Tree-likeness of dendroids and λ-dendroids

by

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By a continuum is meant a compact, connected, metric space. A dendroid is an arcwise connected, hereditarily unicoherent continuum, [4]. A λ-dendroid is an hereditarily decomposable, hereditarily unicoherent continuum, [5]. A continuum M is said to be tree-like if it is degenerate or if, for every positive number ε, there is an ε-map throwing M onto a finite tree, and arc-like if, for every positive number ε, there is an ε-map throwing M onto an arc. It has not been known that every dendroid is tree-like. In this note, we establish a theorem (Theorem 1) from which it follows that every dendroid and every λ-dendroid is tree-like. Our Theorem 1 is also used to establish (Theorem 2) that, if the intersection of two tree-like continua is connected and non-empty, then their union is also tree-like. This latter theorem is analogous to Ingram's theorem [7] that, if the intersection of two arc-like continua is connected and non-empty, and if their union is atriodic, then their union is arc-like.

Bing has shown ([2], Theorem 11) that every non-degenerate, hereditarily decomposable, hereditarily unicoherent, atriodic continuum is arc-like; and Fugate has shown, [9], that every non-degenerate hereditarily unicoherent, atriodic continuum each of whose indecomposable subcontinua are arc-like is itself arc-like. Theorem 1 of this paper is analagous to Fugate's above mentioned theorem. It follows from ([2], Theorem 6) that every planar λ-dendroid is tree-like (if it be observed that each subcontinuum of a λ-dendroid is a λ-dendroid and no planar λ-dendroid separates the plane). Fugate has shown, [8], that certain (not necessarily planar) dendroids, called smooth dendroids, are tree-like.

LEMMA 1. *Suppose that M is a decomposable, unicoherent continuum, P is a connected, one dimensional polyhedron, and f is an essential map of M into P. Then there is a proper subcontinuum M' of M such that $f|M'$ is essential.*

Proof. Suppose the contrary. Let H and K denote two proper subcontinua of M such that $M = H \cup K$ and let m_0 denote a point of the subcontinuum $H \cap K$ of M. Let $p_0 = f(m_0)$. Denote by X the universal covering space of P, with projection π, and denote by x_0 a point of X such that $\pi(x_0) = p_0$. Since $f|H$ and $f|K$ are both homotopic to a constant, it follows from the covering homotopy theorem, [6], that there exist continuous mappings T_H: $H \to X$ and T_K: $K \to X$ such that $T_H(m_0) = T_K(m_0) = x_0$, $\pi T_H = f|H$, and $\pi T_K = f|K$. Suppose that $T_H|(H \cap K) \neq T_K|(H \cap K)$. Let $Z = \{z \in H \cap K | T_H(z) = T_K(z)\}$. Then $m_0 \in Z$ and Z is a closed proper subset of $H \cap K$.

Let $y_1, y_2, y_3, \ldots$ be a sequence of points of $(H \cap K)\setminus Z$ converging to a point $y \in Z$. For each n, $T_H(y_n) \neq T_K(y_n)$. Let O denote an open subset of X containing $T_H(y) = T_K(y)$ such that $\pi|O$ is a homeomorphism. There exists a positive integer N such that, if $n > N$, $T_H(y_n) \in O$ and $T_K(y_n) \in O$. Then, if $n > N$, $\{T_H(y_n)\} = [\pi^{-1}\pi T_H(y_n)] \cap O = [\pi^{-1}\pi T_K(y_n)] \cap O = \{T_K(y_n)\}$, a contradiction. Thus $T_H(H \cap K) = T_K(H \cap K)$. Hence, there is a transformation f^*: $M \to X$ such that $f^*|H = T_H$ and $f^*|K = T_K$; f^* is continuous; and $\pi f^* = f$.

Since X is a 1-complex (infinite) which contains no simple closed curve and $f^*[M]$ is a compact continuum lying in X, $f^*[M]$ is a tree and, thus, is contractible. Then πf^* is inessential. But $\pi f^* = f$ which is essential, a contradiction.

