Trees, valuations and the Green-Lazarsfeld set.

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Thomas Delzant*
Département de Mathématiques, Université de Strabourg
7 rue R. Descartes F-67084 Strasbourg

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1 Introduction.

The aim of this paper is the study of the relationship between two objects, the Green-Lazarsfeld set and the Bieri Neumann Strebel invariant, which appear simultaneously in 1987 ([GL] , [BNS]). Let us recall some basic definitions.

Let Γ be a finitely generated group, and K be a field. A 1-character χ is an homomorphism from Γ to K^* ; in this article we will only consider 1-characters, and call them characters. A character χ is called exceptional if $H^1(\Gamma, \chi) \neq 0$, or more geometrically if χ can be realized as the linear part of a fixed point free affine action of Γ on a K-line.

The set of exceptional characters, $E^1(\Gamma, K)$ is a subset of the abelian group $\mathrm{Hom}(\Gamma, K^*)$, and our aim is to understand its geometry, in particular if Γ is the fundamental group of a Kähler manifold.

Motivated by the pioneering work of M. Green and R. Lazarsfeld, algebraic geometers studied the case where $K = \mathbb{C}$ is the field of complex numbers, and $\Gamma = \pi_1(X)$ is the fundamental group of a projective or more generally a Kähler manifold. In this case, the geometry of $\mathrm{Hom}(\Gamma, K^*)$is well understood : it is the union of a finite set, made up with torsion characters, and a finite set of translates of subtori. This result has been conjectured by A. Beauville and F. Catanese, ([Be]) proved by C. Simpson [Si 2]) for projective manifolds, and extended by F. Campana to the Kähler case (see [Ca] for a detailed introduction). The main tools used by C. Simpson were the flat hyper-Kähler structure of $\mathrm{Hom}(\Gamma, \mathbb{C}^*)$ and the Schneider-Lang theorem in transcendence theory. Another proof, model theoretic, has been proposed by R. Pink and D. Roessler [PR].

The definition of a exceptional class in the sense of Bieri Neumann Strebel is easier to explain in the case of an integral cohomology class (an element of $H^1(\Gamma, \mathbb{Z})$). Such a class is *exceptional* if it can be realized as the translation class of a parabolic, non loxodromic action of Γ in some tree.

*Email: `delzant@math.u-strasbg.fr`

The link between these two notions, explained in the next paragraph, can be sketched as follows. Let χ be a Green-Lazarsfeld character of Γ. Suppose $\chi(\Gamma)$ is not contained in the ring of algebraic integers of K. There exists a discrete non archimedian valuation on K such that $v \circ \chi$ is a non trivial homomorphism to $\mathbb{Z}$. It appears that $v \circ \chi$ is an exceptional class in the sense of Bieri Neumann Strebel. More precisely, on can find a parabolic action of Γ on the Bruhat Tits tree of K_v, with translation length $v \circ \chi$.

Due to the work of C. Simpson [Si 3], M. Gromov and R. Shoen [GS], exceptional cohomology classes on Kähler manifold are well understood (see also [De] for a detailed study of the BNS invariant of a Kähler group). Let X be a Kähler manifold, and ω an exceptional class ; there exists a holomorphic map F from X to a hyperbolic Riemann orbifold Σsuch that ω belongs to $F^*H^1(\Sigma, \mathbb{Z})$. Recall that a complex 2-orbifold Σ is a Riemann surface S marked by a finite set of marked points $\{(q_1, m_1) \ldots, (q_n, m_n)\}$, where the $m_i's$ are integers $\geqslant 2$. A map $F : X \to \Sigma$ is called holomorphic if it is holomorphic in the usual sense, and for every q_i the multiplicity of the fiber $F^{-1}(q_i)$ is divisible by m_i. The main result of this paper is a description of the (generalized) Green Lazarsfeld set of $\pi_1(X)$ in terms of the finite list of its fibrations on hyperbolic 2-orbifolds.

Theorem *Let Γ be the fundamental group of a Kähler manifold X, $(F_i, \Sigma_i)_{1 \leqslant i \leqslant n}$ the family of fibration of X over hyperbolic 2-orbifolds. Let K be a field of characteristic p (if $p = 0$, $K = \mathbb{C}$), $\bar{F}_p \subset K$ the algebraic closure of F_p in K. Then $E^1(\Gamma, K)$ is the union of a finite set of torsion characters (contained in $E^1(\Gamma, \bar{F}_p)$ if $p > 0$) and the union $\bigcup_{1 \leqslant i \leqslant n} F_i^* E^1(\pi_1^{\mathrm{orb}}(\Sigma_i), K^*)$.*

Remarks a) Let $\Sigma = (S; (q_i, m_i)_{1 \leqslant i \leqslant n})$ a hyperbolic 2-orbifold, and $\Gamma = \pi_1^{\mathrm{orb}}(\Sigma)$ its fundamental group. Then, by a simple computation (see prop.22), one checks that $E^1(\pi_1^{\mathrm{orb}}(\Sigma), K^*) = \mathrm{Hom}(\pi_1^{\mathrm{orb}}(\Sigma, K^*)) = (K^*)^{2g} \times \Phi$, where Φ is a finite abelian group, unless $g = 1$ and for all i, $m_i \not\equiv 0(\mathrm{char}\, K)$. If $g = 1$ and for all i $m_i \not\equiv 0(\mathrm{char}\, K)$, $E^1(\pi_1^{\mathrm{orb}}(\Sigma), K^*)$ is finite, made of torsion characters.

