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TREATMENT OF CERENKOV RADIATION FROM ELECTRIC AND MAGNETIC CHARGES IN DISPERSIVE AND DISSIPATIVE MEDIA *

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ABSTRACT

A rigorous treatment of the problem of Cerenkov radiation from fast moving electric and magnetic charges is presented. This is based on the rigorous solution of Maxwell's equations in a general dispersive medium possessing dielectric and magnetic properties and with, and without, dissipation. It is shown that the fields are completely determined by one scalar function. Expressions for the exact fields are obtained. From the asymptotic fields all the relevant properties of Cerenkov radiation are reproduced. In particular, it is shown that in the absence of dissipation the energy in each mode travels with the phase velocity of that mode. For a dissipative medium the electric field develops a longitudinal component and the energy propagates at an angle to the phase velocity. Application to the case of a Tachyon shows that it must emit Cerenkov radiation in vacuum. An estimate is given for the resulting linear density of emitted radiation. Finally, two suggestions are made for the experimental detection of magnetic charges and electric dipole moments of elementary particles based upon the Cerenkov radiation which they would emit in dispersive media.

1. INTRODUCTION

Cerenkov radiation is such a well-established phenomenon in physics as to constitute one of the important means of charged particle detection in high energy physics (Jelley 1958). Although it had been noticed and studied prior to the work of Cerenkov (1957) its real discovery and the establishment of its properties were only achieved through Cerenkov's classic experiments. They were carried out during a five-year period starting from 1934.

While Cerenkov's experiments were still in progress, Frank and Tamm(1937) published their classic theory explaining his findings. This theory soon gained complete verification through further experiments by Cerenkov (1937). A still more rigorous version of this theory was later published by Tamm (1939).

Later treatments of Cerenkov radiation have followed either of two paths. Fermi (1940) follows the treatment of Frank and Tamm which consists essentially in finding the field of the particle in its rest frame and then translating it in order to obtain it in the lab. The other approach is illustrated by the treatment given by Landau and Lifshitz (1960). They obtain the linear energy density radiated by the particle as a limiting case from the energy loss suffered by a charged particle in matter.

This latter method suffers, however, from the limitation of not being able to provide the field of the particle . This entails loss of relevant information concerning the polarization of the radiation, its direction of propagation , the signal velocity, etc. Whereas the former approach does not suffer from this limitation, it nevertheless needs to be generalized in two senses. First, to include cases of actual magnetization , with the magnetic permeability depending on frequency. This becomes important when we try to find the Cerenkov radiation from a ma-

gnetic monopole - a topic of great interest for elementary particle experimentalists seeking to detect magnetic monopoles. Second, to take into account not only dispersion but also dissipation. This is important for two reasons: first, there is no dispersion without dissipation ; second, the results we obtain in this case are quite unexpected and warrant consideration.

Motivated by these considerations we present here a consistent and rigorous treatment of Cerenkov radiation based on exploiting standard results of Fourier transforms. This treatment is as general as it is simple.

The arrangement of material is as follows. In sections 2 and 3 we present the basic formalism and derive the electromagnetic potentials for an electric charge. In section 4 we do the same for a magnetic charge. After treating the case for $V < c'$ in section 5, we treat Cerenkov radiation in sections 6 and 7 . In section 8 we make an application to Tachyons and in the following section, 9, we treat the case of a magnetic charge. In section 10 we give the full treatment for the case of a dissipative medium for both electric and magnetic charges. In section 11, we dedicate to the suggestion of possible ways for detecting magnetic charges and electric dipoles through their Cerenkov radiation. In section 12 we present our conclusions.

2. THE FIELD EQUATIONS

We consider electric charge distributions within a medium which possesses both dielectric and magnetic properties. It will then be characterized by a dielectric constant ϵ and a magnetic permeability μ both of which will be functions of the frequency and will satisfy the conditions:

$$\epsilon(\omega) = \epsilon(-\omega) \qquad \mu(\omega) = \mu(-\omega) \tag{2.1}$$

At this stage we are neglecting dissipation, which allows us to treat ϵ and μ as real quantities. As we mentioned in the introduction, we will treat the general case with dissipation in section 10 below.

We start with Maxwell's equations expressed in Gaussian units:

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho \qquad \vec{\nabla}\times\vec{H} - \frac{1}{c}\frac{\partial\vec{D}}{\partial t} = \frac{4\pi}{c}\vec{J}$$
$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0 \qquad (2.2).$$

In order to be able to handle the dispersion in the medium (both electric and magnetic) we must deal with the Fourier transforms of the fields. These we define as follows, illustrated for the case of the electric displacement:

$$\vec{D}(\vec{x},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\vec{D}(\vec{x},\omega)\, e^{-i\omega t}\, d\omega$$
$$\vec{D}(\vec{x},\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\vec{D}(\vec{x},t)\, e^{i\omega t}\, dt \qquad (2.3)$$

The reality of $\vec{D}(\vec{x},t)$ imposes the following condition on $\vec{D}(\vec{x},\omega)$: $\vec{D}^{*}(\vec{x},\omega) = \vec{D}(\vec{x},-\omega)$. The polarizability and magnetizability of the medium will then be expressed as follows:

$$\vec{D}(\vec{x},\omega) = \epsilon(\omega)\,\vec{E}(\vec{x},\omega)$$
$$\vec{B}(\vec{x},\omega) = \mu(\omega)\,\vec{H}(\vec{x},\omega) \qquad (2.4)$$

The Electromagnetic Potentials

We introduce the electromagnetic potentials through their usual definition:

$$\vec{B} = \vec{\nabla}\times\vec{A}$$
$$\vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\varphi \qquad (2.5)$$

Substituting their Fourier tranforms into the Fourier- tranformed Maxwell's equations, we obtain the following differential equations for $\vec{A}(\vec{x},\omega)$ and $\varphi(\vec{x},\omega)$:

$$-\left(\vec{\nabla}^2 + \frac{\omega^2}{(c^2/\mu\epsilon)}\right)\vec{A} + \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} - i\frac{\omega}{c/\mu\epsilon}\varphi\right) = \frac{4\pi}{c}\mu\vec{J}$$
$$i\frac{\omega}{c}\vec{\nabla}\cdot\vec{A} - \vec{\nabla}^2\varphi = 4\pi\frac{\rho}{\epsilon} \qquad (2.6)$$

In the usual manner we decouple these two equations making use of gauge invariance which arises from the freedom which we still have in defining the divergence of the vector potential (Hylleraas 1970). We fix the gauge by the requirement:

$$\vec{\nabla}\cdot\vec{A} - i\frac{\omega}{c/\mu\epsilon}\varphi = 0 \qquad (2.7)$$

This is the form which the Lorentz condition acquires in the present problem. To make this more apparent, we write down its Fourier transform:

$$\vec{\nabla}\cdot\int\vec{A}(\vec{x},\omega)\, e^{-i\omega t}\, d\omega + \frac{1}{c}\frac{\partial}{\partial t}\int\frac{\varphi(\vec{x},\omega)}{c'^2(\omega)/c^2}\, e^{-i\omega t}\, d\omega = 0 \qquad (2.7a)$$

where

$$c'(\omega) \equiv \frac{c}{\sqrt{\mu(\omega)\epsilon(\omega)}} \qquad (2.8)$$

is the velocity of light in the medium. The equations for the potentials then take the following form:

$$\left[\vec{\nabla}^2 + \frac{\omega^2}{c'^2(\omega)}\right]\vec{A}(\vec{x},\omega) = -\frac{4\pi}{c}\mu(\omega)\vec{J}(\vec{x},\omega)$$
$$\left[\vec{\nabla}^2 + \frac{\omega^2}{c'^2(\omega)}\right]\varphi(\vec{x},\omega) = -4\pi\frac{\rho(\vec{x},\omega)}{\epsilon(\omega)} \qquad (2.9)$$

The potentials satisfy then the inhomogenous Helmholtz equation.

The Fields of a Moving Charged Particle

The case which interests us here is that of the electromagnetic field of a particle of charge e which moves in the medium with constant velocity v. We take the z- axis to point along the direction of motion of the particle.

The charge and current densities for the particle will then be given by:

$$\rho(\vec{x},t) = \frac{e}{v}\,\delta(x)\,\delta(y)\,\delta\left(\frac{z}{v}-t\right)$$
$$\vec{J}(\vec{x},t) = \vec{k}\; e\,\delta(x)\,\delta(y)\,\delta\left(\frac{z}{v}-t\right) \qquad (2.10)$$

Since the charge and current densities depend on z and t through the combination ($\frac{z}{v} - t$), we expect that the same should be true for the fields and by consequence for the potentials. Furthermore, from the form of $\vec{J}(\vec{x},t)$ we expect that the vector potential will be likewise of the form:

$$\vec{A}(\vec{x},t) = \vec{k}\; A(\vec{x},t) \qquad (2.11)$$

Combining these two results we obtain:

$$\vec{\nabla}\cdot\vec{A}(\vec{x},t) = \frac{\partial A(\vec{x},t)}{\partial z} = -\frac{1}{v}\,\frac{\partial A(\vec{x},t)}{\partial t}.$$

Applied to the Fourier transform, this equation becomes:

$$\vec{\nabla}\cdot\vec{A}(\vec{x},\omega) = i\,\frac{\omega}{v}\,A(\vec{x},\omega) .$$

Putting this relation in the gauge condition, namely eq. (2.7), we obtain the following relation between the potentials:

$$\frac{1}{\mu(\omega)}\,A(\vec{x},\omega) = \beta\,\epsilon(\omega)\,\varphi(\vec{x},\omega) \qquad (2.12)$$

This relation means that the potentials as well as the fields are determined by just one scalar function. We call this function $\Psi(\vec{x},\omega)$ and we define it in the following manner:

$$\frac{1}{\mu(\omega)}\,A(\vec{x},\omega) = \beta\,\epsilon(\omega)\,\varphi(\vec{x},\omega) \equiv \frac{1}{\sqrt{4\pi}}\,\frac{e}{c}\,\Psi(\vec{x},\omega) \qquad (2.13)$$

which then has the advantage of making it dimensionless. By putting the above result into either of eqs.(2.9) and using the corresponding equation from (2.10) we obtain the differential equation satisfied by $\Psi(\vec{x},\omega)$. This has the following form:

$$\left[\vec{\nabla}^2 + \frac{\omega^2}{c'^2(\omega)}\right]\Psi(\vec{x},\omega) = -4\pi\,\delta(x)\,\delta(y)\,e^{i\frac{\omega z}{v}} \qquad (2.14)$$

The scalar dimensionless function which determines the potentials and fields is then seen to satisfy the Helmholtz equation with a line source along the z-axis which varies sinusoidally with z . The constant in this Helmholtz equation, which plays the role of a wave-number, is given by:

$$k^2 = \frac{\omega^2}{c'^2(\omega)} \qquad (2.15)$$

3. THE FORM OF THE ELECTROMAGNETIC POTENTIALS

We seek solutions to the Helmholtz equation (2.14) which will lead to radiation. Hence, we will use in solving it the Green function which possesses the boundary conditions suitable to this end. This is given by (Jackson 1975)

$$G(\vec{x},\vec{x}') = \frac{e^{\pm ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \qquad (3.1)$$

where

$$k = \frac{\omega}{c'(\omega)}$$

can acquire both positive and negative values. In the standard manner the + sign in the exponential of the Green function will lead to outgoing-wave solutions and the - sign to incoming-wave solutions.

The respective solutions for the scalar function $\Psi(\vec{x},\omega)$ are then given by:

$$\Psi^{\pm}(\vec{x},\omega) = 2\,e^{i\frac{\omega z}{v}}\int_0^{\infty} d\xi\,\cos\left(\frac{\omega\xi}{v}\right)\frac{e^{\pm i\frac{\omega}{c'(\omega)}\sqrt{\rho^2+\xi^2}}}{\sqrt{\rho^2+\xi^2}} \qquad (3.2)$$

where ρ is the cylindrical coordinate given by

$$\rho = \sqrt{x^2+y^2}\,.$$

The Scalar Function

We now take the Fourier transform of eq. (3.2) in order to arrive at the function $\psi^{\pm}(\vec{x}, t)$. After some standard manipulations we cast this in the following form:

$$\psi^{\pm}(\vec{x},t) = \frac{4}{\sqrt{2\pi}}\int_0^{\infty} d\omega \left[\cos\omega\left(\frac{z}{V}-t\right)\int_0^{\infty} d\xi \cos\left(\frac{\omega\xi}{V}\right)\frac{\cos\left(\frac{\omega}{c'(\omega)}\sqrt{\rho^2+\xi^2}\right)}{\sqrt{\rho^2+\xi^2}} \mp \right.$$

$$\left. \mp \sin\omega\left(\frac{z}{V}-t\right)\int_0^{\infty} d\xi \cos\left(\frac{\omega\xi}{V}\right)\sin\left(\frac{\omega}{c'(\omega)}\sqrt{\rho^2+\xi^2}\right)\Big/\sqrt{\rho^2+\xi^2}\right] \tag{3.3}$$

In arriving at this expression we have made use of eq. (2.1). The two integrals in the square brackets are cosine Fourier transforms. They are of a standard form which is evaluated in tables of such transforms. This then enables us to obtain directly and simply the spectral decomposition for this function and hence for the potentials and fields. Using such tables (Erdélyi 1954) we find that these integrals have two different expressions depending on whether the velocity of the particle is greater or smaller than the velocity of light in the medium. With the help of eq. (2.8) we find that these two cases correspond to the following two respective inequalities:

$$\epsilon(\omega)\mu(\omega) \gtrless \frac{c^2}{V^2}.$$

They are then seen to be restrictions on the domain of integration over the frequency. The final results have the following form:

Case 1: $V < c'(\omega)$

$$\psi^{\pm}(\vec{x},t) = \frac{4}{\sqrt{2\pi}}\int_{k^2(\omega)<\frac{1}{\beta^2}}^{\infty} d\omega \cos\omega\left(\frac{z}{V}-t\right)K_0\left(\frac{\omega\rho}{\gamma' V}\right), \rho>0 \tag{3.4}$$

where

$$\begin{aligned} k^2(\omega) &\equiv \epsilon(\omega)\mu(\omega) \\ \beta &= \frac{V}{c} \\ \gamma' &= \frac{1}{\sqrt{1-\beta'^2(\omega)}} \\ \beta'(\omega) &= \frac{V}{c'(\omega)} \end{aligned} \tag{3.5}$$

and K_0 is the modified Bessel function of the third kind of order zero.

