, say $a_{\ell-1}$ and $a_{\ell-2}$ in J_1 such that

$$|a_{\ell-1} - a_{\ell-2}| \leq \varepsilon^{\mu^3}. \tag{B.27}$$

We eliminate the point $a_{\ell-1}$ from the collection and set

$$J_2 = \{1,\ldots,\ell-2\}.$$

We then proceed to Step 3, and we use the same argument with $\ell-2$ points.

More generally, at step q we are left with $\ell-q+1$ points $a_1,\ldots,a_{\ell-q+1}$ such that

$$|a_{\ell-j-1} - a_{\ell-j}| \leq \varepsilon^{\mu^{j+1}}, \text{ for } j = 0,\ldots,q-1. \tag{B.28}$$

Arguing as in Step 1 or Step 2, we distinguish two cases.

Case 1. – We have

$$|a_i - a_j| \geq \varepsilon^{\mu^{q+1}} \text{ for any } i \neq j, \; i,j \text{ in } \{1,\ldots,\ell-q+1\}.$$

In view of (B.26) we then have, for $j \in \{\ell-q+2,\ldots,\ell\}$

$$|a_j - a_{\ell-q+1}| \leq \sum_{j=0}^{q-1} \varepsilon^{\mu^{j+1}} \leq 2\varepsilon^{\mu^q}$$

if ε is sufficiently small. Hence, the proof is complete with

$$k = q, \quad J = J_q = \{1, \dots, \ell - q + 1\}.$$

Case 2. – We have for two points in J_q, say $a_{\ell-q+1}$ and $a_{\ell-q}$

$$|a_{\ell-q+1} - a_{\ell-q}| < \varepsilon^{\mu^{q+1}}.$$

We remove the point $a_{\ell-q}$ of the collection and proceed with step $q+1$.

Since at each step we remove one point from J_q, the proof is completed in at most N steps.

We apply the previous Lemma to the collection $\{a_i\}$ of points given by Proposition 1. This yields us a family $\{a_i\}_{i\in J}$ of points with

$$\text{Card } J \leq \ell$$

having properties (B.23), (B.24) and (B.25). We next turn to a suitable choice of radius ρ around the vortices : actually for technical reasons we have to introduce two different radius ρ_0 and ρ_1. We begin with ρ_0 :

LEMMA B.8. – *There exists a radius* $\rho_0 \in \left(2\varepsilon^{\mu^k}, \varepsilon^{\mu^{k+\frac{1}{2}}}\right)$ *such that (if* ε *is sufficiently small)*

$$\sum_{i\in J} \rho_0 \int_{\partial B(a_i,\rho_0)} e_\varepsilon(v) \leq C_4 \tag{B.29}$$

and

$$|v(z)| \geq 1 - \frac{2}{|\log\varepsilon|^2} \quad \textit{on } \partial B(a_i,\rho_0), \textit{ for } i \in J. \tag{B.30}$$

Proof. – Let $\xi_0 \in \left(1 - \frac{2}{|\log\varepsilon|^2}, 1 - \frac{1}{|\log\varepsilon|^2}\right)$ be the number obtained in Lemma B.6. Consider next, the subset $\mathcal{B}$ of $\left(\varepsilon^{\mu^k}, \varepsilon^{\mu^{k+1}}\right)$ defined by

$$\mathcal{B} = \left\{\rho \in \left(\varepsilon^{\mu^k}, \varepsilon^{\mu^{k+1}}\right), \text{ such that } \inf_{z\in \underset{i\in J}{\cup} \partial B(a_i,\rho)} |v(z)| \geq \xi_0\right\}.$$

We claim that

$$\text{meas}\, \mathcal{B} \leq \ell\big(\mathcal{H}^1(T(\xi_0))\big) \leq \ell C_0 \varepsilon |\log\varepsilon|^5. \tag{B.31}$$

Indeed the set $T(\xi_0)$ consists in an union of smooth curves, which do not intersect. We keep only the maximal curves, in $T(\xi_0)$, *i.e.* if one curve encloses another one, we keep only the exterior one. Let T^k be the collection of maximal curves in $T(\xi_0)$ and let D^k be the domain enclosed by T^k. We have

$$|v(z)| \geq \xi_0, \quad \forall z \in \mathbb{R}^2 \setminus \bigcup_k D^k.$$

For $i \in J$, let (α_i^k, β_i^k) be the smallest interval such that

$$D^k \subset B(a_i, \beta_i^k) \setminus B(a_i, \alpha_i^k)$$

so that

$$\mathcal{B} \subset \bigcup_{i \in J} B(a_i^k, b^k).$$

Since

$$b_i^k - a_i^k \leq \mathcal{H}^1(T^k)$$

the conclusion (B.31) then follows immediately.

