TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 354, Number 10, Pages 4085–4108
S 0002-9947(02)02984-7
Article electronically published on May 20, 2002

TREES AND BRANCHES IN BANACH SPACES

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ABSTRACT. An infinite dimensional notion of asymptotic structure is considered. This notion is developed in terms of trees and branches on Banach spaces. Every countably infinite countably branching tree $\mathcal{T}$ of a certain type on a space X is presumed to have a branch with some property. It is shown that then X can be embedded into a space with an FDD (E_i) so that all normalized sequences in X which are almost a skipped blocking of (E_i) have that property. As an application of our work we prove that if X is a separable reflexive Banach space and for some $1 < p < \infty$ and $C < \infty$ every weakly null tree $\mathcal{T}$ on the sphere of X has a branch C-equivalent to the unit vector basis of ℓ_p, then for all $\varepsilon > 0$, there exists a subspace of X having finite codimension which $C^2 + \varepsilon$ embeds into the ℓ_p sum of finite dimensional spaces.

1. INTRODUCTION

Consider the following problem:

> State an intrinsic condition which is necessary and sufficient for a separable Banach space X to be isomorphic to a subspace of $(\bigoplus F_n)_{\ell_p}$, $1 < p < \infty$, the ℓ_p-sum of finite dimensional spaces F_n, $n \in \mathbb{N}$.

Thinking about that problem, one might come up with the following property, which is clearly shared by all spaces isomorphic to subspaces of $(\bigoplus F_n)_{\ell_p}$, $\dim F_n < \infty$, for $n \in \mathbb{N}$:

$(*)$ For some $K < \infty$, every weakly null normalized sequence in X has a subsequence which is K-equivalent to the unit vector basis of ℓ_p.

Note that James' space J [Ja1] satisfies property $(*)$ for $p = 2$, but since it is not reflexive it cannot be isomorphic to a subspace of some ℓ_2-sum of finite dimensional spaces.

Taking this example into account, we could ask whether or not a reflexive and separable Banach space which enjoys property $(*)$ is isomorphic to a subspace of an ℓ_p-sum of finite dimensional spaces. In [J2] W. B. Johnson gave a positive answer in the case that X is a subspace of some L_p-space and asked for an answer in the general case.

More generally, a recurrent theme in Banach space theory takes the following form. One has some property (P) for sequences in Banach spaces and one assumes that in a given separable infinite dimensional Banach space X, every normalized weakly null sequence (or perhaps every normalized block basis of a given basis for X) admits a subsequence with (P). One then tries to deduce that X has some other

Received by the editors October 10, 2000 and, in revised form, November 7, 2001.
2000 *Mathematics Subject Classification.* Primary 46B03.
Research supported by NSF.

property (Q). In this paper we consider a stronger hypothesis on X, namely that every countably infinitely branching tree of ω-levels of some type admits a branch with property (P). And we shall show that the condition $(*)$ should be strengthened in this manner. By a tree in a Banach space X we mean a family $(x_A)_{A\in[\mathbb{N}]^{<\omega}} \subset X$ indexed over the finite subsets of $\mathbb{N}$, which we denote by $[\mathbb{N}]^{<\omega}$. A *normalized weakly null tree* is a family $(x_A)_{A\in[\mathbb{N}]^{<\omega}} \subset S_X$ with the property that the successors of each node, i.e., the sequences $(x_{\{n\}\cup A)})_{n\in\mathbb{N}}$ with $A \in [\mathbb{N}]^{<\omega}$, are weakly null. By a *branch* of a tree $(x_A)_{A\in[\mathbb{N}]^{<\omega}}$ we mean a sequence of the form $(x_{A_k})_{k=0}^\infty$ with A_n being an initial segment of A_{n+1} and $A_{n+1} \setminus A_n$ a singleton for $n \in \mathbb{N}$. Note that if (x_n) is a sequence in X, we can define a tree by letting $x_A = x_{\max A}$, and note that the set of all branches of (x_A) is the set of all subsequences of (x_n).

Therefore the hypothesis that every normalized weakly null tree has a branch with property (P) is stronger than the property that every normalized weakly null sequence of X has a subsequence with property (P), and is sometimes the right hypothesis to conclude that X has property (Q).

An example of this type is the solution to the problem raised above.

In Theorem 4.1 we will prove the following quantitative version: If X is reflexive and there exist $1 < p < \infty$ and $C < \infty$ so that every normalized weakly null tree in X admits a branch C-equivalent to the unit vector basis of ℓ_p, then for all $\varepsilon > 0$ there exists a finite codimensional subspace of X which $C^2 + \varepsilon$-embeds into some space $(\bigoplus F_i)\ell_p$, with $\dim F_n < \infty$, for $n \in \mathbb{N}$. Hence this characterizes when a reflexive space embeds into such a sum. On the other hand, we show in Example 4.2, that there are reflexive Banach spaces enjoying property $(*)$ which are not isomorphic to some subspace of $(\bigoplus F_n)_{\ell_p}$ for any sequence (F_n) of finite dimensional spaces.

The motivation for working with branches of trees in place of subsequences comes also from the notion of asymptotic structure ([MT], [MMT]), the recent paper of N.J. Kalton [K], and [KOS]. In its simplest version, suppose (E_i) is an FDD for X (finite dimensional decomposition; see the beginning of section 3) and let $k \in \mathbb{N}$. Then the k^{th}-asymptotic space of X with respect to (E_i) may be described as the smallest closed set C_k of normalized bases of length k with the property that every countably infinitely branching tree of k levels (i.e., the index set of such a tree consists of the subsets of $\mathbb{N}$ with no more than k elements) in S_X whose successors to each node are all block bases of (E_n) must admit, for every $\varepsilon > 0$, a branch $1 + \varepsilon$-equivalent to some member of C_k.

Moreover, given $\varepsilon_n \downarrow 0$, one can then block (E_n) into an FDD (F_n) with the property that for all k any normalized skipped block basis $(x_i)_1^k$ of $(F_n)_{n=k}^\infty$ is $1 + (\varepsilon_k)$-equivalent to a member of C_k [KOS]. We cannot achieve this in the infinite setting, $k = \omega$. There is in general no unique infinite asymptotic structure C_ω. However, if C is big enough so that every such ω-level tree has a branch in C, then one can produce for $\varepsilon > 0$ a blocking (F_n) of (E_n) so that all normalized skipped block bases of (F_n) starting after F_1 are in $\overline{C_\varepsilon}$, the pointwise closure (in the product topology of the discrete topology on S_X) of $\frac{\varepsilon}{2^n}$-perturbations of elements of C. This is done in section 3. (We note that an in-between ordinal notion of asymptotic structure for $\alpha < \omega_1$ has been considered in [W], using the generalized Schreier sets S_α.)

Actually we need to study more general forms of asymptotic structure than that w.r.t. an FDD. We consider the version where one uses arbitrary finite codimensional subspaces rather than just the tail subspaces of a given FDD. While this version is coordinate free, we show in section 3 that one may embed X into a space with an FDD in such a way that the two notions coincide. Section 2 contains our preliminary work and terminology. In section 5 we apply our results to the more general notion of V.D. Milman's [Mi] spectra of a function. We are indebted to W.B. Johnson for showing us the proof of Lemma 3.1.

2. Games in a Banach space X

Assume that X is a separable Banach space of infinite dimension. The set of all subspaces of X having finite codimension is denoted by $\mathrm{cof}(X)$. S_X^ω and S_X^k, $k \in \mathbb{N}$, denote, respectively, the set of all infinite sequences in the unit sphere S_X of X and the set of all sequences in S_X of length k.

For a set $\mathcal{A} \subset S_X^\omega$ or $\mathcal{A} \subset S_X^k$ we consider the following $\mathcal{A}$*-game* between two players, having infinitely many, respectively k, rounds:

Player I chooses $Y_1 \in \mathrm{cof}(X)$
Player II chooses $y_1 \in S_{Y_1}$
Player I chooses $Y_2 \in \mathrm{cof}(X)$
Player II chooses $y_2 \in S_{Y_2}$
...

Player I wins if the resulting sequence (y_i) is in $\mathcal{A}$.

Note that by replacing a set $\mathcal{A} \subset S_X^k$, $k \in \mathbb{N}$, by $\mathcal{A} \times S_X^\omega$, we need only consider games with infinitely many steps.

We say that *Player I has a winning strategy in the* $\mathcal{A}$*-game* if the following condition $\mathrm{W}_I(\mathcal{A})$ holds:

$$(\mathrm{W}_I(\mathcal{A}))\quad \begin{cases} \text{There is a family of finite codimensional subspaces of } X \\ \qquad \big(Y_{(x_1,x_2,\dots,x_\ell)}\big)_{(x_1,x_2,\dots,x_\ell)\in\bigcup_{j=0}^\infty S_X^j}, \qquad S_X^0=\{\emptyset\}, \\ \text{indexed over all finite sequences in } S_X \text{, so that, if} \\ (x_n)_{n\in\mathbb{N}} \text{ satisfies the recursive condition} \\ (1) \qquad x_1\in S_{Y_\emptyset},\ \text{and, for } n\ge 2,\ x_n\in S_{Y_{(x_1,\dots,x_{n-1})}}, \\ \text{then } (x_n)\in\mathcal{A}. \end{cases}$$

The following proposition can be deduced immediately from the definition of $(\mathrm{W}_I(\mathcal{A}))$.

Proposition 2.1. *The set of all $A \subset S_X^\omega$ for which Player I has a winning strategy is closed with respect to taking finite intersections.*

Similarly, we say that Player II has a winning strategy if

$$(\mathrm{W}_{II}(\mathcal{A}))$$
$$\begin{cases} \text{There is a family in } S_X \\ \qquad \big(x_{(Y_1,Y_2,\dots,Y_\ell)}\big)_{(Y_1,Y_2,\dots,Y_\ell)\in\bigcup_{j=1}^\infty \mathrm{cof}^j(X)}, \\ \text{indexed over all finite sequences in } \mathrm{cof}(X) \text{ (of length at least 1), so that} \\ (2) \qquad x_{(Y_1,Y_2,\dots,Y_\ell)}\in S_{Y_\ell} \text{ if } \ell\in\mathbb{N} \text{ and } Y_1,\dots,Y_\ell\in\mathrm{cof}(X), \text{ and} \\ (3) \qquad \text{for every sequence } (Y_i)_{i\in\mathbb{N}}\subset\mathrm{cof}(X),\ \big(x_{(Y_1,Y_2,\dots,Y_i)}\big)_{i=1}^\infty\notin\mathcal{A}. \end{cases}$$

Remark. Informally, $(\mathrm{W}_I(\mathcal{A}))$ means the following:

$$\exists Y_1 \in \mathrm{cof}(X) \forall y_1 \in S_{Y_1} \exists Y_2 \in \mathrm{cof}(X) \forall y_2 \in S_{Y_2} \ldots \text{ so that } (y_i) \in \mathcal{A}.$$

Since this is an infinite phrase (unless we considered a game of finitely many draws), it has to be defined in a more formal way, as was done in $(\mathrm{W}_I(\mathcal{A}))$.

It is not true in general that an $\mathcal{A}$-game is determined, i.e., that either Player I or Player II has a winning strategy. Note that this would mean that if the above infinite phrase is false then we can formally negate it.

From a result of D. A. Martin [Ma] it follows that if $\mathcal{A}$ is a Borel set with respect to the product topology of the discrete topology in S_X, then the $\mathcal{A}$-game is determined. We actually will only need a special case of this theorem which is much easier (see [GS] or section 1 of [Ma]).

Proposition 2.2. *For every $\mathcal{A} \subset S_X^\omega$, $(W_I(\mathcal{A}))$ and $(W_{II}(\mathcal{A}))$ are mutually exclusive, and if $\mathcal{A}$ is closed with respect to the product of the discrete topology, then it follows that the failure of $(W_I(\mathcal{A}))$ implies $(W_{II}(\mathcal{A}))$.*

We furthermore note that both statements remain true if we change the game to a game in which Player I has to choose his spaces among some given subset $\Gamma \subset \mathrm{cof}(X)$ and/or Player II has to choose his vectors among a subset $D \subset S_X$ or can choose his vector in some neighborhood of S_{Y_n}, with Y_n being the n-th choice of Player I.