Case 2: $V > c'(\omega)$

$$\psi^{\pm}(\vec{x},t) = \frac{\sqrt{2\pi}}{2}\left[\int_{k^2(\omega)>\frac{1}{\beta^2}}^{\infty} d\omega\, e^{\pm i\left[\omega\left(\frac{z}{V}-t\right)\pm\frac{\pi}{2}\right]} H_0^{(1)}\left(\frac{\omega\rho}{\gamma' V}\right) + c.c.\right], \rho>0 \tag{3.6}$$

where now γ' is given by:

$$\gamma' = \frac{1}{\sqrt{\beta'^2(\omega)-1}}$$

and $H_0^{(1)}$ is the first Hankel function of order zero. We have used the complex Hankel functions rather than the real Bessel ones in this expression in order to exploit their convenient asymptotic forms when calculating the Cerenkov radiation.

We note that eqs. (3.4) and (3.6) and all the relations which we derive from them below hold for . Having stated this, we will dispense with explicit reference to it in the actual formulas.

The Electromagnetic Potential

Referring to eq. (2.13) we see that the vector potential and the scalar potential will be obtained from the respective expressions for the scalar function by multiplying the integral by $\left(\frac{1}{\sqrt{2\pi}}\frac{e}{c}\mu(\omega)\right)$ and $\left(\frac{1}{\sqrt{2\pi}}\frac{e}{c}\frac{1}{\beta\epsilon(\omega)}\right)$ respectively. We give these results:

Case 1: $V < c'(\omega)$

$$\begin{aligned} A^{\pm}(\vec{x},t) &= \frac{2}{\pi}\frac{e}{c}\int_{k^2(\omega)<\frac{1}{\beta^2}}^{\infty} d\omega\, \mu(\omega)\cos\left(\frac{z}{V}-t\right)K_0\left(\frac{\omega\rho}{\gamma' V}\right) \\ \varphi^{\pm}(\vec{x},t) &= \frac{2}{\pi}\frac{e}{\beta c}\int_{k^2(\omega)<\frac{1}{\beta^2}}^{\infty} d\omega\, \frac{1}{\epsilon(\omega)}\cos\left(\frac{z}{V}-t\right)K_0\left(\frac{\omega\rho}{\gamma' V}\right) \end{aligned} \tag{3.7}$$

Case 2: $V > c'(\omega)$

$$A^{\pm}(\vec{x},t) = \frac{e}{2c}\left[\int_{K^2(\omega)>\frac{1}{\beta^2}}^{\infty} d\omega\, \mu(\omega)\, e^{\pm i\left[\omega\left(\frac{z}{v}-t\right)\pm\frac{\pi}{2}\right]} H_0^{(1)}\left(\frac{\omega\rho}{\gamma' v}\right) + c.c.\right]$$

$$\varphi^{\pm}(\vec{x},t) = \frac{e}{2\beta c}\left[\int_{K^2(\omega)>\frac{1}{\beta^2}}^{\infty} d\omega\, \frac{1}{\epsilon(\omega)}\, e^{\pm i\left[\omega\left(\frac{z}{v}-t\right)\pm\frac{\pi}{2}\right]} H_0^{(1)}\left(\frac{\omega\rho}{\gamma' v}\right) + c.c.\right] \quad (3.8)$$

It will be noticed that $\mu(\omega)$ and $\epsilon(\omega)$ occur in $A(\vec{x},t)$ and $\varphi(\vec{x},t)$ in a rather asymmetric fashion: one occuring in the numerator whereas the other occurs in the denominator. This is due to the historical accident in which the magnetic induction was defined in terms of the magnetic field. Actually it is $\vec{B}$ which plays the role similar to that of $\vec{E}$. It would then be more consistent to define $\vec{H}$ in terms of $\vec{B}$ which would then make the susceptibilities occur in the same manner in the potentials.

4. THE CASE OF A MAGNETIC CHARGE

Before proceeding with the above deiscussion we will derive the potentials for a magnetic current and charge density: $(\vec{J}'(\vec{x},t), \rho'(\vec{x},t))$. We use the prime in order to distinguish these from the corresponding electric current and charge density. We need these results in order to calculate the Cerenkov radiation for a magnetic monopole, which is of interest for experimental efforts to detect them. They also constitute a convenient starting point for deriving the fields of a magnetic dipole.

Maxwell's equations in this case have the following form:

$$\vec{\nabla}\cdot\vec{D} = 0 \qquad \vec{\nabla}\times\vec{H} - \frac{1}{c}\frac{\partial\vec{D}}{\partial t} = 0$$
$$\vec{\nabla}\cdot\vec{B} = -4\pi\rho' \qquad \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = \frac{4\pi}{c}\vec{J}' \quad (4.1)$$

We note the difference in sign between Poisson's equation and the corresponding equation for magnetic charge. Conservation of magnetic charge requires a difference in sign between this equation and that containing the magnetic current density. We have adopted here the sign convention of Recami and Mignani (Recami 1977).

As in the electric case we reduce the homogeneous equations to the status of identities via the introduction of the potentials. However, on account of eq. (2.4) the second homogeneous equation above can only be satisfied in its Fourier-transformed version. To this end we write Maxwell's equations (4.1) in their Fourier-transformed form taking eq. (2.4) into account. These are given as follows:

$$\epsilon(\omega)\vec{\nabla}\cdot\vec{E}(\vec{x},\omega) = 0 \qquad \frac{1}{\mu(\omega)}\vec{\nabla}\times\vec{B}(\vec{x},\omega) + i\frac{\omega}{c}\epsilon(\omega)\vec{E}(\vec{x},\omega) = 0$$
$$\vec{\nabla}\cdot\vec{B}(\vec{x},\omega) = -4\pi\rho'(\vec{x},\omega) \qquad \vec{\nabla}\times\vec{E}(\vec{x},\omega) - i\frac{\omega}{c}\vec{B}(\vec{x},\omega) = \frac{4\pi}{c}\vec{J}'(\vec{x},\omega) \quad (4.2)$$

The first homogeneous equation is satisfied by the standard form:

$$\vec{E}(\vec{x},t) = \vec{\nabla}\times\vec{A}'(\vec{x},t) \quad (4.3).$$

Then the second homogenous equation will be satisfied via the choice

$$\vec{B}(\vec{x},\omega) = -i\,\epsilon(\omega)\mu(\omega)\frac{\omega}{c}\vec{A}'(\vec{x},\omega) + \vec{\nabla}\varphi'(\vec{x},\omega) \quad (4.4).$$

The sign in eq. (4.3) fixes the sign of the first term on the R.H.S. in the above equation. We fix the sign of the second term by the requirement that the wave equation for each potential will have the same sign in front of the source term. This equation acquires the following form for $\vec{B}(\vec{x},t)$:

$$\vec{B}(\vec{x},t) = \frac{1}{c}\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\frac{d\omega}{\sqrt{2\pi}}\epsilon(\omega)\mu(\omega)\vec{A}'(\vec{x},\omega)e^{-i\omega t} + \vec{\nabla}\varphi'(\vec{x},t) \quad (4.5)$$

We will have more to say about this equation in what follows.

Proceeding in the standard fashion we substitute the above equations into the remaining two inhomogeneous Maxwell equations to obtain:

$$-\left(\vec{\nabla}^2 + \frac{\epsilon(\omega)\mu(\omega)}{c^2}\omega^2\right)\vec{A}'(\vec{x},\omega) + \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}'(\vec{x},\omega) - i\frac{\omega}{c}\varphi'(\vec{x},\omega)\right) = \frac{4\pi}{c}\vec{J}'(\vec{x},\omega)$$

$$-i\,\frac{\epsilon(\omega)\mu(\omega)}{c}\omega\,\vec{\nabla}\cdot\vec{A}'(\vec{x},\omega) + \vec{\nabla}^2\varphi'(\vec{x},\omega) = -4\pi\rho'(\vec{x},\omega) \qquad (4.6).$$

We now de-couple these two equations by fixing the gauge through the condition:

$$\vec{\nabla}\cdot\vec{A}'(\vec{x},\omega) - i\,\frac{\omega}{c}\,\varphi'(\vec{x},\omega) = 0 \qquad (4.7).$$

It will be seen that this is just the Fourier transform of the Lorentz condition:

$$\vec{\nabla}\cdot\vec{A}'(\vec{x},t) + \frac{1}{c}\,\frac{\partial\varphi'(\vec{x},t)}{\partial t} = 0 \quad .$$

There thus seems to be a complementarity between this case and that of the electric case. There the potentials can be defined in four-dimensional space-time through covariant equations. However, the gauge condition can be defined only via its Fourier-transformed form. However, in the present case, the gauge is fixed through the Lorentz condition, the potentials can only be introduced via their Fourier-transformed form.

Applying the gauge condition, eq. (4.7) to eq. (4.6), we reduce them to the following standard form:

$$\left[\vec{\nabla}^2 + \frac{\epsilon(\omega)\mu(\omega)}{c^2}\omega^2\right]\vec{A}'(\vec{x},\omega) = -\frac{4\pi}{c}\vec{J}'(\vec{x},\omega)$$

$$\left[\vec{\nabla}^2 + \frac{\epsilon(\omega)\mu(\omega)}{c^2}\omega^2\right]\varphi'(\vec{x},\omega) = -4\pi\rho'(\vec{x},\omega) \qquad (4.8).$$

These are the same as eqs. (2.9), except that the susceptibilities do not occur on th R.H.S. Again this is traceable to the mode in which the potentials occur in eq. (4.4). It is worth noting here that $\vec{A}'(\vec{x},\omega)$ is an axial vector field and $\varphi'(\vec{x},\omega)$ is a pseudoscalar field. Hence, they are distinct from the fields $\vec{A}(\vec{x},\omega)$ and $\varphi(\vec{x},\omega)$. But both fields propagate with the same velocity as can be seen by comparing the above equations with eq. (2.9). Furthermore, it transpires from this comparison that the solution for the primed potentials can be obtained from that for

the unprimed ones via the substitution:

$$\mu(\omega)\,\vec{J}(\vec{x},\omega) \to \vec{J}'(\vec{x},\omega)$$

$$\frac{1}{\epsilon(\omega)}\,\rho(\vec{x},\omega) \to \rho'(\vec{x},\omega) \qquad (4.9)$$

The Fields of a Moving Magnetic Charge

From what we have just said these fields are obtained from the fields for the electric charge via the preceeding transformation. Thus eq. (2.13) takes the following form in our case:

$$\vec{A}'(\vec{x},\omega) = \beta\,\varphi'(\vec{x},\omega) = \frac{1}{\sqrt{2\pi}}\,\frac{e'}{c}\,\Psi'(\vec{x},\omega) \qquad (4.10).$$

This means that A' and φ' differ by just a factor of β . Hence, we just quote one of them in what follows. Thus upon comparison with eqs. (3.4) and (3.6) we obtain the following expressions for A' .

Case 1: $v < c'(\omega)$

$$A'^{\pm}(\vec{x},t) = \frac{2}{\pi}\,\frac{e'}{c}\int_{K^2(\omega)<\frac{1}{\beta^2}}^{\infty} d\omega\,\cos\omega\left(\frac{z}{v}-t\right) K_0\left(\frac{\omega\rho}{\gamma' v}\right) \qquad (4.11)$$

Case 2: $v > c'(\omega)$

$$A'^{\pm}(\vec{x},t) = \frac{1}{2}\,\frac{e'}{c}\left[\int_{K^2(\omega)>\frac{1}{\beta^2}}^{\infty} d\omega\, e^{\pm i\left[\omega\left(\frac{z}{v}-t\right)\pm\frac{\pi}{2}\right]} H_0^{(1)}\left(\frac{\omega\rho}{\gamma' v}\right) + c.c.\right] \qquad (4.12)$$

In the above e' is the magnetic charge which, it is good to remember, is a pseudoscalar quantity.

5. THE FIELDS FOR THE CASE $v < c'(\omega)$

We will first treat this case for three reasons. First, we will need these fields when we come to treat the case of dissipation later on. Second, we will need to derive the fields for the moving magnetic charge. Third, by reducing these fields to the case of no dispersion and comparing with the standard results, we verify the validity of our present method.

The case of a Moving Electric Charge

From eq. (2.2) and the space-time dependence of the potentials exhibited in eq.(3.4), we conclude that the fields have the following form:

$$\vec{E}(\vec{x},t) = \vec{\rho}_1 E_t(\rho,z,t) + \vec{k}\, E_\ell(\rho,z,t)$$
$$\vec{B}(\vec{x},t) = \vec{\varphi}_1\, B(\rho,z,t) \qquad (5.1)$$

In the above $(\vec{\rho}_1, \vec{\varphi}_1, \vec{k})$ are the three orthonormal cylindrical vectors. The subscripts t and ℓ on he components of the electric field stand for transverse and longitudianl components, respectively. The explicit form of the components of the fields is given by the following:

$$E_\ell(\rho,z,t) = \frac{2}{\pi}\frac{e}{c^2}\int_{K^2(\omega)<\frac{1}{\beta'^2}}^{\infty} d\omega\, \frac{\omega}{(\beta'\gamma')^2}\, \mu(\omega) \sin\omega\left(\frac{z}{v}-t\right) K_0\left(\frac{\omega\rho}{\gamma' v}\right)$$

$$E_t(\rho,z,t) = \frac{2}{\pi}\frac{e}{c^2}\int_{K^2(\omega)<\frac{1}{\beta'^2}}^{\infty} d\omega\, \frac{\omega}{\beta'^2\gamma'}\, \mu(\omega) \cos\omega\left(\frac{z}{v}-t\right) K_1\left(\frac{\omega\rho}{\gamma' v}\right) \qquad (5.2)$$

$$B(\rho,z,t) = \frac{2}{\pi}\frac{e}{\beta c^2}\int_{K^2(\omega)<\frac{1}{\beta'^2}}^{\infty} d\omega\, \frac{\omega}{\gamma'}\, \mu(\omega) \cos\omega\left(\frac{z}{v}-t\right) K_1\left(\frac{\omega\rho}{\gamma' v}\right)$$

Since the modified Bessel functions occurring in the fields fall off exponentially with large ρ, we see that the charge can give off no radiation in this case as it is to be expected.

