



Traversable asymptotically flat wormholes in Rastall gravity

Journal:	<i>Canadian Journal of Physics</i>
Manuscript ID	cjp-2017-0040.R1
Manuscript Type:	Article
Date Submitted by the Author:	30-Apr-2017
Complete List of Authors:	Moradpour, H.; Research Institute for Astronomy and Astrophysics of Maragha, Sadeghnezhad, Naser; Research Institute for Astronomy and Astrophysics of Maragha (RIAAM), P.O. Box 55134-441, Maragha, Iran Hendi, Seyed; Shiraz University, Physics
Keyword:	Wormholes, curvature-matter coupling theory, traversability, Modified gravity, asymptotically flat condition
Is the invited manuscript for consideration in a Special Issue? :	N/A

 SCHOLARONE™
 Manuscripts

Only

Traversable asymptotically flat wormholes in Rastall gravity

H. Moradpour^{1*}, N. Sadeghnezhad^{1†}, S. H. Hendi^{2‡}

¹ *Research Institute for Astronomy and Astrophysics of Maragha (RIAAM), P.O. Box 55134-441, Maragha, Iran*

² *Physics Department and Biruni Observatory, Shiraz University, Shiraz 71454, Iran*

There are some gravitational theories in which the ordinary energy-momentum conservation law is not valid in the curved spacetime. Rastall gravity is one of the known theories in this regard which includes a non-minimal coupling between geometry and matter fields. Equipped with the basis of such theory, we study the properties of traversable wormholes with flat asymptotes. We investigate the possibility of exact solutions by a source with the baryonic matter state parameter. Our survey indicates that Rastall theory has considerable effects on the wormhole characteristics. In addition, we study various case studies and show that the weak energy condition may be met for some solutions. We also give a discussion regarding to traversability of such wormhole geometry with phantom sources.

I. INTRODUCTION

Wormholes as the backbone of interstellar travels [1, 2], should be traversable [3, 4]. Some primary solutions for traversable wormholes have also been derived by M. Visser [5]. It is also argued that a phantom energy may support the traversable wormholes [6–8]. Moreover, it is shown that wormholes and black holes are convertible structures and in fact, their physics are so close to each other [9–15]. These structures are also studied in modified theories of gravity [16, 17] such as the braneworld scenario [18, 19], conformal Weyl gravity [20], the $f(R)$ gravity [21, 22], and the curvature-matter coupling framework [23, 24] (for a detailed review see [25]). In addition, one may use cut-and-paste method to construct a thin-shell wormhole which needs an exotic matter that violates the null energy condition. Following this surgical technique, various thin-shell wormhole solutions have been addressed in the context of different gauge-gravity theories [26–38].

In the curvature-matter theory of gravity [39–41], while the divergence of energy-momentum tensor is not always zero, geometry and matter fields are coupled to each other in a non-minimal way. Indeed, quantum effects in curved spacetimes such as the particle production process [42–46] may motivate us to consider non-divergenceless energy-momentum sources and thus modify the general relativity theory. On the other hand, there is another modification to the Einstein's theory proposed by P. Rastall [47], which also couples the geometry to the matter fields in a non-minimal way [48–50]. As it has been argued by Rastall [47], we only test the energy-momentum conservation law in our laboratory and thus the flat space. Indeed, to generalize the condition of null covariant derivative of energy-momentum tensor from flat spacetime to the curved spacetime is the simplest assumption to get general relativity. Therefore, it is not forbidden to relax this condition which leads to a modified general relativity theory [47].

Although the Rastall field equations are more simple than those of the curvature-matter theory, it is in agreement with some observational data on the Hubble parameter and the age of the universe [51] meaning that it could be free of the entropy and age problems of the standard cosmology [52]. In addition, the Rastall theory leads to better consistency with the observational data of the matter dominated era against the Einstein field equations [53]. Observational data on the helium nucleosynthesis also supports this theory [54]. More studies on the cosmological features of the Rastall theory including its consistency with various cosmic eras can be found in [48, 55, 56]. Finally, it is worthwhile mentioning that this theory provides an appropriate platform to investigate the gravitational lensing [57, 58]. In addition, abelian-Higgs strings in a phenomenological Rastall model have been analyzed in Ref. [59]. Moreover, Rastall gravity has been investigated in the context of the Gödel-type universe with a perfect fluid matter [60] and it was shown that the geodesics of particles does not altered.

Moreover, similar to the curvature-matter theory of gravity, in the Rastall theory the divergence of energy-momentum tensor does not always vanish in the curved spacetime [47], and therefore, the energy-momentum conservation law is not always valid. In fact, the curvature-matter theory is a kind of $f(R)$ gravity, in which matter and geometry are coupled to each other in a non-minimal way [39–41], and its lagrangian differs from that of the Rastall theory [49, 50]. As we have mentioned, the wormholes structures are addressed in the curvature-matter coupling framework [23–25]. Therefore, since the Rastall theory differs from the curvature-matter coupling framework

* hn.moradpour@gmail.com

† nsadegh@riaam.ac.ir

‡ hendi@shirazu.ac.ir

[39–41, 49, 50], it is useful to investigate the structure of wormholes in this framework in order to get new aspects of wormholes, Rastall theory and in fact, the effects of considering a source (an energy-momentum tensor) with non-zero divergence on the wormholes and spacetime structures.

Here, we will introduce some traversable asymptotically flat wormholes in the Rastall framework and study their physical properties. Moreover, we are interested in studying the effects of considering an energy-momentum tensor, supporting the background, with non-zero divergence on the wormhole structures and their properties. In order to achieve such goal, we first review some properties of the Rastall theory, and then try to get the energy conditions in the Rastall theory. Besides, taking the Newtonian limit, we derive a dimensionless parameter for the Rastall theory which helps us in simplifying and classifying calculations in this theory, meanwhile, some mathematical properties of traversable asymptotically flat wormholes are also addressed. In addition, we study some physical properties of the energy-momentum tensor supporting the mentioned geometry in the Rastall theory. Our study shows that the wormhole parameters are affected by the parameters of Rastall theory.

The paper is organized as follows. In the next section, we review the Rastall theory and point out some of its mathematical and physical properties. In section (III), we address some mathematical properties of asymptotically traversable wormholes. Sections (IV) and (V) include some examples for the traversable asymptotically flat wormholes in the Rastall theory. We also study the properties of energy-momentum tensor, supporting the geometry in this theory, as well as the relation between the Rastall's and wormhole's parameters throughout the forth and fifth sections. The last section is devoted to the summary. Units of $c = \hbar = 1$ are considered in this paper.

II. A BRIEF REVIEW ON THE RASTALL THEORY

Rastall questioned the validity of the energy-momentum conservation law in the four dimensional spacetime [47]. His hypothesis ($T^{\mu\nu}_{;\mu} \neq 0$) leads to a modification to the Einstein field equations in agreement with various observational data [51–58]. Based on the Rastall's theory [47], if the spacetime is filled by a source with T^{μ}_{ν} , then

$$T^{\mu\nu}_{;\mu} = \lambda R^{;\nu}, \quad (1)$$

where R and λ are the Ricciscalar of the spacetime and Rastall parameter, respectively. This equation leads to [47–54]

$$G_{\mu\nu} + \kappa\lambda g_{\mu\nu}R = \kappa T_{\mu\nu}, \quad (2)$$

which can finally be written as

$$G_{\mu\nu} = \kappa S_{\mu\nu}, \quad (3)$$

where κ is the Rastall gravitational coupling constant, and $S_{\mu\nu}$ is the effective energy-momentum tensor defined as

$$S_{\mu\nu} = T_{\mu\nu} - \frac{\kappa\lambda T}{4\kappa\lambda - 1} g_{\mu\nu}. \quad (4)$$

Therefore, solutions for the Einstein field equations are also valid in the Rastall theory, if only we consider $S_{\mu\nu}$ as the new energy-momentum tensor, for which we have

$$\begin{aligned} S_0^0 &\equiv -\rho^e = -\frac{(3\kappa\lambda - 1)\rho + \kappa\lambda(p_r + 2p_t)}{4\kappa\lambda - 1}, \\ S_1^1 &\equiv p_r^e = \frac{(3\kappa\lambda - 1)p_r + \kappa\lambda(\rho - 2p_t)}{4\kappa\lambda - 1}, \\ S_2^2 = S_3^3 &\equiv p_t^e = \frac{(2\kappa\lambda - 1)p_t + \kappa\lambda(\rho - p_r)}{4\kappa\lambda - 1}. \end{aligned} \quad (5)$$