For a more detailed description of these variations of the $\mathcal{A}$-game we refer to Proposition 2.3, where we discuss the existence of winning strategies. In that proposition we will show that we can reduce the game to a game in which Player I, assuming he has a winning strategy, can determine a countable collection of finite codimensional spaces before the game starts, then make his choices among this countable collection and still win the game.

We need the following notion of trees, and some terminology.

Definition. $[\mathbb{N}]^{<\omega}$ denotes the set of nonempty finite subsets of $\mathbb{N}$, and $[\mathbb{N}]^{\le k}$ denotes the nonempty subsets of $\mathbb{N}$ of cardinality at most k. These are regarded as countably branching trees of infinite length, respectively, of length k, under the order $A \le B$ if A is an initial segment of B. A *countably branching tree of infinite length in S_X* is a family $(x_A)_{A\in[\mathbb{N}]^{<\omega}}$ in S_X, where the order is that induced by $[\mathbb{N}]^{<\omega}$.

Similarly, a *countably branching tree of length $k \in \mathbb{N}$ in S_X* is a family $(x_A)_{A\in[\mathbb{N}]^{\le k}}$ in S_X.

Since these are the only kinds of trees we will consider, we will simply refer to them as *trees of infinite or finite length in S_X*.

If $(x_A)_{A\in[\mathbb{N}]^{<\omega}}$ or $(x_A)_{A\in[\mathbb{N}]^{\le k}}$ is a tree and $A \in [\mathbb{N}]^{<\omega} \cup \{\emptyset\}$, or $A \in [\mathbb{N}]^{\le k-1} \cup \{\emptyset\}$ respectively, we call the sequence $(x_{A\cup\{n\}})_{n>\max A}$ *the A-node* of that tree.

If (n_i) is an increasing sequence in $\mathbb{N}$ of infinite length, respectively of length k, we call the sequence $(x_{\{n_1,\ldots,n_i\}})_{i=1}^\infty$, respectively $(x_{\{n_1,\ldots,n_i\}})_{i=1}^k$, a *branch of the tree*.

Assume that $(x_A)_{A\in[\mathbb{N}]^{<\omega}}$ or $(x_A)_{A\in[\mathbb{N}]^{\le k}}$ is a tree of infinite length or length k, respectively, and $\mathcal{I} \subset [\mathbb{N}]^{<\omega}$, or $\mathcal{I} \subset [\mathbb{N}]^{\le k}$ has the following property:

a) $\mathcal{I}$ is hereditary, i.e., if $A \in \mathcal{I}$, and $\emptyset \ne B$ is an initial segment of A, then $B \in \mathcal{I}$.

b) Assume that $A \in \mathcal{I} \cup \{\emptyset\}$, and that $\mathrm{card}(A) < k$, if we consider the case of a tree of length k. Then there are infinitely many direct successors of A in $\mathcal{I}$, i.e., the set $\{n \in \mathbb{N} : A \cup \{n\} \in \mathcal{I}\}$ is infinite.

Then we call the family $(x_A)_{A\in\mathcal{I}}$ a *subtree of* (x_A). Note that in that case we can relabel the family $(x_A)_{A\in\mathcal{I}}$ as a tree $(y_A)_{A\in[\mathbb{N}]^{<\omega}}$ or $(y_A)_{A\in[\mathbb{N}]^{\leq k}}$, respectively, so that every node and every branch of $(x_A)_{A\in\mathcal{I}}$ is a node or branch, respectively, of (y_A) and vice versa.

If (Y_n) is a decreasing sequence of finite codimensional subspaces of X, we call a tree (x_A) (indexed over $[\mathbb{N}]^{<\omega}$ or $[\mathbb{N}]^{\leq k}$) a (Y_n)-*block tree* if for every $A \in [\mathbb{N}]^{<\omega}$, respectively every $A \in [\mathbb{N}]^{\leq k}$, $x_A \in S_{Y_{\max A}}$.

Let $\delta_i \in (0, 1]$, for $i \in \mathbb{N}$, $\delta_i \searrow 0$. We call a tree $(x_A)_{A\in[\mathbb{N}]^{<\omega}}$ of infinite length in S_X a (δ_i)-*approximation of a* (Y_n)-*block tree*, if

$$\mathrm{dist}(x_A, S_{Y_{\max A}}) < \delta_{\mathrm{card}\,A}, \text{ whenever } A \in [\mathbb{N}]^{<\omega}.$$

If $\mathcal{T}$ is a topology on X (for example the weak topology), we call a tree $\mathcal{T}$-*null* if every node is a $\mathcal{T}$-null sequence.

Remark. For a sequence $(x_n) \subset X$ we can define a tree $(x_A)_{A\in[\mathbb{N}]^{<\omega}}$ by setting $x_A := x_{\max A}$, for $A \in [\mathbb{N}]^{<\omega}$. Note that then the set of all subsequences of (x_n) coincides with the set of all branches of $(x_A)_{A\in[\mathbb{N}]^{<\omega}}$.

We will be interested in conditions of the following form, and relate them to the existence of winning strategies of the above discussed games.

> *Assume that all trees all of whose nodes have a certain property (A) (for example, being weakly null) have a branch with a certain property (B) (for example, being equivalent to the unit vector basis of ℓ_p).*

From the above, such a condition is a strengthening of the following assumption:

> *All normalized sequences having property (A) have a subsequence with property (B).*

Continuing with our notation, if $\mathcal{A} \subset S_X^\omega$ and $\varepsilon > 0$, we let

$$\mathcal{A}_\varepsilon = \big\{(x_i) \subset S_X : \exists (y_i) \in \mathcal{A}, \|x_i - y_i\| < \varepsilon/2^i \text{ for all } i \in \mathbb{N}\big\}$$

and let $\overline{\mathcal{A}_\varepsilon}$ be the closure of $\mathcal{A}_\varepsilon$ with respect to the product of the discrete topology. We note that for $\varepsilon, \delta > 0$

$$\overline{(\overline{\mathcal{A}_\varepsilon})_\delta} \subset \overline{\mathcal{A}_{\varepsilon+\delta}}\ . \tag{4}$$

If $Y \in \mathrm{cof}(X)$ and $\delta > 0$, then

$$(S_Y)_\delta = \{x \in S_X : \|x - y\| < \delta \text{ for some } y \in S_Y\}\ .$$

Let $\varepsilon > 0$, $\Gamma \subseteq \mathrm{cof}(X)$ and $D \subseteq S_X$. We define what it means to say Player I has a winning strategy for $\mathcal{A} \subset S_X^\omega$ given that Player I can only choose $Y \in \Gamma$ or that II can only choose elements of D.

$(\mathrm{W}_I(\mathcal{A},\Gamma,\varepsilon))$

$$\begin{cases} \text{There exists a family} \\ \qquad \big(Y_{(x_1,x_2,\dots,x_\ell)}\big)_{(x_1,x_2,\dots,x_\ell)\in\bigcup_{j=0}^\infty S_X^j} \subset \Gamma\ , \\ \text{so that for every sequence } (x_n)_{n\in\mathbb{N}} \text{ satisfying the} \\ \text{recursive condition} \\ (5) \qquad x_1 \in (S_{Y_\emptyset})_{\varepsilon/2},\ \text{and, for } n\ge 2,\ x_n \in (S_{Y_{(x_1,\dots,x_{n-1})}})_{\varepsilon/2^n} \\ \text{one has } (x_n)\in\mathcal{A}. \end{cases}$$

Remark. It is easy to see by (4) that for any $\varepsilon,\delta>0$,

$$(\mathrm{W}_I(\overline{\mathcal{A}_\varepsilon},\{Y_n\},\varepsilon)) \Rightarrow (\mathrm{W}_I(\overline{\mathcal{A}_{\varepsilon+\delta}},\{\tilde{Y}_n\},\varepsilon))$$

whenever $\{\tilde{Y}_n\}\subseteq \mathrm{cof}(X)$ is a *refinement of* $\{Y_n\}$, by which we mean that

$$\forall\ Y\in\{Y_n\}\ \forall\ \delta>0\ \exists\ \tilde{Y}\in\{\tilde{Y}_n\} \text{ with } S_{\tilde{Y}}\subset (S_Y)_\delta\ .$$

$(\mathrm{W}_I(\mathcal{A},D,\varepsilon))$

$$\begin{cases} \text{There is a family} \\ \qquad \big(Y^{(\varepsilon)}_{(x_1,\dots,x_\ell)}\big)_{(x_1,\dots,x_\ell)\in\bigcup_{j=0}^\infty D^j} \subset \mathrm{cof}(X)\ , \\ \text{so that for any sequence } (x_n) \text{ such that } x_n\in D \text{ and} \\ \qquad x_n \in (S_{Y^{(\varepsilon)}_{(x_1,\dots,x_{n-1})}})_{\varepsilon/2^n}\ ,\ n=1,2,\dots, \\ \text{one has } (x_n)\in\mathcal{A}. \end{cases}$$

Proposition 2.3. 1. *If $\mathcal{B}$ is a countable collection of subsets of S_X^ω, then there is a decreasing sequence (Y_n) in $\mathrm{cof}(X)$ so that the following are equivalent for each $\mathcal{A}\in\mathcal{B}$:*

a) $\forall\varepsilon>0\quad (W_I(\overline{\mathcal{A}_\varepsilon}))$.

b) $\forall\varepsilon>0\quad (W_I(\overline{\mathcal{A}_{2\varepsilon}},\{Y_n\},\varepsilon))$.

c) *For every $\varepsilon>0$ every $(\varepsilon/2^n)$-approximation to a (Y_n) block tree of infinite length in S_X has a branch in $\overline{\mathcal{A}_\varepsilon}$.*

d) *For every $\varepsilon>0$ every (Y_n) block tree of infinite length in S_X has a branch in $\overline{\mathcal{A}_{2\varepsilon}}$.*

2. *If X has a separable dual, then $(Y_n)\subset\mathrm{cof}(X)$ can be chosen so that the equivalences in 1 hold for all subsets $\mathcal{A}\subset S_X^\omega$. In that case it follows that, for any $\mathcal{A}\subset S_X^\omega$, 1(a) is equivalent to*

e) *For every $\varepsilon>0$ every weakly null tree of infinite length in S_X has a branch in $\overline{\mathcal{A}_\varepsilon}$.*

Proof of Proposition 2.3: Let D be a countable dense set in S_X. Using (4), we note that for any $\mathcal{A}\subset S_X^\omega$ and any $\varepsilon>0$ it follows that

$$(\mathrm{W}_I(\overline{\mathcal{A}_\varepsilon}))\Rightarrow(\mathrm{W}_I(\overline{\mathcal{A}_{2\varepsilon}},D,\varepsilon))\ . \tag{6}$$

Assuming now that for all $\varepsilon>0$ the condition $(\mathrm{W}_I(\overline{\mathcal{A}_{2\varepsilon}},D,\varepsilon))$ is satisfied, we can choose a countable subset of $\mathrm{cof}(X)$,

(7)

$$\Gamma_{\mathcal{A}}=\big\{Y^{(\varepsilon)}_{(x_1,\dots,x_\ell)}:\varepsilon>0 \text{ rational },x_n\in D \text{ and } x_n\in(Y^{(\varepsilon)}_{(x_1,\dots,x_{n-1})})_{\varepsilon/2^n} \text{ for } n\in\mathbb{N}\big\},$$

and observe that

(8) $\forall\varepsilon>0\quad (\mathrm{W}_I(\overline{\mathcal{A}_{2\varepsilon}},D,\varepsilon)) \implies$ there exists a countable $\Gamma\subset\mathrm{cof}(X)$ so that

$$\forall\varepsilon>0,\quad (\mathrm{W}_I(\overline{\mathcal{A}_{2\varepsilon}},\Gamma,D,\varepsilon)),$$

where $(\mathrm{W}_I(\overline{\mathcal{A}_{2\varepsilon}},\Gamma,D,\varepsilon))$ is defined just like $(\mathrm{W}_I(\overline{\mathcal{A}_{2\varepsilon}},\Gamma,\varepsilon))$ with the difference that the family $\big(Y_{(x_1,x_2,\dots,x_\ell)}\big)$ is indexed over $\bigcup_{j=0}^\infty D^j$.

Using standard approximation arguments and the fact that D is dense in S_X, we observe for any $\Gamma\subset\mathrm{cof}(X)$ and any $\mathcal{A}\subset S^\omega$

$$(\mathrm{W}_I(\overline{\mathcal{A}_{2\varepsilon}},\Gamma,D,\varepsilon))\Rightarrow(\mathrm{W}_I(\overline{\mathcal{A}_{3\varepsilon}},\Gamma,\varepsilon))\Rightarrow(\mathrm{W}_I(\overline{\mathcal{A}_{3\varepsilon}}))\ . \tag{9}$$

Finally, assume that $\tilde\Gamma\subset\mathrm{cof}(X)$ is a refinement of $\Gamma\subset\mathrm{cof}(X)$. Then by (4) it follows that for $\varepsilon>0$

$$(\mathrm{W}_I(\overline{\mathcal{A}_{\varepsilon}},\Gamma))\Rightarrow(\mathrm{W}_I(\overline{\mathcal{A}_{2\varepsilon}},\tilde\Gamma)). \tag{10}$$

Let $\mathcal{B}$ be any countable collection of subsets of S_X^ω. For $\mathcal{A}\in\mathcal{B}$, if for all $\varepsilon>0$ $(\mathrm{W}_I(\overline{\mathcal{A}_{\varepsilon}}))$ is true, let Γ_A be as in (7), and, otherwise, set $\Gamma_{\mathcal{A}}=\{X\}$. Since $\bigcup_{\mathcal{A}\in\mathcal{B}}\Gamma_{\mathcal{A}}$ is countable we can choose a decreasing sequence $(Y_n)\subset\mathrm{cof}$ which is a refinement of $\bigcup_{\mathcal{A}\in\mathcal{B}}\Gamma_{\mathcal{A}}$.

From (6)–(10) we deduce that for all $\mathcal{A}\in\mathcal{B}$

$$\forall\varepsilon>0\quad \mathrm{W}_I(\overline{\mathcal{A}_{\varepsilon}})\iff\forall\varepsilon>0\quad \mathrm{W}_I(\overline{\mathcal{A}_{2\varepsilon}},\{Y_n\},\varepsilon)\ .$$

Now $\mathrm{W}_I(\overline{\mathcal{A}_{2\varepsilon}},\{Y_n\},\varepsilon)$ says that Player I has a winning strategy in the $\overline{\mathcal{A}_{2\varepsilon}}$-game, even if he has to choose his finite codimensional subspaces among $\{Y_n\}$, and even if Player II "can cheat a little bit" by choosing his vectors in $(S_{Y_n})_{\varepsilon/2^n}$. From Proposition 2.2 we deduce that this is equivalent to the condition that Player II does not have a winning strategy, which means that every $(\varepsilon/2^n)$ approximation to a (Y_n)-block tree has a branch in $\overline{\mathcal{A}_{2\varepsilon}}$.

We therefore have proven the equivalence of (a), (b) and (c). Note also that (c)$\Rightarrow$(d) is trivial, and since (d) means that Player II has no winning strategy even if Player I has to choose form the set $\{Y_n\}$, it follows that (d) implies (a).

In order to prove the second part of the proposition we note that X has a separable dual we can find a *universal* countable refinement, i.e., a countable refinement of the whole set $\mathrm{cof}(X)$. Indeed, choose a dense sequence (ξ_n^*) in S_{X^*} and let

$$Y_n=\mathcal{N}(\xi_1^*,\xi_2^*,\dots,\xi_n^*)=\{x\in X:\forall i\in\{1,\dots,n\}\quad \xi_i^*(x)=0\}\ .$$

Second, note that in this case every (Y_n)-block tree is weakly null, and, conversely, that for $\delta_i\searrow 0$, every weakly null tree $(x_A)_{A\in[\mathbb{N}]^{<\omega}}$ has a subtree $(y_A)_{A\in[\mathbb{N}]^{<\omega}}$ which is a (δ_i)-*approximation* of a (Y_n)-block tree. $\square$

3. A fundamental combinatorical result

For the games in X, introduced in Section 2, we want to discuss how a winning strategy of Player I or Player II can be formulated in terms of a coordinate system on X.

Recall that a Banach space Z has an FDD (F_i), where, for $i\in\mathbb{N}$, F_i is a finite dimensional subspace of Z, if every $z\in Z$ can be written in a unique way as $z=\sum_{i=1}^\infty z_i$ with $z_i\in F_i$, for all $i\in\mathbb{N}$. In this case we write $Z=\bigoplus_{i=1}^\infty F_i$ and denote by $\mathrm{c}_{00}(\bigoplus_{i=1}^\infty F_i)$ the dense linear subspace of Z consisting of all finite linear combinations of vectors x_i, $x_i\in F_i$. For $m\le n$ we denote by $P_{\bigoplus_{i=m}^n F_i}$ the canonical projection from Z onto $\bigoplus_{i=m}^n F_i$.

Using a result of W. B. Johnson, H. Rosenthal and M. Zippin [JRZ], we derive the following Lemma.

Lemma 3.1. *Let (Y_n) be a decreasing sequence of subspaces of X, each having finite codimension. Then X is isometrically embeddable into a space Z having an FDD (E_i) so that the following conditions hold (we identify X with its isometric image in Z):*

a) $c_{00}(\bigoplus_{i=1}^{\infty} E_i) \cap X$ *is dense in* X.
b) *For every $n \in \mathbb{N}$ the finite codimensional subspace $X_n = \bigoplus_{i=n+1}^{\infty} E_i \cap X$ is contained in Y_n.*
c) *There is a $c > 0$ such that for every $n \in \mathbb{N}$, there is a finite set $D_n \subset S_{\bigoplus_{i=1}^{n} E_i^*}$ such that whenever $x \in X$*

$$\|x\|_{X/Y_n} = \inf_{y \in Y_n} \|x - y\| \le c \max_{w^* \in D_n} w^*(x) \ . \tag{11}$$

From (a) it follows that $c_{00}(\bigoplus_{i=n+1}^{\infty} E_i) \cap X$ is a dense linear subspace of X_n.

Moreover, if X has a separable dual, (E_i) can be chosen to be shrinking (every normalized block sequence in Z with respect to (E_i) converges weakly to 0, or, equivalently, $Z^* = \bigoplus_{i=1}^{\infty} E_i^*$), and if X is reflexive Z can also be chosen to be reflexive.

Remark. We will prove that X is isomorphic to a space $\tilde{X}$ having the above properties. Then we consider on $\tilde{X}$ the norm $\|I(\cdot)\|_X$, where $I : \tilde{X} \to X$ is an isomorphism, and extend this norm to all of Z. We might lose monotonicity, or bimonotonicity, and we will not be able to assume that the constant c in (c) can be chosen close to the value 1. But for later purposes we are more interested in an isometric embedding.

Proof of Lemma 3.1. We consider the following three cases. If X is a reflexive space we can choose according to [Z] a reflexive space Z with an FDD (F_i) which contains X. If the dual X^* is separable we can use again a result in [Z] and choose a space Z having a shrinking FDD (F_i). In the general case we choose Z to be a $C(K)$-space containing X, K compact and metric (for example $K = B_{X^*}$ endowed with the w^*-topology), and choose an FDD (F_i) for Z.

We first write Y_n as the null space $\mathcal{N}(U_n)$ of a finite dimensional space $U_n \subset X^*$. We choose a finite set in S_{U_n} which norms all elements of X/Y_n up to a factor $1/2$, and choose for each element of this set a Hahn-Banach extension to an element in Z^*. We denote the set of all extensions by D_n, and we let V_n be the finite dimensional subspace of Z^* generated by D_n. We will produce an FDD (E_i) for Z so that $D_n \subset \bigoplus_{i=1}^{n} E_i^*$. Hence (c) will hold.

Now

$$Y_n = \mathcal{N}(V_n) \cap X, \text{ with } V_n \subset Z^* \text{ and } \dim(V_n) < \infty \ . \tag{12}$$

Second, we choose a subspace $\tilde{W}_n \subset X$, $\dim(\tilde{W}_n) = \dim(U_n) < \infty$, so that X is a complemented sum of Y_n and $\tilde{W}_n$, $X = Y_n \oplus \tilde{W}_n$. Note that in general we do not have control over the norm of the projection onto Y_n. Given a dense countable subset (ξ_n) in S_X, we inflate $\tilde{W}_i$ to $W_i = \mathrm{span}(\tilde{W}_i \cup \{\xi_1, \dots, \xi_i\})$. Thus the closure of $\bigcup_{i=1}^{\infty} W_i$ is X.

Then we choose as follows a separable subspace $\tilde{Z}$ of Z^* which is 1-complemented in Z^*, Z-norming, and contains all the spaces V_n, $n \in \mathbb{N}$. In the case that X has a separable dual (thus also Z^* is separable) we simply take $\tilde{Z} = Z^*$. In the general case we let $\tilde{Z}$ be a separable L_1-space containing a Z-norming set, all the spaces V_n, and all the spaces F_n^* (considered as subspaces of Z^*).

For $n \in \mathbb{N}$ let $P_n : Z \to \bigoplus_{i=1}^n F_i$ be the projection from Z onto $\bigoplus_{i=1}^n F_i$, and let $T_n : Z^* \to \tilde{Z}$ be the adjoint P_n^* if X^* is separable. In the general case we choose (T_n) to be a sequence of projections of norm 1 from Z^* onto a finite dimensional subspace of $\tilde{Z}$ with the property $T_1(Z^*) \subset T_2(Z^*) \subset T_3(Z^*) \dots$, so that $\bigcup_n T_n(Z^*)$ is dense in $\tilde{Z}$ (as a separable L_1-space $\tilde{Z}$ is complemented in Z^* and has an FDD).

We are now in the situation of Lemma 4.2 of [JRZ], i.e., the following statements hold:

$$P_n^*(Z^*) \subset \tilde{Z} \text{ and } T_n(Z^*) \subset \tilde{Z}\ , \tag{13}$$

$$\lim_{n\to\infty} P_n(z) = z, \quad \lim_{n\to\infty} T_n(y^*) = y^* \text{ for all } z \in Z, y^* \in \tilde{Z}, \text{ and} \tag{14}$$

$$K := \sup_n \|T_n\| \vee \sup_n \|P_n\| < \infty. \tag{15}$$

We conclude from Lemma 4.2 in [JRZ] that:

($*$) Let E and F be finite dimensional subspaces of X and $\tilde{Z}$ respectively. Then there is a projection Q on Z with finite dimensional range so that the following three conditions (16), (17) and (18) hold:

$$Q|_E = \mathrm{Id}|_E \text{ and } Q^*|_F = \mathrm{Id}|_F, \tag{16}$$

$$Q^*(Z^*) \subset \tilde{Z}, \tag{17}$$

$$\|Q\| \le 4(K + K^2). \tag{18}$$

Using ($*$), we can proceed as in the proof of Theorem 4.1 in [JRZ] to inductively define for each $n \in \mathbb{N}$ a finite dimensional projection (Q_n) on Z so that for all $1 \le i, j \le n$

$$Q_i Q_j = Q_j Q_i = Q_{i \wedge j}\ , \tag{19}$$

$$Q_i(X) \supset \bigcup_{s=1}^{i} W_s\ , \tag{20}$$

$$\tilde{Z} \supset Q_i^*(Z^*) \supset \bigcup_{s=1}^{i} V_s \quad (\text{in particular, } D_i \subset Q_i^*(Z^*)), \tag{21}$$

$$\|Q_i\| \le 4(K + K^2)\ . \tag{22}$$

Indeed, for $n = 1$ we apply ($*$) to $E = W_1$ and $F = V_1$. If $Q_1, Q_2, \dots, Q_{n-1}$ are chosen we apply ($*$) to $E = [Q_{n-1}(Z) \cup W_n]$ and $F = \mathrm{span}(Q_{n-1}^*(Z^*) \cup V_n)$. We deduce (20), (21) and (22), and we observe that for $i < n$, $Q_n \circ Q_i = Q_i$ and $Q_n^* \circ Q_i^* = Q_i^*$. Since for $z \in Z$ and $z^* \in Z^*$ the second equality implies that

$$\langle Q_i \circ Q_n(z), z^* \rangle = \langle z, Q_n^* Q_i^*(z^*) \rangle = \langle z, Q_n^*(z^*) \rangle = \langle Q_i(z), z^* \rangle\ ,$$

we also deduce that $Q_i \circ Q_n = Q_i$.

Now we let $E_i = (Q_i - Q_{i-1})(Z)$ $(Q_0 = 0)$ and deduce from (19) and (22) that (E_i) is an FDD of a subspace of Z which, by (20) still contains X. (20) also implies that $\mathrm{c}_{00}(\bigoplus F_i) \cap X$ is dense in X. Putting $X_n = \bigoplus_{i=n+1}^\infty F_i \cap X$, we note that for $x \in X_n$ and $z^* \in V_n$ it follows from (21) that $\langle z^*, x \rangle = \langle Q_n^*(z^*), x \rangle = \langle z^*, Q_n(x) \rangle = 0$, and thus, that $X_n \subset \mathcal{N}(V_n) \cap X = Y_n$.

We also deduce that for $n \in \mathbb{N}$, $\mathrm{c}_{00}(\bigoplus_{i=n+1}^\infty F_i) \cap X$ is dense in X_n, using the following lemma, which seems to be folklore. □

Lemma 3.2. *If Y is a linear and dense subspace of X and $\tilde{X}$ has finite codimension in X, then $\tilde{X} \cap Y$ is also dense in $\tilde{X}$.*

Proof. Let $F \subset X$ be a subspace of dimension $\dim(X/\tilde{X})$, admitting a continuous projection $Q : X \to F$, so that $(\mathrm{Id} - Q)(X) = \tilde{X}$.

Let $x \in \tilde{X}$. By assumption we find a sequence $(y_n) \subset Y$ converging to x. Let V be the (finite dimensional) vector space generated by $(Q(y_n))_{n\in\mathbb{N}}$, and choose a basis of V of the form $\{Q(y_{n_1}), \dots, Q(y_{n_\ell})\}$. We represent each vector $Q(y_n)$ as

$$Q(y_n) = \sum_{i=1}^{\ell} \lambda_i^{(n)} Q(y_{n_i}) \ ,$$

and put $x_n = y_n - \sum_{i=1}^{\ell} \lambda_i^{(n)} y_{n_i}$. Note that $x_n \in Y$ and that $Q(x_n) = 0$, for all $n \in \mathbb{N}$. Furthermore it follows, since $\lim_{n\to\infty} \|Q(y_n)\| = 0$ and since $(Q(y_{n_i}))_{i=1}^{\ell}$ is basis of V, that $\lim_{n\to\infty} \lambda_i^{(n)} = 0$ for all $1 \le i \le \ell$. Therefore it follows that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x$. $\square$

We are now ready to state and prove the main result of this section. If a Banach space Z has an FDD (E_i), we will call a sequence (z_i) in Z *a block sequence with respect to* (E_i), if for some $0 = k_0 < k_1 < k_2 < \dots$, for every $i \in \mathbb{N}$, $z_i \in \bigoplus_{j=1+k_{i-1}}^{k_i} E_j$. We will call a tree $(z_A)_{A\in[\mathbb{N}]^{<\omega}}$ or $(z_A)_{A\in[\mathbb{N}]^{\le k}}$ in S_Z an (E_i)*-block tree* if every node is a block sequence with respect to (E_i). In a similar way, given $\delta_n \downarrow 0$, we define trees which are (δ_n) approximations to (E_i)-block trees.

(G_i) is a *blocking* of (E_i) if there exist integers $0 = m_0 < m_1 < \cdots$ so that $G_i = \bigoplus_{j=m_{i-1}+1}^{m_i} E_j$ for all i. $(x_n) \subseteq S_Z$ is a *skipped block w.r.t.* (G_i) if

(SB) for some sequence $1 = k_0 < k_1 < \cdots$ in $\mathbb{N}$, $x_n \in \bigoplus_{j=k_{n-1}+1}^{k_n-1} G_j$ for all n.

If $\delta = (\delta_i)$ with $\delta_i \searrow 0$ and $(x_n) \subseteq S_Z$, we say (x_n) is a (δ_i)*-skipped block w.r.t.* (G_i) if

(δ-SB) for some sequence $1 = k_0 < k_1 < \cdots$ in $\mathbb{N}$,

$$\|(\mathrm{Id} - P_{\bigoplus_{j=k_{n-1}+1}^{k_n-1} G_j})x_n\| < \delta_n \quad \text{for all} \quad n \ .$$

Theorem 3.3. *Let $\mathcal{B}$ be a countable collection of subsets of S_X^ω. Then there exists an isometric embedding of X into a space Z having an FDD (E_i), so that for $\mathcal{A} \in \mathcal{B}$ the following are equivalent.*

a) $\forall \varepsilon > 0 \quad (W_I(\overline{\mathcal{A}_\varepsilon}))$.

b) *For every $\varepsilon > 0$ there are a blocking (G_i) of (E_i) and a sequence $\delta_i \searrow 0$ so that for every sequence $(x_n) \subset S_X$ satisfying (δ-SB) w.r.t. (G_i) we have $(x_n) \in \overline{\mathcal{A}_\varepsilon}$.*

c) *For every $\varepsilon > 0$ there is a blocking (G_i) of (E_i) so that for every sequence $(x_n) \subset S_X$ (SB) w.r.t. (G_i) we have $(x_n) \in \overline{\mathcal{A}_\varepsilon}$.*

If X has a separable dual, then (E_i) can be chosen to be shrinking and independent from $\mathcal{B}$, and, furthermore, if X is reflexive, Z can be chosen to be reflexive. In these cases, a) *is equivalent to*

d) *For every $\varepsilon > 0$ every weakly null tree in S_X has a branch in $\overline{\mathcal{A}_\varepsilon}$.*

Remark. Note that Theorem 3.3 means the following. Assume that for all $\varepsilon > 0$ Player I has a winning strategy for the $\overline{\mathcal{A}_\varepsilon}$-game. Then, given $\varepsilon > 0$, Player I can embed X into a space with an appropriate FDD (F_i), and use the following strategy:

Take $Y_1 = \bigoplus_{i=2}^{\infty} F_i \cap X$.
If Player II has chosen the vector x_{n-1} in the $n-1$st round,
choose $N \in \mathbb{N}$ so that $\|P_{\bigoplus_{i=N}^{\infty} F_i}(x_{n-1})\| < \delta_n$ and put
$Y_n = \bigoplus_{i=N+1} F_i \cap X$.

The proof of Theorem 3.3 also gives the following. Suppose $X \subseteq Z$, where Z has an FDD (E_i), and suppose Player I is only allowed to choose subspaces in $\Gamma = \{X \cap \bigoplus_{i=n}^{\infty} E_i : n \in \mathbb{N}\}$. Then a) and b) are equivalent for all $\mathcal{A}$.

Proof of Theorem 3.3. We first choose a decreasing sequence of finite codimensional spaces (Y_n) in X so that for each $\mathcal{A} \in \mathcal{B}$ the equivalences (a) $\iff$ (b) $\iff$ (c) $\iff$ (d), and, if X^* is separable, (d) $\iff$ (e), of Proposition 2.3 hold. Then we choose the space with an FDD (E_i) as in Lemma 3.1.

We note that trivially (b) of the statement of Theorem 3.3 implies (c). Since the conclusion of Lemma 3.1 implies that every (X_n)-block tree (recall, $X_n = \bigoplus_{i=n+1}^{\infty} E_i \cap X$) has, for a given sequence $\delta_i \searrow 0$, a subtree which is a (δ_i)-approximation of an (E_i)-block tree for which some branch is (SB) w.r.t. (G_i), condition (c) implies condition (a) (Player II cannot have a winning strategy). If X^* is separable, the statement (a) $\iff$ (d) is exactly the statement of the second part of Proposition 2.3.

Thus, it remains to verify the implication (a)$\Rightarrow$(b).

Let $\varepsilon > 0$ and $\mathcal{A} \in \mathcal{B}$. We put $\eta_i = \varepsilon/c2^{i+2}$, where the constant $c > 1$ comes from the conclusion of Lemma 3.1 (c).

Claim. Every tree $(x_A)_{A\in[\mathbb{N}]^{<\omega}}$ in S_X having the property that

$$x_A \in X \cap \big(S_{\bigoplus_{i=\max A+1}^{\infty} E_i}\big)_{\eta_{\mathrm{card}\, A}} \text{ whenever } A \in [\mathbb{N}]^{<\omega} \tag{23}$$

is an $(\varepsilon/2^n)$-approximation to a (Y_n)-block tree, and therefore must have a branch in $\overline{\mathcal{A}_{2\varepsilon}}$ (Proposition 2.3 (a) $\iff$ (c)).

Remark. Note that it is in general not true that if $x \in X \cap \big(S_{\bigoplus_{i=m}^{n} E_i}\big)_\delta$, then we will be able to approximate x by an element in $X_{m-1} = \bigoplus_{j=m}^{\infty} E_j \cap X$ up to some $r(\delta)$, which converges to 0 if δ tends to 0, and which only depends on δ, but not on m and n. But condition (c) of Lemma 3.1 will ensure that we can at least approximate x by an element of Y_n, up to a fixed multiple of δ.

In order to prove the claim it suffices to show that

$$(*) \qquad \begin{cases} \text{Let } \delta > 0 \text{ and } x \in X \cap \big(S_{\bigoplus_{i=n+1}^{\infty} E_i}\big)_\delta . \\ \text{Then there is a } y \in S_{Y_n} \text{ with } \|x - y\| \le 4\delta c. \end{cases}$$

In order to verify the claim we can assume without loss of generality that $\delta < 1/2c$ (otherwise the claim is trivial). Choose $u \in \bigoplus_{i=n}^{\infty} E_i$ and $v \in Z$, $\|v\| < \delta$, so that $x = u + v$. From Lemma 3.1(c) we deduce (recall that $D_n \subset \bigoplus_{i=1}^{n} E_i^*$) that

$$\|x\|_{X/Y_n} \le c \max_{w^* \in D_n} w^*(x) = c \max_{w^* \in D_n} w^*(v) < c\delta .$$

We can therefore write $x = \tilde{y} + d$, with $\tilde{y} \in Y_n$ and $d \in X$ satisfying $\|d\| < c\delta$. Since $\|x\| = 1$, we have $1 - c\delta < \|\tilde{y}\| < 1 + c\delta$. Letting $y = \tilde{y}/\|\tilde{y}\|$, this implies that $\|x - y\| \le 4c\delta$, and finishes the proof of $(*)$.

We next show that there is an increasing sequence $N_i \subset N$ so that if we let $G_i = \bigoplus_{s=1+N_{i-1}}^{N_i} E_s$, then for every sequence $(x_k) \subset S_X$ for which there exist integers $m_0 = 1 < m_1 < \cdots$ so that

$$\operatorname{dist}(x_k, \bigoplus_{s=1+m_{k-1}}^{m_k-1} G_s) = \operatorname{dist}(x_k, \bigoplus_{i=1+N_{1+m_{k-1}}}^{N_{m_k-1}} E_i) < \eta_k, \qquad k \in \mathbb{N},$$

we have $(x_k) \in \overline{\mathcal{A}_{4\varepsilon}}$.

Since for all $x \in S_X$ it follows that (K depends on the basis constant of (E_i))

$$\operatorname{dist}(x, \bigoplus_{i=m+1}^{n} E_i) \leq K \|(\mathrm{Id} - P_{\bigoplus_{i=m+1}^{n} E_i})(x)\|,$$

this will finish the proof of b), taking $\delta_i = \eta_i / K$.

For $\overline{N} = (N_i)_{i=1}^\infty \in [\mathbb{N}]^\omega$ (the set of infinite subsequences of $\mathbb{N}$) we put ($N_0 = 0$)

$$F_i^{\overline{N}} = \bigoplus_{j=1+N_{i-1}}^{N_i} E_i, \qquad i = 1, 2, \dots, \tag{24}$$

$$\mathcal{F}^{\overline{N}} = \big\{ (x_i)_{i=1}^\infty \subset S_X : \forall i \in \mathbb{N} \quad \operatorname{dist}(x_i, F_{2i}^{\overline{N}}) < \eta_i \big\}. \tag{25}$$

Remark. For $\overline{N} \in [\mathbb{N}]^\omega$ and every $(z_i)_{i=1}^\infty \subset S_X$ having the property that

$$\operatorname{dist}(S_{\bigoplus_{j=1+m_{i-1}}^{m_i-1} F_j^{\overline{N}}}, z_i) < \eta_i, \qquad i = 1, 2, \dots,$$

for some sequence $1 \leq m_0 < m_1 < m_1 + 1 < m_2 < m_2 + 1 < m_3 < \dots$ there is a sequence $\overline{M} \in [\overline{N}]^\omega$ so that $(z_i) \in \mathcal{F}^{\overline{M}}$.

Indeed, let $\tilde{z}_i \in S_{\bigoplus_{j=1+m_{i-1}}^{m_i-1} F_j^{\overline{N}}}$ for $i \in \mathbb{N}$ be such that $\|\tilde{z}_i - z_i\| < \eta_i$, and put $M_{2i-1} = N_{m_{i-1}}$ and $M_{2i} = N_{m_i-1}$. Then it follows that

$$\tilde{z}_i \in S_{\bigoplus_{j=1+m_{i-1}}^{m_i-1} F_j^{\overline{N}}} = S_{\bigoplus_{s=N_{1+m_{i-1}}}^{N_{m_i-1}} E_s} = S_{F_{2i}^{\overline{M}}}.$$

Thus, $(z_i) \in \mathcal{F}^{\overline{M}}$.

Completion of the proof of Theorem 3.3. We put

$$\mathcal{C} = \{ \overline{N} \in [\mathbb{N}]^\omega : \mathcal{F}^{\overline{N}} \subset \overline{\mathcal{A}_{4\varepsilon}} \} .$$

It is easy to see that $\mathcal{C}$ is closed in the pointwise topology on $[\mathbb{N}]^\omega$, since $\overline{\mathcal{A}_{4\varepsilon}}$ is closed with respect to the product of the discrete topology on S_X^ω.

By the infinite version of Ramsey's theorem (cf. [O]) we deduce that one of the following two cases occurs:

Either there exists an $\overline{N} \in [\mathbb{N}]^\omega$ so that $[\overline{N}]^\omega \subset \mathcal{C}$,

or there exists an $\overline{N} \in [\mathbb{N}]^\omega$ so that $[\overline{N}]^\omega \subset [\mathbb{N}]^\omega \setminus \mathcal{C}$.

If the first alternative occurs, we are finished by the above remark. Assuming the second alternative, we will show that there is a tree in S_X satisfying (23) without any branch in $\overline{\mathcal{A}_{2\varepsilon}}$. This would be a contradiction, which would imply that the second alternative cannot occur.

If we assume the second alternative, we can pick for each $\overline{M} \in [\overline{N}]^\omega$ a sequence $(y_i^{\overline{M}})_{i=1}^\infty \in \mathcal{F}^{\overline{M}}$ which is not in $\mathcal{C}$. Let $\overline{N} = \{N_1, N_2, \dots\}$.

Note that for any $\overline{M} \subset \{N_3, N_4, \dots\}$,

$$y_1^{(N_1, N_2, \overline{M})} \in S_X \cap \big(S_{\bigoplus_{i=1+N_1}^{N_2} E_i} \big)_{\eta_1} .$$

Here $(N_1, N_2, \overline{M})$ is the infinite sequence starting with N_1 and N_2 and then consisting of the elements of $\overline{M}$).

Using the finite version of Ramsey's theorem and the compactness of $S_{\bigoplus_{i=1+N_1}^{N_2} E_i}$ we can find a vector

$$x_{\{1\}} \in S_X \cap \big(S_{\bigoplus_{i=1+N_1}^{N_2} E_i}\big)_{\eta_1}$$

and an $\overline{M}^{(1)} \subset \{N_3, N_4, \dots\}$ such that

$$\|x_{\{1\}} - y_1^{(N_1,N_2,\overline{M})}\| < 2\eta_1 \text{ for all } \overline{M} \in [\overline{M}^{(1)}]^\omega \ . \tag{26}$$

Doing the same procedure again, we can find an

$$x_{\{2\}} \in S_X \cap \left(S_{\bigoplus_{i=1+N_1^{(2)}}^{N_2^{(2)}} E_i}\right)_{\eta_1}$$

and an $\overline{M}^{(2)} \subset [\overline{M}_{(1)}]^\omega$ so that

$$\|x_{\{2\}} - y_1^{N_1^{(2)},N_2^{(2)},\overline{M}}\| < 2\eta_1 \text{ for all } \overline{M} \in [\overline{M}^{(2)}]^\omega \ ,$$

where $N_1^{(2)}$ and $N_2^{(2)}$ are the first two elements of the sequence $\overline{M}^{(1)}$. Proceeding this way, we construct a sequence $x_{\{i\}}$ and a decreasing sequence $(\overline{M}^{(i)})$ of infinite subsequences of $\overline{N}$ so that

$$x_{\{i\}} \in S_X \cap \big(S_{\bigoplus_{j=1+N_1^{(i)}}^{N_2^{(i)}} E_j}\big)_{\eta_1},$$

$$\|x_{\{i\}} - y^{(N_1^{(i)},N_2^{(i)},\overline{M})}\| < \eta_1, \text{ for all } \overline{M} \in [\overline{M}^{(i)}]^\omega \ .$$

This sequence will be the first level of a tree and the beginning of the following level by level recursive construction of this tree.

Assume that for some ℓ and every $A \in [\mathbb{N}]^{\le \ell}$ we have chosen an $x_A \in S_X$, a pair of natural number $N_1^{(A)}$ and $N_2^{(A)}$, and a sequence $\overline{M}^{(A)} \in [\{N \in \overline{N} : N > N_2^{(A)}\}]^\omega$ so that the following conditions (27) and (28) are satisfied:

(27) If $A \in [\mathbb{N}]^{<\ell} \cup \{\emptyset\}$ and $n > m > \max A,$ then

$$N_1^{(A)} < N_2^{(A)} < N_1^{(A\cup\{m\})} < N_2^{(A\cup\{m\})} < N_1^{(A\cup\{n\})} < N_1^{(A\cup\{n\})},$$

$$[N_1^{(\emptyset)} = N_2^{(\emptyset)} = 0],$$

$$\overline{M}^{(A)} \supset \overline{M}^{(A\cup\{n\})}.$$

(28) If $n_1 < n_2 < \ldots < n_\ell$ are in $\mathbb{N},$ we put

$$A_j = \{n_1, n_2, \dots, n_j\} \text{ for } j = 1, 2, \dots, \ell \ .$$

Then:

$$x_{A^{(j)}} \in S_X \cap \big(S_{\bigoplus_{s=N_1^{(A_j)}+1}^{N_2^{(A_j)}} E_s}\big)_{\eta_j},$$

$$\|x_{A^{(j)}} - y^{(N_1^{(A_1)},N_2^{(A_1)},\dots N_1^{(A_j)},N_2^{(A_j)},\overline{M})}\| < \eta_j,$$

whenever $\overline{M} \in [\overline{M}^{(A_j)}]^\omega$.

Then we can choose for $A \in [\mathbb{N}]^\ell$ the elements $x_{A\cup\{1+\max A\}}$, $x_{A\cup\{2+\max A\}}$, etc., the numbers $N_1^{(A\cup\{1+\max A\})}$, $N_2^{(A\cup\{1+\max A\})}$, $N_1^{(A\cup\{2+\max A\})}$, $N_2^{(A\cup\{2+\max A\})}$, etc. and the sets $\overline{M}^{(A\cup\{1+\max A\})}$, $\overline{M}_1^{(A\cup\{2+\max A\})}$, etc. exactly in the same way we chose $x_{\{1\}}$, $x_{\{2\}}$, etc., and the numbers $N_1^{(1)}, N_2^{(1)}, N_1^{(2)}, N_2^{(2)}$, etc. for the first level.

The condition (28) implies that for every branch (z_n) of the constructed tree there is an $\overline{M} \in [\mathbb{N}]^\omega$ so that $\|z_n - y_n^{\overline{M}}| \le 2\eta_n$ for all $n \in \mathbb{N}$. Since $(y_n^{\overline{M}}) \notin \overline{\mathcal{A}_{4\varepsilon}}$, it follows that (recall that $\eta_n \le \varepsilon/2^n$) $(z_n) \notin \overline{\mathcal{A}_{2\varepsilon}}$, which is a contradiction and finishes the proof. □

4. Subspaces of $(\bigoplus_{i=1}^\infty F_i)_p$

The purpose of this section is to use Theorem 3.3 to produce an intrinsic characterization of a necessary and sufficient condition that ensures a given Banach space X will embed into an ℓ_p-sum of finite dimensional spaces.

Let $1 \le p < \infty$ and let F_i be a finite dimensional space for $i \in \mathbb{N}$. *The ℓ_p-sum of (F_i)*, denoted by $(\sum F_i)_p$, is the space of all sequences (x_i) with $x_i \in F_i$ for $i = 1, 2, \ldots$ such that

$$\|(x_i)\|_p = \left(\sum_{i=1}^\infty \|x_i\|_{F_i}\right)^{1/p} < \infty \ .$$

Theorem 4.1. *Assume that X is reflexive and that there are $1 < p < \infty$ and $C > 1$ so that every weakly null tree in S_X has a branch which is C-equivalent to the unit vector basis of ℓ_p.*

Then X is isomorphic to a subspace of an ℓ_p-sum of finite dimensional spaces.

More precisely, for any $\varepsilon > 0$ there exists a finite codimensional subspace $\tilde{X}$ of X so that $\tilde{X}$ is $(C^2 + \varepsilon)$-isomorphic to a subspace of an ℓ_p-sum of finite dimensional spaces.

Before we start the proof, some remarks are in order.

Remark. The assumption that X is reflexive is necessary. Indeed, James' space J [Ja1] is not reflexive but has the property that every weakly null tree in S_J has a branch which is 2-equivalent to the unit vector basis of ℓ_2. Actually every normalized skipped block with respect to the shrinking basis of J is 2-isomorphic to the unit vector basis of ℓ_2. Since every ℓ_2 sum of finite dimensional spaces must be reflexive, J cannot be isomorphic to a subspace of such a space.

In [KW] Kalton and Werner showed a special version of the above result. They proved the conclusion of Theorem 4.1 (with $C = 1$) under the condition that X does not contain a copy of ℓ_1 and every weakly null type is an ℓ_p type. This means that for every $x \in S_X$ and every normalized weakly null sequence $(x_n) \subset S_X$, for $t > 0$ one has

$$\lim_{n\to\infty} \|x + tx_n\| = (1 + t^p)^{1/p} \ . \tag{29}$$

In [KW] it was shown that this condition implies that X must be reflexive, and it is easy to see that it also implies the hypothesis of Theorem 4.1 with $C = 1 + \varepsilon$ for any $\varepsilon > 0$.

Second, let us explain the reason for C^2 rather than C in the conclusion of Theorem 4.1. A normalized basis (x_i) is C-equivalent to the unit vector basis of ℓ_p if there exist constants A, B with $AB \le C$ and

$$(*) \qquad A^{-1}\Big(\sum_{i=1}^{\infty} |a_i|^p\Big)^{1/p} \le \Big\|\sum_{i=1}^{\infty} a_i x_i\Big\| \le B\Big(\sum_{i=1}^{\infty} |a_i|\Big)^{1/p}$$

for all scalars (a_i). If we had the hypothesis that every weakly null tree in S_X admitted a branch (x_i) with this property, then we could obtain the conclusion of Theorem 4.1 with C^2 replaced by C. However, the constants A, B above could vary with each such tree, and so we can only use $(*)$ with A and B replaced by C. In this case we only get C^2-embedding into ℓ_p.

We also note that Kalton [K] proved the following analogous theorem for c_0: Let X be a separable Banach space not containing ℓ_1. If there exists $C < \infty$ so that every weakly null tree in S_X has a branch C-equivalent to the unit vector basis of c_0, then X embeds into c_0.

W.B. Johnson [J2] showed that in the case $X \subseteq L_p$ $(1 < p < \infty)$, if there exists $K < \infty$ so that every normalized sequence in X has a subsequence K-equivalent to the unit vector basis of ℓ_p, then X embeds into ℓ_p. The tree hypothesis of Theorem 4.1 cannot in general be weakened to the subsequence condition, as the following example shows. (Theorem 4.1 and this example solve some questions raised in [J2].)

Example 4.2. Let $1 < p < \infty$. There exists a reflexive space X with an unconditional basis such that for all $\varepsilon > 0$ every normalized weakly null sequence in X admits a subsequence $1 + \varepsilon$-equivalent to the unit vector basis of ℓ_p. Yet X is not a subspace of an ℓ_p-sum of finite dimensional spaces.

Proof. Fix $1 < q < p$. We define $X = (\sum X_n)_p$, where each X_n is given as follows. X_n will be the completion of $c_{00}([\mathbb{N}]^{\le n})$ under the norm

$$\|x\|_n = \sup\Big\{\Big(\sum_{i=1}^{m} \|x|_{\beta_i}\|_q^p\Big)^{1/p} : (\beta_i)_1^m \text{ are disjoint segments in } [\mathbb{N}]^{\le n}\Big\}.$$

By a *segment* we mean a sequence $(A_i)_{i=1}^k \in [\mathbb{N}]^{\le n}$ with $A_1 = \{n_1, n_2, \dots, n_\ell\}$, $A_2 = \{n_1, n_2, \dots, n_\ell, n_{\ell+1}\}$, $\dots$, $A_k = \{n_1, n_2, \dots, n_\ell, n_{\ell+1}, \dots, n_{\ell+k-1}\}$, for some $n_1 < n_2 < \cdots < n_{\ell+k-1}$. Thus a segment can be seen as an interval of a branch (with respect to the usual partial order in $[\mathbb{N}]^{\le n}$), while a branch is a maximal segment.

Clearly the *node basis* $(e_A^{(n)})_{A \in [\mathbb{N}]^{\le n}}$ given by $e_A(B) = \delta_{(A,B)}$ is a 1-unconditional basis for X_n. Furthermore, the unit vector basis of ℓ_q^n is 1-equivalent to $(e_{A_i}^{(n)})_1^n$, if $(A_i)_1^n$ is any branch of $[\mathbb{N}]^{\le n}$.

Thus no extension of the tree $(e_A^{(n)})_{A \in [\mathbb{N}]^{\le n}}$ to a weakly null tree of infinite length in S_X has a branch whose basis distance to the ℓ_p unit vector basis is closer than $\mathrm{dist}_b(\ell_p^{(n)}, \ell_q^{(n)}) = n^{\frac{1}{q} - \frac{1}{p}} \to \infty$ for $n \to \infty$. Since it is clear that in every subspace Y of an ℓ_p sum of finite dimensional spaces every weakly null tree in S_Y must have a branch equivalent (for a fixed constant) to the unit vector basis of ℓ_p, it follows that X cannot be embedded into a subspace of an ℓ_p-sum of finite dimensional spaces.

Also each X_n is isomorphic to ℓ_p, and thus X is reflexive.

It remains to show that if (x_j) is a normalized weakly null sequence in X and $\varepsilon > 0$, then a subsequence is $1 + \varepsilon$-equivalent to the unit vector basis of ℓ_p. By a gliding hump argument it suffices to prove this in a fixed X_n. We proceed by induction on n.

For $n = 1$ the result is clear since X_1 is isometric to ℓ_p. Assume the result has been proved for X_{n-1}. By passing to a subsequence and perturbing we may assume that $(x_i)_1^\infty$ is a normalized block basis of the node basis for X_n.

Let $\varepsilon_i \downarrow 0$ rapidly. For $j \in \mathbb{N}$ let P_j be the basis projection of X_n onto $[e_A : A \in [\mathbb{N}]^{\le n}, \min A = j]$. Passing to a subsequence, we may assume that $\lim_{i\to\infty} \|P_j x_i\|_n = a_i$, and from the definition of $\|\cdot\|_n$ we have $(a_i)_{i=1}^\infty \in B_{\ell_p}$. Choose $a_0 \ge 0$ so that $(a_i)_{i=1}^\infty \in S_{\ell_p}$.

Passing to a subsequence of (x_i), we may assume that there exist integers $1 = N_0 < N_1 < \cdots$ so that the following conditions hold:

(i) $x_i(\{j\}) \ne 0 \Rightarrow j \in [N_i, N_{i+1})$.

(ii) $P_j x_i = 0$ for $j \ge N_{i+1}$.

(iii) $\displaystyle \Big\| \sum_{j\in[N_i,N_{i+1})} P_j x_i \Big\|_n = \Big(\sum_{j\in[N_i,N_{i+1})} \|P_j x_i\|_n^p \Big)^{1/p}$ is within ε_i of a_0.

(iv) If $j \in [N_i, N_{i+1})$, $i \ge 1$, then if $a_j \ne 0$, $(a_j^{-1} P_j x_\ell)_{\ell > i}$ is $1+\varepsilon_j$-equivalent to the unit vector basis of ℓ_p.

(v) If $j \in [N_0, N_1)$ and $a_j \ne 0$, then $(a_j^{-1} P_j x_\ell)_{\ell=1}^\infty$ is $1+\varepsilon_j$-equivalent to the unit vector basis of ℓ_p.

(vi) $\displaystyle \Big(\sum_{N_1}^\infty a_j^p \Big)^{1/p} < \varepsilon_1$,

(vii) If $j \in [N_0, N_1)$ and $a_j = 0$, then $\|P_j x_i\|_n \le \varepsilon_i$ for all i.

(viii) If $j \in [N_i, N_{i+1})$ and $a_j = 0$, then $\|P_j x_\ell\|_n < \varepsilon_\ell$ for $\ell > i$.

Conditions (iv) and (v) use the induction hypothesis and the fact that for all j, $\mathrm{span}(\{e_{\{j\}\cup A)} : A \in [\mathbb{N}]^{n-1}, \min A > j\})$ is isometric to X_{n-1}. Our conditions are sufficient to yield (for suitably small ε_j's) that (x_i) is $1+\varepsilon$-equivalent to the unit vector basis of ℓ_p. We omit the standard yet tedious calculations. □

For the proof of Theorem 4.1 we need a result which was shown in [KOS]. It is based on a trick of W. B. Johnson [J2], where part (a) was shown.

Lemma 4.3 (Lemma 5.1 in [KOS])**.** *Let X be a subspace of a space Z having a boundedly complete FDD (F_n), and assume X is w^* closed (since (F_n) is boundedly complete, Z is naturally a dual space). Then for all $\varepsilon > 0$ and $m \in \mathbb{N}$ there exists an $n > m$ such that if $x = \sum_1^\infty x_i \in B_X$ with $x_i \in F_i$ for all i, then there exists $k \in (m, n]$ with*

a) *$\|x_k\| < \varepsilon$ and*

b) *$\mathrm{dist}(\sum_{i=1}^{k-1} x_i, X) < \varepsilon$.*

Corollary 4.4. *Let X be a subspace of the reflexive space Z and let (F_i) be an FDD for Z. Let $\delta_i \downarrow 0$. There exists a blocking (G_i) of (F_i) given by $G_i = \bigoplus_{j=N_{i-1}+1}^{N_i} F_j$ for some $0 = N_0 < N - 1 < \cdots$ with the following property. For all $x \in S_X$ there exist $(x_i)_1^\infty \subseteq X$ and $t_i \in (N_{i-1}, N_i]$ for $i \in \mathbb{N}$ so that:*

a) $\displaystyle x = \sum_{i=1}^\infty x_i.$

b) *For $i \in \mathbb{N}$, either $\|x_i\| < \delta_i$ or $\|P_{\bigoplus_{j=t_{i-1}+1}^{t_i-1} F_j}(x_i) - x_i\| < \delta_i \|x_i\|$.*

c) *For $i \in \mathbb{N}$, $\|P_{\bigoplus_{j=t_{i-1}+1}^{t_i-1} F_j} x - x_i\| < \delta_i$.*

Proof. We choose an appropriate sequence $\varepsilon_i \downarrow 0$ depending upon (δ_i) and the basis constant K of (F_i). N_1 is chosen by the lemma for $\varepsilon = \varepsilon_1$ and $m = 1$. We choose $N_2 > N_1$ by the lemma for $\varepsilon = \varepsilon_2$ and $m = N_1$, and so on.

If $x \in S_X$, then for $i \in \mathbb{N}$ the lemma yields $t_i \in (N_{i-1}, N_i]$ with $\|P_{F_{t_i}}(x)\| < \varepsilon_i$ and $z_i \in X$ with $\|P_{\bigoplus_{j=1}^{t_i-1} F_j}(x) - z_i\| < \varepsilon_i$. We then let $x_1 = z_1$ and, for $i > 1$, $x_i = z_i - z_{i-1}$. Thus $\sum_{i=1}^n x_i = z_n \to x$, and so a) holds.

To see c), we note the following:

$$\begin{aligned}\|P_{\bigoplus_{j=t_{i-1}+1}^{t_i-1} F_j}(x) - x_i\| &\leq \|P_{\bigoplus_{j=1}^{t_i-1} F_j}(x) - z_i\| + \|P_{\bigoplus_{j=1}^{t_{i-1}} F_j}(x) - z_{i-1}\| \\ &< \varepsilon_i + 2\varepsilon_{i-1}\ .\end{aligned}$$

Thus

$$\begin{aligned}\|P_{\bigoplus_{j=t_{i-1}+1}^{t_i-1} F_j}(x_i) - x_i\| &= \|(\mathrm{Id} - P_{\bigoplus_{j=t_{i-1}+1}^{t_i-1} F_j})(x_i - P_{\bigoplus_{j=t_{i-1}+1}^{t_i-1} F_j} x)\| \\ &< (2K+1)(\varepsilon_i + 2\varepsilon_{i-1}),\end{aligned}$$

which can be made less than δ_i^2. This yields b). □

Remark. The proof yields that the conclusion of the corollary remains valid for any further blocking of the G_i's (which would redefine the N_i's).

Proof of Theorem 4.1. We first show that X embeds into $(\sum G_n)_{\ell_p}$ for some sequence (G_n) of finite dimensional spaces. Then to obtain the $C^2 + \varepsilon$ estimate we adapt an averaging argument similar to the one of [KW].

Applying Theorem 3.3 to the set

$$\mathcal{A} = \{(x_i) \in S_X^\omega : (x_i) \text{ is } C\text{-equivalent to the unit vector basis of } \ell_p\},$$

we find a reflexive space Z with an FDD (F_i) with basis constant K which isometrically contains X and $\delta_i \downarrow 0$ so that whenever $(x_i) \subseteq S_X$ satisfies

$$\|P_{\bigoplus_{j=n_{i-1}+1}^{n_i-1} F_j}(x_i) - x_i\| < \delta_i \tag{30}$$

for some sequence $1 = n_0 < n_1 < \cdots$ in $\mathbb{N}$ it follows that (x_i) is $2C$-equivalent to the unit vector basis of ℓ_p. Let $G_i = \bigoplus_{j=N_{i-1}+1}^{N_i} F_j$ be the blocking given by Corollary 4.4.

Let $x \in S_X$, $x = \sum \bar{x}_i$ with $\bar{x}_i \in G_i$ for all i. Choose (x_i) and $(t_i) \subseteq \mathbb{N}$ as in Corollary 4.4. It follows from (30) that (for δ_i's sufficiently small)

$$(3C)^{-1} \leq \Big(\sum \|x_i\|^p\Big)^{1/p} \leq 3C$$

and

$$(4C)^{-1} \leq \bigg(\sum_i \|P^{t_i-1}_{\bigoplus_{j=t_{i-1}+1} F_j} x\|^p\bigg)^{1/p} \leq 4C\ .$$

Let $y_i = P_{\bigoplus_{j=t_{i-1}+1}^{t_i-1} F_j} x$.

Since

$$\frac{1}{2(K+1)} \max(\|y_i\|, \|y_{i+1}\|) - \delta_i \leq \|\bar{x}_i\| \leq (2K+1)\|y_i\| + \delta_i,$$

it follows that X embeds isomorphically into $(\sum G_i)_{\ell_p} \equiv W$.

We now renorm W so as to contain X isometrically. Thus W has (G_i) as an FDD, and there exists $\tilde{C}$ such that if (w_i) is any block basis of a permutation of (G_i), then

$$\tilde{C}^{-1}(\sum \|w_i\|^p)^{1/p} \le \|\sum w_i\| \le \tilde{C}(\sum \|w_i\|^p)^{1/p} \ . \tag{31}$$

We repeat the first part of the proof. Let $\varepsilon > 0$. From Theorem 3.3 we may assume that there exist $\delta_i \downarrow 0$ so that if $(x_i) \subseteq S_X$ satisfies

$$\|P_{\bigoplus_{j=n_{i-1}+1}^{n_i-1} G_j}(x_i) - x_i\| < \delta_i \tag{32}$$

for some $1 = n_0 < n_1 < \cdots$, then (x_i) is $C+\varepsilon$-equivalent to the unit vector basis of ℓ_p. Moreover we may assume that this is valid for any further blocking of (G_j). From now on we will replace X by the finite codimensional subspace $\bigoplus_{i=2}^\infty G_i \cap X$ and W by $\bigoplus_{i=2}^\infty G_i$, and replace G_i by G_{i+1}. We will show that this new X can be $C^2+\varepsilon$-embedded into an ℓ_p sum of finite dimensional spaces.

Let $H_i = \bigoplus_{j=N_{i-1}+1}^{N_i} G_j$ be the blocking given by Corollary 4.4. Thus (for appropriately small δ_i's) from (32) and Corollary 4.4 we have that if $x \in S_X$ there exist $t_i \in (N_{i-1}, N_i]$ such that

(33)
$$(C+2\varepsilon)^{-1}\Big(\sum_{i=1}^\infty \Big\|\sum_{j=t_{m_{i-1}}+1}^{t_{m_i}} x_j\Big\|^p\Big)^{1/p} \le \|x\| \le (C+2\varepsilon)\Big(\sum_{i=1}^\infty \Big\|\sum_{j=t_{m_{i-1}}+1}^{t_{m_i}} x_j\Big\|^p\Big),$$

where $x = \sum x_i$ is the expansion of X w.r.t. the FDD (G_j) for W.

Choose $M \in \mathbb{N}$ so that

$$\frac{\tilde{C}2^{1/p}}{M} \le \varepsilon \ \text{ and } \ (C+2\varepsilon)^{-1} - \frac{\tilde{C}^2}{M^{1/p}} \ge (C+3\varepsilon)^{-1} \ . \tag{34}$$

For $i = 1, 2, \dots, M$ and $j = 0, 1, 2, \dots$ set $L(i,j) = \bigoplus_{s=(j-1)M+i+1}^{jM+i-1} H_s \subseteq W$ (using $H_n = \{0\}$ if $n \le 0$) and let $Y_i = (\bigoplus_{j=0}^\infty L(i,j))_p$. Let $Y = (\bigoplus_{i=1}^M Y_i)_p$. We shall prove that X $C^2 + \eta(\varepsilon)$-embeds into Y, where $\eta(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$, which will complete the proof.

To do this we first define maps $T_i : X \to Y_i$ for $1 \le i \le M$. If $x = \sum x_j$ is the expansion of x w.r.t. (H_j), we let

$$T_i x = \sum_{s=1}^\infty \Big(\sum_{u=(s-1)M+i+1}^{sM+i-1} x_s\Big) \in \Big(\bigoplus_{s=1}^\infty L(i,s)\Big)_p = Y_i \ .$$

Let $1 \le i \le M$ and $x \in S_X$, $x = \sum x_j$ as above. Write $x_j = \sum_{u=N_{j-1}+1}^{N_j} x(j,u)$ as the expansion of $x_j \in H_j$ w.r.t. (G_i). Let $(t_i) \subseteq \mathbb{N}$ be given by Corollary 4.4 (w.r.t. (G_j)). From several applications of the triangle inequality and (31) and

(33) we have

$$\begin{aligned}
\|T_i(x)\| &= \left[\sum_{j=0}^{\infty} \| \sum_{s=(j-1)M+i+1}^{jM+i-1} x(s)\|^p\right]^{1/p} \\
&\le \left[\sum_{j=0}^{\infty} \| \sum_{u=t_{(j-1)M+i}}^{N_{(j-1)N+i}} x((j-1)M+i,u) + \sum_{s=(j-1)M+i+1}^{jM+i-1} x(s) \right. \\
&\qquad\qquad \left. + \sum_{u=1+N_{jM+i-1}}^{t_{jM+i}} x(jM+i,u)\|^p\right]^{1/p} \\
&\quad + \left[\sum_{j=0}^{\infty} \| \sum_{u=t_{(j-1)M+i}}^{N_{(j-1)N+i}} x((j-1)M+i,u) + \sum_{u=1+N_{jM+i-1}}^{t_{jM+i}} x(jM+i,u)\|^p\right]^{1/p} \\
&\le (C+2\varepsilon)\|x\| + \left[\sum_{j=0}^{\infty} \sum_{u=1+N_{jM+i-1}}^{N_{jM+i}} \|x(jM+i,u)\|^p\right]^{1/p} \\
&\le (C+2\varepsilon)\|x\| + \tilde{C}\| \sum_{s=i(\mathrm{mod}\, M)} x_s\| \ .
\end{aligned}$$

Similarly one has

$$\|T_i x\| \ge (C+2\varepsilon)^{-1}\|x\| - \tilde{C}\Big\| \sum_{s=i(\mathrm{mod}\, M)} x_s\| \ .$$

Finally we define $T : X \to Y = (\sum_1^M Y_i)$ by $Tx = \frac{1}{M^{1/p}} \sum_{i=1}^M T_i x$. Note that

$$\begin{aligned}
\|Tx\| &\le \frac{1}{M^{1/p}}(C+2\varepsilon)\left(\sum_{i=1}^M \|x\|^p\right)^{1/p} + \frac{\tilde{C}}{M^{1/p}}\left(\sum_{i=1}^M \Big\| \sum_{j=i(\mathrm{mod}\, M)} x_j\Big\|\right) \\
&\le (C+2\varepsilon)\|x\| + \frac{\tilde{C}^2}{M^{1/p}}\|x\| < (C+3\varepsilon)\|x\|
\end{aligned}$$

using (31) and (34).

Similarly one deduces that $\|T(x)\| \ge \frac{1}{C+3\varepsilon}\|x\|$ for $x \in X$. $\square$

Remark. The proof of Theorem 4.1 had two steps. In the first we started with an embedding of X into a certain reflexive space Z with an FDD (F_i), and showed that (F_i) can be blocked to an FDD (G_i) so that X is isomorphic to a subspace of $(\bigoplus G_i)_{\ell_p}$. In that step we could not deduce any bound for the constant of that isomorphism. In the second step we "inflated" $(\bigoplus G_i)_{\ell_p}$ to the space $(\bigoplus_{i=1}^M \bigoplus_{j\ne i(\mathrm{mod}\, M)} G_j)_{\ell_p}$ and showed that this space contains a finite codimensional subspace which is $C^2+\varepsilon$-equivalent to X.

The following example shows that even if the space X has a basis to begin with, it is in general not possible to pass to a blocking (F_n) of that basis and deduce that for some n_0 the identity is a $C^2+\varepsilon$-isomorphism between $\bigoplus_{n=n_0}^\infty F_n$ and $(\bigoplus_{n=n_0}^\infty F_n)_{\ell_p}$.

Example 4.5. Let $\mathcal{D}$ be the set of all sequences (D_n) of pairwise disjoint subsets of $\mathbb{N}$, so that for each $n \in \mathbb{N}$, D_n is either a singleton or it is of the form $D_n = \{k, k+1\}$ for some $k \in \mathbb{N}$. We give ℓ_2 the following equivalent norm $|||\cdot|||$:

$$|||x||| = \sup_{(D_n)\in\mathcal{D}} \left(\sum_{n=1}^{\infty} \left(\sum_{j\in D_n} |x_j| \right)^2 \right)^{1/2},$$

whenever $x = (x_j) \in \ell_2$.

It is easy to see that every normalized skipped block $(x^{(n)})$ in $X = (\ell_2, |||\cdot|||)$ is isometrically equivalent to the ℓ_2 unit vector basis. Thus the assumptions of Theorem 4.1 are satisfied for any $C > 1$. On the other hand, for any blocking (F_n) of the unit vector basis (e_i) of X it follows for any n and $N_n = \max\{N | e_N \in F_n\}$ that $e_{N_n+1} \in F_{n+1}$ and that the span of e_{N_n} and e_{N_n+1} is isometric to ℓ_1^2. Therefore the norm of the identity between $(\bigoplus_{n=2}^\infty F_n)_{\ell_2}$ and $(\bigoplus_{n=2}^\infty F_n)_{\ell_2, |||\cdot|||}$ is at least $\sqrt{2}$.

The following result shows that the property that every normalized weakly null tree contains a branch which is C-equivalent to the ℓ_p unit vector basis dualizes. It can be seen as the isomorphic version of Theorem 2.6. in [KW].

Corollary 4.6. *Assume X is a reflexive Banach space. For $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ the following statements are equivalent.*

a) *There is a $C \geq 1$ so that every normalized weakly null tree in X has a branch which is C-equivalent to the unit vector basis of ℓ_p.*
b) *There are a $C \geq 1$, a finite codimensional subspace $\tilde{X}$ of X, a sequence of finite dimensional spaces $(E_i)_{i=1}^\infty$, and an operator $T : \tilde{X} \to (\bigoplus_{i=1}^\infty E_i)_{\ell_p}$ such that $C^{-1}\|x\| \leq \|T(x)\| \leq C\|x\|$ for all $x \in \tilde{X}$.*
c) *There is a $C \geq 1$ such that every normalized weakly null tree in X^* has a branch which is C-equivalent to the unit vector basis of ℓ_q.*
d) *There are a $C \geq 1$, a finite codimensional subspace Y of X^*, a sequence of finite dimensional spaces $(E_i)_{i=1}^\infty$, and an operator $T : Y \to (\bigoplus_{i=1}^\infty E_i)_{\ell_q}$ such that $C^{-1}\|x\| \leq \|T(x)\| \leq C\|x\|$ for all $x \in Y$.*

Proof. The implications (a)$\Rightarrow$(b) and (c)$\Rightarrow$(d) follow from Theorem 4.1 and its proof. If we prove (b)$\Rightarrow$(c), then (d)$\Rightarrow$(a) will follow.

Assume that $C \geq 1$, $(E_i)_{i=1}^\infty$, $\tilde{X} \subset X$ and $T : \tilde{X} \to Z = (\bigoplus_{i=1}^\infty E_i)_{\ell_p}$ are given as in the statement of (b). By passing to the renorming $|||\cdot|||$, $|||x||| = \|T(x)\|$, for $x \in \tilde{X}$ we can assume without loss of generality that $\tilde{X}$ is isometric to a subspace of Z.

We will show that $\tilde{X}^*$ satisfies the condition (c). Since $\tilde{X}^*$ is isomorphic to a subspace of X^* of finite codimension, the claim will follow.

Thus let $E : \tilde{X} \to (\bigoplus_{i=1}^\infty E_i)_{\ell_p}$ be an isometric embedding and let $(x_A^*)_{A\in[\mathbb{N}]^{<\omega}}$ be a normalized weakly null tree in $\tilde{X}^*$.

We will need the following observation.

Claim. If (x_n^*) is a normalized and weakly null sequence in $\tilde{X}^*$, then there are normalized weakly null sequences (z_n^*) and (x_n) in Z^* and $\tilde{X}$ respectively such that $E^*(z_n^*) = x_n^*$ and $x_n^*(x_n) = 1$ for $n \in \mathbb{N}$.

To see this, use the Hahn-Banach theorem to choose a normalized sequence $(z_n^*)_{n\in\mathbb{N}}$ in Z^* so that $E^*(z_n^*) = x_n^*$. The sequence (z_n^*) is weakly null. Indeed, otherwise we could choose a $y^* \in Z^*$, $y^* \neq 0$, a subsequence $(z^*_{n_k})$ and a weakly null sequence (y^*_k) in Z^* so that $z^*_{n_k} = y^* + y^*_k$ for all $k \in \mathbb{N}$. Thus, $x^*_{n_k} = E^*(y^*) + E^*(y^*_k)$, which implies that $E^*(y^*) = 0$ and therefore that $E^*(y^*_k) = x^*_{n_k}$. Since $\limsup_{k\to\infty} \|y^*_k\| = \limsup_{k\to\infty}(\|z^*_{n_k}\|^q - \|y^*\|^q)^{(1/q)} < 1$, we get a contradiction.

Then we choose $(x_n) \in \tilde{X}$ so that $x^*_n(x_n) = 1$. By a similar argument we have that (x_n) is also weakly null.

Using the claim, we can find a normalized weakly null tree $(z^*_A)_{A\in[\mathbb{N}]^{<\omega}}$ in Z^* and a normalized weakly null tree $(x_A)_{A\in[\mathbb{N}]^{<\omega}}$ in $\tilde{X}$ so that $E^*(z^*_A) = x^*_A$ and $x^*_A(x_A) = 1$ for $A \in [\mathbb{N}]^{<\omega}$.

Given an $\varepsilon > 0$, we can choose a branch $(x^*_n) = (x^*_{A_n})$ so that $(z^*_{A_n})$ is $(1+\varepsilon)$-equivalent to the unit vector basis of ℓ_q, and (x_{A_n}) is $(1+\varepsilon)$-equivalent to the unit vector basis of ℓ_p. This easily implies that $(x^*_{A_n})$ is $(1+\varepsilon)$-equivalent to the unit vector basis of ℓ_q. □

Remark. Johnson and Zippin [JZ] proved the following. Let $C_p = (\bigoplus_{i=1}^\infty E_i)_{\ell_p}$, where (E_i) is dense, in the Banach-Mazur sense, in the set of all finite dimensional spaces. Then X embeds into C_p if and only if X^* embeds into C_q (where $\frac{1}{p}+\frac{1}{q}=1$). Thus Corollary 4.6 could be deduced from [JZ] and Theorem 4.1 (and [JZ] could be deduced from the corollary and theorem).

Furthermore, the proof of Corollary 4.6 yields some quantitative information. If a) holds, then b) is true with C replaced by $C+\varepsilon$ for any $\varepsilon > 0$. If b) holds, then c) is valid with C replaced by $C^2+\varepsilon$.

5. Spectra and asymptotic structures

In [Mi] Milman introduced the notion of the *spectrum of a function* defined on S^n_X. Let (M,ρ) be a compact metric space and let $f : S^n_X \to M$ be Lipschitz. $\sigma(f)$ is defined to be the set of all $a \in M$ for which the following condition (35) is true:

$$\begin{aligned} \forall\,\varepsilon > 0\,\forall\, Y_1 \in \mathrm{cof}(X)\,\exists\, y_1 \in S_X\,\forall\, Y_2 \in \mathrm{cof}(X)\,\exists\, y_2 \in S_X \\ \dots\,\forall\, Y_n \in \mathrm{cof}(X)\,\exists\, y_n \in S_X \text{ such that} \\ \rho(f(y_1,y_2,\dots,y_n),a) < \varepsilon. \end{aligned} \tag{35}$$

In terms of the game we introduced in Section 2, $\sigma(f)$ is the set of all $a \in M$ such that for any $\varepsilon > 0$ Player II has a winning strategy in the $\mathcal{A}^\varepsilon$-game, where

$$\mathcal{A}^\varepsilon = \{(y_i)_{i=1}^n \in S^n_X : \rho(a, f(y_1,\dots,y_n)) > \varepsilon\}$$

(which means that Player II is able to get $f(y_1,\dots,y_n)$ arbitrarily close to a).

As mentioned in [Mi], one can also define the spectrum relative to any *filtration* $\mathcal{S} \subset \mathrm{cof}(X)$, meaning that $\mathcal{S}$ has the property that if $X, Y \in \mathcal{S}$ there is a $Z \in \mathcal{S}$ for which $X \cap Y \supset Z$. The spectrum of f relative to S is the set $\sigma(f,\mathcal{S})$ of all $a \in M$ for which

$$\begin{aligned} \forall\,\varepsilon > 0\,\forall\, Y_1 \in \mathcal{S}\,\exists\, y_1 \in S_{Y_1}\,\forall\, Y_2 \in \mathcal{S}\,\exists\, y_2 \in S_{Y_2}\dots\forall\, Y_n \in \mathcal{S}\,\exists\, y_n \in S_{Y_n} \text{ with} \\ \rho(f(y_1,y_2,\dots,y_n),a) < \varepsilon. \end{aligned} \tag{36}$$

It is obvious that $\sigma(f,\mathcal{S}) \subset \sigma(f,\tilde{\mathcal{S}})$ whenever $\tilde{\mathcal{S}} \subset \mathcal{S}$. In particular, it follows that $\sigma(f) \subset \sigma(f,\mathcal{S})$ for any filtration $\mathcal{S}$.

If X is a subspace of a space Z with FDD (E_i), then we can consider the filtration $\mathcal{S} = \{X \cap \bigoplus_{i=n}^\infty F_i : n \in N\}$, and we write $\sigma(f,(F_i)) = \sigma(f,\mathcal{S})$.

On one hand, the unrelativized spectrum $\sigma(f)$ seems to be the right concept to study geometric and structural properties of X, since it is "coordinate free". On the other hand, spectra with respect to an FDD are combinatorically easier to use and understand.

But from Theorem 3.3 we deduce that $\sigma(f)$ is equal to the spectrum with respect to a certain FDD (of some superspace).

Proposition 5.1. *Let $f : S_X^n \to M$ be Lipschitz. Then*

$$\sigma(f) = \bigcap\{C : C \text{ is a closed subset of } M \text{ and } (W_I(f^{-1}(C)))\} \ . \tag{37}$$

Moreover, for any $\varepsilon > 0$, $(W_I(f^{-1}(C))_\varepsilon)$.

Furthermore, X can be embedded into a space Z with FDD (F_i) so that for every $\varepsilon > 0$ there are a $\delta > 0$ and an $M_0 \in \mathbb{N}$ with the following property.

Whenever $M_0 < M_1 < M_2 < \cdots < M_n$ and $(x_i)_{i=1}^n \subseteq S_X$ satisfies

$$d\Big(x_i, S_{\bigoplus_{j=1+M_{i-1}}^{M_i-1} F_j} \cap X\Big) < \delta \quad \text{for } i = 1, \dots, n,$$

then $\rho(f(x_1, x_2, \dots, x_n), \sigma(f)) < \varepsilon$.

In the case that X^* is separable, $\sigma(f)$ is the minimal closed subset of M such that for any $\varepsilon > 0$ any weakly null tree in S_X of length n has a branch $(x_1, \dots, x_n)$ so that $\rho(f(x_1, \dots, x_n), \sigma(f)) < \varepsilon$.

Proof. Let $\mathcal{C}$ denote the set of all closed subsets of M for which $(\mathrm{W}_I(f^{-1}(C)))$ holds. For $a \in M$ we denote the ε-neighborhood by $U_\varepsilon(a)$ and observe the following equivalences:

$$\begin{aligned}
&a \notin \sigma(f)\\
&\iff \exists\, \varepsilon > 0\, \exists\, Y_1 \in \mathrm{cof}(X)\, \forall\, y_1 \in S_{Y_1} \dots \exists Y_n \in \mathrm{cof}(X)\, \forall\, y_n \in S_{Y_n}\\
&\qquad\quad \rho(f(y_1, \dots, y_n), a) > \varepsilon\\
&\iff \exists\, \varepsilon > 0 \quad (\mathrm{W}_I(f^{-1}(M \setminus U_\varepsilon(a)))\\
&\iff \exists\, C \in \mathcal{C}\,, \qquad a \notin C.
\end{aligned}$$

Thus $\sigma(f) = \bigcap\{C : C \in \mathcal{C}\}$. If $\eta > 0$, then $M \setminus (\sigma(f))_\eta$ is compact and is contained in the open covering $\bigcup_{C \in \mathcal{C}} M \setminus C$. Thus there exists a finite $\tilde{\mathcal{C}} \subset \mathcal{C}$ such that $M \setminus (\sigma(f))_\eta \subset \bigcup_{C \in \tilde{\mathcal{C}}} M \setminus C$, and thus $(\sigma(f))_\eta \supset \bigcap_{C \in \tilde{\mathcal{C}}} C$, which implies by Proposition 2.1 that Player I has a winning strategy for $f^{-1}((\sigma(f))_\eta)$. By the uniform continuity of f, η can be chosen small enough so that $f^{-1}((\sigma(f))_\eta)_n$ is contained in a given neighborhood of $f^{-1}(\sigma(f))$, which finishes the proof of the first part. The remainder of the proposition follows easily from Theorem 3.3. $\square$

A special example of spectra was considered by Milman and Tomczak [MT], the *asymptotic structure of* X. A finite dimensional space E together with a normalized monotone basis $(e_i)_1^n$ is called an *element of the n^{th} asymptotic structure of* X, and we write $(E, (e_i)_{i=1}^n) \in \{X\}_n$, if

$$\begin{aligned}
&\forall\, \varepsilon > 0\, \forall\, Y_1 \in \mathrm{cof}(X)\, \exists\, y_1 \in S_X \dots \exists\, Y_n \in \mathrm{cof}(X)\, \exists\, y_n \in S_X\\
&\mathrm{dist}_b((y_i)_{i=1}^n, (e_i)_{i=1}^n) < 1 + \varepsilon,
\end{aligned} \tag{38}$$

where dist_b denotes the basis distance, i.e., if $(e_i)_{i=1}^n$ and $(f_i)_{i=1}^n$ are bases of E and F respectively, then $\mathrm{dist}_b((e_i)_{i=1}^n, (f_i)_{i=1}^n)$ is defined to be $\|T\| \cdot \|T^{-1}\|$ where $T : E \to F$ is given by $T(e_i) = f_i$ for $i = 1, \dots, n$. Note that the space $(M_n, \log \mathrm{dist}_b)$

of all normalized bases of length n and basis constant not exceeding a fixed constant is a compact metric space.

Therefore from Proposition 5.1 and the usual diagonalization argument we deduce the following corollary (cf. [KOS]).

Corollary 5.2. *X can be embedded into a space Z with FDD (F_i) so that for every $k \in \mathbb{N}$, whenever $k = M_0 < M_1 < M_2 < \cdots < M_k$ and*

$$x_i \in S_{\bigoplus_{j=1+M_{i-1}}^{M_i-1} F_j} \cap X \quad \text{for } i = 1, 2, \dots, k\ ,$$

then $\mathrm{dist}_b((x_i)_{i=1}^k, \{X\}_k) < 1 + \varepsilon$.

In the case that X^* is separable, $\{X\}_k$ is the minimal closed subset of M_k such that for any $\varepsilon > 0$ any weakly null tree in S_X of length n has a branch $(x_1, \dots, x_k)$ so that $\mathrm{dist}_b((x_i)_{i=1}^k, \{X\}_k) < 1 + \varepsilon$.

An interesting case is when the asymptotic structure of X is as small as possible.

Theorem 5.3. *Let X be a separable reflexive Banach space with $|\{X\}_2| = 1$. Then there exists $p \in (1, \infty)$ such that X embeds into the ℓ_p-sum of finite dimensional spaces. Moreover, for all $\varepsilon > 0$ there exists a finite codimensional subspace X_0 of X which $1 + \varepsilon$-embeds into the ℓ_p-sum of finite dimensional spaces.*

Proof. Since there exists $1 \le p \le \infty$ such that the unit vector basis of ℓ_p^2 is in $\{X\}_2$ (see [MMT]), we have that $\{X\}_2$ must be this unit vector basis. In turn, this condition (see [MMT] or [KOS]) implies that X contains an isomorph of ℓ_p (c_0 if $p = \infty$), and so $1 < p < \infty$.

Let $X \subseteq Z$, a reflexive space with an FDD (E_n). The condition on $\{X\}_2$ yields that for all $\varepsilon > 0$ there exists n so that if $x_1 \in S_X \cap [E_i]_{i=n}^\infty$ then there exists m so that if $x_2 \in S_X \cap [E_i]_{i=m}^\infty$ then $(x_i)_1^2$ is $1 + \varepsilon$-equivalent to the unit vector basis of ℓ_p^2. From this it follows that X satisfies the hypothesis of Theorem 4.1 with $C = 1$, and thus the theorem follows. $\square$

The following problem remains open. We say X is *Asymptotic* ℓ_p if there exists $K < \infty$ so that for all k and all $(x_i)_1^k \in \{X\}_k$, $(x_i)_1^k$ is K-equivalent to the unit vector basis of ℓ_p. An FDD (E_n) for a space Z is *asymptotic* ℓ_p if there exists $K < \infty$ so that, for all k, if $(x_i)_1^k$ is a block sequence of $(E_i)_k^\infty$ in S_Z, then $(x_i)_1^k$ is K-equivalent to the unit vector basis of ℓ_p.

Problem 5.4. Let X be a reflexive Asymptotic ℓ_p space for some $1 < p < \infty$. Does X embed into a space Z with an asymptotic ℓ_p FDD?

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