The Fields in the Absence of Dispersion

In this case, β', γ', ϵ and μ will be independent of ω. We take them out from underneath the integral sign in eqs. (4.2) above. The resulting Fourier transforms can then be evaluated to give the following results (Erdéyli 1954):

$$E_\ell(\rho,z,t) = \frac{e}{\epsilon}\gamma' \frac{(z-vt)}{\left[\rho^2+\gamma'^2(z-vt)^2\right]^{3/2}}$$

$$E_t(\rho,z,t) = \frac{e}{\epsilon}\gamma' \frac{\rho}{\left[\rho^2+\gamma'^2(z-vt)^2\right]^{3/2}} \qquad (5.3)$$

$$B(\rho,z,t) = \mu e \beta\gamma' \frac{\rho}{\left[\rho^2+\gamma'^2(z-vt)^2\right]^{3/2}}$$

These are expressions for the fields which would be obtained by the standard methods for a particle in a non-dispersive medium (Jackson 1975).

In anticipation of our results for the case of Cerenkov radiation which we will discuss below, we note the following relation between the electric and the magnetic fields:

$$\vec{B}(\vec{x},t) = \left(\frac{c}{c'}\right)^2 \vec{\beta}\times\vec{E}(\vec{x},t) \qquad (5.4)$$

where $\vec{\beta} = \frac{\vec{v}}{c}$.

This shows that both fields are of the same order of magnitude. Since the effect of the factor β above is balanced by that coming from$(c/c')^2$.

6. THE CASE OF AN ELECTRIC CHARGE MOVING WITH $V > c'(\omega)$

In the present case we must be careful not to ignore the causality condition which requires that the field must not overtake the particle. However, as we will find below, this is best expressed following our discussion of the asymptotic fields. From eq. (3.6) we obtain the exact expression for the fields. These are given as follows:

$$E_\ell^\pm(\rho,z,t) = \mp \frac{e}{2\beta^2 c^2}\left[\int_{K^2(\omega)>\frac{1}{\beta^2}}^{\infty} d\omega \, \frac{\omega}{\epsilon(\omega)\gamma'^2(\omega)} e^{\pm i\omega\left(\frac{z}{V}-t\right)} H_0^{(1)}\left(\frac{\omega\rho}{\gamma' V}\right) + c.c.\right]$$

$$E_t^\pm(\rho,z,t) = \frac{e}{2\beta^2 c^2}\left[i\int_{K^2(\omega)>\frac{1}{\beta^2}}^{\infty} d\omega \, \frac{\omega}{\epsilon(\omega)\gamma'(\omega)} e^{\pm i\omega\left(\frac{z}{V}-t\right)} H_1^{(1)}\left(\frac{\omega\rho}{\gamma' V}\right) + c.c.\right] \tag{6.1}$$

$$B(\rho,z,t) = \frac{e}{2\beta c^2}\left[i\int_{K^2(\omega)>\frac{1}{\beta^2}}^{\infty} d\omega \left(\frac{c}{c'(\omega)}\right)^2 \frac{\omega}{\epsilon(\omega)\gamma'(\omega)} e^{\pm i\omega\left(\frac{z}{V}-t\right)} H_1^{(1)}\left(\frac{\omega\rho}{\gamma' V}\right) + c.c.\right]$$

The Asymptotic Fields

To study the radiation from such particles we must use the asymptotic form of the fields for large ρ. Using then the standard asymptotic forms for the Hankel functions, we obtain the following expressions for the fields.

$$E_\ell^\pm(\rho,z,t) = \mp\frac{2}{\sqrt{2\pi}}\frac{e}{\beta^{3/2}\sqrt{c^3\rho}}\int_{K^2(\omega)>\frac{1}{\beta^2}}^{\infty} d\omega \, \frac{1}{\epsilon(\omega)\gamma'(\omega)}\sqrt{\frac{\omega}{\gamma'(\omega)}}\cos\left[\omega\left(\frac{\rho}{\gamma' V} \pm \left(\frac{z}{V}-t\right)\right) - \frac{\pi}{4}\right]$$

$$E_t^\pm(\rho,z,t) = \frac{2}{\sqrt{2\pi}}\frac{e}{\beta^{3/2}\sqrt{c^3\rho}}\int_{K^2(\omega)>\frac{1}{\beta^2}}^{\infty} d\omega \, \frac{1}{\epsilon(\omega)}\sqrt{\frac{\omega}{\gamma'(\omega)}}\cos\left[\omega\left(\frac{\rho}{\gamma' V} \pm \left(\frac{z}{V}-t\right)\right) - \frac{\pi}{4}\right] \tag{6.2}$$

$$B(\rho,z,t) = \frac{2}{\sqrt{2\pi}}\frac{e}{\sqrt{\beta c^3\rho}}\int_{K^2(\omega)>\frac{1}{\beta^2}}^{\infty} d\omega \left(\frac{c}{c'(\omega)}\right)^2 \frac{1}{\epsilon(\omega)}\sqrt{\frac{\omega}{\gamma'(\omega)}}\cos\left[\omega\left(\frac{\rho}{\gamma' V} \pm \left(\frac{z}{V}-t\right)\right) - \frac{\pi}{4}\right]$$

To put these results in a more significant form, we introduce the following set of orthonormal vectors:

$$\vec{n}_\pm(\omega) = \frac{\pm\vec{\rho}_1 + \gamma'(\omega)\vec{k}}{\beta'(\omega)\gamma'(\omega)}$$

$$\vec{\ell}_\pm(\omega) = \frac{\gamma'(\omega)\vec{\rho}_1 \mp \vec{k}}{\beta'(\omega)\gamma'(\omega)} \tag{6.3}$$

Together with the unit vector $\vec{\varphi}_1$, these form an orthonormal set satisfying

$$\vec{n}_\pm \times \vec{\ell}_\pm = \vec{\varphi}_1 \tag{6.4}$$

and its cyclic permutations. With the use of these vectors the fields acquire the following forms:

$$\vec{E}^\pm(\rho,z,t) = \frac{2}{\sqrt{2\pi}}\frac{e}{\sqrt{\beta^3 c^3\rho}}\int_{K^2(\omega)>\frac{1}{\beta^2}}^{\infty} d\omega \, \vec{\ell}_\pm(\omega)\frac{\beta'(\omega)}{\epsilon(\omega)}\sqrt{\frac{\omega}{\gamma'(\omega)}}\cos\left[\omega\left(\frac{\vec{n}_\pm(\omega)\cdot\vec{x}}{c'(\omega)} - t\right) \mp \frac{\pi}{4}\right] \tag{6.5}$$

$$\vec{B}^\pm(\rho,z,t) = \vec{\varphi}_1\frac{2}{\sqrt{2\pi}}\frac{e}{\sqrt{\beta c^3\rho}}\int_{K^2(\omega)>\frac{1}{\beta^2}}^{\infty} d\omega \left(\frac{c}{c'(\omega)}\right)^2 \frac{1}{\epsilon(\omega)}\sqrt{\frac{\omega}{\gamma'(\omega)}}\cos\left[\omega\left(\frac{\vec{n}_\pm(\omega)\cdot\vec{x}}{c'(\omega)} - t\right) \mp \frac{\pi}{4}\right]$$

Using an obvious notation we express the relation between $\vec{B}$ and $\vec{E}$ as follows:

$$\vec{B}^{\pm}(\rho,z,t) = \vec{\beta} \times \int_{K^2(\omega) > \frac{1}{\beta^2}}^{\infty} d\omega \left(\frac{c}{c'(\omega)}\right)^2 \vec{E}^{\pm}(\rho,z,t,\omega) \qquad (6.6)$$

Again in this case as in the preceeding case of $V < c'(\omega)$ given by eq. (5.4) both factors β and $\left(\frac{c}{c'}\right)^2$ combine to neutralize each other's effect on the magnitude of the magnetic field leaving it of the same order of magnitude as that of the electric field.

Interpretation of the above results

The preceeding results give us the spectral decomposition of the fields in the radiation or asymptotic zone. The field is then given as the sum of waves of the form:

$$\vec{E}^{\pm}(\rho,z,t,\omega) = \vec{\ell}_{\pm}(\omega) \frac{2e}{\sqrt{2\pi\beta^3 c^3 \rho}} \frac{\beta'}{\epsilon} \sqrt{\frac{\omega}{\gamma'}} \cos\left[\omega\left(\frac{\vec{n}_{\pm}\cdot\vec{\rho}}{c'} - t\right) \mp \frac{\pi}{4}\right]$$

$$\vec{B}^{\pm}(\rho,z,t,\omega) = \beta \left(\frac{c'(\omega)}{c}\right)^2 \vec{k} \times \vec{E}(\rho,z,t,\omega) \qquad (6.8)$$

We use the form (6.2) for the field which involves no loss of generality for our interpretation. Considering the outgoing wave for the time being we see that the above component propagates along the direction $\vec{n}_+(\omega)$ and possesses wave-number $K_+(\omega)$ given by:

$$k_+(\omega) = \frac{\omega}{c'(\omega)} .$$

This means that this wave propagates with the phase-velocity $c'(\omega)$ in accordance with our expectations. From the cylindrical symmetry of the problem we conclude that $\vec{n}_+(\omega)$ lies on the surface of a cone about the direction of motion of the particle. We show this schematically in figure 1. The wave fronts then lie on the cone which is orthogonal to this and is generated by $\vec{\ell}_+(\omega)$ as we show in the same diagram. The angle made by this generator and the negative z-axis is the Cerenkov angle. It is given by:

$$\sin\theta_c(\omega) = \left|\vec{k} \times \vec{\ell}_+(\omega)\right| = \frac{1}{\beta'} = \frac{c'(\omega)}{V} \qquad (6.9)$$

which is the well-known result. Since the electric field is polarized parallel to $\vec{\ell}_+(\omega)$ and the magnetic field is polarized parallel to $\vec{n}_+(\omega) \times \vec{\ell}_+(\omega)$ i.e. parallel to $\vec{\varphi}_1$, this is a transverse wave. This is sketched in figure 1.

We must next discuss the causality condition. This requires that the outgoing field cannot overtake the particles. In other words, we must stipulate that the field should vanish at points in space and time for which :

$$z > Vt .$$

This condition will apply to each spectral component of the field. For the component $\vec{E}(\rho,z,t,\omega)$ this means that the field must vanish outside of the Cerenkov cone with apex at the position of the particle and with Cerenkov angle given by eq. (6.9). Hence, the causality condition requires that each spectral component of the field must vanish outside its respective

Cerenkov cone with apex at the position of the particle (and moving with it) . Inside this cone it is given by eq. (6.8). Thus our solution given above must be supplemented with this statement of the causality condition.

Another limiting condition which we must take into account is that arising from the domain of integration over the frequency, namely:

$$K^2(\omega) = \epsilon(\omega)\mu(\omega) > \frac{1}{\beta^2} \qquad (6.10) .$$

The general behaviour of ϵ with ω is quoted by Jelley (1958). As for $\mu(\omega)$ its variation with the frequency is either similar to that of $\epsilon(\omega)$ or it does not vary much from unity (Landau and Lifshitz 1960). In discussing eq.(6.10) we are justified then in assuming that $K^2(\omega)$ behaves also in a general fashion similar to $\epsilon(\omega)$. Then the investigation of the behaviour of $\epsilon(\omega)$ leads us to the conclusion that the values of ω which would satisfy eq. (6.10) will occur in bands of finite extension and that this equation determines a maximum frequency beyond which it cannot be satisfied. Applied to the Cerenkov angle this means that its allowed values come also in bands and, what is most significant, it has a maximum value which is determined by the value of β . Calling this angle:

$$\vartheta_{c\,max.}(\beta) ,$$

we conclude that the Cerenkov cone with this angle and with apex at the position of the particle constitutes an envelope for the field: the field outside of it is zero.

The Incoming Solution

This is similar to the case of the outgoing field except that now the field converges towards the particle with phase velocity $c'(\omega)$ along the directions given by $\vec{n}_-(\omega)$. The Cerenkov cones have now the positive z -axis as their common axis and the electric field is polarized along $\vec{\ell}_-$.

7. THE RADIATED ENERGY

The Poynting Flux

In a dispersive medium the Poynting flux is given by (Landau and Lifshitz 1960)

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{H} .$$

We have so far established that the fields are given by the following spectral decompositions

$$\vec{E}^{\pm}(\rho,z,t) = \int_{K^2(\omega) > \frac{1}{\beta^2}}^{\infty} d\omega \, \vec{\ell}_{\pm}(\omega) \, E^{\pm}(\rho,z,t,\omega)$$

$$\vec{B}^{\pm}(\rho,z,t) = \int_{K^2(\omega) > \frac{1}{\beta^2}}^{\infty} d\omega \, \vec{\varphi}_{\pm} \, B^{\pm}(\rho,z,t,\omega) \qquad (7.1) ,$$

where the functions $E^{\pm}$, $B^{\pm}$ are given by eqs. (6.5). They satisfy the following relation:

$$B^{\pm}(\rho,z,t,\omega) = \frac{c}{c'(\omega)} E^{\pm}(\rho,z,t,\omega) \qquad (7.2) .$$

With the use of eqs.(2.4) we can write the above equation as follows:

$$H^{\pm}(\rho,z,t,\omega) = \sqrt{\frac{\epsilon(\omega)}{\mu(\omega)}} \, E^{\pm}(\rho,z,t,\omega) \qquad (7.3) .$$

Using these relations we readily write the Poynting flux in the following form:

$$\vec{S}^{\pm} = \int_{K^2(\omega) > \frac{1}{\beta^2}}^{\infty} d\omega \, \vec{c}'^{\pm}(\omega) \frac{1}{8\pi} \Big[\sqrt{\epsilon(\omega)\mu(\omega)} \, E^{\pm}(\rho,z,t,\omega) \int_{K^2(\omega') > \frac{1}{\beta^2}} d\omega' \sqrt{\frac{\epsilon(\omega')}{\mu(\omega')}} \, E^{\pm}(\rho,z,t,\omega') + \mu(\omega) H^{\pm}(\rho,z,t,\omega) \int_{K^2(\omega') > \frac{1}{\beta^2}} d\omega' \, H^{\pm}(\rho,z,t,\omega') \Big] \qquad (7.4) .$$

Comparison of the integrand with the usual expression for the energy density in dispersive media leads us to the following interpretation. In each mode of the field the quantity in square brackets above represents the energy density stored in that mode. This energy is then transported with the phase velocity of the field which is equal to:

$$\vec{c}'^{\pm}(\omega) = \vec{n}_{\pm}(\omega)\, c'(\omega) .$$

In accordance with the findings of Schiff (Motz 1953) we find that the energy is transported with the phase velocity of that mode.

