Tree-likeness of hereditarily equivalent continua

by

H. Cook (Houston, Tex.)

A continuum is a compact, connected, metric space. An hereditarily equivalent continuum is a non-degenerate continuum which is homeomorphic to each of its non-degenerate subcontinua.

In [5], S. Mazurkiewicz raised the question: "Un continu dans l'espace à m dimensions qui est homéomorphe de tout continu qu'il contient, est-il necessairement un arc simple?" Moïse, [6], has answered this question in the negative by showing that the pseudo-arc is an indecomposable, planar, hereditarily equivalent continuum; however, G. W. Henderson, [3], has shown that each decomposable, hereditarily equivalent continuum is an arc. The pseudo-arc and the arc are the only known hereditarily equivalent continua. The author has heard, conversationally, speculation that (especially in light of the proof of D. W. Henderson, [2], that there is a continuum each non-degenerate subcontinuum of which is infinite dimensional) there may be an infinite dimensional, hereditarily equivalent continuum. Lemma 3 of this paper shows that this is not the case.

A tree-like continuum is one which, for every positive number ε, can be ε-mapped onto a finite tree. Each tree-like continuum is one-dimensional. Whyburn has shown, [8], that each planar, hereditarily equivalent continuum is tree-like. In this note it is shown that every hereditarily equivalent continuum is tree-like.

LEMMA 1. *Suppose that M is an hereditarily equivalent continuum and ε is a positive number. Then there is a homeomorphism h of M onto a proper subcontinuum of M such that, for each point x of M, the distance from x to $h(x)$ is less than ε.*

Proof. Let G be an uncountable monotonic collection of non-degenerate subcontinua of M (e.g., for some point p, let G be the collection to which g belongs if, and only if, for some positive number δ, g is the component containing p of the closed δ-neighborhood of p.) For each element g of G, let h_g denote a homeomorphism of M onto g. Let $X = \{h_g|\ g \in G\}$. Then X is a subset of M^M, the space of all mappings (i.e. continuous transformations) of M into M, and, thus, is separable and metric. Since every uncountable subset of X has a limit element, there is an element k of G such that h_k is a limit element of $\{h_g|\ g \in G$ and $g \subset k\}$ and of $\{h_g|\ g \in G$ and $k \subset g\}$, ([7], Theorem 6, p. 3). Thus there is a sequence $\{g_i\}$ of distinct elements of G which are proper subcontinua of k such that the sequence $\{h_{g_i}\}$ converges to h_k. Then the sequence $\{h_{g_i} \circ h_k^{-1}\}$ converges to the identity mapping of k onto itself and, for each i, $h_{g_i} \circ h_k^{-1}(k)$ is a proper subcontinuum, g_i, of k. Since M is homeomorphic to k, there also exists a sequence $\{f_i\}$ of homeomorphisms, each throwing M onto a proper subcontinuum of M such that the sequence $\{f_i\}$ converges to the identity mapping of M onto M. Thus, Lemma 1 is true.

LEMMA 2. *If M is an hereditarily equivalent continuum, every mapping of M into a connected one-dimensional polyhedron is homotopic to a constant.*

Proof. Suppose that Y is connected, one-dimensional polyhedron and there is an essential mapping of M into Y. Since the space Y^M of all mappings of M into Y is an ANR ([4], p. 260), if we show that there is an essential mapping of M into Y which is a limit point (in Y^M) of the set of inessential mapping of M into Y, we will have achieved a contradiction. Let f be a mapping of M onto Y which is essential; there is a subcontinuum K of M such that $f|K$ is essential but, if K' is a proper subcontinuum of K, $f|K'$ is inessential. Since K is homeomorphic to M, there is an essential mapping g of M into Y such that, if M' is a proper subcontinuum of M, $g|M'$ is inessential. Let $\varepsilon > 0$. Since g is uniformly continuous, there is a positive number δ such that, if h is a mapping of M into M at a distance in M^M less than δ from the identity mapping of M onto M then $[g|h(M)] \circ h$ is a distance less than ε from g in Y^M. But there is a homeomorphism h of M onto a proper subcontinuum M' of M which moves no point of M a distance greater than δ and $g|M'$ is inessential. Thus, the distance in Y^M from g to $[g|M'] \circ h$ is less than ε and is inessential. Thus g is an essential mapping of M into Y which is a limit point in Y^M of the set of inessential mappings of M into Y, our contradiction. Hence, every mapping of M into a one-dimensional polyhedron is inessential.

LEMMA 3. *Every hereditarily equivalent continuum is one-dimensional.*

Proof. Suppose M is an hereditarily equivalent continuum and $\dim M > 1$. Then ([4], p. 271) there is a subcontinuum M' of M and an essential mapping f of M' onto a circle. But M' is hereditarily equivalent, a contradiction to Lemma 2.

THEOREM. *Every hereditarily equivalent continuum is tree-like.*

Proof. Every hereditarily equivalent continuum is a one-dimensional continuum each mapping of which onto a one-dimensional polyhedron is inessential and, thus, [1], Theorem 1, is tree-like.

References

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THE UNIVERSITY OF HOUSTON

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