Set $\mathcal{A} = \left(\varepsilon^{\mu^k}, \varepsilon^{\mu^{k+\frac{1}{2}}}\right) \setminus \mathcal{B}$. We have by Fubini's theorem

$$\text{(B.32)} \qquad \sum_{i \in J} \int_{\mathcal{A}} \left[\int_{\partial B(a_i,\rho)} e_\varepsilon(v) \right] d\rho \leq E_\varepsilon(v) \leq C_0(|\log \varepsilon|).$$

Assume now by contradiction, that for any $\rho \in \mathcal{A}$

$$\text{(B.33)} \qquad \rho \sum_{i \in J} \int_{\partial B(a_i,\rho)} e_\varepsilon(v) \geq \frac{1}{\mu^k - \mu^{k+\frac{1}{2}}} 4C_0.$$

Integrating this inequality, we obtain

$$\text{(B.34)} \qquad \int_{\mathcal{A}} d\rho \left(\sum_{i \in J} \int_{\partial B(a_i,\rho)} e_\varepsilon(v) \right) \geq \left(\int_{\mathcal{A}} \frac{d\rho}{\rho} \right) \frac{4C_0}{\mu^k - \mu^{k+\frac{1}{2}}}.$$

We have

(B.35)

$$\int_{\mathcal{A}} \frac{d\rho}{\rho} \geq \int_{2\varepsilon^{\mu^k} - \text{meas}\, B}^{\varepsilon^{\mu^{k+\frac{1}{2}}}} \frac{d\rho}{\rho} = \left(\mu^k - \mu^{k+\frac{1}{2}}\right) |\log \varepsilon| - \log\left(2 - \frac{\text{meas}\, B}{\varepsilon^{\mu^k}}\right)$$
$$> \frac{1}{2}\left(\mu^k - \mu^{k+\frac{1}{2}}\right) |\log \varepsilon|, \text{ if } \varepsilon \text{ is sufficiently small}$$

where, for the last inequality, we have used (B.31). Hence going back to (B.34) we are led to

$$\int_{\mathcal{A}} d\rho\Big(\sum_{i\in J}\int_{\partial B(a_i,\rho)} e_\varepsilon(v)\Big) \geq 2C_0(|\log\varepsilon|)$$

which contradicts (B.32). Hence (B.33) cannot hold for any $\rho \in \mathcal{A}$ and the existence of ρ_0 is established.

An important consequence of Lemma B.8 is the following

PROPOSITION B.1. – *Assume that*

(B.36) $$0 < \mu < \frac{1}{5}.$$

Then if ε is sufficiently small

(B.37) $$\sum_{i\in J} |d_i| \leq 2.$$

There are only two possibilities, setting $d_i = \deg(a_i)$

i) $\sum_{i\in J} |d_i| = 0$, *and all vortices have degree zero.*

ii) $\sum_{i\in J} |d_i| = 2$, *and v_ε has two vortices, say a_1 and a_2, with degrees $+1$ and -1 respectively. All other vortices have degree zero.*

Proof. – Arguing as in the proof of Lemma II.2, we may show using (B.29), (B.30) and Theorem IX.3 of [BBH], that

(B.38) $$\int_{B(a_i,\rho_0)} e_\varepsilon(v) \geq \pi|d_i|\log\frac{\rho_0}{\varepsilon} - C$$

where C is some constant. Hence, since $\rho_0 \geq 2\varepsilon^{\mu^k}$ we obtain

$$\begin{aligned}\sum_{i\in J}\int_{B(a_i,\rho)} e_\varepsilon(v) &\geq \pi\Big(\sum_{i\in J}|d_i|\Big)\log\frac{\rho_0}{\varepsilon} - C\\ &\geq \pi(1-\mu)\Big(\sum_{i\in J}|d_i|\Big)|\log\varepsilon| - C.\end{aligned}$$

On the other hand, by assumption (13) we have

$$\int_{\mathbb{R}^2} e_\varepsilon(v) \leq 2\pi|\log\varepsilon| + \Lambda$$

so that if ε is sufficiently small

$$(1-\mu)\sum_{i\in J}|d_i| < \frac{5}{2},$$

which implies, assuming (B.36), that

$$\sum_{i\in J}|d_i| \leq 2.$$

For the second assertion of the proposition it suffices to observe, that

$$\sum_{i\in J} d_i = 0, \quad . \tag{B.39}$$

otherwise the energy of v would be infinite. The conclusion then follows easily, since $d_i \in \mathbb{Z}$.