Furthermore, from the preceeding results we find that:

$$\vec{S}^{\pm} = \int_{K^2(\omega)>\frac{1}{\beta^2}}^{\infty} d\omega\, \vec{c}'^{\pm}(\omega)\, \frac{1}{4\pi}\sqrt{\epsilon(\omega)\mu(\omega)}\, E^{\pm}(\rho,z,t,\omega) \int_{K^2(\omega')>\frac{1}{\beta^2}} d\omega' \sqrt{\frac{\epsilon(\omega')}{\mu(\omega')}}\, E^{\pm}(\rho,z,t,\omega')$$

$$= \int_{K^2(\omega)>\frac{1}{\beta^2}}^{\infty} d\omega\, \vec{c}'^{\pm}(\omega)\, \frac{1}{4\pi}\, \mu(\omega) H^{\pm}(\omega) \int_{K^2(\omega')>\frac{1}{\beta^2}}^{\infty} d\omega'\, H^{\pm}(\rho,z,t,\omega') .$$

This means that the energy is carried equally by the electric field and the magnetic field.

The Time Integral of the Poynting Flux

The expression for the Poynting flux which we derived has the following explicit asymptotic form:

$$\vec{S}^{\pm}(\rho,z,t) = \frac{1}{(2\pi)^2}\frac{e^2}{\beta c^3 \rho} \iint_{K^2(\omega),\,K^2(\omega')>\frac{1}{\beta^2}}^{\infty} d\omega\, d\omega'\, \vec{c}'^{\pm}(\omega)\mu(\omega)\sqrt{\frac{\omega\omega'}{\gamma'(\omega)\gamma'(\omega')}} \cdot$$

$$\cdot\left\{ \pm \sin\left[\left(\frac{\omega \vec{n}_{\pm}(\omega)}{c'(\omega)} + \frac{\omega' \vec{n}_{\pm}(\omega')}{c'(\omega')}\right)\cdot\vec{\lambda} - (\omega+\omega')t\right] + \right.$$
$$\left. + \cos\left[\left(\frac{\omega \vec{n}_{\pm}(\omega)}{c'(\omega)} - \frac{\omega' \vec{n}_{\pm}(\omega')}{c'(\omega')}\right)\cdot\vec{\lambda} - (\omega-\omega')t\right]\right\} . \quad (7.6)$$

This expression is greatly simplified when we integrate it over the time.

Thus at any point (ρ, z) the field arrives at times characterized by:

$$t \geq \frac{\rho + \gamma'(\omega_{max}) z}{\beta'(\omega_{max})\gamma'(\omega_{max})c'(\omega_{max})} = t_{min} \quad (7.6).$$

Before this time the field and the energy flux at this point are zero. Starting at this time the radiative energy starts to arrive at this point. Consequently, if we want to find the energy crossing a unit area at this point over a time interval we must evaluate the following integral:

$$\int_{t_{min}}^{t} S^{\pm}(\rho,z,t)\, dt .$$

We take t large enough (eventually infinite) so that $(t - t_{min})$ will contain several periods belonging to any frequency contributing to the above integral. Then in the time integral of eq. (7.5) the term containing the sine function gives no contribution. The term containing the cosine function will give a contribution only when $\omega = \omega'$ and in that case the integral will be independent of the values of ρ and z. Hence, if we want to integrate the Poynting flux at (ρ, z) for all the duration of the field there we take $t = \infty$ and because of the independence of the time integral on (ρ, z) we can take the limit $z \to -\infty$, which means setting $t_{min} = -\infty$. Hence,

$$\int_{-\infty}^{\infty} \vec{S}^{\pm}(\rho,z,t)\, dt = \frac{1}{2\pi}\frac{e^2}{\beta c^3 \rho} \int_{K^2(\omega)>\frac{1}{\beta^2}}^{\infty} d\omega\, \vec{c}'^{\pm}(\omega)\mu(\omega)\frac{\omega}{\gamma'(\omega)} \quad (7.7)$$

Energy Radiated Per Length of path

The energy radiated radially away from the path of the particle which crosses a cylindrical shell of radius ρ and elevation dz is then given by

$$dU^{+} = \int_0^{2\pi} d\varphi \left(\int_{-\infty}^{\infty} \vec{S}^{+}(\rho,z,t)\, dt\right)\cdot\vec{\rho}\, dz .$$

From this we obtain the standard result:

$$\frac{dU^+}{dz} = \left(\frac{e}{c}\right)^2 \int\limits_{\kappa^2(\omega) > \frac{1}{\beta^2}}^{\infty} d\omega \quad \omega\mu(\omega)\left(1 - \frac{1}{\beta'^2(\omega)}\right) \qquad (7.8).$$

If we had considered the incoming solution, then we would have obtained the same result except with a negative sign in front of the R.H.S. This would mean that the particle is absorbing energy from the medium.

The Reaction Force

Since the particle is losing energy through radiation, this means that the medium is effectively executing a force on the particle, the reaction force, which is directed opposite to its direction of motion. The radiated energy can then be visualized as equal to the work done by the particle against this force. We readily find this force as follows:

$$Fv = F\frac{dz}{dt} = \frac{dU^+}{dt}.$$

Thus,

$$F = \frac{dU^+}{dz} = \left(\frac{e}{c}\right)^2 \int\limits_{\kappa^2(\omega) > \frac{1}{\beta^2}}^{\infty} d\omega \quad \omega\mu(\omega)\left(1 - \frac{1}{\beta'^2(\omega)}\right) \qquad (7.9) \; .$$

In the case of the incoming wave solution, the force on the particle is along its direction of motion. It is thus the absorption by the particle of the incoming radiation which gives rise to a driving force on it.

8. CERENKOV RADIATION IN A NON-DISPERSIVE MEDIUM

Our present method gives us the fields in straightforward manner when the medium is non-dispersive. In this case the susceptibilities will no more be functions of the frequency. We can then evaluate the integrals for the potentials because they will then be cosine or sine - Fourier transforms of Bessel functions (Erdéyli 1954). Thus the vector potential given in eq.(3.6) will now have the following form.

$$A^{\pm}(\rho,z,t) = -\frac{e\mu}{c}\int_0^{\infty} d\omega \left[\cos\omega\left(\frac{z}{v}-t\right)Y_0\left(\frac{\omega\rho}{\gamma' v}\right) \pm \sin\omega\left(\frac{z}{v}-t\right)J_0\left(\frac{\omega\rho}{\gamma' v}\right)\right].$$

Considering for the time being the outgoing field, we apply to it the standard results for Fourier cosine and sine transforms. We find that this field vanishes whenever,

$$z > vt.$$

This is the causality condition which, so far, we had to impose a priori on the field but which we recover directly in the present case. For values of z satisfying

$$z < vt,$$

we find that the field vanishes outside the backward Cerenkov cone i.e. that with apex at the position of theparticle and opening behind it. Within this cone the vector potential is given by the expression:

$$A^+(\rho,z,t) = 2e\mu\beta\gamma' \frac{1}{\sqrt{\gamma'^2(z-vt)^2-\rho^2}}.$$

We recapitulate this result as follows:

$$A^+(\rho,z,t) = 2e\mu\beta\gamma' \frac{1}{\sqrt{\gamma'^2(z-vt)^2-\rho^2}} \begin{cases} 1 & \text{within the backward Cerenkov cone} \\ 0 & \text{everywhere else} \end{cases} \qquad (8.1).$$

For the incoming field we obtain a similar relation:

$$\vec{A}^{-}(\rho,z,t) = 2e\mu\beta\gamma' \frac{1}{\sqrt{\gamma'^2(z-vt)^2-\rho^2}} \begin{cases} 1 & \text{within the forward Cerenkov cone} \\ 0 & \text{everywhere else} \end{cases} \qquad (8.2)\ .$$

The scalar potential is then given by:

$$\varphi^{\pm}(\rho,z,t) = \frac{1}{\beta\epsilon\mu} A^{\pm}(\rho,z,t) \qquad (8.3)\ .$$

The Fields

From what we have just proved, the fields vanish outside the respective cones. Within these cones they have the same expression for either the outgoing field or the incoming field. These expressions are given as follows:

$$\vec{E}(\rho,z,t) = -2\frac{e\gamma'}{\epsilon} \frac{\vec{\rho}+\vec{k}(z-vt)}{\left[\gamma'^2(z-vt)^2-\rho^2\right]^{3/2}}$$

$$\vec{B}(\rho,z,t) = -\vec{q}\, 2e\mu\beta\gamma' \frac{\rho}{\left[\gamma'^2(z-vt)^2-\rho^2\right]^{3/2}} \qquad (8.4)\ .$$

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Considering the form (8.4) for the fields or that given by (8.1) for the potentials we find that they all become infinite on the Cerenkov cone. This new singularity in the field comes about because we have negelcted dispersion. The backward cone is defined by:

$$\frac{\rho}{\gamma'} + (z-vt) < 0\ .$$

If now we formally find the limit for the direction of the electric field as we approach this cone, we find it to be $-\vec{\ell}_+$. The field will then be polarized along the Cerenkov cone.

The Poynting Flux

We readily find this to be:

$$\vec{S}(\rho,z,t) = \frac{1}{\pi}\frac{e^2}{\epsilon}\beta\gamma'^2 c\left(\vec{k}\rho - \vec{\rho}_1(z-vt)\right)\frac{\rho}{\left[\gamma'^2(z-vt)^2-\rho^2\right]^{3/2}} \qquad (8.6)\ .$$

We must remember that this is to be supplemented with the condition which holds for the fields; namely, that the above is the expression within the respective Cerenkov cone. Outside the cone $\vec{S}$ vanishes. On the cone the Poynting flux is singular. But if we formally approach the cone so as to find the direction of $\vec{S}$ there, then we will readily find:

$$\lim_{\frac{\rho}{\gamma'}+(z-vt)\to 0} \vec{S}^{+} = \lim_{\frac{\rho}{\gamma'}+(z-vt)\to 0} \frac{1}{8\pi}\left[\epsilon\, \vec{E}^2(\rho,z,t) + \mu\, \vec{H}^2(\rho,z,t)\right]\vec{c}\,'_{+} \qquad (8.7)\ .$$

This result again demonstrates what we have proved before; namely, that the energy travels normal to the Cerenkov cone and with a velocity equal to the phase velocity. This will become more evident in what follows.

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The Asymptotic Fields

We start with the asymptotic expressions for the fields in the case of dispersion which are given in eq. (6.5). We illustrate the treatment by considering the outgoing electric field. In the absence of dispersion (6.5) acquires the following form:

$$\vec{E}^{+}(\rho,z,t) = \vec{\ell}_{+} \frac{2}{\sqrt{2\pi}} \frac{(e\beta'/\epsilon)}{\sqrt{\beta^3 c^3 \gamma' \rho}} \int_0^\infty d\omega \sqrt{\omega}\, \cos\left[\omega\left(\frac{\vec{n}_+\cdot\vec{x}}{c'} - t\right) - \frac{\pi}{4}\right] \qquad (8.8)\ .$$

This is the value of the field within the Cerenkov cone i.e. for:

$$\frac{\vec{n}_+\cdot\vec{x}}{c'} - t < 0\ .$$

Everywhere else it is zero. We re-write the integal as follows:

$$\int_0^\infty d\omega \sqrt{\omega}\, \cos\left[\omega\left(\frac{\vec{n}_+\cdot\vec{x}}{c'} - t\right) - \frac{\pi}{4}\right] = \left[\frac{1}{2} e^{-i\frac{\pi}{4}} \int_0^\infty d\omega \sqrt{\omega}\, e^{-i\omega\left(t - \frac{\vec{n}_+\cdot\vec{x}}{c'}\right)} + c.c.\right]\ .$$

We consider the integral I in the complex η-plane defined as follows:

$$I = \int_0^\infty d\eta \sqrt{\eta}\, e^{-ia\eta}\ , \quad a\ \text{real} > 0\ .$$

We consider the complex η-plane with a cut along the negative real axis. Then the integrand above will be an analytic function. By considering a contour C as shown in figure 2, we find that:

$$I = -\int_C d\eta \sqrt{\eta}\, e^{-ia\eta}\ ,$$

where the contour C coincides with the negative imaginary axis, as seen in the figure. Along C:

$$\eta = e^{-i\frac{\pi}{2}} \xi\ , \quad \xi\ \text{real} > 0\ .$$

$$I = -e^{i\frac{\pi}{4}} \int_0^\infty d\xi \sqrt{\xi}\, e^{-a\xi}\ .$$

From this we obtain:

$$\int_0^\infty d\omega \sqrt{\omega}\, \cos\left[\omega\left(t - \frac{\vec{n}_+\cdot\vec{x}}{c'}\right) - \frac{\pi}{4}\right] = -\int_0^\infty d\omega \sqrt{\omega}\, e^{-\omega\left(t - \frac{\vec{n}_+\cdot\vec{x}}{c'}\right)}\ .$$

This is a standard Laplace transform which we take from the standard tables (Erdéyli 1954). The result we find is the following:

$$\int_0^\infty d\omega \sqrt{\omega}\, \cos\left[\omega\left(t - \frac{\vec{n}_+\cdot\vec{x}}{c'}\right) - \frac{\pi}{4}\right] = -\frac{\sqrt{\pi}}{2} \frac{1}{\left[t - \frac{\vec{n}_+\cdot\vec{x}}{c'}\right]^{3/2}}\ .$$

Putting this in eq.(8.8) we obtain:

$$\vec{E}^{+}(\rho,z,t) = -\vec{\ell}_{+} \frac{1}{\sqrt{2}} \frac{e\beta'/\epsilon}{\sqrt{\beta^3 c^3 \gamma'}} \frac{1}{\sqrt{\rho}} \frac{1}{\left[t - \frac{\vec{n}_+\cdot\vec{x}}{c'}\right]^{3/2}}\ .$$

We obtain the magnetic field via eq. (6.6). We state the result for both the outgoing and the incoming fields:

$$\vec{E}^{\pm}(\rho,z,t) = -\vec{\ell}_{\pm} \frac{1}{\sqrt{2}} \frac{e\beta\gamma'}{\epsilon} \frac{1}{\sqrt{\rho}} \frac{1}{\left[\pm\gamma'(vt-z) - \rho\right]^{3/2}}$$

$$\vec{B}^{\pm}(\rho,z,t) = -\vec{\varphi} \frac{1}{\sqrt{2}} e\mu\beta\gamma' \frac{1}{\sqrt{\rho}} \frac{1}{\left[\pm\gamma'(vt-z) - \rho\right]^{3/2}} \qquad (8.9)\ .$$

We see that the polarization of the field agrees with what we concluded above. Again the field is singular on the Cerenkov cone. But this is an artificial singularity due to the fact that we have neglected the dispersion.

The Poynting Flux

It is sufficient to consider the outgoing field case. The case of the incoming field is quite similar.

$$\vec{S}^{+}(\rho,z,t) = \vec{C}'_{+} \left(\frac{\epsilon \left|\vec{E}^{+}(\rho,z,t)\right|^2 + \mu \left|\vec{H}^{+}(\rho,z,t)\right|^2}{8\pi} \right) \qquad (8.10)\ .$$

This is the definite proof that the energy travels with the phase velocity and in a direction normal to the Cerenkov cone.

Radiated Energy per Unit Path Length

Explicitly the Poynting flux has the following form:

$$\vec{S}^{+}(\rho,z,t) = \vec{n}_{+} \frac{1}{8\pi} \frac{e^2}{c^2} \frac{\mu}{\beta'\gamma'} \frac{1}{\rho} \frac{1}{\left[t - \frac{\vec{n}_+\cdot\vec{x}}{c'}\right]^3} \qquad (8.11)\ .$$

$$\frac{dU^+}{dz} = \int_{\vec{n}\cdot\vec{\lambda}/c'}^{\infty} dt \; 2\pi \vec{\rho}\cdot\vec{S}^+ \; ,$$

where the lower limit of integration is imposed by the causality condition, which causes the field to vanish outside the backward Cerenkov cone. We readily find the following result:

$$\frac{dU^+}{dz} = \frac{1}{8}\frac{e^2}{(\vec{n}\cdot\vec{\lambda})^2}\frac{1}{\epsilon}\left(1-\frac{1}{\beta'^2}\right)\lim_{\zeta\to 0}\frac{1}{\zeta} \qquad (8.12),$$

where ζ is a dimensionless variable. This result is also singular. However, this may be remedied as we shall show in the following discussion.

The Case of Tachyons

All the considerations which we have presented in this section hold unchanged when the medium is the vacuum. We only need to put each susceptibility equal to unity and delete all primes. We will then be dealing with tachyons. In the literature there has been disagreement as to wether such particles will emit Cerenkov radiation (Mignani 1974). From the above considerations we conclude that such particles, if they exist, will indeed emit such radiation.

As to the difficulty arising from the singularity of the field on the Cerenkov cone this is of a technical and not of a fundamental nature. As we have seen it arises out of our neglect of dispersion. In the present case it indicates to us that the proper treatment of tachyons must take vacuum polarization into account. Hence, a theory of tachyons must be second-quantized from the very first.

Thus in such a theory a tachyon will have a finite probability

for radiation emission per unit time. From this we obtain a radiation mean - free path λ. This then provides us with a length with which we can regularize eq.(8.12). We write this eq. as follows:

$$\frac{dU^+}{dz} = \frac{1}{8}\frac{e^2}{\lambda^2}\left(1-\frac{1}{\beta^2}\right)\frac{1}{(\vec{n}\cdot\vec{\lambda}/\lambda)^2}\lim_{\zeta\to 0}\frac{1}{\zeta^2} .$$

Now, $\vec{n}\cdot\vec{\lambda}$ is just the distance from the origin measured normal to the Cerenkov cone. In the radiation zone we can take this, in the limit, to infinity. Introducing the dimensionless symbol ζ for $(\vec{n}\cdot\vec{\lambda}/\lambda)$, we see that:

$$\frac{dU^+}{dz} = \frac{1}{8}\frac{e^2}{\lambda^2}\left(1-\frac{1}{\beta^2}\right)\left(\lim_{\zeta\to\infty}\frac{1}{\zeta^2}\cdot\lim_{\zeta\to 0}\frac{1}{\zeta^2}\right)$$

$$= \frac{1}{8}\frac{e^2}{\lambda^2}\left(1-\frac{1}{\beta^2}\right),$$

which is a finite quantity.

Again such a theory will have a direct bearing on the question of the detection of the radiation from such a particle. Some authors have considered the inability to detect such a radiation as evidence that tachyons do not emit Cerenkov radiation (Mignani 1974). However, as we show in a subsequent publication, in a second-quantized theory, tachyonic charged Fermions interact with axial-vector photons. Hence, the Cerenkov radiation coming from tachyons will be in the form of such photons.

The readiness then with which such radiation can be detected depends on the strength of the interaction between such photons and matter.

9. THE FIELDS OF A MOVING MAGNETIC CHARGE

We do not need to calculate these results all over again. Since by comparing the results of section 4 with those of section 3 we find that the fields for the magnetic charge are related to those for the electric charge as follows:

$$\vec{E}_{mag}(\rho,z,t) = \int_{K^2(\omega)\gtrless\frac{1}{\beta^2}}^{\infty} d\omega \, \frac{1}{\mu(\omega)} \vec{B}_{elect}(\rho,z,t,\omega)$$

$$\vec{B}_{mag}(\rho,z,t) = \int_{K^2(\omega)\gtrless\frac{1}{\beta^2}} d\omega \, \epsilon(\omega) \vec{E}_{elect}(\rho,z,t,\omega) \qquad (9.1)\,,$$

and with e replaced by e' on the R.H.S. We quote the results.

Case 1: $V < c'(\omega)$

The Exact Expressions for the Fields

$$\vec{E}(\rho,z,t) = \vec{q}_1 \frac{2}{\pi}\frac{e'}{\beta c^2}\int_{K^2(\omega)<\frac{1}{\beta^2}}^{\infty} d\omega \frac{\omega}{\gamma'(\omega)} \cos\omega\left(\frac{z}{v}-t\right) K_1\left(\frac{\omega\rho}{\gamma'V}\right)$$

$$\vec{B}(\rho,z,t) = -\frac{2}{\pi}\frac{e'}{\beta^2c^2}\left\{\vec{\rho}_1 \int_{K^2(\omega)<\frac{1}{\beta^2}}^{\infty} d\omega \frac{\omega}{\gamma'(\omega)}\cos\omega\left(\frac{z}{v}-t\right)K_1\left(\frac{\omega\rho}{\gamma'V}\right) + \vec{k}\int_{K^2(\omega)<\frac{1}{\beta^2}}^{\infty} d\omega \frac{\omega}{\gamma'^2(\omega)}\sin\omega\left(\frac{z}{v}-t\right)K_0\left(\frac{\omega\rho}{\gamma'V}\right)\right\} \qquad (9.2)$$

The Fields in the Absence of Dispersion

$$\vec{E}(\rho,z,t) = \vec{q}_1\, e'\beta\gamma' \frac{\rho}{\left[\rho^2+\gamma'^2(z-Vt)^2\right]^{3/2}}$$

$$\vec{B}(\rho,z,t) = -e'\gamma'\frac{\vec{\rho}+\vec{k}(z-Vt)}{\left[\rho^2+\gamma'^2(z-Vt)^2\right]^{3/2}} \qquad (9.3)\,.$$

Case 2: $V > c'(\omega)$

The Exact Expressions for the Fields

$$E^{\pm}(\rho,z,t) = \frac{e'}{2\beta c^2}\left[i\int_{K^2(\omega)>\frac{1}{\beta^2}}^{\infty} d\omega \frac{\omega}{\gamma'(\omega)} e^{\pm i\omega\left(\frac{z}{v}-t\right)} H_1^{(1)}\left(\frac{\omega\rho}{\gamma'V}\right)+c.c.\right] \qquad (9.4)$$

$$B_{\ell}^{\pm}(\rho,z,t) = \pm\frac{e'}{2\beta^2c^2}\left[\int_{K^2(\omega)>\frac{1}{\beta^2}}^{\infty} d\omega \frac{\omega}{\gamma'^2(\omega)} e^{\pm i\omega\left(\frac{z}{v}-t\right)} H_0^{(1)}\left(\frac{\omega\rho}{\gamma'V}\right)+c.c.\right]$$

$$B_{t}^{\pm}(\rho,z,t) = -\frac{e'}{2\beta^2c^2}\left[i\int_{K^2(\omega)>\frac{1}{\beta^2}}^{\infty} d\omega \frac{\omega}{\gamma'(\omega)} e^{\pm i\omega\left(\frac{z}{v}-t\right)} H_1^{(1)}\left(\frac{\omega\rho}{\gamma'V}\right)+c.c.\right]$$

The Asymptotic Fields

$$\vec{E}^{\pm}(\rho,z,t) = \vec{q}_1 \frac{2}{\sqrt{2\pi}}\frac{e'}{\sqrt{\beta c^3\rho}}\int_{K^2(\omega)>\frac{1}{\beta^2}}^{\infty} d\omega \sqrt{\frac{\omega}{\gamma'(\omega)}} \cos\left[\omega\left(\frac{\vec{n}_{\pm}(\omega)\cdot\vec{\lambda}}{c'(\omega)}-t\right)\mp\frac{\pi}{4}\right]$$

$$\vec{B}^{\pm}(\rho,z,t) = -\frac{2}{\sqrt{2\pi}}\frac{e'}{\sqrt{\beta^3c^3\rho}}\int_{K^2(\omega)>\frac{1}{\beta^2}} d\omega\, \vec{\ell}_{\pm}(\omega)\beta'(\omega)\sqrt{\frac{\omega}{\gamma'(\omega)}}\cos\left[\omega\left(\frac{\vec{n}_{\pm}(\omega)\cdot\vec{\lambda}}{c'(\omega)}-t\right)\mp\frac{\pi}{4}\right] \qquad (9.5)\,.$$

The discussion given in section 6 above concerning the causality condition also holds here. The outgoing field vanishes outside the backward Cerenkov cone and the incoming field vanishes outside the foreward Cerenkov cone.

The Poynting Flux

Again all the discussion concerning the general form of the Poynting flux and the propagation of the energy with velocity $\vec{C}'_{\pm}(\omega)$ carries over to the present case unchanged.

Explicitly, the Poynting flux has the following form:

$$\vec{S}^{\pm}(\rho,z,t) = \frac{2}{(2\pi)^2}\frac{e'^2}{\beta c^2 \rho}\iint\limits_{k^2(\omega),\,k^2(\omega') > \frac{1}{\beta^2}}^{\infty} d\omega\, d\omega'\, \vec{c}'^{\pm}(\omega)\,\epsilon(\omega)\sqrt{\frac{\omega\omega'}{\gamma'(\omega)\gamma'(\omega')}}\,\cdot$$

$$\cdot\left\{\cos\left[\left(\frac{\omega\,\vec{n}_{\pm}(\omega)\cdot\vec{x}}{c'(\omega)}-t\right)\mp\frac{\pi}{4}\right]\cos\left[\left(\frac{\omega'\,\vec{n}_{\pm}(\omega')\cdot\vec{x}}{c'(\omega')}-t\right)\mp\frac{\pi}{4}\right]\right\} \tag{9.6}$$

This is the equivalent of eq. (7.5). It differs from it through the replacement of $\mu(\omega)$ by $\epsilon(\omega)$ within the integrand. Consequently, by analogy with eq. (7.7), we have:

$$\int_{-\infty}^{\infty}\vec{S}^{\pm}(\rho,z,t)\,dt = \frac{1}{2\pi}\frac{e'^2}{\beta c^2\rho}\int\limits_{k^2(\omega)>\frac{1}{\beta^2}}^{\infty} d\omega\, \vec{c}'^{\pm}(\omega)\,\epsilon(\omega)\frac{\omega}{\gamma'(\omega)} \tag{9.7}$$

Then from eq. (7.8), we obtain:

$$\frac{dU^{t}}{dz} = \left(\frac{e'}{c}\right)^2\int\limits_{k^2(\omega)>\frac{1}{\beta^2}}^{\infty} d\omega\;\; \omega\,\epsilon(\omega)\left(1-\frac{1}{\beta'^2(\omega)}\right) \tag{9.8}$$

The expression for the reaction force on the particle will likewise be:

$$F = \frac{dU^{t}}{dz} = \left(\frac{e'}{c}\right)^2\int\limits_{k^2(\omega)>\frac{1}{\beta^2}} d\omega\;\; \omega\,\epsilon(\omega)\left(1-\frac{1}{\beta'^2(\omega)}\right) \tag{9.9}$$

Cerenkov Radiation in a Non-Dispersive Medium

We obtain the expression for the fields from eqs. (8.4):

$$\vec{E}(\rho,z,t) = -\vec{q}_1\, e'\beta\gamma'\frac{\rho}{\left[\gamma'^2(z-vt)^2-\rho^2\right]^{3/2}}$$

$$\vec{B}(\rho,z,t) = 2e'\gamma'\frac{\vec{\rho}+\vec{k}\,(z-vt)}{\left[\gamma'^2(z-vt)^2-\rho^2\right]^{3/2}} \tag{9.10}$$

From eq.(8.9) we obtain the asymptotic fields:

$$\vec{E}^{\pm}(\rho,z,t) = -\vec{q}_1\frac{1}{\sqrt{2}}\frac{e'\beta\gamma'}{\sqrt{\rho}}\frac{1}{\left[\pm\gamma'(vt-z)-\rho\right]^{3/2}}$$

$$\vec{B}^{\pm}(\rho,z,t) = \vec{\ell}_{+}\frac{1}{\sqrt{2}}\frac{e'\beta\gamma'}{\sqrt{\rho}}\frac{1}{\left[\pm\gamma'(vt-z)-\rho\right]^{3/2}} \tag{9.11}$$

By analogy with eqs. (8.11) and (8.12), we obtain:

$$\vec{S}^{+}(\rho,z,t) = \vec{n}_{+}\frac{1}{8\pi}\left(\frac{e'}{c}\right)^2\frac{\epsilon}{\beta\gamma'}\frac{1}{\rho}\frac{1}{\left[t-\frac{\vec{n}_{+}\cdot\vec{x}}{c'}\right]^3} \tag{9.12}$$

$$\frac{dU^{t}}{dz} = \frac{1}{8}\left(\frac{e'}{\vec{n}_{+}\cdot\vec{x}}\right)^2\frac{1}{\mu}\left(1-\frac{1}{\beta'^2}\right)\lim_{\zeta\to 0}\frac{1}{\zeta^2} \tag{9.13}$$

For a tachyonic magnetic charge in vacuum we thus obtain:

$$\frac{dU^{t}}{dz} = \frac{1}{8}\left(\frac{e'}{\lambda}\right)^2\left(1-\frac{1}{\beta^2}\right) \tag{9.14}$$

This is identical in form with the respective expression for an electric tachyon.

10. THE FIELDS IN THE PRESENCE OF BOTH DISPERSION AND DISSIPATION

A. A Moving Electric Charge

So far we have considered the somewhat idealized case of a dispersive medium without dissipation. However, in rality there is never a case of dispersion without some dissipation. Hence, the rigorous treatment of the fields of a particle moving in a dispersive medium must take dissipation into account.Our present method is ideally suited to obtaining the solution in this general case.

The basic modification introduced by dissipation concerns the susceptibilities.These will no more be real but will become complex quantities which satisfy the following generalization of eq. (2.1):

$$\epsilon(\omega) = \epsilon^*(-\omega)$$
$$\mu(\omega) = \mu^*(-\omega) \qquad (10.1).$$

This has an immediate consequence; that we cannot now interpret $c'(\omega)$ as the phase velocity of the electromagnetic field in the medium, because this is now a complex quantity. However, the treatment which we have presneted in section 2 above still holds intact. In particular, the fields are still given by the scalar function $\Psi(\vec{x},\omega)$ which satisfies the inhomogenous Helmholtz equation similar to eq. (2.14), namely:

$$\left(\vec{\nabla}^2 + \frac{\omega^2}{c^2}K^2(\omega)\right)\Psi(\vec{x},\omega) = -4\pi\,\delta(x)\,\delta(y)\,e^{i\frac{\omega z}{V}} \qquad (10.2),$$

except now $K^2(\omega)$ is a complex quantity satisfying:

$$K^2(\omega) = K^{2*}(-\omega) \qquad (10.3).$$

In what follows we will need to consider the square roots of $K^2(\omega)$. We will use the symbol $K(\omega)$ for the root which occurs in the upper half plane. The other root of $K^2(\omega)$, which will, by consequence, lie in the lower half plane, will then be given by $-K(\omega)$.

The Green Function

In the usual manner the Green function for the above Helmholtz equation will be given by the following expression:

$$G(\vec{x},\vec{x}') = \frac{-i}{\pi}\frac{1}{|\vec{x}-\vec{x}'|}\int_{-\infty}^{\infty}\frac{k}{\left[k^2 - \frac{\omega^2}{c^2}K^2(\omega)\right]}e^{ik|\vec{x}-\vec{x}'|}\,dk .$$

However, due to the presence of dissipation the poles of the integrand in the complex-k plane lie off the path of integration. This then fixes the

above function to be:

$$G(\vec{x},\vec{x}') = \frac{e^{i\left(\frac{\omega}{c}K(\omega)|\vec{x}-\vec{x}'|\right)}}{|\vec{x}-\vec{x}'|} \qquad (10.3).$$

This contrasts with the case of no dissipation. There the poles lay right on the path of integration. The two solutions which we then obtained corresponded to the two modes in which we could lift these poles off the path of integration.

But the physical reason for this behaviour lies in the fact that when dissipation is absent Maxwell's equations will be invariant under the operation of time-reversal. This would imply that if we have a solution with outgoing waves, then its transform under time-reversal, which would represent incoming waves, will be a solution also. In the present case, because of dissipation time-reversal invariance is violated. This is exhibited explicitely by the present circumstance where our problem admits only an outgoing-wave solution without a corresponding incoming-wave solution.

The Dimensionless Scalar Function

Corresponding to eq. (3.2), we now have just one solution. This is given by:

$$\Psi(\vec{x},\omega) = 2e^{i\frac{\omega z}{V}}\int_0^{\infty} d\zeta\,\cos\left(\frac{\omega\zeta}{V}\right)\frac{e^{i\frac{\omega K(\omega)}{c}\sqrt{\rho^2+\zeta^2}}}{\sqrt{\rho^2+\zeta^2}} \qquad (10.4),$$

where all the symbols retain their meaning as in that equation. But now $K(\omega)$ is a complex number with positive imaginary part.

From this we have for the function $\Psi(\vec{x},t)$ the following expression:

$$\Psi(\vec{x},t) = \frac{2}{\sqrt{2\pi}}\left[\int_0^{\infty} d\omega\, e^{i\omega\left(\frac{z}{V}-t\right)}\int_0^{\infty} d\zeta\,\cos\left(\frac{\omega\zeta}{V}\right)\frac{e^{i\frac{\omega K(\omega)}{c}\sqrt{\rho^2+\zeta^2}}}{\sqrt{\rho^2+\zeta^2}}\right. \qquad (10.5).$$

As before in evaluating this function we would have to distinguish between two cases which would correspond to the cases $K^2(\omega) \gtrless \frac{1}{\beta^2}$ which we encountered before. The obvious generalization of these cases to our present situation is to require:

$$Re.\ K^2(\omega) \gtrless \frac{1}{\beta^2}\,.$$

In section 3 we derived eq. (3.4) from eq. (3.3) by using results given in the standard tables for Fourier transforms. The latter equation corresponds to our eq. (10.6). Indeed, they have identical structure except for the complex nature of $K(\omega)$ in the present case. We cannot use the tables directly in our case now. However, considering eq. (10.5) as the analytic continuation of eq.(3.3) to complex values of $K(\omega)$, we will argue that $\Psi(\vec{x},t)$ will just be the analytic continuation of eq.(3.4) to complex values of $K(\omega)$. In particular we will take this to be:

$$\Psi(\vec{x},t) = \frac{2}{\sqrt{2\pi}}\left[\int_{Re.K^2(\omega)<\frac{1}{\beta^2}}^{\infty} d\omega\, e^{i\omega\left(\frac{z}{v}-t\right)} K_0\left(\frac{\omega\rho}{\gamma' v}\right) + c.c.\right] \quad ,\rho>0 \qquad (10.6)\,.$$

Correspondingly, the analytic continuation of eq. (3.5) will be:

$$\Psi(\vec{x},t) = \frac{\sqrt{2\pi}}{2}\left[\int_{Re.K^2(\omega)>\frac{1}{\beta^2}} d\omega\, e^{i\left[\omega\left(\frac{z}{v}-t\right)+\frac{\pi}{2}\right]} H_0^{(1)}\left(\frac{\omega\rho}{\gamma' v}\right) + c.c.\right] ,\rho>0 \qquad (10.7)\,.$$

Our justification for arriving at the solution via this procedure of analytic continuation lies in the uniqueness of the solution of Maxwell's equations. A direct verification for this comes from the instance that for the case of $v < c'(\omega)$ the solution has been obtained by Fermi through the use of other methods (Fermi 1940). Our result, obtained by the present procedure is identical to Fermi's result.

The calculation of the potentials and the fields follows the same procedure which we have used above. We will then be contented with just

quoting the results.

The Fields and Potentials for a Moving Electric Charge

Case 1: $Re.\ K^2(\omega) < \frac{1}{\beta^2}$

The Potentials

$$A(\vec{x},t) = \frac{1}{\pi}\left(\frac{e}{c}\right)\left[\int_{Re.K^2(\omega)<\frac{1}{\beta^2}}^{\infty} d\omega\, \mu(\omega)\, e^{i\omega\left(\frac{z}{v}-t\right)} K_0\left(\frac{\omega\rho}{\gamma' v}\right) + c.c.\right]$$

$$\varphi(\vec{x},t) = \frac{1}{\pi}\frac{e}{\beta c}\left[\int_{Re.K^2(\omega)<\frac{1}{\beta^2}}^{\infty} d\omega\, \frac{1}{\epsilon(\omega)}\, e^{i\omega\left(\frac{z}{v}-t\right)} K_0\left(\frac{\omega\rho}{\gamma' v}\right) + c.c.\right] \qquad (10.8)$$

The Fields

$$E_\ell(\rho,z,t) = \frac{1}{i\pi}\frac{e}{\beta^2 c^2}\left[\int_{Re.K^2(\omega)<\frac{1}{\beta^2}}^{\infty} d\omega\, \frac{\omega}{\epsilon(\omega)\gamma'^2(\omega)}\, e^{i\omega\left(\frac{z}{v}-t\right)} K_0\left(\frac{\omega\rho}{\gamma' v}\right) - c.c.\right]$$

$$E_t(\rho,z,t) = \frac{1}{\pi}\frac{e}{\beta^2 c^2}\left[\int_{Re.K^2(\omega)<\frac{1}{\beta^2}}^{\infty} d\omega\, \frac{\omega}{\epsilon(\omega)\gamma'(\omega)}\, e^{i\omega\left(\frac{z}{v}-t\right)} K_1\left(\frac{\omega\rho}{\gamma' v}\right) + c.c.\right] \qquad (10.9)\,.$$

$$B(\rho,z,t) = \frac{1}{\pi}\frac{e}{\beta c^2}\left[\int_{Re.K^2(\omega)<\frac{1}{\beta^2}}^{\infty} d\omega\, \frac{\omega\mu(\omega)}{\gamma'(\omega)}\, e^{i\omega\left(\frac{z}{v}-t\right)} K_1\left(\frac{\omega\rho}{\gamma' v}\right) + c.c.\right]$$

These are the fields given in eqs. (18) and (19) in Fermi's article which we cite above.

Case 2: $\mathrm{Re}\, K^2(\omega) > \frac{1}{\beta^2}$

In this case the potentials will have identical form to that given in eq. (3.6) and the fields will have the same form given in eq. (6.1), except now ϵ, μ and γ' will all be complex and the latter quantity is defined as follows:

$$\gamma' = \frac{1}{\sqrt{\beta^2 K^2(\omega) - 1}} .$$

In particular, the magnetic field must be written in the following form:

$$B(\rho, z, t) = \frac{e}{2\beta c^2}\left[i \int_{\mathrm{Re}\, K^2(\omega) > \frac{1}{\beta^2}}^{\infty} d\omega\, \omega \frac{\mu(\omega)}{\gamma'(\omega)} e^{i\omega\left(\frac{z}{v} - t\right)} H_1^{(1)}\left(\frac{\omega\rho}{\gamma' v}\right) + \mathrm{c.c.}\right] .$$

<u>The Asymptotic Fields</u>

These have the following form:

$$\vec{E}(\rho, z, t) = \frac{i}{\sqrt{2\pi}} \frac{e}{\sqrt{\beta c^3 \rho}} \left[\int_{\mathrm{Re}\, K^2(\omega) > \frac{1}{\beta^2}}^{\infty} d\omega\, \vec{\ell}_+(\omega) \frac{K(\omega)}{\epsilon(\omega)} \sqrt{\frac{\omega}{\gamma'(\omega)}}\, e^{i\left[\omega\left(\frac{\vec{n}_+(\omega)\cdot\vec{\lambda}}{c/K(\omega)} - t\right) - \frac{\pi}{4}\right]} + \mathrm{c.c.}\right]$$

$$\vec{B}(\rho, z, t) = \frac{\vec{\varphi}_1}{2\pi} \frac{e}{\sqrt{\beta c^3 \rho}} \left[\int_{\mathrm{Re}\, K^2(\omega) > \frac{1}{\beta^2}}^{\infty} d\omega\, \mu(\omega) \sqrt{\frac{\omega}{\gamma'(\omega)}}\, e^{i\left[\omega\left(\frac{\vec{n}_+(\omega)\cdot\vec{\lambda}}{c/K(\omega)} - t\right) - \frac{\pi}{4}\right]} + \mathrm{c.c.}\right] \tag{10.10}$$

The vectors $\vec{n}_+(\omega)$, $\vec{\ell}_+(\omega)$ are unit complex vectors defined

in a similar way to the real unit vectors which we introduced above via eq.(6.3). They are defined as follows:

$$\vec{n}_+(\omega) = \frac{\vec{\rho}_1 + \gamma'(\omega)\vec{k}}{\beta K(\omega)\gamma'(\omega)}$$
$$\vec{\ell}_+(\omega) = \frac{\gamma'(\omega)\vec{\rho}_1 - \vec{k}}{\beta K(\omega)\gamma'(\omega)} \tag{10.11}$$

They satisfy the usual cyclic realtion:

$$\vec{n}_+(\omega) \times \vec{\ell}_+(\omega) = \vec{\varphi}_1 .$$

Although the above results look formally like the results for the case of no dissipation, yet in actuality they lead to quite different physical results. We start by considering the spatial term in the phase-factor; namely:

$$\frac{\vec{n}_+(\omega)\cdot\vec{\lambda}}{c/K(\omega)} = \frac{1}{\beta c}\left(\frac{\rho}{\gamma'} + z\right) .$$

Since the present case must lead to the previous one in the limit of no dissipation, we expect to find in the phase-factor a term which would lead to outgoing waves. The velocity of these waves cannot depend on β. This condition proves very restrictive. To express this in the most convenient form, we introduce the following notation:

$$K^2(\omega) = K_1^2(\omega) + i K_2^2(\omega)$$
$$\frac{1}{\gamma'} = \eta e^{i\delta} \tag{10.12}$$

From the definition of γ' we then have the relation:

$$\left(1 - \eta^2 \sin^2\delta\right) + \eta^2 \cos^2\delta = \beta^2 K_1^2(\omega) .$$

With the help of this relation we introduce now the following two real orthonormal vectors:

$$\vec{n}'(\omega) = \frac{\eta\cos\delta}{\beta K_1}\vec{\rho}_1 + \frac{\sqrt{1-\eta^2\sin^2\delta}}{\beta K_1}\vec{k}$$

$$\vec{\ell}'(\omega) = \frac{\sqrt{1-\eta^2\sin^2\delta}}{\beta K_1}\vec{\rho}_1 - \frac{\eta\cos\delta}{\beta K_1}\vec{k} \qquad (10.13),$$

which satisfy the cyclic relation:

$$\vec{n}'(\omega)\times\vec{\ell}'(\omega) = \vec{\varphi}_1 .$$

Using these new vectors, we find that the only separation of the spatial term in the phase-factor which would give rise to outgoing waves with phase velocity independent of β is the following:

$$\frac{\vec{n}_+(\omega)\cdot\vec{x}}{c/K(\omega)} = \left[\frac{\vec{n}'}{c/K_1} + \frac{1-\sqrt{1-\eta^2\sin^2\delta}}{\beta c}\vec{k} + i\,\frac{\eta\sin\delta}{\beta c}\vec{\rho}_1\right]\cdot\vec{x} \qquad (10.14) .$$

The first term on the R.H.S. tells us that the electromagnetic radiation from the particle travels along $\vec{n}'(\omega)$. This gives a Cerenkov angle ϑ_c equal to:

$$\vartheta_c = \sin^{-1}\frac{\sqrt{1-\eta^2\sin^2\delta}}{\beta K_1} \qquad (10.15) .$$

The phase velocity of these waves is given by:

$$c'(\omega) = \frac{c}{K_1(\omega)} \qquad (10.16) .$$

Remembering the definition of $K_1(\omega)$ given by eq. (10.12) above, we find from this the relation:

$$\beta'^2(\omega) = \beta^2 . \mathrm{Re}\, K^2(\omega) ,$$

and the condition for Cerenkov radiation becomes:

$$\mathrm{Re}.\, K^2(\omega) > \frac{1}{\beta^2} .$$

This then is another proof of the correctness of our procedure of analytic continuation which we have followed in this section. Furthermore, if we suppress dissipation, then eqs. (10.15) and (10.16) will go over onto the respective equations for the case of no dissipation.

The remaining two terms on the R.H.S. of eq. (10.14) are a direct consequence of the presence of dissipation and they would vanish in its absence. The first term among them gives a standing wave along the z -axis which acts as a modulation on the Cerenkov radiation causing its amplitude to become z -dependent. The last term gives rise to the radial attenuation of the wave caused by the dissipation.

We now turn to the polarization of the field. This also is modified in a major manner by the presence of dissipation. The complex polarization vector $\vec{\ell}_+(\omega)$ now has the following form:

$$K(\omega)\vec{\ell}_+(\omega) = \frac{\beta'(\omega)}{\beta}\left\{\frac{\sqrt{1-\eta^2\sin^2\delta} + e^{i\delta}\eta^2\cos\delta}{(\beta K_1)^2}\vec{\ell}' + \frac{\eta\cos\delta - e^{i\delta}\eta\sqrt{1-\eta^2\sin^2\delta}}{(\beta K_1)^2}\vec{n}'\right\}$$

$$= \frac{\beta'(\omega)}{\beta}\left\{\frac{\sin\vartheta_c + e^{i\delta}\eta\cos\vartheta_c}{\beta'}\vec{\ell}' + \frac{\cos\vartheta_c - e^{i\delta}\eta\sin\vartheta_c}{\beta'}\vec{n}'\right\} \qquad (10.17)$$

$$\equiv \frac{\beta'}{\beta}\left[\Gamma_t(\omega)\,e^{i\delta_t(\omega)}\vec{\ell}' + \Gamma_\ell(\omega)\,e^{i\delta_\ell(\omega)}\vec{n}'\right]$$

where we have made use of eqs. (10.15) and (10.16). The angle δ is the parameter which characterizes the dissipation. For no dissipation it must vanish. To first order in δ we have:

$$\Gamma_t(\omega)\,e^{i\delta_t(\omega)} = \frac{1}{\beta'^2}\left(1 + i\eta^2\delta + O(\delta^2)\right) \qquad (10.18) .$$

$$\Gamma_\ell(\omega)\,e^{i\delta_\ell(\omega)} = \frac{1}{\beta'^2}\left(-i\eta\delta + O(\delta^2)\right)$$

As we see from eq. (10.17) above the presence of dissipation brings about a fundamental modification of the electric field. It causes it to develop a longitudinal component. For weak dissipation this component will be quite small, being proportional to δ.

In order to write the field in a significant manner, we introduce some more notation:

$$\epsilon(\omega) \equiv |\epsilon(\omega)|\, e^{i\delta_\epsilon(\omega)}$$

$$\mu(\omega) \equiv |\mu(\omega)|\, e^{i\delta_\mu(\omega)}$$

$$\frac{1}{\lambda_t(\omega)} \equiv \frac{\omega}{\beta c}\sqrt{1-\beta'^2\sin^2\vartheta_c}$$

$$\frac{1}{\lambda_\ell(\omega)} \equiv \frac{\omega}{\beta c}\left(1-\beta'\sin\vartheta_c\right) \qquad (10.18a)\ .$$

The fields then acquire the following form:

$$\vec{E}(\rho,z,t) = \frac{1}{\sqrt{2\pi}}\frac{e}{\sqrt{\beta^3c^3\rho}}\left\{\int\limits_{\mathrm{Re}\,K^2(\omega)>\frac{1}{\beta^2}}^{\infty} d\omega\, \frac{\beta'(\omega)}{|\epsilon(\omega)|}\sqrt{\frac{\omega}{|\gamma'(\omega)|}}\, e^{-\frac{\rho}{\lambda_t}} \cdot e^{i\left[\omega\left(\frac{\vec{n}'\cdot\vec{z}}{c'}-t\right)+\frac{z}{\lambda_\ell}+\frac{\delta}{2}-\delta_\epsilon-\frac{\pi}{4}\right]} \cdot \left(\Gamma_t e^{i\delta_t}\vec{\ell}'+\Gamma_\ell e^{i\delta_\ell}\vec{n}'\right) + c.c.\right\} \qquad (10.19)\ ,$$

$$\vec{B}(\rho,z,t) = \vec{q}_1\frac{1}{\sqrt{2\pi}}\frac{e}{\sqrt{\beta c^3\rho}}\left\{\int\limits_{\mathrm{Re}\,K^2(\omega)>\frac{1}{\beta^2}}^{\infty} d\omega\, |\mu(\omega)|\sqrt{\frac{\omega}{|\gamma'(\omega)|}}\, e^{-\frac{\rho}{\lambda_t}} \cdot e^{i\left[\omega\left(\frac{\vec{n}'\cdot\vec{z}}{c'}-t\right)+\frac{z}{\lambda_\ell}+\frac{\delta}{2}+\delta_\mu-\frac{\pi}{4}\right]} + c.c.\right\}.$$

The Poynting Flux

We just quote the result corresponding to eq. (7.7). This is given as follows:

$$\int_{-\infty}^{\infty}\vec{S}\,dt = \frac{1}{2\pi}\frac{e^2}{\beta^2c^2\rho}\int\limits_{\mathrm{Re}\,K^2(\omega)>\frac{1}{\beta^2}}^{\infty} d\omega\left[\cos(\vartheta_c-\vartheta')\,\vec{n}'+\sin(\vartheta_c-\vartheta')\,\vec{\ell}'\right] \cdot e^{-\frac{2\rho}{\lambda_t}}\sqrt{\cos^2\delta_\epsilon+\eta^2\cos^2(\delta-\delta_\epsilon)}\,\frac{\omega}{|\epsilon\gamma'|}.$$

The angle ϑ' is defined as follows:

$$\vartheta' = \sin^{-1}\frac{\cos\delta_\epsilon}{\sqrt{\cos^2\delta_\epsilon+\eta^2\cos^2(\delta-\delta_\epsilon)}}\ .$$

The energy travels along the direction $\vec{s}(\omega)$ given by:

$$\vec{s}(\omega) = \cos(\vartheta_c-\vartheta')\,\vec{n}' + \sin(\vartheta_c-\vartheta')\,\vec{\ell}'\ .$$

This is rotated by the angle $(\vartheta_c-\vartheta')$ away from the direction in which the wave fronts travel. Thus the energy travels along a direction different from that of the phase-velocity. It makes an angle $(\vartheta_c-\vartheta')$ with the phase velocity. In the limit of no dissipation $\vartheta'\to\vartheta_c$ as we see from the definition of these angles. Since the magnitude of the velocity must be equal to c' in the limit of no dissipation, we conclude that it must be so also in the case of dissipation. Because there is no other way in which we can manipulate our above result in order to put it in the form of eq. (7.7). Consequently, we have the final result:

$$\int_{-\infty}^{\infty}\vec{S}\,dt = \frac{1}{2\pi}\frac{e^2}{\beta^3c^3\rho}\int\limits_{\mathrm{Re}\,K^2(\omega)>\frac{1}{\beta^2}}^{\infty} d\omega\, c'(\omega)\,\vec{s}(\omega)\, e^{-\frac{2\rho}{\lambda_t}}\,\frac{\sqrt{\cos^2\delta_\epsilon+\eta^2\cos^2(\delta-\delta_\epsilon)}\,\beta'\omega}{|\epsilon\gamma'|} \qquad (10.20)$$

From this result we obtain the expression for the radiated linear energy density:

$$\frac{dV}{dz} = \left(\frac{e}{\beta c}\right)^2 \int_{Re.\,k^2(\omega) > \frac{1}{\beta^2}}^{\infty} d\omega \; \frac{\omega \cos(\delta - \delta_\epsilon)}{\left|\sqrt{\epsilon(\omega)}\; \gamma'(\omega)\right|^2} \; e^{-\frac{2\rho}{\lambda_t}} \tag{10.21}$$

This reduces to eq. (8.8) in the limit of no dissipation. It is also identical to the result given by Budini (1953) using a different method of calculation.

Finally, we may mention that the exact expression for the fields which we present in this section should be very useful in calculating the energy loss of charged particles of very high velocity in matter. They should play the same role for particles with $V > c'$ as that played by the fields given by eq. (10.9) in Fermi's calculation which we cited above.

B. A Moving Magnetic Charge

The potentials and fields can be obtained from the case with no dissipation in an identical manner to the electric case. We will be content to give the results.

Case 1: $V < c'(\omega)$

$$A'(\rho, z, t) = \frac{1}{\pi} \frac{e'}{c} \left[\int_{Re.\,k^2(\omega) < \frac{1}{\beta^2}}^{\infty} d\omega \; e^{i\omega\left(\frac{z}{V} - t\right)} K_0\!\left(\frac{\omega\rho}{\gamma' V}\right) + c.c. \right]$$

$$E\,(\;, z, t) = \frac{1}{\pi} \frac{e'}{\beta c^2} \left[\int_{Re.\,k^2(\omega) < \frac{1}{\beta^2}}^{\infty} d\omega \; \frac{\omega}{\gamma'(\omega)} \; e^{i\omega\left(\frac{z}{V} - t\right)} K_1\!\left(\frac{\omega\rho}{\gamma' V}\right) + c.c. \right] \tag{10.22}$$

$$B_t(\rho, z, t) = \frac{-1}{\pi} \frac{e'}{\beta^2 c^2} \left[\int_{Re.\,k^2(\omega) < \frac{1}{\beta^2}}^{\infty} d\omega \; \frac{\omega}{\gamma'(\omega)} \; e^{i\omega\left(\frac{z}{V} - t\right)} K_1\!\left(\frac{\omega\rho}{\gamma' V}\right) + c.c. \right] \tag{10.22}$$

$$B_\ell(\rho, z, t) = \frac{i}{\pi} \frac{e'}{\beta^2 c^2} \left[\int_{Re.\,k^2(\omega) < \frac{1}{\beta^2}}^{\infty} d\omega \; \frac{\omega}{\gamma'^2(\omega)} \; e^{i\omega\left(\frac{z}{V} - t\right)} K_0\!\left(\frac{\omega\rho}{\gamma' V}\right) - c.c. \right]$$

Case 2: $V > c'(\omega)$

In this case the potentials and the fields will be given by the same expressions as before, except that all quantities containing the susceptibilities will now be complex. This is identical to the case of the electric charge which we treated above. We just quote the result for the asymptotic fields:

$$\vec{E}(\rho, z, t) = \vec{\varphi}_1 \frac{1}{\sqrt{2\pi}} \frac{e'}{\sqrt{\beta c^3 \rho}} \left[\int_{Re.\,k^2(\omega) > \frac{1}{\beta^2}}^{\infty} d\omega \sqrt{\frac{\omega}{|\gamma'(\omega)|}} \; e^{-\frac{\rho}{\lambda_t}} \cdot e^{i\left(\omega\left(\frac{\vec{n}'\cdot\vec{R}}{c'} - t\right) + \frac{z}{\lambda_\ell} + \frac{\delta}{2} - \frac{\pi}{4}\right)} + c.c. \right] \tag{10.23}$$

$$\vec{B}(\rho, z, t) = \frac{-1}{\sqrt{2\pi}} \frac{e'}{\sqrt{\beta^3 c^3 \rho}} \left[\int_{Re.\,k^2(\omega) > \frac{1}{\beta^2}}^{\infty} d\omega \; \beta'(\omega) \sqrt{\frac{\omega}{|\gamma'(\omega)|}} \; e^{-\frac{\rho}{\lambda_t}} \cdot e^{i\left(\omega\left(\frac{\vec{n}'\cdot\vec{R}}{c'} - t\right) + \frac{z}{\lambda_\ell} + \frac{\delta}{2} - \frac{\pi}{4}\right)} \cdot \left(\Gamma_t e^{i\delta_t} \vec{\ell}' + \Gamma_\ell e^{i\delta_\ell} \vec{m}'\right) + c.c. \right].$$

The Poynting Flux

$$\int_{-\infty}^{\infty} \vec{S}\, dt = \frac{1}{2\pi} \frac{e'^2}{\beta^3 c^3 \rho} \int_{Re.K^2(\omega) > \frac{1}{\beta^2}}^{\infty} d\omega \; c'(\omega) \vec{s}'(\omega) \, e^{-\frac{2\rho}{\lambda_b}} \frac{\sqrt{\cos^2\delta_\mu + \eta^2 \cos^2(\delta - \delta_\mu)}\; \beta' \omega}{|\mu\gamma'|} \qquad (10.24)$$

where,

$$\vec{s}'(\omega) = \cos(\theta_c - \theta'')\, \vec{n}' + \sin(\theta_c - \theta'')\, \vec{l}'$$

$$\theta'' = \sin^{-1} \frac{\cos\delta_\mu}{\sqrt{\cos^2\delta_\mu + \eta^2 \cos^2(\delta - \delta_\mu)}} ,$$

and all the other symbols retain the same meaning which we have used for them before.