In every cases, the Green Lazarsfeld set is the union of a finite set of torsion characters and a finite set of abelian groups which are translates of tori ; this is our generalization of Simpson's theorem.

b) The main tool used by Simpson to prove his theorem [Si 2]was the study of algebraic triple tori ; if $\mathrm{char}\, K \neq 0$ no such a structure is available. Our proof furnishes a geometric (i.e. non arithmetic) alternative to Simpson's proof in the case of characteristic 0. In fact, in this case ($\mathrm{car}\, K = 0$) our method proves that $E^1(\Gamma, K)$ is made with a finite set of *integral* characters (in the sense of Bass [Ba]), and the union $\bigcup_{1 \leqslant i \leqslant n} F_i^* \mathrm{Hom}(\pi_1^{\mathrm{orb}}(\Sigma_i), K^*)$; the conclusion follows from the study of the the absolute value $|\chi|$ of exceptional characters, which was already done by A. Beauville[Be].

In a recent preprint, [CS] ,C. Simpson and K. Corlette study the variety of characters of a Kähler group Γ, $\mathrm{Hom}^{\mathrm{ss}}(\Gamma, \mathrm{PSl}(2, \mathbb{C}) / \mathrm{PSL}(2, \mathbb{C})$ from a very similar point of view ; they prove in particular that a Zariski dense representation of a Kähler group which is not *integral* in the sense of Bass factorizes through a fibration over a hyperbolic 2-orbifold. Their proof is based on the same idea as ours : if a representation ρ is not integral, there exists a valuation on the field generated by $\rho(\Gamma)$ such that the action of Γ on the Bruhat Tits building is non elementary. The conclusion follows by applying the theory of Gromov Shoen on harmonic maps with value in a tree. Using Simpson's work on Higgs bundles they prove further a rigid representation come from a complex variation of Hodge structure.

In paragraph 2, we explain the relationship between the Green-Lazarsfeld and Bieri-Neumann-Strebel invariants ; in paragraph 3 we study the Green-Lazarsfeld set of a metabelian group : a finiteness result on this set is established. These two paragraphs are purely group theoretic, and no Kähler structure is mentioned. In the paragraph 4 we prove the main result.

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2 From an affine action on a line to a parabolic action on a tree.

2.1 Affine action on the line : the Green-Lazarsfeld set

Let K be a field. The affine group of transformation of a K-line, $\mathrm{Aff}_1(K)$, is isomorphic to $K^* \ltimes K$. We identify this group with the set of upper triangular $(2,2)$matrices $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ with values in K.

Let Γ be a finitely generated group. An affine action of Γ on the line is a morphism $\rho : \Gamma \to \mathrm{Aff}_1(K)$. One can write $\rho(g) = \begin{pmatrix} \chi(g) & \theta(g) \\ 0 & 1 \end{pmatrix}$. The linear part of ρ is an homomorphism $\chi : \Gamma \to K^*$. Its translation part $\theta : \Gamma \to K$ is a 1-cocycle of Γ with value in χ, i.e. a function which satisfies $\theta(gh) = \theta(g) + \chi(g)\theta(h)$. The representation ρ is conjugate to a diagonal representation if and only if $\rho(\Gamma)$ fixes a point $\mu \in K$, or equivalently if and only if there exists a $\mu \in K$ such that $\theta(g) = \mu(-1 + \chi(g))$ is a coboundary.

Definition 1 *A character $\chi \in \mathrm{Hom}(\Gamma, K^*)$is exceptional if it can be realized as the linear part of a fixed point free affine action of Γ on the line, i.e if $H^1(\Gamma, \chi) \neq 0$. The set of exceptional characters $E^1(\Gamma, K)$ is called the Green-Lazarsfeld set of* Γ.

2.2 Parabolic action on a tree : the Bieri Neumann Strebel invariant.

Let T be a simplicial tree. We endow T with its natural simplicial metric, and think of T as a complete geodesic space. Let us recall the definitions of the boundary of T, and of the Busemann cocyle associated to a point in this boundary.

A ray in T is an isometric map $r : [a, +\infty[\to T$. Two rays $r : [a, +\infty[\to T$, $s : [b, +\infty[\to T$ are equivalent (or asymptotic) if they coincide after a certain time : there exists a', b' s.t. for all $t \geqslant 0$ $r(a' + t) = s(b' + t)$. The boundary of T, denoted ∂T, is the set of equivalence classes of rays. If $\alpha \in \partial T$ and $r : [a, +\infty[\to T$ represents α, for every point x, the function $t \to d(x, r(t)) - t$ is eventually constant. Its limit $b_r(x)$ is called the Busemann function of r. If s is equivalent to r, the difference $b_r - b_s$ is a constant.

Definition 2 *(Busemman cocyle) . Let* Γ *be a group acting on* T*, and* $\alpha \in \partial T$*. If* Γ *fixes* α*, one define an homomorphism, the Busemann cocyle, by the formula :*

$$\omega_\alpha : \Gamma \to \mathbb{Z}$$

$$\omega_\alpha(g) = b_r \circ g - b_r$$

Definition 3 *(Exceptional classes) The action of* Γ *is called parabolic if it fixes some point at infinity. It is called exceptional if fixes a unique point at infinity, and if the associated Busemann cocycle is not trivial. A class* $\omega \in H^1(\Gamma, \mathbb{Z})$ *is exceptional if it can be realized as the Busemann cocycle of an exceptional action of* Γ *in some tree. The set of exceptional classes is denoted* $\mathcal{E}^1(\Gamma, \mathbb{Z})$.

Remark 1 A *topological* definition of a exceptional class can also be given, in the case where Γ is finitely presented. Let $\Gamma = \pi_1(X)$,where X is a compact manifold, and let ω be some class in $H^1(\Gamma, \mathbb{Z})$. One represents ω by a closed 1-form w on X and consider a primitive $F : \tilde{X} \to \mathbb{R}$ of the lift of w to the universal cover of X. Then ω is exceptional iff $F \geqslant 0$ has several components on which F is unbounded (see [Bi] , [Le] , [Bro]).

Remark 2 The notion of an exceptional class, defined by Bieri Neumann Strebel and studied by several authors, in particular [Bro] , [Le] ,is more general : it concerns homomorphism with value in $\mathbb{R}$ and can be defined along the same lines, using $\mathbb{R}$-trees instead of combinatorial trees. Our point of view is that of Brown ; it is interesting to remark that [Bro] , [BNS]and [GL] are published in the same issue of the same journal, but apparently nobody remarked that [Bro] and [GL] studied the same object from a different point of view. This remark justify the choice of our title.

2.3 Discrete valuations and Bruhat-Tits trees.

In this paragraph we fix a field K. Let $v : K^* \to \mathbb{Z}$ be a discrete non archimedian valuation on K. Bruhat and Tits [BT]constructed a tree T_v with an action of $\mathrm{PGL}(2, K)$. One should think of the action of $\mathrm{PGL}(2, K)$ of T_v as an analogue of the action of $\mathrm{PGL}(2,\mathbb{C})$ on the hyperbolic space of dimension 3 ; we recall below some basic facts about this action (see [Se] for a detailed study).

Let $O_v \subset K$ denote the valuation ring $v \geqslant 0$. The *vertices* of T_v are the homothety classes of O_v-lattices, i.e. free O_v-modules of rank 2, in K^2 . The *boundary* of this tree is the projective line $P^1(\bar{K}_v)$ over the v-completion of K.

By the general theory of lattices, if Λ, Λ' are two lattices, one can find a O_v-base of Λ such that, in this base, Λ' is generated by $(t^a, 0)$ and $(0, t^b)$ for some t with $v(t) = 1$; hence up to homothety by $(1,0)$ and $(0, t^n)$, for $n = b - a$. Then the distance between Λ, and Λ' is $|n|$, and the segment between Λ and Λ' is the set of lattices generated by $(1,0)$ and$(0,t^k), k = 1, n$. More generally if l, l' are two different lines in K^2, considered as points in ∂T_v, the geodesic from l to l' is the set of product of lattices in l and l'.

The matrix $g_u = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ fixes the lattice Λ_n generated by $(1,0)$ and $(0, t^n)$ for $n \leqslant v(u)$. The matrix $g_u = \begin{pmatrix} t^n & u \\ 0 & 1 \end{pmatrix}$ transforms Λ_m to Λ_{m+n} if $m + n \leqslant v(u)$.

Acting on T_v the Borel sub-group $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ is *parabolic* : it fixes an end of T_v (namely the line generated by the first basis vector), but neither a point of T_v nor a pair of points of ∂T_v.

The Busemann cocyle of this parabolic subgroup is $b\left(\begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix}\right) = v(\alpha)$.

The relation between the Green-Lazarsfeld set and the Bieri-Neumann-Strebel invariant is now simple to explain.

Proposition 1 *Let $\chi \in H^1(\Gamma, K^*)$. Suppose that $\chi \in E^1(\Gamma, K^*)$ and let $\theta \in H^1(\Gamma, \chi) \neq 0$. Let $\rho : \Gamma \to \mathrm{Gl}(2, K)$ be defined by $\rho(g) = \begin{pmatrix} \chi(g) & \theta(g) \\ 0 & 1 \end{pmatrix}$. If $v \circ \chi \in H^1(\Gamma, \mathbb{Z})$ is not 0, ρ is an exceptional action on T_v.*

Proof. By construction the action of Γ on T_v fixes a point at infinity. It contains an hyperbolic element as $v \circ \chi \neq 0$, but the action cannot fix a line : the other point in the boundary $P^1(\bar{K}_v)$ would be fixed by the group Γ, and ρ would be conjugate to diagonalizable action. The orbit of any point of Γ is therefore a minimal tree which is not a line. $\square$

3 Metabelian groups

If Γ is a group, let $\Gamma' = [\Gamma, \Gamma]$ its derived group. Recall that a group is *metabelian* if Γ' is abelian, or $\Gamma^2 = (\Gamma')'$ is trivial. If Γ is a f.g. group, Γ/Γ^2 is metabelian.

3.1 The Green-Lazarsfeld set of a metabelian group.

If K is a field, the Green-Lazarsfeld set $E^1(\Gamma, K)$ of the group Γ only depends on its metabelianized Γ/Γ^2 as it only depends of the set of representation of Γ

in the metabelian group $\mathrm{Aff}_1(K) = K^* \ltimes K$.

Let Γ be a metabelian group. We write $1 \to [\Gamma, \Gamma] \to \Gamma \to Q \to 1$, where $Q = \Gamma / [\Gamma, \Gamma]$ is the abelianized group, and $[\Gamma, \Gamma]$ is abelian. As an abelian group, $M = [\Gamma, \Gamma]$ is not necessary f.g, however we can let Q acts on $[\Gamma, \Gamma]$ by conjugation, so that M can be promoted as a $\mathbb{Z}Q$ module. The following fact is basic and well-known.

Lemma 1 *The module M is finitely generated as a $\mathbb{Z}Q$ module.*

If $g_1, \ldots\ldots g_r$ are generators of Γ, the commutators $h_{ij} = [g_i, g_j]$ generate $[\Gamma, \Gamma]$ as a $\mathbb{Z}Q$ module : if $[g, h]$ if $h = ab$ we have $[g, h] = [g, ab] = gag^{-1}a^{-1}agbg^{-1}b^{-1}a^{-1} = [g, a]\, a\, [g, b]\, a^{-1} = [g, a]\, a_* [g, b]$, and the result follows by induction. $\square$

Theorem 1 *Let Γ be a finitely generated group. Given a prime number p (p might be 0), there exists a finite number of fields K_ν of characteristic p and of finite transcendence degree over F_p (if $p = 0$, set $F_p = \mathbb{Q}$) and characters $\xi_\nu : \Gamma \to K_\nu^*$ such that :*

1. *$H^1(\Gamma, \xi_\nu) \neq 0$, i.e. $\xi_\nu \in E^1(\Gamma, K_\nu)$*
2. *If K is a field of characteristic p and $\chi \in E^1(\Gamma, K)$ a Green-Lazarsfeld character, then there exists an index ν s.t. $\ker \chi \supset \ker \xi_\nu$.*

Proof Let F_p be the field with p elements and $F_p[Q]$ the group ring of Q with F_p coefficients. Let $M_p = [\Gamma, \Gamma] \otimes F_p, \mathcal{J} \subset F_p[Q]$ the annihilator of M_p, and $A = F_p[Q]/\mathcal{J}$. As Q is a finitely generated abelian group, isomorphic to $\mathbb{Z}^r \times \Phi$, with Φ finite abelian, A is a noetherian ring. Thus A admits a finite number of *minimal* prime ideals $(\mathfrak{p}_\nu)_{1 \leqslant \nu \leqslant \nu_0}$. Let k_i be the field of fraction of $A/\mathfrak{p}_i$, and ξ_i be the natural character $\Gamma \to Q \to A/\mathfrak{p}_i \to k_i$. Up to re-ordering the list of these ideals, we may assume that for $1 \leqslant i \leqslant \nu_1, H^1(\Gamma, \xi_i) \neq 0$.

The theorem 8 is a consequence of the following :

Lemma 2 *Let $\chi \in E^1(\Gamma, K)$ be an exceptional character, $\chi \neq 1$, and let $\mathfrak{p}$ be a minimal prime ideal contained in $\ker \chi$. Then, the character $\xi_\mathfrak{p}$ belongs to $E^1(\Gamma, k_\mathfrak{p}^*)$, i.e. $H^1(\Gamma, \xi) \neq 0$.*

Let$M_\mathfrak{p} = M \otimes A_\mathfrak{p}$, and $M_0 = M \otimes_A K = M_\mathfrak{p}/\mathfrak{p}M_p$. Note that M_0 is a finitely generated $k_\mathfrak{p}$ vector space, on which Γ acts by homotheties : the action of g is the homothety of ratio $\xi(g)$. Let $\pi : [\Gamma, \Gamma] \to M_0$ the canonical map. We shall prove that $H^1(\Gamma, M) \neq 0$.

For some $g_0 \in \Gamma, \xi(g_0)$ is not 1 (as an element of $k_\mathfrak{p}$) : if not $\Gamma = \ker \xi_p$ so $\chi = 1$.

The map $\Gamma \to M_0$ defined by $c(g) = \pi(g_0 g g_0^{-1} g^{-1})$ satisfies $c(gh) = \pi(g_0 g h g_0^{-1} h^{-1} g^{-1}) = \pi(g_0 g g_0^{-1} g^{-1}) + \pi(g g_0 h g_0^{-1} h^{-1} g^{-1}) = c(g) + \xi(g)\pi(g_0 h g_0^{-1} h^{-1}) = c(g) + \xi(g)c(h)$. Therefore c is a 1-cocycle of Γ with value in M.

Let us prove, by contradiction, that the cohomology class of c is not 0.

For every $m \in M_0$, $c(m) = (\xi(g_0)m - m) = (\xi(g_0) - 1)m$. If $c = 0$, as $\xi(g_0) \neq 1$, then $M_0 = 0$. But if $M_0 = 0, M_{\mathfrak{p}}/\mathfrak{p}M_p = 0$,i.e. $\mathfrak{p}M_{\mathfrak{p}} = M_{\mathfrak{p}}$, and $M_{\mathfrak{p}} = 0$ by the Nakayama lemma ($\mathfrak{p}$ is the unique maximal ideal of $A_{\mathfrak{p}}$), i.e. $M = \mathfrak{p}M$. But $\mathfrak{p} \subset \ker \chi$, so this would implies that $M \otimes_A K = 0$ and $H^1(\Gamma, \chi) = 0$.

If this cocyle is a coboundary we could find some $m \in M_0$ s.t. $c(g) = (1 - \xi(g))m$, but $c(g_0) = 0$, and $\xi(g_0) \neq 1$, so c would be 0.

In order to prove lemma 9, we see that, for every linear map $l = M_0 \to K$, $l \circ c$ is a non trivial $1-$cocycle.

This proves theorem 8.

□

Remark 3 The previous proof is a combination of arguments by [BG] and [Bre]. In their remarkable paper R. Bieri and J. Groves describe the BNS invariant of a metabelian group in terms of the finite set of field k_ν and characters ξ_ν for a finite set of primes p (the primes p for which $[\Gamma, \Gamma]$ has p-torsion). For every such a field and every valuation on it, $v \circ \xi_\nu$ is exceptional. This provide a map from the cone of valuations on $k_\nu$to the BNS set. This set turns out to be the union of the images of these cones. In [Bre] , Breuillard proves along the same lines, that a metabelian not virtually nilpotent group admits a non trivial affine action.

4 Fundamental groups of Kähler manifolds.

4.1 Fibering a Kähler manifold.

For the general study of orbifolds and their fundamental groups, we refer to W. Thurston [Th] chap. 13. Complex 2-orbifolds are 2-orbifolds with singularities modeled on the quotient of the unit disk by the action of $\mathbb{Z}/n\mathbb{Z}$. The usefulness of this notion in our context of (fibering complex manifolds to Riemann surfaces) has been pointed out by C. Simpson [Si 1] .

Definition 4 *Complex 2-orbifold, and holomorphic maps. A complex 2-orbifold Σ is a Riemann surface S marked by a finite set of marked points $\{(q_1, m_1) \dots, (q_n, m_n)\}$, where the $m_i's$ are integers $\geqslant 2$.*

Let X be a complex manifold, $f : X \to \Sigma$ a map. Let $x \in X, q = f(x)$. Let $m \in \mathbb{N}^*$ be the multiplicity of q, so that there exists an holomorphic map $u : D(0, r) \subset \mathbb{C} \to (\Sigma, q)$ which is a ramified cover of order m of a neighborhood of q. Then,f is called holomorphic at x, if there exists a neighborhood U of q and a lift $\tilde{f} : U \to D$, holomorphic at x such that $f = u \circ \tilde{f}$.

Definition 5 *Fundamental group. Let $\Sigma = (S; \{(q_1, m_1) \dots, (q_n, m_n)\})$be a $2-$orbifold. Let $q \in S \setminus \{(q_1, m_1) \dots, (q_n, m_n)\}$. The fundamental group -in the sense of orbifolds- of Σ at the point p is the quotient $\pi_1^{\mathrm{orb}}(\Sigma, p) = \pi_1(S \setminus \{q_1, \dots q_n\}) / \ll$*

$\gamma_i^{m_i} \gg$, *where* γ_i *is the class of homotopy (well defined up to conjugacy) of a small circle turning once around* q_i, *and* $\ll \gamma_i^{m_i} \gg$ *is the normal subgroup generated by all the conjugates of* $\gamma_i^{m_i}$.

Example 1 (This is the main example, see [Th] chap. 13) Let $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ be a uniform (discrete co-compact) lattice. The quotient $S = D/\Gamma$ of the unit disk by the action of Γ is a Riemann surface. If $p \in D$, its stabilizer is a finite hence cyclic subgroup of $\mathrm{PSL}(2, \mathbb{R})$. Modulo the action of Γ there are only a finite set of points $\{q_1, \dots . q_n\}$with non trivial stabilizers of order m_i. The quotient orbifold is $\Sigma = (S; \{(q_1, m_1) \dots, (q_n, m_n)\})$. One proves that $\Gamma = \pi_1^{\mathrm{orb}}(\Sigma)$. An orbifold is called hyperbolic if it is obtained in this way ; an orbifold is hyperbolic if and only if its Euler characteristic $\chi^{\mathrm{orb}}(\Sigma) = \chi(S) - \Sigma_{1 \leqslant i \leqslant n}(1 - \frac{1}{m_i})$ is non positive.

The following definition is useful to understand the structure of Kähler groups (see [ABCKT]).

Definition 6 *A Kähler manifold* X *fibers if there exists a pair* (Σ, F) *where* $\Sigma = (S; \{(q_1, m_1) \dots, (q_n, m_n)\})$ *is a hyperbolic 2-orbifold, and* $F : X \to \Sigma$ *an holomorphic map with connected fibers. Two such maps* $F : X \to \Sigma, F' : X' \to \Sigma'$ *are equivalent if the fibers of* F *and* F' *are the same and images in* Σ *and* Σ' *of singular fibers have same order. In this case there exists an holomorphic isomorphism from* S *to* S' *which maps singular points of* S *to singular points of* S' *preserving the multiplicity.*

Let $\pi : X \to S$ be an holomorphic map from a compact complex surface to a curve. If $q \in S$ is a singular value of π , the analytic set $\pi^{-1}(q)$ can be decomposed in a finite union of irreducible sets, (D_i). Away from a set of complex dimension $n - 2$ in D_i, hence of complex codimension 2 in X, the map p can by written $\pi(z_1, \dots . z_n) = z_1^{d_i}$, where d_i is the multiplicity of D_i. The multiplicity of the fiber $\pi^{-1}(q)$ is by definition $m = \mathrm{pgcd}(d_i)$. Let Σ be the orbifold whose underlying space is S, singular points are singular values of π with corresponding multiplicity.

Lemma 3 $\pi : X \to \Sigma$ *is holomorphic.*

By construction, locally in the neighborhood of a point of $\pi^{-1}(q), \pi(x) = f_1^{d_1} \dots . f_k^{d_k} + \mathrm{cte}$, with $m | \,\mathrm{pgcd}\, d_i$ □

The following finiteness theorem is well-known in the smooth case, and implicit in the litterature at several places ; we give below a short proof based on the hyperbolic geometry of hyperbolic orbifolds.

Theorem 2 *Let* X *be a compact complex manifold. There exists, up to equivalence, a finite set of pair* (Σ_i, F_i) *where* Σ_i*is a complex hyperbolic* 2*-orbifold,* $F_i : X \to \Sigma_i$ *is holomorphic with connected fibers* □.

Let us give a proof of this (well known) fact based on the Kobayashi-hyperbolicity of a hyperbolic 2 orbifold :there exist no holomorphic map from $\mathbb{C}$ to an hyperbolic 2-orbifold as there exists no holomorphic map from $\mathbb{C}$ to the unit disk. Thus, by the Bloch principle, as X is compact there exists a uniform bound on the differential of an holomorphic map $F : X \to \Sigma$. Therefore the set of pairs (F, Σ) is *compact* (two such orbifold are ε-close if they are close for the Gromov-Hausdorff topology, i.e. there exists a map between them which is isometric up to an error of ε). But this compact space has only isolated points : if $F_1 : X \to \Sigma_1$ is given, and, and the (Gromov-Hausdorff) distance of F to F_1 is smaller than the diameter of Σ_1 (for instance $\leqslant 1/2\,\mathrm{diam}(X)$ where X is endowed the Kobayashi pseudo-metric) all the fibers of F_1 are send by F inside a disk (or an annulus in the case of the Margulis constant)therefore to a constant by the maximum principle ; in other words F factorizes through F_1 and induces an isomorphism between Σ and Σ_1.□

Remark 4 This proof shows that the number of pairs (F, Σ) for a given complex manifold X can be bounded by the Kobayashi diameter of X.

The following is well known (see [Si 1] [CKO]) .

Theorem 3 *Let $F : X \to S$ by an holomorphic map with connected fibers from the complex manifold X to a complex curve S. Let Σ be the orbifold whose singular points are the singular values of p and multiplicity the multiplicity of the corresponding fiber. Let $Y = F^{-1}(b)$ be the fiber of a non singular point of S. Let $\pi_1'(Y)$the image in $\pi_1(X)$ of $\pi_1(Y)$. One has the exact sequence*

$$1 \to \pi_1'(Y) \to \pi_1(X) \to \pi_1^{\mathrm{orb}}(\Sigma) \to 1$$

in particular the kernel of $\pi_* : \pi_1(X) \to \pi_1^{\mathrm{orb}}(B)$ is finitely generated.□

4.2 Valuations.

The next result is a reformulation of a fibration theorem of Gromov-Shoen [GS]and Simpson [Si 3]in terms of the exceptional set in the sense of Bieri Neumann Strebel ; see also [De] for a more general study of the BNS invariant of a Kähler group, where $\omega \in H^1(\Gamma, \mathbb{R})$ rather than $H^1(\Gamma, \mathbb{Z})$).

Theorem 4 *Let $\omega \in H^1(\Gamma, \mathbb{Z})$. Then ω is exceptional iff there exist a hyperbolic orbifold Σ, an holomorphic map $F : X \to \Sigma$ such that $\omega \in F^* H^1(\Sigma, \mathbb{Z})$.*

Let η be a closed holomorphic $(1,0)$ form whose real part is the harmonic representative of ω. Let $\tilde{X}$ the universal cover of X, and $F : \tilde{X} \to \mathbb{R}$ a primitive of $\mathrm{Re}\,\eta$. From the definition (Remark 5) of $\mathcal{E}^1$ we know that $F \geqslant 0$ is not connected ; [Si 3]applies. One can also apply the proof of corollary 9.2 of [GS] to the foliation defined by the complex valued closed $(1,0)$ from whose real part is the harmonic representative of ω .

To prove the converse (which will not be used), one remarks that for every $w \in H^1(\Sigma, \mathbb{Z})$,its pull back to $H^1(\Sigma, \mathbb{Z})$ is exceptional, as $\pi_1^{\mathrm{orb}}(\Sigma)$ is hyperbolic, and the kernel of $\pi_1^{\mathrm{orb}}(\Sigma) \to \mathbb{Z}$ cannot be finitely generated. □

4.3 The Green-Lazarsfeld set of a Kähler group.

Let K be a field. Recall that a character $\chi : \Gamma \to K^*$ is called integral in the sense of Bass [Ba] if $\chi(\Gamma) \subset O$, the ring of algebraic integers of K.

Proposition 2 *Let X be a Kähler manifold, $\chi \in E^1(\Gamma, K^*)$ be a character. If χ is not integral, X fibers over a 2-orbifold Σ such that $\chi \in F^* E^1(\pi_1^{\mathrm{orb}}(\Sigma), K^*)$.*

Proof. Let v be some valuation such that $\omega = v \circ \chi \neq 0$. Let Γ acts on T_v. By prop.6 this action is exceptional. Applying Thm. 18 we get a pair F, Σ such that $\omega \in F^* H^1(\Sigma, \mathbb{Z})$. From the exact sequence of Theorem 17, we see that $\pi_1'(Y)$ is a finitely generated normal subgroup of Γ made up with elliptic elements. As $\pi_1'(Y)$ is *finitely generated*, the subtree of T_v made up with fixed points of $\pi_1'(F)$ is not empty. As $\pi_1'(Y)$ is normal, it is invariant by the action of Γ. Therefore the boundary of this tree contains at least 3 distinct elements. Thus acting on $P^1(K)$ $\pi_1'(Y)$fixes three different points and is the identity : $\pi_1'(Y) \subset \ker \rho$, and ρ descends to some character on $\pi_1^{\mathrm{orb}}(\Sigma)$. □

The following proposition is a reformulation of a result by Beauville [Be] (Cor 3.6), it will be used to study the cohomology class of $v \circ \chi$, for the archimedian valuation $v(z) = \ln |z|$ an $\chi : \Gamma \to \mathbb{C}^*$ a character.

Proposition 3 *Let X be a Kähler manifold, $\chi \in E^1(\Gamma, \mathbb{C}^*)$ be character. If $|\chi| \neq 1$, there exist an holomorphic map $F : X \to \Sigma$ from X to a 2-orbifold Σ such that $\chi \in F^* E^1(\pi_1^{\mathrm{orb}}(\Sigma), K^*)$.*

Combining propositions 19 and 20, we get the description of the GL set of a Kähler manifold in terms of its fibering over hyperbolic 2-orbifolds. It generalizes results by M. Green R. Lazarsfeld [GL], A. Beauville [Be], C. Simpson [Si 2], F. Campana [Ca], R. Pink D. Roessler [PR], who studied the case where the field K is the field of complex numbers.

Theorem 5 *Let Γ be the fundamental group of a Kähler manifold X, $(F_i, \Sigma_i)_{1 \leqslant i \leqslant n}$ the family of fibration of X over hyperbolic 2-orbifolds. Let K be a field of characteristic p (if $p = 0$, $K = \mathbb{C}$), $\bar{F}_p \subset K$ the algebraic closure of F_p in K. Then $E^1(\Gamma, K)$ is made with a finite set of torsion characters (contained in $E^1(\Gamma, \bar{F}_p)$ if $p > 0$) and the union of $F_i^* \operatorname{Hom}(\pi_1^{\mathrm{orb}}(\Sigma_i), K^*)$.*

Proof We shall prove that a character χ which is not in the union$\bigcup F_i^* \operatorname{Hom}(\pi_1^{\mathrm{orb}}(\Sigma_i), K^*)$ must be a torsion character of bounded order. Let us fix such a character χ.

From theorem 8, we know that there exists a finite number of fields K_ν and characters ξ_ν such that $H^1(\Gamma, \xi_\nu) \neq 0$, and for every $\chi \in E^1(\Gamma, K)$ there exists an index ν for which $\ker \xi_\nu \subset \ker \chi$. If ξ_ν is not integral, there exists a 2-orbifold Σ and a holomorphic map $F : X \to \Sigma$ such that $\ker F_* \supset \ker \xi_v$: therefore $\ker F_* \supset \ker \chi$ and $\chi \in F^* E^1(\pi_1^{\mathrm{orb}}(\Sigma))$.

Thus, as $\chi \notin \bigcup F_i^* \operatorname{Hom}(\pi_1^{\mathrm{orb}}(\Sigma_i), K^*)$ χ is integral.

Let us first discuss the case of positive characteristic. If ξ_ν is integral, then $\xi_\nu(\Gamma)$ is made with roots of unity of K_ν. But K_ν is of finite transcendence degree over F_p so admits only a finite number of roots of unity of degree d_ν (see [Ba] for instance). Therefore, χ is a torsion character of order d dividing d_ν.

Suppose now that $\operatorname{char} K = 0$, and ξ_ν is integral. Thus K_ν is a number field, and $\xi_\nu(\Gamma)$ is contained in the ring O_ν of integers of ξ_ν. If $|\xi_\nu| \neq 1$, or if one of its conjugates $\sigma(\xi_\nu)$ has $|\sigma(\xi_\nu)| \neq 1$, as $H^1(\Gamma, \xi_\nu) \neq 0$ we know (prop. 20) that there exists a 2-orbifold Σ and a holomorphic map $F : X \to \Sigma$ such that $\ker F_* \supset \ker \xi_v$; the previous argument apply and proves that $\chi \in F^* E^1(\pi_1^{\mathrm{orb}}(\Sigma))$. Therefore, χ must be a root of unity, by a theorem of Kronecker, of bounded degree d, as the degree of the n-th cyclotomic polynomial goes to infinity with n, and as d divides the degree of K_ν. The rest of the argument is unchanged.

□

Thus, the theorem 21 reduces the computation of $E^1(\Gamma, K^*)$ to the case where Γ is the fundamental group of a 2-orbifold.

Proposition 4 *Let $\Gamma = \pi_1^{\mathrm{orb}}(\Sigma)$, for $\Sigma = (S; (q_i, m_i)_{1 \leqslant i \leqslant n})$ a hyperbolic 2-orbifold then,*

$E^1(\pi_1^{\mathrm{orb}}(\Sigma), K^*) = \mathrm{Hom}(\pi_1^{\mathrm{orb}}(\Sigma, K^*))$ unless $g = 1$ and for all i, $m_i \not\equiv 0(\operatorname{char} K)$.

If $g = 1$ and for all i $m_i \not\equiv 0(\operatorname{char} K)$, $E^1(\pi_1^{\mathrm{orb}}(\Sigma), K^*)$ is finite, made of torsion characters.

Let $\chi : \pi^{\mathrm{orb}}(\Sigma) \to K^*$ be a representation. If $\chi = 1, H^1(\pi_1^{\mathrm{orb}}(\Sigma), K^*) = \mathrm{Hom}(\pi_1^{\mathrm{orb}}(\Sigma), K^*) \neq 0$. If $g > 1$,consider a simple closed curves on S such that c are homologous to 0, which separated S in two compact surface of positive genus S_1, S_2, with common boundary c and such that all singular points are in S_2; if $g = 1$ consider a curve c, which bounds a disk $\bar{D}$ on S containing all singular points q_i, and let $S_1 = S \setminus \mathrm{int}(D)$ be the other component. One consider a representation $\chi : \pi_1^{\mathrm{orb}}(\Sigma) \to K^*$, and note that $\chi(c) = 1$ as c is homologous to 0. We think of χ as a local system on Σ and we will use a Mayer Vietoris exact sequence.

First note that if $\chi|_{\pi(S_1)}$ and $\chi|_{\pi^{\mathrm{orb}}(\Sigma_2)}$ are not 1, then $H^1(\pi_1^{\mathrm{orb}}(\Sigma), K^*) \neq 0$: let $x_0 \in K$, there exists a unique 1-cocyle c such that $c(g) = x_0(1 - \chi(g))$ is $g \in \pi_1(S_1)$, $c(g) = 0$ if $g \in S_2$.

If $\chi|_{S_1} = 1$, as $H^1(S_{1,}, \partial S_1, K) = K^2$,one can find a 1-cocyle c whose restriction on S_2 or D is 0, and restriction on S^1 is not trivial.

We are left to the case $\chi|_{S_2}$ or $\chi|_D = 1$. If $g(S_2) > 0 H^1(\pi_1^{\mathrm{orb}}(\Sigma_2), C, K) \twoheadrightarrow K^{2g}$ and the previous argument apply.

The remaining case is $g = 1$, $\chi|_{\pi_1^{\mathrm{orb}}(D)} = 1$, $\chi|_{\pi_1(S_1)} \neq 1$. Note that in this case χ is a torsion character. Furthermore, $H^1(\pi_1^{\mathrm{orb}}(\Sigma_2), K) = \{(z_1, \ldots, z_n) \in K / m_i z_i = 0\}$. This space is 0 unless $m_i \equiv 0$ (char K) for some i. On the other hand, if $\rho|_{\pi_1(S_1)} \neq 0$ the homomorphism $H^1(\pi_1(\Sigma_1), \rho) \to K$ which sends θ to $\theta(c)$ is an isomorphism. Using the exact sequence of Mayer Vietoris, we see that $H^1(\pi_1^{\mathrm{orb}}(\Sigma), \chi) \neq 0$ if $g > 1$ or $g = 1$ and for some i, m_i divides the characteristic of K. □

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