Here, ρ , p_r and p_t are the energy density and pressures corresponding to T^{μ}_{ν} , respectively. Besides, ρ^e , p_r^e and p_t^e are also the effective energy density and pressures corresponding to S^{μ}_{ν} , respectively. We should note here that ρ^e , p_r^e and p_t^e (effective components) differ from the energy density and pressures components of original energy-momentum source ($T_{\mu\nu}$). In fact, this effective components have some geometrical aspects and they help us in comparing energy-momentum sources satisfying Rastall field equations (2) with those satisfying the Einstein field equations (3). Moreover, as a desired result, the Einstein field equations are reobtained in the appropriate limit $\lambda \rightarrow 0$. Finally, it is worthwhile mentioning that the Einstein solutions of $R = 0$ is also valid in this theory [18, 61]. One can also use Eq. (5) to see that

$$\rho^e + p_r^e = \rho + p_r, \quad \rho^e + p_t^e = \rho + p_t, \quad (6)$$

meaning that whenever the null and weak energy conditions are satisfied by the energy-momentum tensor, the effective energy-momentum tensor will also meet these conditions. It is shown that, in Rastall's framework, if the weak energy condition is met by the energy-momentum tensor, then the second law of thermodynamics is also satisfied by the apparent horizon of the Friedmann-Lemaître-Robertson-Walker universe [62]. More studies on the thermodynamic properties of this theory can be found in [63, 64]. In addition, the dominant energy condition expresses that matter flux should be directed along the timelike and null geodesics, i.e. $\rho \geq 0$ and $\rho \geq |p_i|$ [65]. On the other hand, Raychaudhuri's equation as well as the Focusing theorem lead to the strong energy condition ($\rho^e + p_r^e + 2p_t^e \geq 0$) for the Einstein tensor and thus $S_{\mu\nu}$ [65]. Combining the strong energy condition with Eq. (5), one finds

$$\rho^e + p_r^e + 2p_t^e = \rho + p_r + 2p_t + \frac{2\kappa\lambda}{4\kappa\lambda - 1}(\rho - p_r). \quad (7)$$

Therefore, $\rho + p_r + 2p_t \geq 0$ if $\rho^e + p_r^e + 2p_t^e \geq \frac{2\kappa\lambda}{4\kappa\lambda - 1}(\rho - p_r)$. Besides, since the time-time component of the Ricci tensor (R_{00}) should meet the Newtonian limit [47, 50, 66], we get

$$\frac{\kappa}{4\kappa\lambda - 1}(3\kappa\lambda - \frac{1}{2}) = \kappa_G, \quad (8)$$

where $\kappa_G = 4\pi G$, and therefore, the Einstein coupling constant ($\kappa = \kappa_E \equiv 8\pi G$) is recovered only while $\lambda = 0$ [47, 50]. Solving this equation for λ , one reaches

$$\lambda = \frac{\kappa - 2\kappa_G}{6\kappa^2 - 8\kappa\kappa_G}. \quad (9)$$

Here, it is useful to note that Eqs. (4) and (9) indicate that the dimension of λ should be the inverse of that of κ which means $\lambda\kappa = \gamma$, where γ is a dimensionless constant, we call the Rastall dimensionless parameter. Inserting this result into (8), we obtain

$$\kappa = \frac{8\gamma - 2}{6\gamma - 1}\kappa_G. \quad (10)$$

We finally define the state parameter w and the effective state parameter w_e as

$$w = \frac{p_r}{\rho}, \quad (11)$$

and

$$w_e = \frac{p_r^e}{\rho^e}, \quad (12)$$

respectively. One can use Eqs. (5) and (11) to get

$$\begin{aligned} \rho &= \gamma(p_r^e - \rho^e) + 2\gamma p_t^e + \rho^e, \\ p_r &= \gamma(\rho^e - p_r^e) - 2\gamma p_t^e + p_r^e, \\ p_t &= \gamma(\rho^e - p_r^e) - 2\gamma p_t^e + p_t^e, \end{aligned} \quad (13)$$

and

$$w = \frac{\gamma(\rho^e - p_r^e) - 2\gamma p_t^e + p_r^e}{\gamma(p_r^e - \rho^e) + 2\gamma p_t^e + \rho^e}, \quad (14)$$

for the components of T_{ν}^{μ} and the state parameter, respectively. From now on, for the sake of simplicity, we set $8\pi G = 1$ or equivalently $\kappa_G = \frac{1}{2}$ which leads to

$$\kappa = \frac{4\gamma - 1}{6\gamma - 1}, \quad (15)$$

where we used Eq. (10) to get this result.

III. TRAVERSABLE ASYMPTOTICALLY FLAT WORMHOLES, GENERAL REMARKS

Consider the general form of traversable static spherically symmetric wormholes written as

$$ds^2 = -U(r)dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\Omega^2, \quad (16)$$

in which $b(r)$ and $U(r)$ are called the shape and redshift functions, respectively. Additionally, $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the line element on the two-dimensional sphere with radius r . The zeroth and radial components of Eq. (3) lead to

$$b'(r) = \kappa\rho^e r^2, \quad (17)$$

and

$$\frac{U'(r)}{U(r)} = \frac{\kappa p_r^e r^3 + b(r)}{r(r - b(r))}, \quad (18)$$

respectively. The last equation can be rewritten as

$$\frac{U'(r)}{U(r)} = \frac{r w_e b'(r) + b(r)}{r(r - b(r))}, \quad (19)$$

by using Eq. (12). The final equation that comes from the G_2^2 component, is

$$p_t^e = p_r^e + \frac{r}{2} \left[(p_r^e)' + (\rho^e + p_r^e) \frac{U'(r)}{2U(r)} \right], \quad (20)$$

for which we have also used Eq. (18). In the preceding formulae, the prime sign denotes the derivative with respect to r . Since we are looking for wormhole solutions, the shape function should satisfy the $b(r_0) = r_0$ condition, in which r_0 is the wormhole throat radius. Besides, in order to avoid singularities $U(r)$ should be finite and non-zero everywhere [4]. Moreover, the asymptotically flat condition implies the $(1 - \frac{b(r)}{r}) \rightarrow 1$ and $U(r) \rightarrow 1$ conditions for $r \rightarrow \infty$. The later condition leads to the $1 + z = \frac{1}{\sqrt{U(r_1)}}$ relation for the redshift of a photon which has been emitted at r_1 and is observed at infinity. One can check that, for $\alpha < 1$, $b(r) = r_0 + \beta[(\frac{r}{r_0})^\alpha - 1]$ is a solution which satisfies both the $b(r_0) = r_0$ and $(1 - \frac{b(r)}{r}) \rightarrow 1$ conditions [67, 68]. Therefore, inasmuch as obtain the $b(r) = r_0 + \beta[(\frac{r}{r_0})^\alpha - 1]$ relation is independent of the $U(r)$ function, the mentioned shape function is general and can be employed for every redshift function. Bearing Eq. (17) in mind, we obtain $\rho^e(r) = \frac{\alpha\beta}{\kappa r_0^2} (\frac{r}{r_0})^{\alpha-3}$. Considering the $\phi(r) = r - c$ hypersurface with normal $n_\alpha = \partial_\alpha \phi(r)$, simple calculations lead to $n_\alpha n^\alpha = n_r n^r = 1 - \frac{b(c)}{c}$ at $r = c$ meaning that the $r = c$ hypersurface is null whenever $c = r_0$. Therefore, inasmuch as the wormhole throat is a null hypersurface, one may expect that a radiation source ($w = \frac{1}{3}$) should at least satisfy the throat of traversable wormholes. But, due to the flaring-out condition, this expectation cannot be satisfied in the framework of general relativity [26]. Although the wormhole throat is a null hypersurface, it is not a horizon. The latter is due to this fact that, in order to obtaining wormhole, the redshift function should be finite and non-zero everywhere [4], meaning that the redshift should not be divergent at $r = r_0$. Therefore, for avoiding horizon at $r = r_0$, we should have $U(r_0) \neq 0$.

Now, using Eqs. (17) and (18), one can evaluate the effective radial pressure and density at throat as

$$p_r^e(r_0) = -\frac{1}{\kappa r_0^2} \text{ and } \rho^e(r_0) = \frac{\alpha\beta}{\kappa r_0^3}, \quad (21)$$

respectively. Finally, since $p_r^e(r) = w_e(r)\rho^e(r)$, we get the $w_e(r_0)\beta\alpha = -r_0$ condition. The flaring-out condition also tells us that the shape function should satisfy the $b'(r_0) < 1$ condition, where again the prime denotes the derivative with respect to r [4]. Therefore, the flaring-out condition leads to $\alpha\beta < r_0$. Moreover, since at the throat we have [4]

$$p_r^e(r_0) + \rho^e(r_0) = \rho^e(r_0)(1 + w_e(r_0)) = \frac{b'(r_0) - 1}{\kappa r_0^2}, \quad (22)$$

the flaring-out condition leads to $p_r^e(r_0) + \rho^e(r_0) < 0$ and $p_r^e(r_0) + \rho^e(r_0) > 0$ for $\kappa > 0$ and $\kappa < 0$, respectively. In summary, independent of the value of κ , the wormhole parameters, including α , β , r_0 , and $w_e(r)$ should meet the $w_e(r_0)\beta\alpha = -r_0$ and $\alpha\beta < r_0$ conditions. Thus, for $\alpha\beta > 0$, one can find that $w_e(r_0) < -1$. Moreover, since

we are looking for asymptotically flat solutions, we have $\alpha < 1$. Finally, we should remind here that the effective components do not represent a real fluid. Such geometry has been previously studied in the Einstein and braneworld frameworks [67–71]. In what follows, we investigate some properties of such geometry as well as its corresponding energy-momentum source in the Rastall framework.

Combining Eqs. (6) and (22) with each other, one can easily find

$$p_r(r_0) + \rho(r_0) = \rho(r_0)(1 + w(r_0)) = \frac{b'(r_0) - 1}{\kappa r_0^2}, \quad (23)$$

meaning that if γ meets either the $\gamma < \frac{1}{6}$ or $\frac{1}{4} < \gamma$ condition (or equally $\kappa > 0$), then the flaring-out condition is obtained whenever we have $\rho(r_0)(1 + w(r_0)) < 0$. In this situation, an energy-momentum source with negative energy density and $-1 < \omega$ may support wormhole. In addition, a source with $\omega < -1$ and positive energy (a phantom source) can also support this geometry. Moreover, for negative values of κ (or equally $\frac{1}{6} < \gamma < \frac{1}{4}$), one finds that the flaring-out condition leads to the $\rho(r_0)(1 + w(r_0)) > 0$ condition meaning that a source with positive energy density and state parameter which satisfies the $-1 < \omega$ condition may also support wormholes. For this case, it is also easy to obtain that a source with $\omega < -1$ may support this geometry if it meets the $\rho(r_0) < 0$ condition. We should note that although the negative energy may support wormholes [72, 73], due to their various problems, physicists mostly focus on the phantom solutions [26, 72, 73].

IV. WORMHOLES WITH CONSTANT REDSHIFT FUNCTION

Now, we consider the $U(r) = 1$ case which respects the asymptotically flat condition and also leads to $z = 0$. As we have previously mentioned, since the $b(r_0) = r_0$ and $(1 - \frac{b(r)}{r}) \rightarrow 1$ conditions are enough to obtain the $b(r) = r_0 + \beta[(\frac{r}{r_0})^\alpha - 1]$ relation, we can use this shape function in order to continue our study. From Eqs. (18) and (20), one obtains

$$p_r^e = -\frac{b(r)}{\kappa r^3} = -\frac{r_0 + \beta[(\frac{r}{r_0})^\alpha - 1]}{\kappa r^3}, \quad (24)$$

and

$$p_t^e = -\frac{p_r^e + \rho^e}{2}, \quad (25)$$

respectively, where $\rho^e(r) = \frac{\alpha\beta}{\kappa r_0^3}(\frac{r}{r_0})^{\alpha-3}$. Therefore, for the effective state parameter, we reach

$$w_e(r) = -\frac{b(r)}{r b'(r)} = -\frac{r_0 + \beta[(\frac{r}{r_0})^\alpha - 1]}{\alpha\beta(\frac{r}{r_0})^\alpha}, \quad (26)$$

which, as a check, leads to $w_e(r_0) = -\frac{r_0}{\alpha\beta}$ at the wormhole throat. By combining Eqs. (13), (14) and (15) with the above results, we find

$$\begin{aligned} \rho &= \frac{\alpha\beta(1-2\gamma)(6\gamma-1)}{(4\gamma-1)r_0^3} \left(\frac{r}{r_0}\right)^{\alpha-3}, \\ p_r &= \frac{6\gamma-1}{4\gamma-1} \left[\frac{2\alpha\beta\gamma}{r_0^3} \left(\frac{r}{r_0}\right)^{\alpha-3} + \frac{\beta-r_0-\beta(\frac{r}{r_0})^\alpha}{r^3} \right], \\ p_t &= \frac{6\gamma-1}{4\gamma-1} \left[\frac{\alpha\beta(4\gamma-1)}{2r_0^3(\frac{r}{r_0})^{3-\alpha}} - \frac{\beta-r_0-\beta(\frac{r}{r_0})^\alpha}{2r^3} \right], \end{aligned} \quad (27)$$

and

$$w(r) = \frac{1}{1-2\gamma} \left[2\gamma - \frac{1}{\alpha} + \frac{\beta-r_0}{\alpha\beta(\frac{r}{r_0})^\alpha} \right], \quad (28)$$

for the components of T_ν^μ and the state parameter, respectively.

A. The $0 \leq \alpha < 1$ case

Eq. (28) leads to

$$w(r_0) = \frac{1}{1-2\gamma} \left[2\gamma - \frac{r_0}{\alpha\beta} \right], \quad (29)$$

at the wormhole throat. Moreover, since the wormhole throat is a null hypersurface, one may equal the above state parameter with that of radiation ($\frac{1}{3}$) and get

$$\beta = \frac{3r_0}{\alpha(8\gamma-1)}. \quad (30)$$

Now, since $\alpha\beta < r_0$, simple calculations yield $\frac{1}{2} < \gamma$ meaning that κ is positive, and therefore, based on Eq. (23), the flaring-out condition is met if we have $\rho(r_0) < 0$.

As the second example, we look for solutions that satisfy the $w(r \rightarrow \infty) \rightarrow 0$ condition. Applying the $r \rightarrow \infty$ limit on Eq. (28), one can obtain

$$w = \frac{1}{1-2\gamma} \left[2\gamma - \frac{1}{\alpha} \right]. \quad (31)$$

Bearing the $w(r \rightarrow \infty) \rightarrow 0$ condition in mind, a simple calculation leads to

$$\alpha = \frac{1}{2\gamma}, \quad (32)$$

as the mutual relation between α and γ . Inserting this result into Eq. (28), one obtain

$$w(r) = \frac{2\gamma(\beta - r_0)}{(1-2\gamma)\beta\left(\frac{r}{r_0}\right)^\alpha}. \quad (33)$$

It is useful to note here that γ should meet the $\gamma > \frac{1}{2}$ condition to cover the $0 \leq \alpha < 1$ case. Besides, since $\alpha\beta < r_0$, Eq. (32) implies $\beta < 2\gamma r_0$. In Figs. (1) and (2), energy density, the pressure components and the state parameter are plotted, respectively, in the exterior of a wormhole with radius $r_0 = 1$. It is interesting to note that, unlike the pressure components, energy density is positive for these solutions. In fact, the positivity of energy density is due to the $\beta(1-2\gamma)$ term in Eq. (27) which is positive for $\gamma > \frac{1}{2}$, while $\beta < 0 < \frac{r_0}{\alpha}$. For these solutions, as it is clear from Eq. (33) and Fig. (2), we have $w(r) \rightarrow 0$ at the $r \rightarrow \infty$ limit. The weak energy condition is also violated by the plotted cases.

Now, let us consider situation in which $w(r \rightarrow \infty) \rightarrow \eta$, where η is an arbitrary constant. Therefore, from Eq. (31), we reach at

$$\alpha = \frac{2\gamma}{1-2\gamma\eta(1-2\gamma)}, \quad (34)$$

combined with Eq. (27) to find

$$\begin{aligned} \rho &= \frac{2\gamma\beta(1-2\gamma)(6\gamma-1)}{[1-2\gamma\eta(1-2\gamma)](4\gamma-1)r_0^3} \left(\frac{r}{r_0}\right)^{\frac{2\gamma}{1-2\gamma\eta(1-2\gamma)}-3}, \\ p_r &= \frac{6\gamma-1}{4\gamma-1} \left[\frac{4\gamma^2\beta}{[1-2\gamma\eta(1-2\gamma)]r_0^3} \left(\frac{r}{r_0}\right)^{\frac{2\gamma}{1-2\gamma\eta(1-2\gamma)}-3} + \frac{\beta-r_0-\beta\left(\frac{r}{r_0}\right)^{\frac{2\gamma}{1-2\gamma\eta(1-2\gamma)}}}{r^3} \right], \\ p_t &= \frac{6\gamma-1}{4\gamma-1} \left[\frac{2\gamma\beta(4\gamma-1)}{2[1-2\gamma\eta(1-2\gamma)]r_0^3\left(\frac{r}{r_0}\right)^{\frac{2\gamma}{1-2\gamma\eta(1-2\gamma)}}} - \frac{\beta-r_0-\beta\left(\frac{r}{r_0}\right)^{\frac{2\gamma}{1-2\gamma\eta(1-2\gamma)}}}{2r^3} \right]. \end{aligned} \quad (35)$$

As another example, we consider the $\beta = r_0$ case, leading to $w(r) = \eta$, and find that for $\gamma < \frac{\alpha-1}{4\alpha} < 0$ (and thus $\kappa > 0$), energy density is positive and pressure components are negative. Additionally, one can check to see that the flaring-out condition is also satisfied in this situation. The behavior of non-zero components of energy-momentum tensor have been plotted in Fig. (3) for a wormhole with $a = 0.98$. Finally, we should indicate that since γ is negative and $0 \leq \alpha < 1$, we always have $\eta < -1$ meaning that it is a phantom solution.

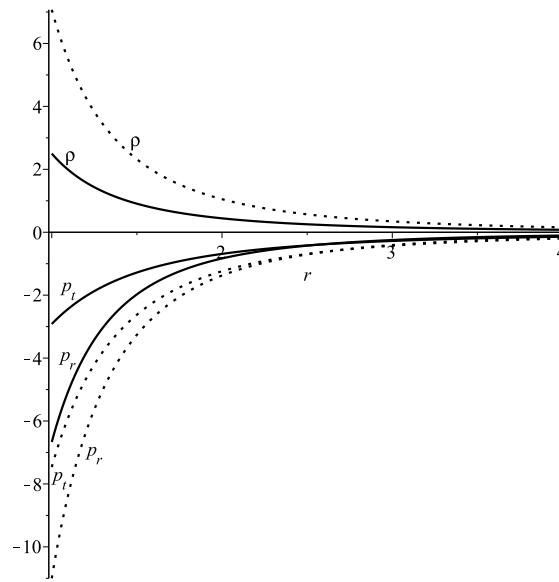


FIG. 1: The plot depicts ρ , p_r and p_t as the functions of radius. Solid lines: $\alpha = \frac{1}{2}$, $\gamma = 1$ and $\beta = -3$. Dot lines: $\alpha = \frac{1}{4}$, $\gamma = 2$ and $\beta = -6$.

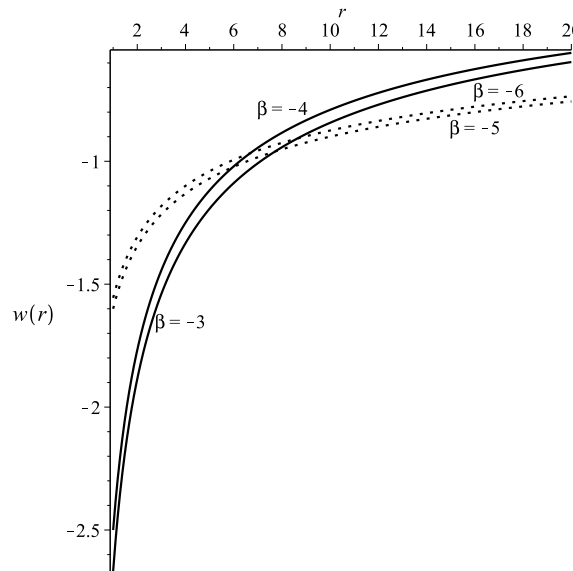


FIG. 2: The plot depicts $w(r)$ function for some values of γ . Solid lines: $\alpha = \frac{1}{2}$ and $\gamma = 1$. Dot lines: $\alpha = \frac{1}{4}$ and $\gamma = 2$.

B. The $\alpha < 0$ case

Here, we investigate wormholes with $\alpha \leq 0$. From Eq. (28), it is apparent that, in order to have a non-divergent state parameter at the $r \rightarrow \infty$ limit, we should have $\beta = r_0$. Therefore, we confine ourselves to the $\beta = r_0$ case and get

$$w = \frac{1}{1 - 2\gamma} \left[2\gamma - \frac{1}{\alpha} \right], \tag{36}$$

for the state parameter as a function of γ and α . In addition, the $\alpha\beta < r_0$ constraint leads to $\alpha < 1$ and therefore, the flaring-out condition ($\alpha\beta < r_0$) is automatically respected by wormholes of $\alpha < 0$ in Rastall's framework. In this situation, $\rho^e(r_0) = \frac{\alpha}{\kappa r_0^2}$, and we can also use Eq. (26) to reach at $w_e = -\frac{1}{\alpha}$. Therefore, for $\alpha < 0$, the effective state

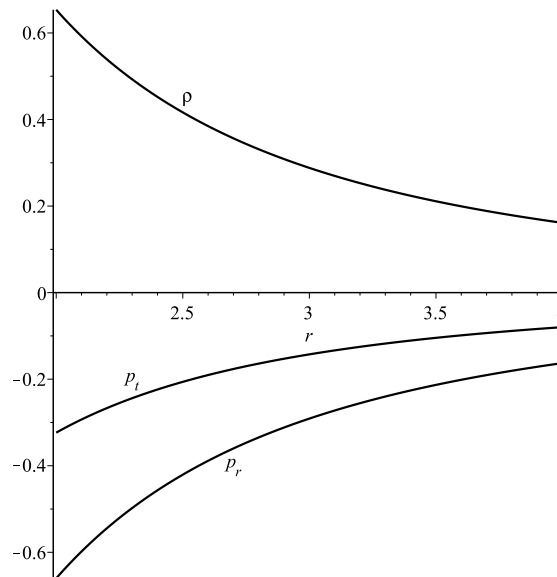


FIG. 3: The plot depicts ρ , p_r and p_t as the functions of radius for $\beta = r_0 = 2$, $\gamma = -0.5$ and $\eta = -1.01$.

parameter meets the $0 < w_e$ condition. Now, using the results obtained from Eq. (22), we find that the flaring-out condition is satisfied. It is due to this fact that, since $1 + w_e > 0$, for positive (negative) κ , we should have $\rho^e(r_0) < 0$ ($\rho^e(r_0) > 0$), a result respected by the $\rho^e(r_0) = \frac{\alpha}{\kappa r_0^2}$ expression.

As the first example, consider the $w = \frac{1}{3}$ case leading to

$$\alpha = \frac{3}{8\gamma - 1}. \quad (37)$$

It can be combined with the $\alpha < 0$ condition to get $\gamma < \frac{1}{8}$ meaning that κ is positive (Eq. 15), and thus, we have $\rho^e(r_0) < 0$ and $\rho^e(r_0) + p^e(r_0) < 0$ (see (Eq. 22)). Therefore, wormholes with $\alpha < 0$ and $\beta = r_0$ may be supported by a fluid with the same state parameter as that of the radiation source ($w = \frac{1}{3}$) in Rastall theory with $\gamma < \frac{1}{8}$. Inserting Eq. (37) into (27) and using (15), we obtain

$$\begin{aligned} p_r(r) &= w\rho(r) = \frac{(6\gamma - 1)(1 - 2\gamma)}{(4\gamma - 1)(8\gamma - 1)r_0^2} \left(\frac{r}{r_0}\right)^{\frac{3}{8\gamma - 1} - 3}, \\ p_t(r) &= 2\frac{(5\gamma - 1)}{(1 - 2\gamma)}p_r(r). \end{aligned} \quad (38)$$

Since $\gamma < \frac{1}{8}$, unlike the transverse pressure, energy density and radial pressure are negative, and therefore, ordinary energy-momentum sources, which have positive energy density, cannot support these solutions.

As the second example, we consider the $w = 0$ case for which Eq. (36) leads to $\alpha = \frac{1}{2\gamma}$, and thus $\gamma < 0$ to respect the $\alpha < 0$ condition. Additionally, from (27) we get

$$\begin{aligned} \rho(r) &= \frac{(6\gamma - 1)(1 - 2\gamma)}{(4\gamma - 1)2\gamma r_0^2} \left(\frac{r}{r_0}\right)^{\frac{1}{2\gamma} - 3}, \\ p_t(r) &= \frac{(6\gamma - 1)}{2(1 - 2\gamma)}\rho(r). \end{aligned} \quad (39)$$

It is apparent that $p_t(r) > 0$ and energy density is negative for $\gamma < 0$ meaning that a dust source (a source with $\rho > 0$ and $w = 0$) cannot support this geometry in Rastall theory.

Finally, since a fluid with $w \leq -\frac{2}{3}$ is needed to describe the current phase of the universe expansion [74], we consider the $w = -\frac{5}{6}$ case. Inserting it into (36), one reaches

$$\alpha = \frac{6}{2\gamma + 5}, \quad (40)$$

and therefore, whenever $\gamma < -\frac{5}{2}$, leading to $\kappa > 0$, the $\alpha < 0$ condition will be satisfied. Combining (40) and (27), we obtain

$$\begin{aligned}\rho(r) &= -\frac{6}{5}p_r(r) = \frac{6(6\gamma-1)(1-2\gamma)}{(4\gamma-1)(2\gamma+5)r_0^2} \left(\frac{r}{r_0}\right)^{\frac{6}{2\gamma+5}-3}, \\ p_t(r) &= \frac{(26\gamma-1)}{12(1-2\gamma)}\rho(r).\end{aligned}\quad (41)$$

As it is obvious, for $\gamma < -\frac{5}{2}$, we have $\rho(r) < 0$ whenever the pressure components are positive. In addition, from Eqs. (36) and (27), one finds

$$\begin{aligned}\rho &= \frac{\alpha(1-2\gamma)(6\gamma-1)}{(4\gamma-1)r_0^2} \left(\frac{r}{r_0}\right)^{\alpha-3}, \\ p_r &= \frac{(6\gamma-1)(2\gamma\alpha-1)}{(4\gamma-1)r_0^2} \left(\frac{r}{r_0}\right)^{\alpha-3}, \\ p_t &= \frac{6\gamma-1}{4\gamma-1} \left[\frac{\alpha(4\gamma-1)}{2r_0^2} \left(\frac{r}{r_0}\right)^{3-\alpha} + \frac{1}{2r_0^2} \left(\frac{r}{r_0}\right)^\alpha \right],\end{aligned}\quad (42)$$

where $\beta = r_0$ has also been used to obtain these results. Since α is negative, energy density is positive whenever γ either meets the $\frac{1}{6} < \gamma < \frac{1}{4}$ or $\frac{1}{2} < \gamma$ condition. In this situation, p_r and w are positive (negative) for $\frac{1}{6} < \gamma < \frac{1}{4}$ ($\frac{1}{2} < \gamma$). As we have previously seen in Eq. (23), for $\kappa < 0$ (or equally $\frac{1}{6} < \gamma < \frac{1}{4}$), the flaring-out condition is satisfied if $\rho(r_0) + p_r(r_0) > 0$. In addition, $\rho(r_0) + p_r(r_0) = \frac{(6\gamma-1)(\alpha-1)}{(4\gamma-1)r_0^2}$ which is positive for $\kappa < 0$ and negative for $\frac{1}{2} < \gamma$ (or equally $\kappa > 0$), and therefore, based on Eq. (23), the flaring-out condition is obtained. In Figs. (4) and (5), we

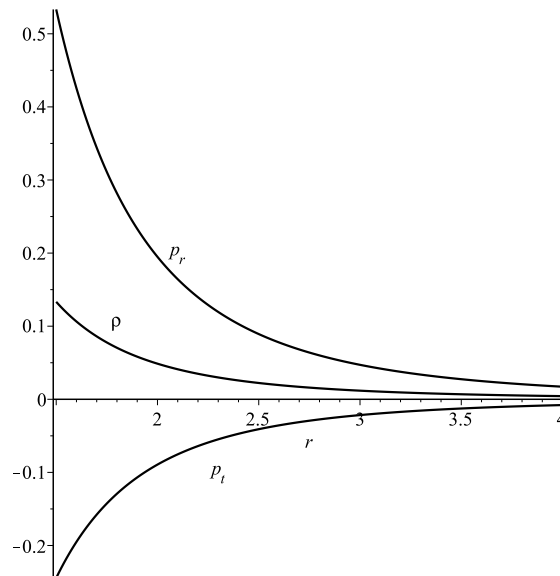


FIG. 4: The plot depicts ρ , p_r and p_t as the functions of radius for $\beta = r_0 = \frac{3}{2}$ and $\alpha = -0.5$.

have plotted the non-zero components of energy-momentum tensor for $\gamma = \frac{1}{5}$ and $\gamma = \frac{3}{2}$, respectively. Using Eq. (28), one finds that, since α is negative, $w = \frac{2\alpha-5}{3\alpha} > \frac{2}{3}$ for $\gamma = \frac{1}{5}$, and moreover, $w = \frac{1-3\alpha}{2\alpha} < -\frac{3}{2}$ (phantom solution), whenever $\gamma = \frac{3}{2}$. Finally, it is worthwhile remembering that since for $\gamma = \frac{1}{5}$ ($\gamma = \frac{3}{2}$) we have $\kappa < 0$ ($\kappa > 0$), the flaring-out condition leads to the $\rho(r_0) + p_r(r_0) > 0$ ($\rho(r_0) + p_r(r_0) < 0$) condition which is met based on the results obtained from the $\rho(r_0) + p_r(r_0) = \frac{(6\gamma-1)(\alpha-1)}{(4\gamma-1)r_0^2}$ relation.

V. WORMHOLES WITH CONSTANT EFFECTIVE STATE PARAMETER

Here, we investigate some properties of a source that supports asymptotically flat wormholes with constant effective state parameter and $b(r) = r_0 + \beta[(\frac{r}{r_0})^\alpha - 1]$, in the Rastall framework. Since the effective state parameter is constant,

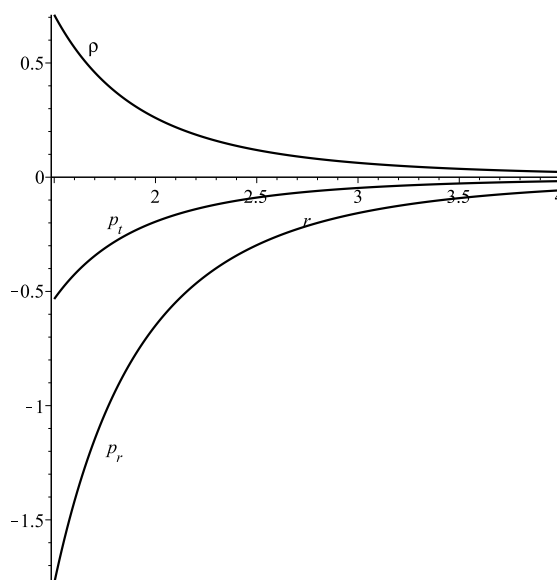


FIG. 5: The plot depicts ρ , p_r and p_t as the functions of radius for $\beta = r_0 = \frac{3}{2}$ and $\alpha = -0.5$.

$w_e(r_0)\alpha\beta = -r_0$ is reduced to $w_e\alpha\beta = -r_0$ which leads to $w_e = -\frac{r_0}{\alpha\beta}$. Using this result and Eq. (19), we get

$$U(r) = C \exp\left(\int \frac{\left(\frac{r}{r_0}\right)^\alpha(\beta - r_0) + r_0 - \beta}{r(r - r_0 + \beta(1 - \left(\frac{r}{r_0}\right)^\alpha))} dr\right), \tag{43}$$

where C is an integration constant and may be found by asymptotically flat condition. Combining Eqs. (12) and (14) with each other, one reaches

$$w(r) = \frac{w_e(1 - \gamma) + \gamma(1 - 2w_e^t)}{\gamma(w_e + 2w_e^t) + 1 - \gamma}, \tag{44}$$

where $w_e^t = \frac{p_t^e}{\rho^e}$, for the state parameter. In addition, from Eqs. (20) and (19) we obtain

$$w_e^t = w_e + \frac{r}{2} \left[\frac{w_e(\rho^e)'}{\rho^e} + (1 + w_e) \frac{b + rw_e b'}{2r(r - b)} \right]. \tag{45}$$

For example, inserting $\alpha = -1$ into Eq. (43), one reaches at

$$U(r) = C(r - r_0)^{\frac{r_0 - 1}{r_0 + \beta}} (r + \beta)^{\frac{\beta^2 - r_0}{\beta(\beta + r_0)}} r^{\frac{1 - \beta}{\beta}}. \tag{46}$$

Therefore, in order to avoid the $r = r_0$ singularity, we should have $r_0 = 1$ and $\beta > -1 = -r_0$ which lead to

$$U(r) = \left(\frac{r + \beta}{r}\right)^{\frac{\beta - 1}{\beta}}, \tag{47}$$

where we also considered the asymptotically flat condition to get the above result.

From now on, for the sake of simplicity, we only focus on the $r_0 = 1$ case yielding $w_e\alpha\beta = -1$, and $\alpha\beta < 1$. Inserting Eq. (17) and $b(r) = 1 + \beta[r^\alpha - 1]$ into Eq. (44) and using the $w_e\alpha\beta = -1$ condition, we finally get

$$w(r) = \frac{\frac{2w_e(1 - \gamma\alpha)}{(w_e + 1)A(r)} + \gamma\left(\frac{2}{(w_e + 1)A(r)} + 1\right)}{\gamma\left(\frac{2w_e\alpha}{(w_e + 1)A(r)} + 1\right) + \frac{2(1 - \gamma)}{(w_e + 1)A(r)}}, \tag{48}$$

in which $A(r) = \frac{(\beta - 1)r^\alpha + 1 - \beta}{r - 1 + \beta(1 - r^\alpha)}$ and in the $r \rightarrow \infty$ limit, $w \rightarrow \frac{w_e + \gamma(1 - w_e\alpha)}{1 - \gamma(1 - w_e\alpha)}$, for $\alpha < 1$. In addition, in the $\kappa = 1$ limit, which leads to $\gamma = 0$ and thus $\lambda = 0$ (15), the result of Einstein theory, i.e. $w(r) \rightarrow w_e$ is reobtained [67].

A. Solutions with asymptotically zero state parameter

For solutions in which state parameter vanishes asymptotically ($w(r \rightarrow \infty) \rightarrow 0$), we get

$$w_e = \frac{\gamma}{\gamma\alpha - 1}, \quad (49)$$

for the effective state parameter as a function of γ and α . Besides, since $w_e\alpha\beta = -1$, one finds

$$\beta = \frac{1 - \gamma\alpha}{\gamma\alpha}, \quad (50)$$

for the β parameter. Bearing the $\alpha\beta < 1$ condition in mind, we can use the above equation to obtain

$$\frac{1 - \gamma}{\gamma} < \alpha, \quad (51)$$

available if γ meets either $\gamma < 0$ or $\frac{1}{2} < \gamma$. In this situation, Eq. (51) gives a lower bound for α , and therefore, we should have $\frac{1-\gamma}{\gamma} < \alpha < 1$ to meet the asymptotically flat condition ($\alpha < 1$). Inserting (50) and (49) into (48), one can obtain

$$w(r) = \frac{\frac{2\gamma(\gamma\alpha-1)}{B(r)(\gamma(\alpha+1)-1)} + \gamma(\frac{2(\gamma\alpha-1)}{B(r)(\gamma(\alpha+1)-1)} + 1)}{\gamma(\frac{2\gamma\alpha}{(\gamma(1+\alpha)-1)B(r)} + 1) + \frac{2(1-\gamma)(\gamma\alpha-1)}{(\gamma(1+\alpha)-1)B(r)}}, \quad (52)$$

in which $B(r) = \frac{(1-2\gamma\alpha)r^\alpha + 2\gamma\alpha - 1}{(r-1)\gamma\alpha + (1-\gamma\alpha)(1-r^\alpha)}$.

For the $\alpha = -1$ case, using Eqs. (50), (17) and (22), one can easily see that the flaring-out condition is satisfied when $\gamma < 0$ leading to $\kappa > 0$. In fact, for $-1 < \gamma < 0$, although $-1 < w_e < 0$, we have $\rho^e < 0$ and therefore, based

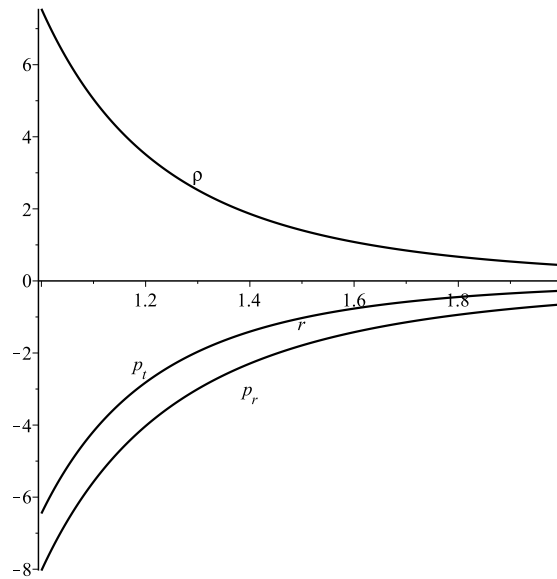


FIG. 6: The plot depicts ρ , p_r and p_t as the functions of radius for $\beta = \frac{1}{w_e} = -\frac{2}{3}$ and $\gamma = 3\alpha = -3$.

on Eq. (22), the flaring-out condition is met. In addition, for $\gamma < -1$, we have $w_e < -1$ and $\rho^e > 0$ meaning that the flaring-out condition is satisfied. Considering a Rastall theory of $\gamma = -3$, we have plotted this case for $\beta = \frac{1}{w_e} = -\frac{2}{3}$, which is a phantom solution, in Fig. (6).

B. Solutions with asymptotically radiation state parameter

In order to get solutions with asymptotically radiation state parameter, following the above recipe, we get

$$w_e = \frac{1 - 4\gamma}{3 - 4\gamma\alpha}, \quad (53)$$

for the effective state parameter. In obtaining this result we used the $(w(r \rightarrow \infty) \rightarrow \frac{1}{3})$ condition. Moreover, combining Eq. (53) with the $\omega_e \alpha \beta = -1$ condition, we reach at

$$\beta = \frac{4\gamma\alpha - 3}{\alpha(1 - 4\gamma)}. \quad (54)$$

Now, one can combine these equations with Eqs. (13) and (48) to get the non-zero components of energy-momentum tensor as well as the state parameter, respectively. The $\beta = 19$, $\gamma = \frac{1}{5}$, $\alpha = -1$ has been plotted in Fig. (7). Here,

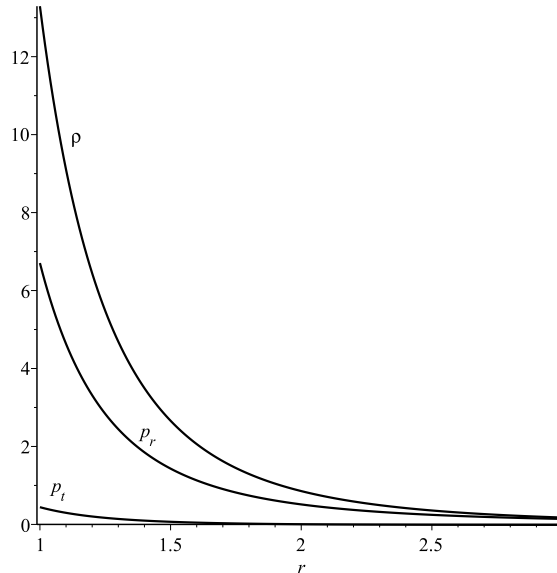


FIG. 7: The plot depicts ρ , p_r and p_t as the functions of radius for $\beta = 19$, $\gamma = \frac{1}{5}$, and $\alpha = -1$.

since $\gamma = \frac{1}{5}$, we have $\kappa = -1 < 0$ and therefore, based on Eq. (23), the flaring-out condition is met if $\rho(r_0) + p_r(r_0) > 0$, a condition obtained by this case. Therefore, for these solutions, it is theoretically possible to respect the flaring-out condition and energy conditions simultaneously.

C. The $\beta = r_0 = 1$ case

Inserting $\beta = r_0 = 1$ into (48), one can obtain

$$w = \frac{2w_e(1 - \gamma\alpha) + 2\gamma}{2w_e\alpha\gamma + 2(1 - \gamma)}, \quad (55)$$

for the state parameter. Moreover, since $\beta = 1$, the $w_e\beta\alpha = -1$ condition leads to $w_e = -\frac{1}{\alpha}$. By substituting this result into the last equation, we arrive at

$$w = \frac{2\gamma\alpha - 1}{(1 - 2\gamma)\alpha}, \quad (56)$$

for the state parameter. It is also obvious that, since $\beta = r_0 = 1$, the $\alpha\beta < 1$ condition is satisfied whenever $\alpha < 1$. Additionally, from Eq. (43), we obtain

$$U(r) = C. \quad (57)$$

Therefore, the asymptotically flat condition implies $C = 1$ and finally, one gets

$$\begin{aligned} \rho(r) &= \frac{\alpha r^{\alpha-3}(6\gamma - 1)}{4\gamma - 1}(1 - 2\gamma), \\ p_r(r) &= \frac{r^{\alpha-3}(6\gamma - 1)}{4\gamma - 1}(2\gamma\alpha - 1), \\ p_t(r) &= \frac{r^{\alpha-3}(6\gamma - 1)}{2(4\gamma - 1)}(\alpha(4\gamma - 1) + 1). \end{aligned} \quad (58)$$

Here, we should note that although these results are similar to those obtained in Sec. (IV B), there is a key difference between these results and those addressed in (IV B). While in Sec. (IV B), we have $\beta = r_0$ where r_0 is an arbitrary quantity, here, $\beta = r_0$ and r_0 must be equal to 1.

As the first example, consider the $w = 0$ case leading to $\alpha = \frac{1}{2\gamma}$, $p_r = 0$, and finally $p_t(r) = \frac{6\gamma-1}{2(1-2\gamma)}\rho$, where $\rho(r) = \frac{(6\gamma-1)(1-2\gamma)}{2\gamma(4\gamma-1)}r^{\frac{1-6\gamma}{2\gamma}}$. For these solutions, since the asymptotically flat condition implies $\alpha < 1$, the Rastall dimensionless parameter should meet the $\frac{1}{2} < \gamma$ condition meaning that the energy density is negative. Therefore, we do not focus on this case further.

As the second example, we consider the $w = \frac{1}{3}$ case. Simple calculations yield $\alpha = \frac{3}{8\gamma-1}$, $\rho(r) = \frac{p_r(r)}{3} = \frac{3(6\gamma-1)(1-2\gamma)}{(4\gamma-1)(8\gamma-1)}r^{\frac{6(1-4\gamma)}{8\gamma-1}}$ and $p_t(r) = \frac{2(6\gamma-1)(5\gamma-1)}{(4\gamma-1)(8\gamma-1)}r^{\frac{6(1-4\gamma)}{8\gamma-1}}$. For these solutions, energy density and radial pressure are positive whenever $\frac{1}{4} < \gamma < \frac{1}{2}$. But, for these values of γ , we have $\kappa > 0$ and therefore, based on Eq. (23), the flaring-out condition is not satisfied.

Using Eq. (56), one can easily find that phantom solutions ($w < -1$) may be obtained whenever one of the below conditions is met:

1. $\gamma > \frac{1}{2}$ and $\alpha < 0$.
2. $\gamma < \frac{1}{2}$ and $0 < \alpha < 1$.

For the first case, we have $\kappa > 0$ and energy density is positive. Moreover, from Eq. (23), it is far from apparent that the flaring-out condition is also satisfied. In Fig. (8), non-zero components of energy-momentum tensor have

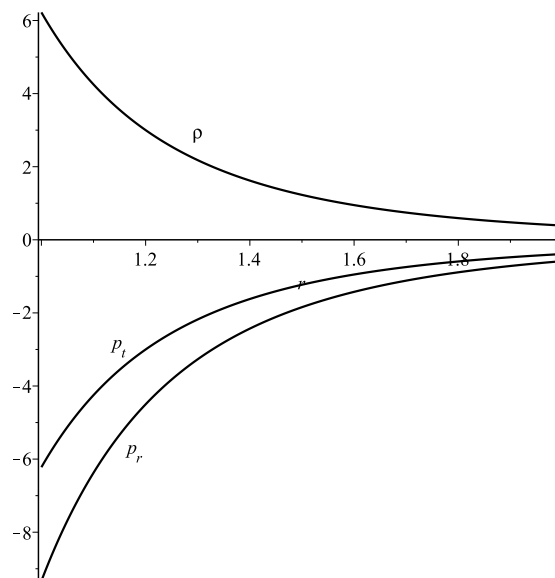


FIG. 8: Here, $\alpha = -1$, $\gamma = \frac{5}{2}$, and thus $w = -\frac{3}{2}$.

been plotted for $\alpha = -1$ and $\gamma = \frac{5}{2}$. It is obvious that the weak energy condition is violated by this phantom solution.

On the other hand, positive energy density is obtainable in the second case if γ either meets $\gamma < \frac{1}{6}$ or $\frac{1}{4} < \gamma < \frac{1}{2}$ again leading to $\kappa > 0$ and $p_r(r) < 0$. The case of $\gamma = \alpha = \frac{1}{3}$ has been depicted in Fig. (9). As it is apparent, although the transverse pressure is positive for this solution, since $\rho + p_r < 0$, the weak energy condition is not met by this solution.

VI. CONCLUSION

After referring to the Rastall theory, we defined the Rastall dimensionless parameter (γ) helping us in simplifying the calculations. In fact, regarding the Newtonian limit, one can find a relation between Rastall gravitational coupling constant (κ) and the Rastall parameter (λ) on one hand, and the Newtonian gravitational coupling constant (κ_G) on the other hand. Indeed, κ and λ are unknowns in this theory and they are only constrained by the Newtonian limit.

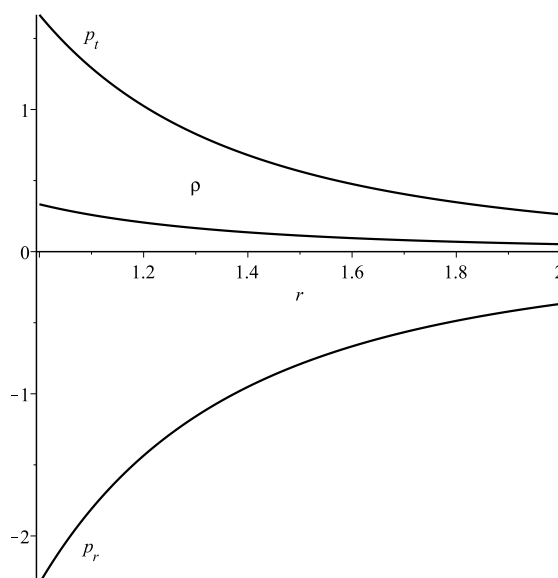


FIG. 9: Here, $\gamma = \alpha = \frac{1}{3}$.

Therefore, by finding a suitable value for γ and using the results of Newtonian limit, one can obtain both κ and λ parameters. It is also obvious that $\lambda = 0$ case leads to $\gamma = 0$ and thus the Einstein field equations are recovered.

Thereinafter, we considered a general form for the shape function of traversable asymptotically flat wormholes and studied some cases. Our results indicate that phantom solutions can be supported by this theory. Moreover, we found out that, depending on the value of γ and thus κ , traversable wormholes may meet both the flaring-out condition and energy conditions in the Rastall theory. Therefore, our study shows that a non-minimal coupling between curvature and matter fields may theoretically support traversable wormholes satisfying energy conditions. In addition, we found that the wormhole parameters (α and β) are affected by the Rastall dimensionless parameter as well as the assumed primary conditions such as the asymptotically zero- or radiation-like state parameter. Moreover, we studied wormholes of $w_e = \text{constant}$ and investigated the properties of the energy-momentum source supporting the geometry in some cases, including solutions with asymptotically dust- or radiation-like state parameter, as well as the solutions with constant state parameter while $\beta = r_0 = 1$. We also investigated the possibility of supporting such geometries by a source of $w \leq -\frac{2}{3}$.

Finally, it is important to study the effects of the Rastall hypothesis on the stability conditions of wormholes. Moreover, the study of charged wormhole structures in the Rastall framework is also interesting. It is also worthwhile to investigate the particle geodesics in the context of obtained wormhole geometry. We leave these subjects for the future works.