The radiated energy density will have the following form:

$$\frac{dU}{dz} = \left(\frac{e'}{\beta c}\right)^2 \int_{Re.K^2(\omega) > \frac{1}{\beta^2}}^{\infty} d\omega \frac{\omega \cos(\delta - \delta_\mu)}{\left|\sqrt{\mu(\omega)}\, \gamma'(\omega)\right|^2} e^{-\frac{2\rho}{\lambda_b}} \qquad (10.25) .$$

11. EXPERIMENTAL CONSIDERATIONS

Magnetic Charge

The introduction of the concept of magnetic charge by Dirac in 1931 (Dirac 1931) has aroused a continuing and widening interest among physicits. This interest has been enhanced through the discovery that such entities do occur in gauge theories (Goddard 1978). Due to the great success which such

theories have played in the recent development of elementary particle physics, deductions from them are considered to be serious enough as to warrant experimental verification. Hence, the many efforts at an experimental detection of these entities (Amaldi 1972).

Cerenkov radiation from such particles can be utilized as a means for detecting them. This has been realized quite some time ago by Tompkins (1965). He suggested two experiments to look for particles carrying magnetic charge by making use of their Cerenkov radiation. In one experiment he proposed to compare the Cerenkov radiation pulses which would be emitted by a magnetic charge in two media with different dielectric constants and negligible dispersion. In the other he suggested the comparison of the radiation pulses which would be registered by such a particle in a Cerenkov counter and in a scintillation counter.

We propose here yet another method to look for a magnetic charge which is based on the Cerenkov radiation emitted by it. But whereas Tompkins ' s method requires no dispersion in the medium our attempts to utilize the dispersion in the medium in order to distinguish magnetic charge from electric charge. It is based on eqs. (7.8) and (9.8) for the linear radiated energy density. In a pure dielectric $\mu=1$ and we obtain the following

$$\frac{dU^t}{dz} = \left(\frac{e}{c}\right)^2 \int_{K^2(\omega) > \frac{1}{\beta^2}}^{\infty} d\omega \;\; \omega \left(1 - \frac{1}{\beta'^2(\omega)}\right) \qquad (11.1)$$

The corresponding relation for a magnetic charge is given by:

$$\frac{dU^t}{dz} = \left(\frac{e'}{c}\right)^2 \int_{K^2(\omega) > \frac{1}{\beta^2}}^{\infty} d\omega \;\; \epsilon(\omega)\, \omega \left(1 - \frac{1}{\beta'^2(\omega)}\right) \qquad (11.2) .$$

These two results differ from each other by the factor $\epsilon(\omega)$ which occurs in the integrand in the magnetic case and is lacking in the electric case. Since $\epsilon(\omega)$ is highly frequency dependent and is appreciably higher than unity for wide ranges of ω this occurence will have two important consequences. First, the Cerenkov spectrum from a magnetic charge will be more frequency dependent and in a definite and predictable manner than that from an electric charge. Second, a magnetic charge would radiate more energy (per unit charge squared) than an electric charge.

Hence, in our view the determination of the spectrum of Cerenkov radiation in a highly dispersive medium should constitute a very effective experimental method for the detection of fast magnetic charges. Measurement of the intensity of the radiated energy is another effective method, except that this depends on what choice we take for the magnetic charge. This then would suggest to us that if magnetic charges are detected via their spectrum, then the magnitude of their charges relative to a known electric charge could be determined via a measurement of the intensity of their Cerenkov radiation.

Electric Dipole Moments

The discovery of the violation of CP- invariance by Cronin and Fitch in 1964 (Christensen 1964) removed the last theoretical obstacle to the existence of electric dipole moments for elementary particles. This gave rise to a great deal of work both theoretical and experimental aimed at establishing such properties for elementary particles (Ramsey 1982). However, since as far back as 1950, Ramsey (Purcell 1950) has always maintained that the establishment of the discrete symmetries should be based on experiment. He has considered that experiments to set a limit on the value of the electric dipole moments for elementary particles would be of the greatest importance towards this end.

The most stringent experimental results have been obtained so far with neutrons for the obvious reason that their total charge is zero. The results for charged particles have been several orders of magnitude less precise. Due to the immense difficulty involved in the measurement of

electric dipole moments of charged particles any new method of measurement should be welcome. We present here a method based on the Cerenkov radiation from an electric dipole.

Due to the smallness of the magnitude of the electric dipole moment - assuming it exists - we must seek measurable quantities which contain it to the first order. This precludes the use of the radiated energy since this involves the square of the electric dipole (Balazs 1956). We then look for interference effects between the charge of the particle, its magnetic dipole moment and its electric dipole moment.

To this end we need the fields of a moving eletric dipole. These are obtained from eq. (6.5) via the procedure of operating on them by the operator:

$$\frac{1}{e}\,\vec{p}\cdot\vec{\nabla}\,,$$

where $\vec{p}$ is the magnetic dipole moment. The result is as follows:

$$\vec{E}^{+}(\rho,z,t) = \frac{2}{\sqrt{2\pi}}\,\frac{1}{\sqrt{\beta^2 c^3 \rho}}\int_{K^2(\omega)>\frac{1}{\beta^2}}^{\infty} d\omega\;\vec{\ell}_{+}(\omega)\,\frac{\vec{p}\cdot\vec{n}_{+}(\omega)}{c'(\omega)}\,\frac{\omega\beta'(\omega)}{\epsilon(\omega)}\sqrt{\frac{\omega}{\gamma'(\omega)}}\;\cdot\cos\left[\omega\left(\frac{\vec{n}_{+}\cdot\vec{\lambda}}{c'}-t\right)+\frac{\pi}{4}\right] \tag{11.2}$$

$$\vec{B}^{+}(\rho,z,t) = \vec{q}_1\,\frac{2}{\sqrt{2\pi}}\,\frac{1}{\sqrt{\beta c^3\rho}}\int_{K^2(\omega)>\frac{1}{\beta^2}}^{\infty} d\omega\;\frac{\vec{p}\cdot\vec{n}_{+}(\omega)}{c'(\omega)}\,\omega\mu(\omega)\sqrt{\frac{\omega}{\gamma'(\omega)}}\;\cos\left[\omega\left(\frac{\vec{n}_{+}\cdot\vec{\lambda}}{c'}-t\right)+\frac{\pi}{4}\right].$$

Similarly, we obtain the fields for a magnetic dipole $\vec{q}$ by operating on eq. (9.5) by the operator:

$$\frac{1}{e'}\,\vec{q}\cdot\vec{\nabla}\,.$$

We obtain the following expressions:

$$\vec{E}^{+}(\rho,z,t) = \vec{q}_1\,\frac{2}{\sqrt{2\pi}}\,\frac{1}{\sqrt{\beta c^3\rho}}\int_{K^2(\omega)>\frac{1}{\beta^2}}^{\infty} d\omega\;\frac{\vec{q}\cdot\vec{n}_{+}(\omega)}{c'(\omega)}\,\omega\sqrt{\frac{\omega}{\gamma'(\omega)}}\;\cdot\cos\left[\omega\left(\frac{\vec{n}_{+}\cdot\vec{\lambda}}{c'}-t\right)+\frac{\pi}{4}\right]$$

$$\vec{B}^{+}(\rho,z,t) = -\frac{2}{\sqrt{2\pi}}\frac{1}{\sqrt{\beta^3 c^3 \rho}}\int_{K^2(\omega)>\frac{1}{\beta^2}}^{\infty} d\omega\, \vec{\ell}_{+}(\omega)\, \frac{\vec{q}\cdot\vec{n}_{+}(\omega)}{c'(\omega)} \cdot \omega\beta'(\omega)\sqrt{\frac{\omega}{\gamma'(\omega)}}\cos\left[\omega\left(\frac{\vec{n}_{+}\cdot\vec{\lambda}}{c'}-t\right)+\frac{\pi}{4}\right] . \tag{11.3}$$

By comparing eq.(11.2) with eq. (6.5) we see that the fields of an electric dipole differ in phase by $\frac{\pi}{2}$ from those of an electric charge. Due to this the interference terms contributing to the total energy radiation vanish. As to the interference terms between the electric dipole and the magnetic dipole, they vanish since their respective fields are orthogonal. Due to either of these two reasons there will also be no interference term in the radiated energy between the charge and its magnetic dipole.

We are left with the electric field. For a particle such as the electron its electric field will be given by the sum of the three contributions coming from eqs.(6.5), (11.2) and (11.3). We consider the first two. They are parallel and they add to the following expression:

$$\vec{E}^{+}(\rho,z,t) = \frac{2}{\sqrt{2\pi}}\frac{e}{\sqrt{\beta^3 c^3 \rho}}\int_{K^2(\omega)>\frac{1}{\beta^2}}^{\infty} d\omega\, \vec{\ell}_{+}(\omega)\, \frac{\beta'(\omega)}{\epsilon(\omega)}\sqrt{\frac{\omega}{\gamma'(\omega)}}\cos\left[\omega\left(\frac{\vec{n}_{+}\cdot\vec{\lambda}}{c'}-t\right)-\frac{\pi}{4}+\delta\right] \tag{11.4}$$

where,

$$\delta \equiv \frac{\vec{p}\cdot\vec{n}_{+}(\omega)}{e\,c'(\omega)/\omega} \ll 1 \tag{11.5}$$

This means that the possession of an electric dipole moment by an electron changes the phase of the elctric field in its Cerenkov radiation by the

angle δ. This angle is linear in the magnitude of the electric dipole moment. Since the Cerenkov radiation is independent of the mass of the radiating particle, the above angle can in principle be determined by the interference between the fields emitted by an electron and a spinless particle such as a π-meson.

Measurement of the Longitudinal Component of $\vec{E}$

In trying to detect the longitudinal component of the electric field in the case of dissipation experimentalists will be forced to consider semi-transparent media for which absorption will be very weak. Considering the case of a pure dielectric we then have the following relations which will characterize this medium:

$$\mu = 1$$
$$\delta_\epsilon \approx \delta \ll 1 \tag{11.6}$$

To the first order in δ, the vector term in eq. (10.19) acquires the following form:

$$\left(\Gamma_t e^{i\delta_t}\vec{\ell}' + \Gamma_\ell e^{i\delta_\ell}\vec{m}'\right) \approx (1+i\eta\delta)\left(\vec{\ell}' - i\eta\delta\,\vec{m}'\right) .$$

From this we see that the ratio of the amplitude of the longitudinal to the transverse field varies as: $\eta\delta$. Thus if a measurement is made on the electric field itself, then the quantity to be measured will be of order δ.

On the other hand, the existence of the longitudinal field can also be ascertained through the determination of the direction of the Poynting flux. This is tantamount to measuring the angle $(\theta_e-\theta')$ which will be given by:

$$(\theta_e-\theta') \approx \frac{\eta}{(1+\eta^2)}\left(\frac{1-\eta^2}{2}\cdot\delta^2 + \frac{1}{2}\delta_\epsilon^2\right) \approx \delta^2 .$$

In this case the quantity to be measured is of order δ^2. However, in measuring energy fluxes a higher accuracy is more readily attainable than in measuring fields of force. Thus this second method might offer better possibilities for the detection of the longitudinal field than the first one.

12. CONCLUSIONS

We have presented a rigorous and complete treatement of Cerenkov rediation in both non-dissipative and dissipative media. Since our method gives the exact expressions for the fields of moving electric and magnetic charges, it allows us to deduce all the relevant properties of this radiation. We have also developped a rigorous method for the calculation of the radiated energy density.

These methods have a wide range of applicability. In applying them to the case of tachyons we derive the asymptotic expressions for the fields and we obtain a regularized expression for the linear density of the energy radiated by them. They can also be applied readily to find the fields of moving electric and magnetic multipoles. We show this explicitely for the case of dipoles.

The case of a dissipative medium warrants the following two remarks . First, our fields for the case $V > C'$ should now be used to calculate the energy loss of fast charged particles in matter in the manner used by Fermi for particles with $V < C'$. Second, an experimental effort should be made to detect the longitudinal componenet which the electric field develops in such a medium.

Finally, it might be of use to mention that if the need arises our fields would facilitate the calculation of the energy loss in matter of fast moving magnetic monopoles.

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FOOTNOTE

* For completeness and to compare with other published results (Balazs 1956) we quote here our result for the energy radiated by an electric dipole with moment p moving along the z-axis with velocity $v = c\,\beta$.

$$\frac{dU}{dz} = \frac{p^2}{\beta^2 c^4} \int_{k^2(\omega) > \frac{1}{\beta^2}}^{\infty} d\omega \left(\cos^2\vartheta + \frac{\beta'^2 - 1}{2} \sin^2\vartheta \right) \mu(\omega)\, \omega^3 \left(1 - \frac{1}{\beta'^2}\right),$$

where ϑ is the angle which the dipole makes with the z-axis. The energy radiated by a magnetic dipole of moment q is obtained from the above via the replacement of p by q and of $\mu(\omega)$ by $\varepsilon(\omega)$.

FIGURE CAPTIONS

Fig. 1 The Cerenkov Cone generated by $\vec{\ell}_+(\omega)$ and the Cone orthogonal to it generated by $\vec{n}_+(\omega)$

FIG. 2 The Contour C in the Complex - n plane

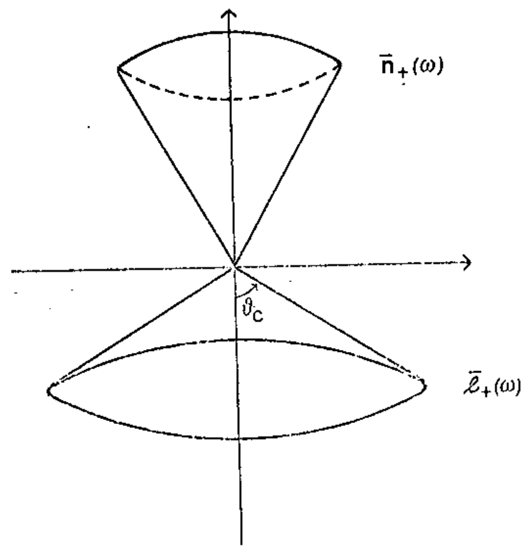

Fig.1

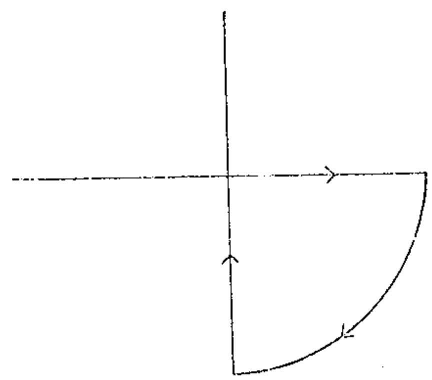

Fig.2

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