In what follows, we will assume that

$$\sum_{i\in J}|d_i| = 2, \tag{B.40}$$

and assume that

$$|a_1 - a_2| \geq \frac{1}{16} \tag{B.41}$$

then we have

LEMMA B.9. – *Assume (B.40) and (B.41) hold. Then, for $\rho \in \left(\rho_0, \varepsilon^{\mu^{k+1}}\right)$ we have*

$$\int_{\Omega_\rho} e_\varepsilon(v) \geq 2\pi \log\frac{1}{\rho} - C \tag{B.42}$$

where C is some constant.

Proof. – The proof is similar to the proof of (II.14). Therefore, we omit it.

Next, set for $i \in J$,

$$\mathcal{C}_i = B\left(a_i, \varepsilon^{\mu^{k+1}}\right) \setminus B(a_i, \rho_0)$$

$$\text{and} \qquad \mathcal{C} = \bigcup_{i\in J} \mathcal{C}_i.$$

We have

LEMMA B.10. – *Assume that (B.40) and (B.41) hold. Then we have, if* $0 < \varepsilon < \frac{1}{2}$

$$\int_C e_\varepsilon(v) \leq 2\pi \log \frac{\varepsilon^{2\mu^{k+1}}}{\rho_0} - C$$

if v *verifies (13).*

Proof. – It suffices to combine (13), (B.38) and (B.42) for $\rho = \varepsilon^{2\mu^{k+1}}$.

Finally, we are going to choose a new radius $\rho_1 \in (\rho_0, \varepsilon^{2\mu^{k+1}})$, having suitable properties for Proposition 2.

LEMMA B.11. – *If* ε *is sufficiently small, then there exists a radius* ρ_1 *in* $(\rho_0, \varepsilon^{2\mu^{k+1}})$, *such that*

$$\text{(B.43)} \quad \rho_1 \sum_{i\in J} \int_{\partial B(a_i,\rho_1)} e_\varepsilon(v) \leq \left(\int_C e_\varepsilon(v) \right) \left[\log \frac{\rho_0}{\varepsilon^{\mu^{k+1}}} + \varepsilon^{1-\mu^k} \right]^{-1}$$

and

$$|v(x)| \geq 1 - \frac{2}{|\log \varepsilon|^2} \quad \textit{on} \quad \underset{i\in J}{\cup} \partial B(a_i, \rho_1).$$

Proof. – The proof is similar to the proof of Lemma B.8. Therefore, we omit it.

We are now in position to complete the proof of Proposition 2.

Proof of Proposition 2 completed. – Choose

$$\mu = \frac{1}{10}, \quad \overline{\mu} = 2\mu^{k+1}, \quad \tilde{\mu} = \mu^{N+1}$$

and

$$\rho = \rho_1 \ \text{and} \ \overset{\ell}{\underset{i=1}{\cup}} \{a_i\} = \underset{i\in J}{\cup} \{a_i\}.$$

Clearly, in view of our previous results, (14) to (18) are satisfied, with these choices, as well as (19) by Proposition B.1. It remains to establish (20).

Assuming $|a_1 - a_2| \geq \frac{1}{16}$, we may use Lemma B.10. Combining it with (B.43), we deduce that

$$\text{(B.44)} \qquad \rho_1 \sum_{i\in J} \int_{\partial B(a_i,\rho_1)} e_\varepsilon(v) \leq 2\pi \left(1 + C |\log \varepsilon|^{-1}\right)$$

where C is some absolute constant. Arguing as in the proof of Lemma II.2, we introduce on $\mathbb{R}^2 \setminus \underset{i\in J}{\cup} B(a_i, \rho_0)$ the map

$$\overline{v} = \frac{v}{|v|}$$

and notice that

$$\rho_1 \int_{\partial B(a_i,\rho_1)} e_\varepsilon(v) \geq \frac{\rho_1}{2} \int_{\partial B(a_i,\rho_1)} |v|^2\, |\nabla \overline{v}|^2$$
$$\geq \frac{1}{2}\ \rho_1 \int_{\partial B(a_i,\rho)} |\nabla \overline{v}|^2 + K_1(\varepsilon)$$

where $K_1(\varepsilon) \to 0$ as $\varepsilon \to 0$ (here we use the fact that $|v| \geq 1 - \frac{2}{|\log \varepsilon|^2}$ on $\partial B(a_i, \rho_1)$). On the other hand, since $\overline{v}$ is S^1−valued,

$$\rho_1 \int_{\partial B(a_i,\rho_1)} |\nabla \overline{v}|^2 \geq 2\pi\, |d_i|^2 .$$