Acknowledgments

We are grateful to the respected referees for their valuable comments. The work of H. Moradpour has been supported financially by Research Institute for Astronomy & Astrophysics of Maragha (RIAAM)

-
- [1] J. A. Wheeler, Phys. Rev. **97**, 511 (1995).
 - [2] J. A. Wheeler, Ann. Phys. **2**, 604 (1957).
 - [3] M. S. Morris, K. S. Thorne and U. Yurtsever, Phys. Rev. Lett. **61**, 1446 (1988).
 - [4] M. S. Morris, K. S. Thorne, Am. J. Phys. **56**, 395 (1988).
 - [5] M. Visser, Phys. Rev. D **39**, 3182 (1989).
 - [6] S. V. Sushkov, Phys. Rev. D **71**, 043520 (2005).
 - [7] F. S. N. Lobo, Phys. Rev. D **71**, 084011 (2005).
 - [8] F. S. N. Lobo, Phys. Rev. D **71**, 124022 (2005).
 - [9] S. A. Hayward, Int. J. Mod. Phys. D **8**, 373 (1999).

- [10] N. S. Kardashev, I. D. Novikov and A. A. Shatskiy, *Int. J. Mod. Phys. D* **16**, 909 (2007).
- [11] P. K. F. Kuhfittig, *Schol. Res. Exch.* 296158 (2008).
- [12] S. V. Sushkov and O. B. Zaslavskii, *Phys. Rev. D* **79**, 067502 (2009).
- [13] K. K. Nandi, Y. Z. Zhang and R. G. Cai, [arXiv:gr-qc/0409085].
- [14] K. K. Nandi, Y. Z. Zhang, R. G. Cai, A. Panchenko, *Phys. Rev. D* **79**, 024011 (2009).
- [15] M. A. Anou, *JCAP*. **07**, 037 (2015).
- [16] T. Harko, F. S. N. Lobo, M. K. Mak and S. V. Sushkov, *Phys. Rev. D* **87**, 067504 (2013).
- [17] R. A. El-Nabulsi, *Can. Jour. Phys.* DOI: 10.1139/cjp-2017-0109 (2017).
- [18] K. A. Bronnikov and Sung-Won Kim, *Phys. Rev. D* **67**, 064027 (2003).
- [19] F. S. N. Lobo, *Phys. Rev. D* **75**, 064027 (2007).
- [20] F. S. N. Lobo, *Class. Quant. Grav.* **25**, 175006 (2008).
- [21] F. S. N. Lobo and M. A. Oliveira, *Phys. Rev. D* **80**, 104012 (2009).
- [22] S. N. Sajadi and N. Riazi, *Prog. Theor. Phys.* **126**, 753 (2011).
- [23] N. M. Garcia and F. S. N. Lobo, *Phys. Rev. D* **82**, 104018 (2010).
- [24] N. M. Garcia and F. S. N. Lobo, *Class. Quant. Grav.* **28**, 085018 (2011).
- [25] F. S. N. Lobo, *AIP. Conf. Proc.* **1458**, 447 (2011).
- [26] M. Visser, *Lorentzian Wormholes*, (AIP Press, Woodbury, NY, USA 1996).
- [27] F. S. N. Lobo and P. Crawford, *Class. Quant. Grav.* **21**, 391 (2004).
- [28] M. G. Richarte and C. Simeone, *Phys. Rev. D* **76**, 087502 (2007); Erratum *Phys. Rev. D* **77**, 089903 (2008).
- [29] G. A. S. Dias and J. P. S. Lemos, *Phys. Rev. D* **82**, 084023 (2010).
- [30] S. Habib Mazharimousavi, M. Halilsoy and Z. Amirabi, *Phys. Lett. A* **375**, 3649 (2011).
- [31] P. Kanti, B. Kleihaus, and J. Kunz, *Phys. Rev. D* **85**, 044007 (2012).
- [32] M. H. Dehghani and S. H. Hendi, *Gen. Rel. Grav.* **41**, 1853 (2009).
- [33] S. H. Hendi, *J. Math. Phys.* **52**, 042502 (2011).
- [34] S. H. Hendi, *Prog. Theor. Phys.* **127**, 907 (2012).
- [35] S. H. Hendi, *Adv. High Energy Phys.* **2014**, 697863 (2014).
- [36] S. Habib Mazharimousavi, M. Halilsoy, *Eur. Phys. J. C* **75**, 271 (2015).
- [37] A. Övgün, [arXiv:1610.08118].
- [38] A. Övgün and I. Sakalli, *Theor. Math. Phys.* **190**, 120 (2017).
- [39] T. Koivisto, *Class. Quant. Grav.* **23**, 4289 (2006).
- [40] O. Bertolami, C. G. Boehmer, T. Harko and F. S. N. Lobo, *Phys. Rev. D* **75**, 104016 (2007).
- [41] T. Harko and F. S. N. Lobo, *Galaxies*, **2**, 410 (2014).
- [42] L. Parker, *Phys. Rev. D* **3**, 346 (1971); *D* **3**, 2546 (1971).
- [43] G. W. Gibbons and S. W. Hawking, *Phys. Rev. D* **15**, 2738 (1977).
- [44] N. D. Birrell, P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).
- [45] L. H. Ford, *Phys. Rev. D* **35**, 2955 (1987).
- [46] Ph. Brax, C. van de Bruck, A. Davis, arXiv:0706.1024 (2007).
- [47] P. Rastall, *Phys. Rev. D* **6**, 3357 (1972).
- [48] C. E. M. Batista, M. H. Daouda, J. C. Fabris, O. F. Piattella, D. C. Rodrigues, *Phys. Rev. D* **85**, 084008 (2012).
- [49] L. L. Smalley, *Il Nuovo Cimento B*, **80**, 42 (1984).
- [50] M. Capone, V. F. Cardone, M. L. Ruggiero, *Journal of Physics: Conference Series*, **222**, 012012 (2010).
- [51] A. S. Al-Rawaf, O. M. Taha, *Phys. Lett. B* 366, 69 (1996).
- [52] J. C. Fabris, R. Kerner, J. Tossa, *Int. J. Mod. Phys. D* 9, 111 (2000).
- [53] A. S. Al-Rawaf, O. M. Taha, *Gen. Rel. Grav.* 28, 935 (1996).
- [54] A. S. Al-Rawaf, *Int. J. Mod. Phys. D* 14, 1941 (2005).
- [55] V. Majernik, *Gen. Rel. Grav.* 35, 1007 (2003).
- [56] A. I. Arbab, *JCAP*. 05, 008 (2003).
- [57] A.-M. M. Abdel-Rahman, *Astrophys. Space Sci.* 278, 383 (2001).
- [58] A.-M. M. Abdel-Rahman, M. H. A. Hashim, *Astrophys. Space Sci.* 298, 519 (2005).
- [59] E. R. Bezerra de Mello, J. C. Fabris and B. Hartmann, [arXiv:1407.3849].
- [60] A. F. Santos and S. C. Ulhoa, [arXiv:1407.4322].
- [61] K. A. Bronnikov, J. C. Fabris, O. F. Piattella and E. C. Santos, *Gen. Relativ. Gravit.* **48**, 162 (2016).
- [62] H. Moradpour, *Phys. Lett. B* **757**, 187 (2016)[arXiv:1601.04529].
- [63] H. Moradpour, I. G. Salako, *AHEP*, 3492796 (2016).
- [64] F. F. Yuan and P. Huang, *Class. Quant. Grav.* 34, 077001(2017).
- [65] E. Poisson, *A Relativists Toolkit* (Cambridge University Press, Cambridge, 2004).
- [66] A. Sheykhi, H. Moradpour and N. Riazi, *Gen. Rel. Grav.* **45**, 1033 (2013).
- [67] F. S. N. Lobo, F. Parsaei and N. Riazi, *Phys. Rev. D* **87**, 084030 (2013).
- [68] Y. Heydarzade, N. Riazi and H. Moradpour, *Can. J. Phys* **93**, 12 (2015).
- [69] F. Parsaei and N. Riazi, *Phys. Rev. D* **91**, 024015 (2015).
- [70] D. A. Tretyakova, B. N. Latosh and S. O. Alexeyev. *Class. Quantum Grav.* **32**, 185002 (2015).
- [71] F. Rahaman, M. Kalam, M. Sarker and S. Chakraborty, *Acta Phys. Pol. B* **40**, 25 (2009).
- [72] S. Hawking, *A Brief History of Time* (Bantam Dell Publishing Group, 1988).

[73] L. H. Ford and T. A. Roman, *Scientific American* 282, 46 (2000).

[74] M. Roos, *Introduction to Cosmology* (John Wiley and Sons, UK, 2003).

For Review Only