THEOREM 1. *Suppose that M is an hereditarily unicoherent continuum such that, if X is an indecomposable subcontinuum of M, X is tree-like. Then M is tree-like.*

Proof. Suppose that $\dim M \geqslant 2$. It follows from a theorem of Alexandroff ([1], p. 170) and a theorem of Mazurkiewicz ([11], Cor. 1) that there is a subcontinuum M_1 of M and an essential map f of M onto a circle J. There is ([10], p. 281), a subcontinuum M_2 of M_1 such that $f|M_2$ is essential but, if M_3 is a proper subcontinuum of M_2, then $f|M_3$ is inessential. Then, since M_2 is unicoherent, it follows from Lemma 1 that M_2 is indecomposable. Then M_2 is tree-like. Then every mapping of M_2 onto a circle is inessential, ([3], Theorem 1), a contradiction. Then $\dim M \leqslant 1$.

If $\dim M = 0$, M is degenerate and, hence, is tree-like.

Suppose $\dim M = 1$ but M is not tree-like. Then, [3, Theorem 1], there is a one-dimensional polyhedron P and an essential map g of M onto P. There is, ([10], p. 281), a sub-continuum M' of M such that $g|M'$ is essential but, if M'' is a proper subcontinuum of M', then $g|M''$ is inessential. Since M' is unicoherent, it follows from Lemma 1 that M' is indecomposable. Then M' is tree-like and, [3, Theorem 1], every mapping of M' into P is inessential, a contradiction. Thus M is tree-like.

COROLLARY. *Every dendroid and every λ-dendroid is tree-like.*

The author has been told that several people (including Fugate) know the following Lemma 2 but, since he has not seen it in print, its proof is included here.

LEMMA 2. *If the continuum M is the union of two hereditarily unicoherent continua H and K whose intersection is a continuum, then M is hereditarily unicoherent.*

Proof. Suppose that X and Y are subcontinua of M and $X \cap Y$ is the union of two mutually exclusive closed sets U and V.

Suppose that $X \subset M\setminus K$, then Y is not a subset of H. One component C_U of $Y \cap H$ intersects both U and $H \cap K$ and one component, C_V, of $Y \cap H$ intersects both V and $H \cap K$. Then X and $C_U \cup C_V \cup (H \cap K)$ are intersecting subcontinua of H whose intersection is not connected, a contradiction. Thus X and Y each intersect both H and K.

Suppose C is a component of $X \cap Y$ which is a subset of $M\setminus K$. Let C_X denote the component containing C of $H \cap X$ and let C_Y denote the component containing C of $H \cap Y$. Then $C_X \cup (H \cap K)$ and $C_Y \cup (H \cap K)$ are intersecting subcontinua of H whose intersection is not connected. Thus, every component of $X \cap Y$ intersects both H and K and, hence, intersects $H \cap K$.

If $X \cap H \cap K$ and $Y \cap H \cap K$ were both connected, then $X \cap Y \cap H \cap K$ would be connected and would intersect every component of $X \cap Y$ and, hence, $X \cap Y$ would be connected. Suppose $X \cap H \cap K$ is the union of two mutually exclusive closed sets L_1 and L_2. Then there is a subcontinuum X_1 of X irreducible from L_1 to L_2; $X_1\setminus(L_1 \cup L_2)$ is connected ([12], Theorem 47, p. 16) and, thus, is a sub-set either of H or of K. Then $(H \cap K) \cup \mathrm{Cl}[X_1\setminus(L_1 \cup L_2)]$ (where Cl denotes closure) is a unicoherent continuum, a contradiction. Similarly, the assumption that $Y \cap H \cap K$ is not connected leads to a contradiction. Thus M is hereditarily unicoherent.