Hence

$$\sum_{i=1}^{2} \rho_1 \int_{\partial B(a_1,\rho_1)} e_\varepsilon(v) \geq 2\pi(1 + K_1(\varepsilon)) . \tag{B.45}$$

Going back to (B.44), we deduce from (B. 45) that

$$\sum_{i=3}^{\ell} \rho_1 \int_{\partial B(a_1,\rho_i)} e_\varepsilon(v) \leq \pi K_2(\varepsilon) \tag{B.46}$$

where $K_2(\varepsilon) \to 0$ as $\varepsilon \to 0$. This establishes (20) for $i \geq 3$. The estimate for $i = 1, 2$ can be obtained similarly.

This completes the proof of Proposition 2.

Finally, we complete this Appendix by a result of independent interest, which will be useful in Section VI and Appendix C.

PROPOSITION B.2. – *Assume f and v verify (8), (9) and (10). Then, there exists a constant C depending only on C_0 and β such that*

$$|v(z)| \leq C, \qquad \forall z \in \mathbb{R}^2 .$$

Proof. – The proof relies on a combination of Lemma B.2 and Lemma B.3.

APPENDIX C

PROPERTIES OF THE FUNCTIONAL L

Proof of Lemma 2. – One may verify that G_h is weakly sequentially continuous on V. Hence, its infimum is achieved. It is then easy to deduce the Euler equation (25). For (27) it suffices to write

$$G_h(v_h) \le G_h(v) = e_\varepsilon(v).$$

Finally for (26), we have

$$L(v) - L(v_h) = \frac{1}{2}\int_{\mathbb{R}^2} \big(iv_{x_1}, v - v_h\big) + \frac{1}{2}\int_{\mathbb{R}^2} \big(i(v - v_n)_{x_1}, v_h - 1\big).$$

We have, by Cauchy-Schwarz inequality

$$\begin{aligned}\left|\int_{\mathbb{R}^2} (iv_{x_1}, v - v_h)\right| &\le \left(\int_{\mathbb{R}^2} |\nabla v|^2\right)^{1/2}\left(\int_{\mathbb{R}^2} |v - v_h|^2\right)^{1/2}\\ &\le 2h\, G_h(v) = h\, E_\varepsilon(v).\end{aligned}$$

Similarly, integrating by parts, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^2} \big(i(v - v_h), v_h - 1)\big)\right| &= \left|\int \big(v - v_h, iv_{h_{x_1}}\big)\right|\\ &\le \left(\int_{\mathbb{R}^2} |\nabla v_h|^2\right)^{1/2}\left(\int_{\mathbb{R}} |v - v_h|^2\right)^{1/2}\\ &\le 2h E_\varepsilon(v).\end{aligned}$$

Since $h = \varepsilon^{1/4}$, combining the above inequalities, we obtain (26).

Proof of Lemma 3. – All the properties described in Lemma 3 are direct consequence of Proposition 2, except (28). Therefore we only have to prove (28). We divide the proof in several steps.

Step 1. – Let $\{a_1, \dots, a_\ell\}$ and ρ be as in Proposition 2. There exists a radius $R_1 > 1$ such that the following properties hold.

$$\bigcup_{i=1}^{\ell} B(a_i, \rho) \subset B(R_1) \tag{C.1}$$

$$\int_{\mathbb{R}^2 \setminus B(R_1)} e_\varepsilon(v_h) + |v_h - 1|^2 \le \varepsilon^2 \tag{C.2}$$

$$\text{(C.3)} \qquad R_1 \int_{\partial B(R_1)} e_\varepsilon(v_h) + |v_h - 1|^2 \le \varepsilon^2 .$$

In particular we have

$$\text{(C.4)} \qquad \left| L(v_h) - \frac{1}{2} \int_{B(R_1)} \left(iv_{h_{x_1}}, v_h - 1\right) \right| \le \frac{1}{2} \varepsilon^2 .$$

Proof. – Since

$$\int_{\mathbb{R}^2} e_\varepsilon(v_h) + |v_h - 1|^2 < +\infty$$

we have

$$\int_{\mathbb{R}^2 \setminus B(R)} e_\varepsilon(v_h) + |v_h - 1|^2 \longrightarrow 0 \text{ as } \varepsilon \to 0 .$$

Hence, there exists some $R_0 > 0$ such that for $R > R_0$

$$\bigcup_{i=1}^{\ell} B(a_i, \rho) \subset B(R)$$

and

$$\int_{\mathbb{R}^2 \setminus B(R)} e_\varepsilon(v_h) + |v_h - 1|^2 \le \varepsilon^2 .$$

Next, we claim that there exists some $R_1 > R_0$ such that (C.3) holds. Indeed, otherwise we would have by Fubini's theorem

$$\int_{\mathbb{R}^2 \setminus B(R_0)} e_\varepsilon(v_h) + |v_h - 1|^2 \ge \varepsilon^2 \int_{R_0}^{+\infty} \frac{dr}{r} = +\infty$$

a contradiction.

Finally (C.4) can be deduced from (C.2) by Cauchy-Schwarz inequality.

Step 2. – Set

$$\eta(z) = |v_h| \text{ if } |v_h| > \frac{1}{2}$$

and $n(z) = \frac{1}{2}$ otherwise. Then, we have

$$\text{(C.5)} \qquad \left| \int_{B(R_1)} \left(iv_{h_{x_1}}, v_h - 1\right) - \left(iv_{h_{x_1}}, \frac{v_h}{|\eta|^2} - 1\right) \right| \le C \varepsilon^{\tilde{\mu}} |\log \varepsilon| .$$

Proof. – Set

$$\Omega_\rho = B(R_1) \setminus \bigcup_{i=1}^{\ell} B(a_i, \rho)$$

so that, on Ω_ρ

$$|v_h(z)| \geq \frac{1}{2} \text{ and } \eta(z) = |v_h|(z).$$

We have

$$\begin{aligned}\left| \int_{\Omega_\rho} \left(iv_{h_{x_1}}, \frac{v_h}{|\eta|^2} - 1 \right) - \left(iv_{h_{x_1}}, v_h - 1 \right) \right| &\leq \left| \int_{\Omega_\rho} \left(iv_{h_{x_1}}, \frac{|\eta|^2 - 1}{|\eta|^2} v_h \right) \right| \\ &\leq 2 \int_{\Omega} |\nabla v_h| \, |\eta^2 - 1| \\ &\leq 2 ||\nabla v_h||_{L^2(\mathbb{R}^2)} \, ||\eta^2 - 1||_{L^2(\mathbb{R}^2)} \\ &\leq C \, \varepsilon |\log \varepsilon|,\end{aligned}$$

where C is some constant. On the other hand, we have, by proposition B.2

$$\begin{aligned}\left| \sum_{i=1}^{\ell} \int_{B(a_i,\rho)} \left(iv_{h_{x_1}}, v_h - 1 \right) \right| &\leq C \sum_{i=1}^{\ell} \int_{B(a_i,\rho)} |\nabla u| \\ &\leq C \rho ||\nabla v_h||_{L^2(\mathbb{R}^2)} \leq C \, \varepsilon^{\tilde{\mu}} |\log \varepsilon|\end{aligned}$$

and a similar estimate holds for

$$\sum_{i=1}^{\ell} \int_{B(a_i,\rho)} \left(iv_{h_{x_1}}, \frac{v_h}{|\eta|^2} - 1 \right).$$

Combining the previous inequalities, we deduce the conclusion (C.5)

Step 3. – We have

$$\left| \int_{B(R_1)} \left(iv_{h_{x_1}}, \frac{v_h}{|\eta|^2} - 1 \right) - 2\pi \left(a_{2,2} - a_{1,2} \right) \right| \leq C \, \varepsilon^{\tilde{\mu}} |\log \varepsilon|^{1/2}. \tag{C.6}$$

Proof. – First note that on Ω_ρ we have $|v_h| \geq \frac{1}{2}$, and hence we may write locally

$$v_h = |\eta| \exp i\varphi$$

where φ is some real function. It then follows that

$$\left(iv_{h_{x_i}}, \frac{v_h}{|\eta|^2} \right) = -\varphi_{x_i} \text{ for } i = 1, 2$$

so that

$$\frac{\partial}{\partial x_1}\Big(iv_{h_{x_2}}, \frac{v_h}{|\eta|^2}\Big) - \frac{\partial}{\partial x_2}\Big(iv_{h_{x_1}}, \frac{v_h}{|\eta|^2}\Big) = 0 \text{ on } \Omega_\rho .$$

We multiply this relation by the function $x_2 : z = x_1 + ix_2 \to x_2$ and integrate by parts on Ω_ρ. This yields

$$\text{(C.7)} \qquad \int_{\Omega_\rho} \Big(iv_{h_{x_1}}, \frac{v_h}{|\eta|^2}\Big) = \sum_{i=1}^{\ell} \int_{\partial B(a_i,\rho)} \Big(iv_{h_\tau}, \frac{v_h}{|\eta|^2}\Big) x_2 \, d\sigma + \int_{\partial B(R_1)} \Big(iv_{h_\tau}, \frac{v_h}{|\eta|^2}\Big) x_2 \, d\sigma .$$

Here τ denote the unit tangent vector to $\partial B(a_i, \rho)$ oriented counter-clock wise. We notice that

$$\text{(C.8)} \qquad \int_{\partial B(a_i,\rho)} \Big(iv_{h_\tau}, \frac{v_h}{|\eta|^2}\Big) d\sigma = -2\pi \deg(a_i)$$

and that

$$\text{(C.9)} \quad \left| \int_{\partial B(a_i,\rho)} \Big(iv_{h_\tau}, \frac{v_h}{|\eta|^2}\Big)(x_2 - a_{i,2}) \, d\sigma \right| \leq 2\rho \int_{\partial B(a_i,\rho)} |\nabla v_h| \, d\sigma$$
$$\text{(by Cauchy-Schwarz)} \quad \leq C\rho \leq C\,\varepsilon^{\tilde{\mu}}$$

where we have made use of (20). We deduce from (C.8) and (C.9) that

$$\text{(C.10)} \quad \left| \sum_{i=1}^{\ell} \int_{\partial B(a_i,\rho)} \Big(iv_{h_\tau}, \frac{v_h}{|\eta|^2}\Big) x_2 \, d\sigma + 2\pi \sum_{i=1}^{\ell} (\deg a_i)\, a_{i,2} \right| \leq C\,\varepsilon^{\tilde{\mu}} .$$

On the other hand, we have

$$\text{(C.11)} \qquad \left| \sum_{i=1}^{\ell} \int_{B(a_i,\rho)} \Big(iv_{h_{x_1}}, \frac{v_h}{|\eta|^2}\Big) \right| \leq C\rho \leq C\,\varepsilon^{\tilde{\mu}} \,|\log\varepsilon|^{1/2}$$

and

$$\text{(C.12)} \qquad \int_{B(R_1)} (iv_{hx_1}, 1) = \int_{\partial B(R_1)} (iv_{h_\tau}, 1)\, x_2 \, d\ell .$$

Combining (C.7), (C.10), (C.11) and (C.12) we are led to

$$\text{(C.13)} \quad \left| \int_{B(R_1)} \Big(iv_{h_{x_1}}, \frac{v_h}{|\eta|^2} - 1\Big) - 2\pi\big(a_{2,2} - a_{1,2}\big) \right| \leq C\,\varepsilon^{\tilde{\mu}} \,|\log\varepsilon|^{1/2} + \left| \int_{\partial B(R_1)} \Big(iv_{h_\tau}, \frac{v_h}{|\eta|^2} - 1\Big) x_2 \, d\sigma \right| .$$

To complete the proof, it remains to estimate the integral on the r.h.s of (C.13). First note that

$$\left|\frac{v_h}{\eta^2} - 1\right| = \frac{1}{|\eta^2|}|(v_h - \eta^2)| \leq 4\,|v_h - 1| + |\eta^2 - 1|$$

so that

$$\text{(C.14)} \quad \int_{\partial B(R_1)} \left|\left(\frac{v_h}{|\eta|^2} - 1\right)\right|^2 \leq 8 \int_{\partial B(R_1)} |v_h - 1|^2 + |\eta^2 - 1|^2 \leq \frac{16}{R_1}\varepsilon^2$$

where we have used estimate (C.3). Hence we have

$$\text{(C.15)} \quad \left|\int_{\partial B(R_1)} \left(iv_{h_\tau}, \frac{v_h}{|\eta|^2} - 1\right) x_2\right| d\sigma \leq R_1\,||\nabla v_h||_{L^2(\partial B(R_1))} \left|\left|\frac{v_h}{|\eta|^2} - 1\right|\right|_{L^2(\partial B(R_1))} \leq C\,\varepsilon^2$$

where we have used (C.3) and (C.14). Going back to (C.13) we derive (C.6).

Step 4. *Proof of Lemma* 3 *completed.* – Combining (C.5) and (C.6), we deduce easily the desired estimate.

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