THEOREM 2. *If the continuum M is the union of two tree-like continua H and K whose intersection is connected, then M is tree-like.*

Proof. By Lemma 2, M is hereditarily unicoherent. Suppose T is an indecomposable subcontinuum of M which is not tree-like. Then T intersects both $M\setminus H$ and $M\setminus K$ and $T \cap H$ and $T \cap K$ are proper subcontinua of T. But $T = (T \cap H) \cup (T \cap K)$ and is, therefore, decomposable, a contradiction. Thus, by Theorem 1, M is tree-like.

References

[1] P. Alexandroff, *Dimensionstheorie, Ein Beitrag zur Geometrie der abgeschlossenen Mengen*, Math. Ann. 106 (1932), pp. 161–238.

[2] R. H. Bing, *Snake-like continua*, Duke Math. J. 18 (1951), pp. 653–663.

[3] J. H. Case and R. E. Chamberlin, *Characterizations of tree-like continua*, Pacific J. Math. 10 (1960), pp. 73–84.

[4] J. J. Charatonik, *Ramification points in the classical sense*, Fund. Math. 51 (1962), pp. 227–252.
[5] — *On decompositions of λ-dendroids*, Fund. Math. 67 (1970), pp. 15–30.
[6] M. L. Curtis, *The covering homotopy theorem*, Proc. Amer. Math. Soc. 7 (1956), pp. 682–684.
[7] W. T. Ingram, *Decomposable circle-like continua*, Fund. Math. 63 (1968), pp. 193–198.
[8] J. B. Fugate, *A sufficient condition that a compact metric continuum be chainable*, to appear.
[9] — *Retracting dendroids onto trees*, Notices Amer. Math. Soc. 15 (1968), pp. 773.
[10] K. Kuratowski, *Topologie* II, Warszawa 1961.
[11] S. Mazurkiewicz, *Sur l'existence des continus indécomposables*, Fund. Math. 25 (1935), pp. 327–328.
[12] R. L. Moore, *Foundations of point set theory*, Amer. Math. Soc. Colloq. Publ., vol. 13, New York, 1962.

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A 2-complex is collapsible if and only if it admits a strongly convex metric

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§ 1. Introduction. A metric d on a compact space X is *strongly convex* if, for any two points $x, y \in X$, there is a unique point $m \in X$ such that $d(x, m) = d(m, y) = \frac{1}{2}d(x, y)$. In the last few years, there has been considerable interest in characterizing the spaces which admit convex metrics. Lelek and Nitka [3] and Rolfsen [4] have shown that cells are the only compact 2 and 3-dimensional spaces which admit strongly convex metrics with the property that no midpoint of x and y is a midpoint of x and y' unless $y = y'$. Rolfsen [4] has further shown that the only compact n-manifold, $n \leqslant 3$, admitting a strongly convex metric is the cell.

It is well known (see [2]) that any compact space which admits a strongly convex metric is contractible, but Sieklucki [5] has demonstrated a contractible 2-complex which admits no strongly convex metric. Joseph Martin conjectured in 1966 that the stronger condition of collapsibility does characterize the 2-complexes which admit strongly convex metrics, and a proof of this is the object of this note. It is interesting to note that this theorem also provides, conversely, a topological characterization of collapsibility in 2-complexes, and thus cannot be directly extended to higher dimensions, for a 3-cell can have a non-collapsible triangulation [1].

§ 2. A collapsible 2-complex admits a strongly convex metric.

DEFINITIONS. All simplices are closed simplices. If $a_1, a_2, \ldots, a_k$ are points in a simplex σ, then $a_1a_2\ldots a_k$ is their convex hull in the linear structure of σ. A *triangle* is a 2-simplex in E^2 with the regular euclidean metric $\|x-y\|$.

All maps are continuous; if X and Y are spaces, the notation $f\colon X \to Y$ denotes a map from X onto Y. If K is a complex, then $K^{(k)}$ denotes the k-skeleton of K.

Let X be a compact space with a strongly convex metric d. Any two points x, y of X are joined in X by a unique arc, the *segment* $\widehat{xy}$, which is isometric to a closed interval of the real line ([2]). A *concave collection for* d is a finite collection T of segments in X satisfying: